# CHAPTER - II

# CHAPTER II
# FUZZY TOPSIS METHOD

Decision making problem is the process of finding the best option from all of the feasible alternatives. In almost all such problems the multiplicity of criteria for judging the alternatives is pervasive. That is, for many such problems, the decision maker wants to solve a multiple criteria decision making (MCDM) problem.

In most of real world situations, usually decision makers are confronted with multiple criteria to be considered before any decision can be made. This is the case of Multi Criteria Decision Making (MCDM); a case with the aim to find the overall preferences among the available alternatives. One of the most popular methods in MCDM is the Technique for Order Preference by Similarity or TOPSIS.

TOPSIS was proposed by Hwang and Yoon in 1981 [29]. In this method, the main concept is that the most preferred alternative should have the shortest distance from the Positive Ideal Solution (PIS) and the longest distance from the Negative Ideal Solution (NIS).

Many real-life decision problems are confronted with unquantifiable, incomplete and non-obtainable information that make precise judgment impossible. This is when fuzzy TOPSIS comes into play where the criteria weights and alternative ratings are given by linguistic variables, expressed by fuzzy numbers.

## SECTION 2.1
## FUZZY TOPSIS METHOD USING TRIANGULAR FUZZY NUMBERS

Let MCDM (Multi Criteria Decision Making) problem has n alternatives $A_1$, $A_2$,..., $A_n$, and m criteria, $C_1$, $C_2$,...,$C_m$. Each alternative will take a consideration with respect to criterion m. The ratings of criteria can be concisely expressed in matrix format as $D = (\tilde{x}_{ij})_{nxm}$ and $W = (\tilde{w}_1,...,\tilde{w}_m)$ , where $\tilde{x}_{ij}$ (i = 1,...,n; j = 1,...,m) and

$\tilde{w}_j$ (j = 1,...,m) are the fuzzy rating of alternative $A_i$ (i = 1,...,n) with respect to criterion $C_j$ (j = 1,...,m) and the weight of criterion $C_j$ (j = 1,...,m), respectively.

The method is calculated using the following steps:

(a) Decision matrix $D = (\tilde{x}_{ij})_{nxm}$, is normalized via Eq. (1):

$$\tilde{r}_{ij} = \frac{\tilde{x}_{ij}}{\sqrt{\sum_{k=1}^{n} \tilde{x}^2{}_{kj}}}, i = 1,\dots,n; j = 1,\dots,m \tag{1}$$

Weighted normalized decision matrix is formed:

$$\begin{aligned} &V = (\tilde{v}_{ij})_{n\times m} \\ &\tilde{v}_{ij} = \tilde{w}_j \tilde{r}_{ij}, i = 1,\dots,n; j = 1,\dots,m \end{aligned} \tag{2}$$

(b) Fuzzy Positive Ideal Solution (FPIS) and Fuzzy Negative Ideal Solution (FNIS) are determined:

$$\begin{aligned} A^+ &= \{\tilde{v}_1^+,\dots,\tilde{v}_m^+\} \\ &= \{(\max_j \tilde{v}_{ij} | j \in \varphi_b), (\min_j \tilde{v}_{ij} | j \in \varphi_c)\} \end{aligned} \tag{3}$$

$$\begin{aligned} A^- &= \{\tilde{v}_1^-,\dots,\tilde{v}_m^-\} \\ &= \{(\min_j \tilde{v}_{ij} | j \in \varphi_b), (\max_j \tilde{v}_{ij} | j \in \varphi_c)\} \end{aligned} \tag{4}$$

where $\varphi_b$ denotes benefit criteria and $\varphi_c$ the cost criteria

(c) The distance of each alternative from Fuzzy Positive Ideal Solution and Fuzzy Negative Ideal Solution are calculated using Euclidean distance formula:

$$d_i^+ = \sqrt{\sum_{j=1}^{m} (\tilde{v}_{ij} - \tilde{v}_j^+)^2}, i = 1,\dots,n \tag{5}$$

$$d_i^- = \sqrt{\sum_{j=1}^{m} (\tilde{v}_{ij} - \tilde{v}_j^-)^2}, i = 1,\dots,n \tag{6}$$

(d) The closeness coefficient of each alternative is calculated as

$$CC_i = \frac{d_i^-}{d_i^+ + d_i^-}, i = 1, \ldots, m \tag{7}$$

(e) By comparing $CC_i$ values, the ranking of alternatives are determined.

**Steps Outlining The Algorithm Of A Group Multi-Criteria Decision Making Using Triangular Fuzzy Numbers**

Step 1: Identify the evaluation criteria (Usually done by a committee of decision-makers)

Step 2: Choose appropriate linguistic variables (based on the importance weight of the criteria) and the linguistic ratings for alternatives with respect to the criteria.

Step 3: Aggregate the weight of the criteria to get the aggregated fuzzy weight $\tilde{w}_j$ of criterion $C_j$ and pool the decision makers' opinions to get the aggregated fuzzy rating $\tilde{x}_{ij}$ of alternative $A_i$ under criterion $C_j$.

Step 4: Construct the fuzzy decision matrix and the normalized fuzzy decision matrix

Step 5: Construct the weighted normalized fuzzy decision matrix

Step 6: Determine the Fuzzy Positive Ideal Solution (FPIS) and Fuzzy Negative Ideal Solution (FNIS)

Step 7: Calculate the distance of each alternative from Fuzzy Positive Ideal Solution (FPIS) and Fuzzy Negative Ideal Solution (FNIS), respectively.

Step 8: Calculate the closeness coefficient of each alternative.

Step 9: Determine the ranking order of all alternatives according to the closeness coefficients.

## SECTION 2.2

## FUZZY TOPSIS METHOD USING TRIANGULAR FUZZY NUMBERS WITH GRADED MEAN INTEGRATION REPRESENTATION

**Definition: 2.1 [12]**

Let $\widetilde{a} = (a, b, c)$ be a triangular fuzzy number. The ***graded mean integration representation*** of $\widetilde{a}$ is defined as $P(\widetilde{a}) = \frac{1}{6}(a + 4b + c)$

**Definition: 2.2 [12]**

Let $\widetilde{a} = (a_1, b_1, c_1)$ and $\widetilde{b} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers. Then the canonical representation of addition and multiplication operations on triangular fuzzy numbers can be defined as follows.

Addition operation $\oplus$:

$$P(\widetilde{a} \oplus \widetilde{b}) = P(\widetilde{a}) + P(\widetilde{b})$$

$$= \frac{1}{6}(a_1 + 4b_1 + c_1 + a_2 + 4b_2 + c_2) \qquad (8)$$

Multiplication operation $\otimes$:

$$P(\widetilde{a} \otimes \widetilde{b}) = P(\widetilde{a}) \text{ x } P(\widetilde{b})$$

$$= \frac{1}{6}(a_1 + 4b_1 + c_1) \text{x} \frac{1}{6}(a_2 + 4b_2 + c_2) \qquad (9)$$

Assume that a group of k users ($D_1$, $D_2$,…., $D_k$) is formed for ranking m alternatives ($A_1$, $A_2$,…, $A_m$) with respect to n criteria ($C_1$, $C_2$,…, $C_n$). Then the decision matrix, $R_t$, given by decision maker, $d_t$, t=1,2,…k, is

$$R_t = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} \tilde{x}_{11t} & \tilde{x}_{12t} & \cdots & \tilde{x}_{1nt} \\ \tilde{x}_{21t} & \tilde{x}_{22t} & \cdots & \tilde{x}_{2nt} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1t} & \tilde{x}_{m2t} & \cdots & \tilde{x}_{mnt} \end{bmatrix} \end{array} \tag{10}$$

$\tilde{r}_{ijt}=(o_{ijt}, p_{ijt}, q_{ijt})$, $\tilde{r}_{ijt} \in R_t$ , i=1,2,…m, j=1,2,…n, t=1,2,…k, denotes the rating of alternative $A_i$ with respect to criterion $C_j$ given by the user $d_t$.

The procedure of the Fuzzy TOPSIS method with graded mean integration representation is stated as follows:

Step 1: Aggregate the importance weights.

Let $\tilde{w}_{jt}=(a_{jt}, b_{jt}, c_{jt})$, j=1,2,…n, t=1,2,…k, be the importance weight of criterion $C_j$ given by the user $d_t$. Then the aggregated crisp weight $W_j$ of criterion $C_j$ is calculated by

$$W_j = \frac{\sum_{t=1}^{k} w'_{jt}}{k} \tag{11}$$

where $w'_{jt}$ is the weight derived from the graded mean integration representation of fuzzy numbers, as illustrated in definition 2.1.

Step 2: Aggregate rating of alternatives.

The following formula is used to obtain the aggregated crisp rating of alternatives $R_{ij}$.

$$R_{ij} = \frac{\sum_{t=1}^{k} r'_{ijt}}{k} \tag{12}$$

where $r'_{ijt}$ is obtained by the graded mean integration representation of fuzzy numbers, as illustrated in definition 2.1.

Step 3: Construct the normalized and weighted decision matrix.

Let $S=[s_{ij}]_{m\times n}$ be the normalized decision matrix. The normalized value

$$s_{ij} = \frac{R_{ij}}{\sqrt{\sum_{i=1}^{m}(R_{ij})^2}} \tag{13}$$

Let $V=[v_{ij}]_{m\times n}$ be the weighted decision matrix. The weighted value $v_{ij}$ is derived from the product of elements in the normalized decision matrix and crisp weights.

$$v_{ij} = W_j\, s_{ij} \tag{14}$$

Step 4: Determine the Fuzzy Positive Ideal Solution (FPIS) and the Fuzzy Negative Ideal Solution (FNIS).

Let I and J be the index sets associated with the alternative set and the criterion set, respectively. The Fuzzy Positive Ideal Solution (FPIS) $A^+$ and the Fuzzy Negative Ideal Solution (FNIS) $A^-$ is

$$A^+ = \{v_1^+, v_2^+ ..., v_n^+\} = \{\max_{i\in I} v_{ij} \mid j \in J\} \tag{15}$$

$$A^- = \{v_1^-, v_2^- ..., v_n^-\} = \{\min_{i\in I} v_{ij} \mid j \in J\} \tag{16}$$

Step 5: Measure the distance of each alternative from the Fuzzy Positive Ideal Solution (FPIS) and the Fuzzy Negative Ideal Solution (FNIS) respectively.

Traditionally, the Euclidean distance was used to measure the distance of each alternative from $A^+$ and $A^-$ as follows.

$$d_i^+ = \sqrt{\sum_{j=1}^{n}(v_{ij} - v_j^+)^2}, i = 1,2,...,m \tag{17}$$

$$d_i^- = \sqrt{\sum_{j=1}^{n}(v_{ij} - v_j^-)^2}, i = 1,2,...,m \tag{18}$$

Based on the weighted Euclidean distance, $A^+$ and $A^-$ is redefined as

$$A^+ = \{s_1^+, s_2^+ ..., s_n^+\} = \{\max_{i \in I} s_{ij} | j \in J\} \tag{19}$$

$$A^- = \{s_1^-, s_2^- ..., s_n^-\} = \{\min_{i \in I} s_{ij} | j \in J\} \tag{20}$$

and then the distance of each alternative from $A^+$ and $A^-$ based on the weighted Euclidean distance is computed as

$$d_i^+ = \sqrt{\sum_{j=1}^{n} W_j \left|s_{ij} - s_j^+\right|^2} \; i = 1,2,...,m \tag{21}$$

$$d_i^- = \sqrt{\sum_{j=1}^{n} W_j \left|s_{ij} - s_j^-\right|^2} \; i = 1,2,...,m \tag{22}$$

However, the use of the Euclidean distance had the problem associated with weight having been calculated twice. From Eq. (21), it is observed that the decision results are overly controlled by weighting.

This problem was resolved by introducing Eq. (23) or Eq. (24) as follows

$$d_i^+ = \sqrt{\sum_{j=1}^{n}(v_{ij}-v_j^+)^2} = \sqrt{\sum_{j=1}^{n}(W_j s_{ij}-W_j s_j^+)^2} = \sqrt{\sum_{j=1}^{n} W_j^2 (s_{ij}-s_j^+)^2} \tag{23}$$

Minkowski distance,

$$L_p^w(x,y) = \left[\sum_{j=1}^{n} w_j \left|x_j - y_j\right|^p\right]^{1/p} \tag{24}$$

where $w_j$ is the weight of importance with respect to the $j^{th}$ criterion and $p \geq 1$. Note that $L_p^w$ with p=2 is known as the weighted Euclidean distance.

Step 6: Calculate the relative closeness coefficient and rank the preference order. The relative closeness coefficient of the $i^{th}$ alternative, $RCC_i$, can be computed by

$$RCC_i = \frac{d_i^-}{d_i^+ + d_i^-} \tag{25}$$

Consequently, the alternatives can be ranked according to $RCC_i$.

## SECTION 2.3

## FUZZY TOPSIS METHOD USING TRAPEZOIDAL FUZZY NUMBERS

Step 1: Establish a decision matrix for ranking.

A Multi-Criteria Decision Making problem can be concisely expressed in matrix format

$$C_1 \quad C_2 \quad \cdots \quad C_n$$

$$R_t = \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn} \end{bmatrix} \tag{26}$$

where $A_1, A_2, \ldots, A_m$ are possible alternatives among which decision makers have to choose, $C_1, C_2, \ldots, C_n$ are criteria with which alternative performance are measured, $\tilde{x}_{ij} = (x_{ij}^a, x_{ij}^b, x_{ij}^c, x_{ij}^d)$ is the fuzzy rating of alternative, $A_i$ with respect to criterion $C_j$.

Step 2: Calculate the normalized decision matrix.

The normalized value $\tilde{n}_{ij} = (n_{ij}^a, n_{ij}^b, n_{ij}^c, n_{ij}^d)$ is

$$\tilde{n}_{ij} = \frac{\tilde{x}_{ij}}{\sqrt{\sum_{i=1}^{m} (S(\tilde{x}_{ij}, 0))^2}}, \text{j=1,2,...,n} \tag{27}$$

where

$$S(\tilde{x}_{ij}, 0) = \frac{x_{ij}^a + x_{ij}^b + x_{ij}^c + x_{ij}^d}{4} \tag{28}$$

Step 3: Calculate the weighted normalized decision matrix.

The weighted normalized value $\tilde{v}_{ij} = (v_{ij}^a, v_{ij}^b, v_{ij}^c, v_{ij}^d)$ is calculated as:

If w is a crisp value:

$$\tilde{v}_{ij} = w_j \times \tilde{n}_{ij} \quad \text{i=1,2,...,m} \quad \text{j=1,2,...,n} \tag{29}$$

where $w_j$ is the weight if the $i^{th}$ criterion, and $\sum_{j=1}^{n} w_j = 1$

If w is a fuzzy value:

$$\begin{aligned}\tilde{v}_{ij} &= (S(\tilde{w}_j,0)) \times \tilde{n}_{ij} \\ &= ((S(\tilde{w}_j,0)) \times n^{a}_{ij}, ((S(\tilde{w}_j,0)) \times n^{b}_{ij}, ((S(\tilde{w}_j,0)) \times n^{c}_{ij}, ((S(\tilde{w}_j,0)) \times n^{d}_{ij}\end{aligned}$$

$$i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n. \qquad (30)$$

where $w_j$ is the weight if the $i^{th}$ criterion, and $\sum_{j=1}^{n} S(\tilde{w}_j,0) = 1; S(\tilde{w}_j,0) \geq 0,\ j=1,2,\ldots,n$

Step 4: Determine the Positive Ideal Solutions and Negative Ideal Solutions

$$A^{+} = \{\tilde{v}_1^{+}, \tilde{v}_2^{+}, \ldots, \tilde{v}_n^{+}\} = \{(\tilde{v}_{uj} | j \in J), (\tilde{v}_{dj} | j \in J')\} \qquad (32)$$

$$A^{-} = \{\tilde{v}_1^{-}, \tilde{v}_2^{-}, \ldots, \tilde{v}_n^{-}\} = \{(\tilde{v}_{dj} | j \in J), (\tilde{v}_{uj} | j \in J')\} \qquad (33)$$

where

$J$ is associated with the positive criteria.

$J'$ is associated with the negative criteria.

u is the value between 1 to m that $\tilde{v}_{uj}$ has the most distance from the crisp number 0, it means $S(\tilde{v}_{uj},0) \geq S(\tilde{v}_{ij},0) \quad, i=1,2,\ldots,m$ (34)

d is the value between 1 to m that $\tilde{v}_{dj}$ has the most distance from the crisp number 0,it means $S(\tilde{v}_{dj},0) \leq S(\tilde{v}_{ij},0) \quad, i=1,2,\ldots,m$ (35)

Step 5: Calculate the separation measures using the n-dimensional Euclidean distance.

The separation of each alternative from the positive-ideal solution is given as

$$d_i^{+} = \sqrt{\sum_{j=1}^{n} (S(\tilde{v}_j^{+}, \tilde{v}_{ij}))^2} \quad, i=1, 2, \ldots, m. \qquad (36)$$

Similarly, the separation from the negative-ideal solution is given as:

$$d_i^- = \sqrt{\sum_{j=1}^{n}(S(\tilde{v}_j^-, \tilde{v}_{ij}))^2} \quad , i=1,2,...,m \tag{37}$$

Step 6: Calculate the relative closeness to the ideal solution.

The relative closeness of the alternative $A_i$ with respect to $A^+$ is defined as

$$cl_i^* = \frac{d_i^+}{d_i^+ + d_i^-}, \ i=1,2,...,m \tag{38}$$

Then ranking of alternative is done.

## SECTION 2.4

## FUZZY TOPSIS METHOD BASED ON ALPHA-LEVEL SETS

Let $X = (\tilde{x}_{ij})_{n \times m}$ be a fuzzy decision matrix characterized by membership functions $\mu_{\tilde{x}_{ij}}(x)$ (i = 1, 2, ..., n;j = 1, 2, ..., m) and $W = (\tilde{w}_1, ..., \tilde{w}_m)$ be fuzzy weights characterized by $\mu_{\tilde{x}_{ij}}(x)$ (j = 1, 2, ...,m).

If all the criteria $C_1$,..., $C_m$, are assessed using the same set of fuzzy linguistic variables, then the fuzzy decision matrix X is of the same dimension and therefore needs no normalization. Otherwise, X has to be normalized.

Let $X = (\tilde{x}_{ij})_{n \times m}$ where $\tilde{x}_{ij} = (a_{ij}, b_{ij}, d_{ij})$ is a triangular fuzzy number, then normalization process is conducted by

$$\tilde{r}_{ij} = (\frac{a_{ij}}{d_j^+}, \frac{b_{ij}}{d_j^+}, \frac{d_{ij}}{d_j^+}), i = 1,2,...,n; j \in \Omega_b \tag{39}$$

$$\tilde{r}_{ij} = \left(\frac{a_j^-}{d_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right), i = 1,2,\ldots,n; j \in \Omega_c \tag{40}$$

where $\Omega_b$ and $\Omega_c$ denotes set of benefit criteria and cost criteria respectively, and

$$d_j^+ = \max_i d_{ij}, \quad j \in \Omega_b \tag{41}$$

$$a_j^- = \min_i a_{ij}, \quad j \in \Omega_c \tag{42}$$

The normalized criteria values/ratings $\tilde{r}_{ij}$ are between zero and one. Then the ideal solution is defined as $A^+=\{1,.,1\}$ and the negative ideal solution is defined as $A^-=\{0,.,0\}$.

If there is no need to normalize the fuzzy decision matrix $X = (\tilde{x}_{ij})_{n\times m}$ then the ideal and the negative ideal solutions is respectively defined as

$$A^+ = \{x_1^+,\ldots,x_m^+\} = \{(\max_j d_{ij} | j \in \Omega_b),(\min_j a_{ij} | j \in \Omega_c)\} \tag{43}$$

$$A^- = \{x_1^-,\ldots,x_m^-\} = \{(\min_j a_{ij} | j \in \Omega_b),(\max_j d_{ij} | j \in \Omega_c)\} \tag{44}$$

Let $(r_{ij})_\alpha = [(r_{ij})_\alpha^L,(r_{ij})_\alpha^U]$ and $(w_j)_\alpha = [(w_j)_\alpha^L,(w_j)_\alpha^U]$ be the alpha- level sets of $\tilde{r}_{ij}$ and $\tilde{w}_j$, respectively. Then the closeness coefficient is calculated by

$$RC_i = \frac{\sqrt{\sum_{j=1}^{m}(w_j r_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(w_j r_{ij})^2} + \sqrt{\sum_{j=1}^{m}(w_j r_{ij} - 1)^2}}, \text{ i= 1, 2, ...,n.} \tag{45}$$

where

$(w_j)_\alpha^L \le w_j \le (w_j)_\alpha^U$, j=1, ...,m.

$(r_{ij})_\alpha^L \le r_{ij} \le (r_{ij})_\alpha^U$, j=1, ...,m.

RC$_i$ is an interval whose lower and upper bounds is captured by the following pair of fractional programming models:

$$(RC_i)_\alpha^L = \mathrm{Min}\frac{\sqrt{\sum_{j=1}^{m}(w_j r_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(w_j r_{ij})^2}+\sqrt{\sum_{j=1}^{m}(w_j r_{ij}-1)^2}} \qquad (46)$$

such that

$(w_j)_\alpha^L \le w_j \le (w_j)_\alpha^U$, j=1, ...,m,

$(r_{ij})_\alpha^L \le r_{ij} \le (r_{ij})_\alpha^U$, j=1, ...,m,

$$(RC_i)_\alpha^U = \mathrm{Max}\frac{\sqrt{\sum_{j=1}^{m}(w_j r_{ij})^2}}{\sqrt{\sum_{j=1}^{m}(w_j r_{ij})^2}+\sqrt{\sum_{j=1}^{m}(w_j r_{ij}-1)^2}} \qquad (47)$$

such that

$(w_j)_\alpha^L \le w_j \le (w_j)_\alpha^U$, j=1, ...,m,

$(r_{ij})_\alpha^L \le r_{ij} \le (r_{ij})_\alpha^U$, j=1, ...,m.

Due to the fact that

$$\frac{\partial RC_i}{\partial r_{ij}}=\frac{r_{ij}\sqrt{\dfrac{\sum\limits_{j=1}^{m}(w_j(r_{ij}-1))^2}{\sum\limits_{j=1}^{m}(w_jr_{ij})^2}}+w_j^2(1-r_{ij})\sqrt{\dfrac{\sum\limits_{j=1}^{m}(w_jr_{ij})^2}{\sum\limits_{j=1}^{m}(w_j(r_{ij}-1))^2}}}{\left(\sqrt{\sum\limits_{j=1}^{m}(w_jr_{ij})^2}+\sqrt{\sum\limits_{j=1}^{m}(w_j(r_{ij}-1))^2}\right)^2}>0,\ j=1,\ldots,\ m, \quad (48)$$

$RC_i$ is therefore a monotonically increasing function of $r_{ij}$ (j=1,.,m), which means $RC_i$ reaches its maximum at $r_{ij}$ = $(r_{ij})_\alpha^U$ and arrives at its minimum when $r_{ij}=(r_{ij})_\alpha^L$.Then the pair of fractional programming models can be simplified as

$$(RC_i)_\alpha^L=\mathrm{Min}\frac{\sqrt{\sum\limits_{j=1}^{m}(w_j(r_{ij})_\alpha^L)^2}}{\sqrt{\sum\limits_{j=1}^{m}(w_j(r_{ij})_\alpha^L)^2+\sqrt{\sum\limits_{j=1}^{m}(w_j(r_{ij})_\alpha^L-1)^2}}} \quad (49)$$

such that $(w_j)_\alpha^L\le w_j\le(w_j)_\alpha^U$, j=1, ...,m.

$$(RC_i)_\alpha^U=\mathrm{Max}\frac{\sqrt{\sum\limits_{j=1}^{m}(w_j(r_{ij})_\alpha^U)^2}}{\sqrt{\sum\limits_{j=1}^{m}(w_j(r_{ij})_\alpha^U)^2+\sqrt{\sum\limits_{j=1}^{m}(w_j(r_{ij})_\alpha^U-1)^2}}} \quad (50)$$

such that $(w_j)_\alpha^L\le w_j\le(w_j)_\alpha^U$, j=1, ...,m.

In the case of no normalization that needs to be carried out, the relative closeness $RC_i$ is determined by the following Non-Linear Programming models:

$$(RC_i)_\alpha^L = \text{Min}\frac{\sqrt{\sum_{j=1}^{m}(w_j((x_{ij})_\alpha^L - x_j^-))^2}}{\sqrt{\sum_{j=1}^{m}(w_j((x_{ij})_\alpha^L - x_j^-))^2} + \sqrt{\sum_{j=1}^{m}(w_j((x_{ij})_\alpha^L - x_j^+))^2}} \qquad (51)$$

such that $(w_j)_\alpha^L \le w_j \le (w_j)_\alpha^U$, j=1, ...,m.

$$(RC_i)_\alpha^U = \text{Max}\frac{\sqrt{\sum_{j=1}^{m}(w_j((x_{ij})_\alpha^U - x_j^-))^2}}{\sqrt{\sum_{j=1}^{m}(w_j((x_{ij})_\alpha^U - x_j^-))^2} + \sqrt{\sum_{j=1}^{m}(w_j((x_{ij})_\alpha^U - x_j^+))^2}} \qquad (52)$$

such that $(w_j)_\alpha^L \le w_j \le (w_j)_\alpha^U$, j=1, ...,m.

where $(x_{ij})_\alpha = [(x_{ij})_\alpha^L, (x_{ij})_\alpha^U]$ are the alpha-level sets of $\tilde{x}_{ij}(i=1,..,n;\ j=1,..,m)$, and $x_j^+$ and $x_j^-$ are the ideal and negative ideal solutions determined by Eqs. (43) and (44), respectively.

By setting different $\alpha$ levels, different alpha-level sets $(RC_i)_\alpha = [(RC_i)_\alpha^L, (RC_i)_\alpha^U]$ is generated by solving the above pair of Non-Linear Programming models (39) and (40).

According to the extension principle [70], $\widetilde{RC}_i$ is finally expressed as $\widetilde{RC}_i = \bigcup_\alpha \alpha \cdot (RC_i)_\alpha = \bigcup_\alpha \alpha[(RC_i)_\alpha^L, (RC_i)_\alpha^U], 0 < \alpha \le 1$

For n alternatives, n fuzzy relative closenesses is determined, which are all expressed by their alpha-level sets and are usually no longer triangular fuzzy numbers.

In order to select a best alternative or rank the n alternatives, these fuzzy relative closenesses need to be defuzzified. The averaging level cuts (ALC) [54] is the simplest defuzzification method based on alpha-level sets. Let $\alpha_1$,., $\alpha_N$ be different alpha levels satisfying $0 < \alpha_1 < .... < \alpha_N = 1$. Then the defuzzified values of $\widetilde{RC}_i$ is determined by

$$(\widetilde{R}C_i)^*_{ALC} = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{(\widetilde{R}C_i)^L_{\alpha_j} + (\widetilde{R}C_i)^U_{\alpha_j}}{2}\right) \quad i = 1, \ldots, n. \tag{53}$$

As a summary, the fuzzy TOPSIS method based on alpha-level sets is summed up as follows:

Step 1: Normalize fuzzy decision matrix $X = (\tilde{x}_{ij})_{n\times m}$ by Eqs. (39) and (40) if necessary.

Step 2: Determine the ideal solution and the negative ideal solution by Eqs. (43) and (44) if necessary.

Step 3: Calculate the alpha-level sets of $\tilde{r}_{ij}$ or $\tilde{x}_{ij}$(i=1,.,n; j=1, .,m) by setting different $\alpha$ levels.

Step 4: Compute the fuzzy relative closeness of each alternative by solving the Non-Linear Programming models (49) and (50) or (51) and (52) for each alpha level.

Step 5: Defuzzify the fuzzy relative closeness by Eq. (53).

Step 6: Rank alternatives in terms of their defuzzified relative closenesses.