International Journal of Mathematical Sciences and Applications
Vol. 1 No. 2 (May, 2011)

www.ijmsa.yolasite.com

SECOND ORDER FUZZY TOPOLOGICAL SPACES - III

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ABSTRACT

In this paper second order fuzzy continuity is defined and it is proved that the associations $\hat{\delta} \rightarrow i_\varepsilon(\hat{\delta})$, $\hat{\delta} \rightarrow i^*(\hat{\delta})$, $\hat{\delta} \rightarrow i(\hat{\delta})$, $\tau \rightarrow (\omega_2(\tau))_\varepsilon$, $\tau \rightarrow \omega_2(\tau)$ and $\tau \rightarrow \omega_2^*(\tau)$ are functorial.

INTRODUCTION

A **fuzzy set** on a set X is a map defined on X with values in I, where I is the closed unit interval [0, 1]. Equivalently fuzzy sets which are named as first order fuzzy sets in this study deal with crisply defined membership functions or degrees of membership. It is doubtful whether, for instance, human beings have or can have a crisp image of membership functions in their minds. Zadeh [8] therefore suggested the notion of a fuzzy set whose membership function itself is a fuzzy set. This leads to the following definition of a second order fuzzy set or a fuzzy set of type 2.

A **second order fuzzy set** on a nonempty set X is a map from X to I^I.

First order fuzzy sets are denoted by f, g, h, . . . and second order fuzzy sets are denoted by $\hat{f}$, $\hat{g}$, $\hat{h}$, ...

In this paper the terms 'fuzzy set' and 'first order fuzzy set' are used synonymously.

Whenever a fuzzy set is considered without mentioning the order, it always refers to a first order fuzzy set.

Similar terminology applies to all concepts related to first order fuzzy sets.

Fundamental definitions and properties of second order fuzzy sets and second order fuzzy topological spaces are introduced in [4]. Six important and interesting connections $¢_1$, $¢_2$, $¢_3$, $¢_4$, $¢_5$ and $¢_6$ between first order and second order fuzzy topological spaces are also studied in [4].

Connections between crisp topological spaces and second order fuzzy topological spaces are given in [5].

With every crisp topology τ on a nonempty set X, three second order fuzzy topologies $\omega_2(\tau)$, $(\omega_2(\tau))_\varepsilon$ and $\omega_2^*(\tau)$ on X are associated. Also with every second order fuzzy topology $\hat{\delta}$ on a nonempty set X, three crisp topologies $i(\hat{\delta})$, $i_\varepsilon(\hat{\delta})$ and $i^*(\hat{\delta})$ on X are associated [5].

In this paper second order fuzzy continuity is defined and it is proved that the associations

$\hat{\delta} \to i_\varepsilon(\hat{\delta})$, $\hat{\delta} \to i^*(\hat{\delta})$, $\hat{\delta} \to i(\hat{\delta})$, $\tau \to (\omega_2(\tau))_\varepsilon$, $\tau \to \omega_2(\tau)$ and $\tau \to \omega_2^*(\tau)$ are functorial.

SECTION 1 : FUNDAMENTAL DEFINITIONS AND NOTATIONS

Definition : 1.1

A second order Chang fuzzy topology $\hat{\delta}$ on a nonempty set X is a collection of second order fuzzy sets on X satisfying the following conditions :

(i) $\hat{0}, \hat{1} \in \hat{\delta}$ where, for any $x \in X$

$\hat{0}(x)$ = the zero function **0** on I

$\hat{1}(x)$ = the constant function **1** on I

(ii) $\hat{f}_\lambda \in \hat{\delta}$ for each $\lambda \in \Lambda$ implies $(\bigvee_{\lambda \in \Lambda} \hat{f}_\lambda) \in \hat{\delta}$

(iii) $\hat{f}_i \in \hat{\delta}$ for i = 1, 2, . . . , m implies $(\bigwedge_{i=1}^{m} \hat{f}_i) \in \hat{\delta}$

The pair $(X, \hat{\delta})$ is called a **second order Chang fuzzy topological space**.

A **second order Lowen fuzzy topology** $\hat{\delta}$ on X is defined by replacing axiom (i) in the above definition by axiom (i)′.

(i)′ All constant function $\hat{\alpha} \in \hat{\delta}$ where, for any $x \in X$, $\hat{\alpha}(x)$ = the constant function α on I. The pair $(X, \hat{\delta})$ is called a **second order Lowen fuzzy topological space**.

The elements of $\hat{\delta}$ are called **second order fuzzy open sets.**

Notation : 1.2

A second order fuzzy topological space means a second order Chang fuzzy topological space.

A second order fuzzy topological space (Lowen) means a second order Lowen fuzzy topological space.

Definition : 1.3 [5]

Let $(X, \hat{\delta})$ be a second order fuzzy topological space. Define

(1) $i_\varepsilon(\hat{\delta})$ to be the topology generated by the collection $\{(A_{\hat{f}})_\varepsilon / \hat{f} \in \hat{\delta}\}$ where for $\varepsilon \in (0, 1)$, $(A_{\hat{f}})_\varepsilon = \{x \in X / \hat{f}(x)^{-1}(\varepsilon, 1] = I\}$.

(2) $i^*(\hat{\delta})$ to be the topology generated by the collection $\{A_{\hat{f}} / \hat{f} \in \hat{\delta}\}$ where $(A_{\hat{f}}) = \{x \in X / \hat{f}(x)^{-1}(\varepsilon, 1] = I$ for some $\varepsilon \in (0, 1)\}$

(3) $i_\varepsilon(\hat{\delta})$ to be the topology having the collection $\{(A_{\hat{f}})_\varepsilon / \hat{f} \in \hat{\delta}, \varepsilon \in (0, 1)\}$ as a sub basis.

Definition : 1.4 [5]

Let (X, τ) be a topological space. Define

(1) $(\omega_2(\tau))_\varepsilon$ to be the second order fuzzy topology generated by $\hat{K}_\varepsilon$ where for $\varepsilon \in (0, 1)$

$$\hat{K}_\varepsilon = \{\hat{f} \in (I^I)^X / (A_{\hat{f}})_\varepsilon \in \tau\}$$

(2) $\omega_2(\tau)$ to be the second order fuzzy topology generated by $\hat{K}$ where

$$\hat{K} = \{\hat{f} \in (I^I)^X / (A_{\hat{f}})_\varepsilon \in \tau, \text{ for every } \varepsilon \in (0, 1)\}.$$

(3) $\omega_2^*(\tau)$ to be the second order fuzzy topology generated by $\hat{K}_*$ where

$$\hat{K}_* = \{\hat{f} \in (I^I)^X / A_{\hat{f}} \in \tau\}.$$

SECTION 2 : SECOND ORDER FUZZY CONTINUITY

Definition : 2.1

Let $(X, \hat{\delta}_1)$, $(Y, \hat{\delta}_2)$ be two second order fuzzy topological spaces. Then a function $\theta : X \to Y$ is said to be **2-f continuous** if the following condition is satisfied :

$$\theta^{-1}(\hat{f}) \in \hat{\delta}_1 \text{ if } \hat{f} \in \hat{\delta}_2.$$

Theorem : 2.2

If $\theta : (X, \hat{\delta}_1) \to (Y, \hat{\delta}_2)$ be 2-f continuous, then

(1) For $\varepsilon \in (0, 1)$, $\theta : (X, i_\varepsilon(\hat{\delta}_1)) \to (Y, i_\varepsilon(\hat{\delta}_2))$ is continuous.

(2) $\theta : (X, i^*(\hat{\delta}_1)) \to (Y, i^*(\hat{\delta}_2))$ is continuous.

(3) $\theta : (X, i(\hat{\delta}_1)) \to (Y, i(\hat{\delta}_2))$ is continuous.

Proof :

(1) Consider a basis element $(A_{\hat{f}})_\varepsilon$ of $i_\varepsilon(\hat{\delta}_2)$,

where $\hat{f} \in \hat{\delta}_2$

$\hat{f} \in \hat{\delta}_2 \Rightarrow \theta^{-1}(\hat{f}) \in \hat{\delta}_1$

Let $\hat{g} = \theta^{-1}(\hat{f})$

$\therefore$ $(A_{\hat{g}})_\varepsilon$ is a basis element of $i_\varepsilon(\hat{\delta}_1)$,

$x \in (A_{\hat{g}})_\varepsilon$

$\Leftrightarrow$ $(\hat{g}(x))^{-1}(\varepsilon, 1] = I$

$\Leftrightarrow$ $[(\theta^{-1}(\hat{f}))(x)]^{-1}(\varepsilon, 1] = I$

$\Leftrightarrow$ $[\hat{f}(\theta(x))]^{-1}(\varepsilon, 1] = I$

$\Leftrightarrow$ $\theta(x) \in (A_{\hat{f}})_\varepsilon$

$\Leftrightarrow$ $x \in \theta^{-1}((A_{\hat{f}})_\varepsilon)$

$\therefore$ $(A_{\hat{g}})_\varepsilon = \theta^{-1}((A_{\hat{f}})_\varepsilon)$

$\therefore$ $\theta^{-1}((A_{\hat{f}})_\varepsilon)$ is a basis element of $i_\varepsilon(\hat{\delta}_1)$

$\therefore$ $\theta : (X, i_\varepsilon(\hat{\delta}_1)) \rightarrow (Y, i_\varepsilon(\hat{\delta}_2))$ is continuous.

Proofs of (2) and (3) are similar.

Theorem : 2.3

If $\theta : (X, \tau) \to (Y, \tau')$ is continuous, then

(1) For $\varepsilon \in (0, 1)$, $\theta : (X, (\omega_2(\tau))_\varepsilon \to (Y, (\omega_2(\tau'))_\varepsilon)$ is 2-f continuous.

(2) $\theta : (X, (\omega_2(\tau)) \to (Y, (\omega_2(\tau'))$ is 2-f continuous.

(3) $\theta : (X, (\omega_2{}^*(\tau)) \to (Y, (\omega_2{}^*(\tau'))$ is 2-f continuous.

Proof :

By definition, a base for $(\omega_2(\tau'))_\varepsilon$ is

$$(\hat{K}_\varepsilon)' = \{\hat{f} \in (I^I)^Y \,/\, (A_{\hat{f}})_\varepsilon \in \tau'\}$$

Let $\hat{f} \in (\hat{K}_\varepsilon)'$

$\therefore$ $(A_{\hat{f}})_\varepsilon \in \tau'$

$\therefore$ $\theta^{-1}((A_{\hat{f}})_\varepsilon) \in \tau$

$\therefore$ $(A_{\theta^{-1}(\hat{f})})_\varepsilon \in \tau$

$\therefore$ $\theta^{-1}(\hat{f}) \in \hat{K}_\varepsilon$ which is a base of $(\omega_2(\tau))_\varepsilon$.

$\therefore$ $\theta : (X, (\omega_2(\tau))_\varepsilon \to (Y, (\omega_2(\tau'))_\varepsilon)$ is 2-f continuous.

Proofs of (2) and (3) are similar.

Theorem : 2.4

If $\theta : (X, \hat{\delta}_1) \to (Y, \hat{\delta}_2)$ is 2-f continuous, then

$\theta : (X, S_2(\hat{\delta}_1)) \to (Y, S_2(\hat{\delta}_2))$ is continuous. (A base for $S_2(\hat{\delta}_2)$ is $\{S_2(\hat{f}) \,/\, \hat{f} \in \hat{\delta}_2\}$

where $S_2(\hat{f}) = \{x \in X \,/\, \hat{f}(x)(\alpha) > 0$ for every $\alpha \in I\}$).

Proof :

Consider a basis element $S_2(\hat{f})$ of $S_2(\hat{\delta}_2)$.

$\hat{f} \in \hat{\delta}_2 \Rightarrow \theta^{-1}(\hat{f}) \in \hat{\delta}_1$

Let $\hat{g} = \theta^{-1}(\hat{f})$

$\therefore$ $S_2(\hat{g})$ is a basis element of $S_2(\hat{\delta}_1)$

$x \in S_2(\hat{g})$

$\Leftrightarrow$ $\hat{g}(x)(\alpha) > 0$, for every $\alpha \in I$

$\Leftrightarrow$ $(\theta^{-1}(\hat{f}))(x)(\alpha) > 0$, for every $\alpha \in I$

$\Leftrightarrow$ $\hat{f}(\theta(x))(\alpha) > 0$, for every $\alpha \in I$

$\Leftrightarrow$ $\theta(x) \in S_2(\hat{f})$

$\Leftrightarrow$ $x \in \theta^{-1}(S_2(\hat{f}))$

$\therefore$ $\theta^{-1}(S_2(\hat{f}))$ is a basis element of $S_2(\hat{\delta}_1)$

$\therefore$ $\theta : (X, S_2(\hat{\delta}_1)) \rightarrow (Y, S_2(\hat{\delta}_2))$ is continuous.

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