Chapter IV

CHAPTER – IV
CIRCULANT TRIANGULAR FUZZY NUMBER MATRICES

Definition : 4.1

A **circulant matrix** is a square matrix generated from a vector as the first row (or column). Successive rows use the same element as the first row, but each such row is circularly shifted by one element. The last element in the first row will be the first element in the second row ... and form cycle permutations.

Definition : 4.2

A Triangular Fuzzy Number Matrix A is said to be **circulant TFNM** if all the elements of A can be determined completely by its first row suppose the first row of A is

$[<a_1^\ell, \tilde{a}_1, a_1^u>, <a_2^\ell, \tilde{a}_2, a_2^u>, \ldots\ldots <a_n^\ell, \tilde{a}_n, a_n^u>]$.

A circulant TFNM is the form of

$$\begin{bmatrix} <a_1^\ell, \tilde{a}_1, a_1^u> & <a_2^\ell, \tilde{a}_2, a_2^u> & \ldots\ldots & <a_{(n-1)}^\ell, \tilde{a}_{(n-1)}, a_{(n-1)}^u> & <a_{(n)}^\ell, \tilde{a}_{(n)}, a_{(n)}^u> \\ <a_{(n)}^\ell, \tilde{a}_{(n)}, a_{(n)}^u> & <a_1^\ell, \tilde{a}_1, a_1^u> & \ldots\ldots & <a_{(n-2)}^\ell, \tilde{a}_{(n-2)}, a_{(n-2)}^u> & <a_{(n-1)}^\ell, \tilde{a}_{(n-1)}, a_{(n-1)}^u> \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ <a_3^\ell, \tilde{a}_3, a_3^u> & <a_4^\ell, \tilde{a}_4, a_4^u> & \ldots\ldots & <a_1^\ell, \tilde{a}_1, a_1^u> & <a_2^\ell, \tilde{a}_2, a_2^u> \\ <a_2^\ell, \tilde{a}_2, a_2^u> & <a_3^\ell, \tilde{a}_3, a_3^u> & \ldots\ldots & <a_n^\ell, \tilde{a}_n, a_n^u> & <a_1^\ell, \tilde{a}_1, a_1^u> \end{bmatrix}$$

Remark : 4.3

It is noted that the TFNM A is circulant if and only if $\tilde{A}_{ij} = \tilde{A}_{(k\oplus i)(k\oplus j)}$ for every i, j, k $\in$ {1, 2, ..., n}, where $\oplus$ is sum modula n. This supply that the elements of the diagonal are all equal.

Remark : 4.4

For a circulant TFNM A we notice that $\tilde{A}_{in} = \tilde{A}_{(i\oplus 1)1}$ and $\tilde{A}_{nj} = \tilde{A}_{1(i\oplus 1)}$ for every i, j $\in$ {1, 2, …, n}.

Remark : 4.5

For a circulant TFNM A we notice that $\tilde{A}_{(i\oplus \overline{n-1})j} = \tilde{A}_{i(j\oplus 1)}$ for every i, j $\in$ {1, 2, …, n}.

Remark : 4.6

For a circulant TFNM A of order n x n with first row

$$[<a_1^{\ell}, \tilde{a}_1, a_1^u>, <a_2^{\ell}, \tilde{a}_2, a_2^u>, <a_3^{\ell}, \tilde{a}_3, a_3^u> \ldots\ldots, <a_n^{\ell}, \tilde{a}_n, a_n^u>].$$

Then the k^{th} column of A is

$$[<a_k^{\ell}, \tilde{a}_k, a_k^u>, <a_{(k-1)}^{\ell}, \tilde{a}_{(k-1)}, a_{(k-1)}^u>, \ldots\ldots, <a_1^{\ell}, \tilde{a}_1, a_1^u>, <a_n^{\ell}, \tilde{a}_n, a_n^u>,$$
$$<a_{(n-1)}^{\ell}, \tilde{a}_{(n-1)}, a_{(n-1)}^u>, \ldots\ldots, <a_{(k+1)}^{\ell}, \tilde{a}_{(k+1)}, a_{(k+1)}^u>]$$

Theorem : 4.7

An n x n TFNM A is circulant if and only if A C_n = C_n A, where C_n is the permutation matrix of unit TFNM.

$$C_n = \begin{bmatrix} <0,0,0> & <0,0,0> & \ldots\ldots & <0,0,0> & <0,1,0> \\ <0,1,0> & <0,0,0> & \ldots\ldots & <0,0,0> & <0,0,0> \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ <0,0,0> & <0,0,0> & \ldots\ldots & <0,0,0> & <0,0,0> \\ <0,0,0> & <0,0,0> & \ldots\ldots & <0,1,0> & <0,0,0> \end{bmatrix}$$

Proof

Let A be a TFNM and P = A C_n, then $\tilde{P}_{ij} = \sum_{k=1}^{n} (\tilde{A}_{ik}\, \tilde{C}_{kj})$. Since, only c_{1n} is < 0, 1, 0 > and all other elements of the first row of C_n is < 0, 0, 0 >. We get $\tilde{P}_{ij} = \tilde{A}_{i(j\oplus 1)}$.

Similarly, if T = C_n A, then $\widetilde{T}_{ij}$ = $\widetilde{A}_{(i\oplus \overline{n-1})j}$. So, by Remark 4.5 $\widetilde{P}_{ij} = \widetilde{T}_{ij}$ for all i, j $\in$ n. Hence A C_n = C_n A. So, A is circulant TFNM. Converse is straight forward.

Example : 4.8

Let A and C be two circulant TFNMs of order 3 x 3, where

$$A = \begin{bmatrix} <2,3,4> & <4,6,7> & <3,6,7> \\ <3,6,7> & <2,3,4> & <4,6,7> \\ <4,6,7> & <3,6,7> & <2,3,4> \end{bmatrix} \text{ and}$$

$$C = \begin{bmatrix} <0,0,0> & <0,0,0> & <0,1,0> \\ <0,1,0> & <0,0,0> & <0,0,0> \\ <0,0,0> & <0,1,0> & <0,0,0> \end{bmatrix}$$

$$\text{Then } A\,C = \begin{bmatrix} <4,6,7> & <3,6,7> & <2,3,4> \\ <2,3,4> & <4,6,7> & <3,6,7> \\ <3,6,7> & <2,3,4> & <4,6,7> \end{bmatrix} \text{ and}$$

$$C\,A = \begin{bmatrix} <4,6,7> & <3,6,7> & <2,3,4> \\ <2,3,4> & <4,6,7> & <3,6,7> \\ <3,6,7> & <2,3,4> & <4,6,7> \end{bmatrix}$$

Thus A C = C A.

Theorem : 4.9

For the circulant TFNMs A and B

(i) A + B is a circulant TFNM.

(ii) A′ is a circulant TFNM.

(iii) A B is also a circulant TFNM. In particular, A^k is also a circulant TFNM.

(iv) A A′ is circulant TFNM.

Proof

(i) Proof is straightforward.

(ii) Since A is circulant TFNM then A commutes with C_n.

So, A C_n = C_n A.

Transposing both sides of $A\,C_n = C_n\,A$, we get $C_n'\,A' = A'\,C_n'$

or, $C_n\,C_n'\,A' = C_n\,A'\,C_n'$

or, $A' = C_n\,A'\,C_n'$ [since, $C_n'\,C_n = I = C_n\,C_n'$]

$A'\,C_n = C_n\,A'\,C_n'\,C_n = C_n\,A'$

So, A' is circulant TFNMs.

(iii) Since, A and B are circulant TFNMs, each of A and B commutes with C_n.

Hence, A B commutes with C_n.

So, by Remark 4.5 and Theorem 4.7 we get A B is circulant TFNM.

(iv) Similar to (iii).

Theorem : 4.10

If A and B are circulant TFNMs then A B = B A.

Proof

Let A B = C and B A = D then both C and D are circulant by theorem 4.9 (iii) and their first rows are

$[<c_1^\ell, \tilde{c}_1, c_1^u>, \ <c_2^\ell, \tilde{c}_2, c_2^u>, \ <c_3^\ell, \tilde{c}_3, c_3^u>, \ \ldots\ldots \ <c_n^\ell, \tilde{c}_n, c_n^u>]$ and

$[<d_1^\ell, \tilde{d}_1, d_1^u>, \ <d_2^\ell, \tilde{d}_2, d_2^u>, \ <d_3^\ell, \tilde{d}_3, d_3^u>, \ \ldots\ldots \ <d_n^\ell, \tilde{d}_n, d_n^u>]$

respectively.

Then the mean value of the k^{th} element of the first row of C and D are respectively.

$$\tilde{c}_k = \left[\sum_{p=1}^{k} (\tilde{a}_p\,\tilde{b}_{(k-p+1)})\right] + \left[\sum_{p=k+1}^{n} (\tilde{a}_p\,\tilde{b}_{(n-p+k+1)})\right]$$

$$= (\tilde{a}_1\,\tilde{b}_k) + (\tilde{a}_2\,\tilde{b}_{(k-1)}) + (\tilde{a}_3\,\tilde{b}_{(k-2)}) + \ldots\ldots + (\tilde{a}_{(k-1)}\,\tilde{b}_2)$$

$$+ (\tilde{a}_k\,\tilde{b}_1) + (\tilde{a}_{(k+1)}\,\tilde{b}_n) + \ldots\ldots + (\tilde{a}_{(k+2)}\,\tilde{b}_{(n-1)}) + \ldots\ldots +$$

$$(\tilde{a}_{(n-1)}\,\tilde{b}_{(k+2)}) + (\tilde{a}_n\,\tilde{b}_{(k+1)})$$

$$\tilde{d}_k = \left[\sum_{p=1}^{k} (\tilde{b}_p \, \tilde{a}_{(k-p+1)})\right] + \left[\sum_{p=k+1}^{n} (\tilde{b}_p \, \tilde{a}_{(n-p+k+1)})\right]$$

$$= (\tilde{a}_1 \tilde{b}_k) + (\tilde{a}_2 \tilde{b}_{(k-1)}) + (\tilde{a}_3 \tilde{b}_{(k-2)}) + \ldots\ldots + (\tilde{a}_{(k-1)} \tilde{b}_2)$$

$$+ (\tilde{a}_k \tilde{b}_1) + (\tilde{a}_{(k+1)} \tilde{b}_n) + \ldots\ldots + (\tilde{a}_{(k+2)} \tilde{b}_{(n-1)}) + \ldots\ldots +$$

$$(\tilde{a}_{(n-1)} \tilde{b}_{(k+2)}) + (\tilde{a}_n \tilde{b}_{(k+1)})$$

It can be easy to see that $\tilde{c}_k = \tilde{d}_k$

The left hand spread of the k^{th} element of the first row of C and D are

$$c_k^{\ell} = \sum_{p=1}^{k} (a_p^{\ell} \, \tilde{b}_{(k-p+1)} + \tilde{a}_p \, b_{(k-p+1)}^{\ell}) + \sum_{p=k+1}^{n} (a_p^{\ell} \, \tilde{b}_{(n-p+k+1)} + \tilde{a}_p \, b_{(n-p+k+1)}^{\ell})$$

$$= (a_1^{\ell} \tilde{b}_k + \tilde{a}_1 \tilde{b}_k^{\ell}) + (a_2^{\ell} \tilde{b}_{(k-1)} + \tilde{a}_2 \tilde{b}_{(k-1)}^{\ell}) + \ldots\ldots +$$

$$(a_{(n-1)}^{\ell} \tilde{b}_{(k+2)}) + (\tilde{a}_{(n-1)} \tilde{b}_{(k+2)}^{\ell}) + (a_n^{\ell} \tilde{b}_{(k+1)} + \tilde{a}_n b_{(k+2)}^{\ell})$$

$$d_k^{\ell} = \sum_{p=1}^{k} (b_p^{\ell} \, \tilde{a}_{(k-p+1)} + \tilde{b}_p \, a_{(k-p+1)}^{\ell}) + \sum_{p=k+1}^{n} (b_p^{\ell} \, \tilde{a}_{(n-p+k+1)} + \tilde{b}_p \, a_{(n-p+k+1)}^{\ell})$$

$$= (b_1^{\ell} \tilde{a}_k + \tilde{a}_1 b_k^{\ell}) + (b_2^{\ell} \tilde{a}_{(k-1)} + \tilde{b}_2 a_{(k-1)}^{\ell}) + \ldots\ldots +$$

$$(b_{(n-1)}^{\ell} \tilde{a}_{(k+2)}) + (\tilde{b}_{(n-1)} a_{(k+2)}^{\ell}) + (b_n^{\ell} \tilde{a}_{(k+1)} + \tilde{b}_n a_{(k+2)}^{\ell})$$

It can be easy to see that $c_k^{\ell} = d_k^{\ell}$.

Similarly we can see, the right hand spread of the k^{th} element of the first row of C and D are equal i.e., $c_k^u = d_k^u$.

Since C and D are circulant, we have $\tilde{C}_{ij} = \tilde{D}_{ij}$ and hence the theorem is proved.

Theorem : 4.11

A circulant TFNM A is symmetric iff $\tilde{A}_{1i} = \tilde{A}_{1(n-i+2)}$ for every $i \in \{1, 2, \ldots, n\}$.

Proof

Let A be symmetric, then

$\widetilde{A}_{1i} = \widetilde{A}_{(1 \oplus k)\,(i \oplus k)} = \widetilde{A}_{i1} = \widetilde{A}_{(i \oplus k)\,(1 \oplus k)}$ for every i, k $\in$ {1, 2, ... n}.

Taking k = n – i, then

$$\widetilde{A}_{(1 \oplus (n-i))(i \oplus (n-i))} = \widetilde{A}_{(i \oplus (n-i))\,(1 \oplus (n-i))}$$

$$= \widetilde{A}_{n(n-i+1)} = \widetilde{A}_{1(n-i+2)} \text{ [by remark 4.4]}$$

Conversely, suppose $\widetilde{A}_{1i} = \widetilde{A}_{1(n-i+2)}$ for every i $\in$ {1, 2, ... n}, then $\widetilde{A}_{i1} = \widetilde{A}_{(i \oplus k)\,(1 \oplus k)}$ for every i, k $\in$ {1, 2, …, n}.

Taking k = n – i, we get $\widetilde{A}_{i1} = \widetilde{A}_{n(n-i+1)} = \widetilde{A}_{1(n-i+2)} = \widetilde{A}_{1i}$.

But since A is circulant and $\widetilde{A}_{1i} = \widetilde{A}_{i1}$. We have $\widetilde{A}_{ij} = \widetilde{A}_{ji}$. for every i, k $\in$ {1, 2, …, n} and A is symmetric.

Theorem : 4.12

If a TFNMs A is circulant, then E A is symmetric where E is a permutation matrix of unit TFNM and the form of E is

$$E = \begin{bmatrix} <0,0,0> & <0,0,0> & \ldots\ldots & <0,0,0> & <0,1,0> \\ <0,0,0> & <0,0,0> & \ldots\ldots & <0,1,0> & <0,0,0> \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ <0,0,0> & <0,1,0> & \ldots\ldots & <0,0,0> & <0,0,0> \\ <0,1,0> & <0,0,0> & \ldots\ldots & <0,0,0> & <0,0,0> \end{bmatrix}$$

Proof

Let R = E A, then $\widetilde{R}_{ij} = \sum_{k=1}^{n} \widetilde{E}_{ik}\, \widetilde{A}_{kj}$ for all i, j, k = 1, 2, …, n. Since, E is a permutation matrix of unit TFNM and only the elements $\widetilde{E}_{1n}$, $\widetilde{E}_{1(n-1)}$, $\widetilde{E}_{1(n-2)}$, … $\widetilde{E}_{n1}$ are < 0, 1, 0 > and all other elements are < 0, 0, 0 >, we get

$$\widetilde{R}_{ij} = \sum_{k=1}^{n} \widetilde{E}_{ik}\, \widetilde{A}_{kj} = \widetilde{A}_{(n-i+1)\,j}$$

Now, since A is circulant, we know $\tilde{R}_{ij} = \tilde{A}_{(n-i+1)j} = \tilde{A}_{((n-i+1)\oplus k)(k \oplus j)}$ for all i, j, k ∈ {1, 2, …. n}.

When, k = i, then

$\tilde{R}_{ij} = \tilde{A}_{(n-i+1)j} = \tilde{A}_{(n\oplus 1)(i\oplus j)} = \tilde{A}_{1(i\oplus j)}$ and

$\tilde{R}_{ji} = \tilde{A}_{(n-j+1)i} = \tilde{A}_{((n-j+1)\oplus k)(k\oplus i)}$ for i, j, k ∈ {1, 2, …, n}.

Taking k = j, then

$$\tilde{R}_{ji} = \tilde{A}_{(n-j+1)i}$$
$$= \tilde{A}_{(n\oplus 1)(i\oplus j)}$$
$$= \tilde{A}_{1(i\oplus j)}$$

Hence $\tilde{R}_{ij} = \tilde{R}_{ji}$ and thus R is symmetric.

Theorem : 4.13

Let A be a circulant TFNM of order n x n. Then

(i) adj A is also circulant TFNM.

(ii) If A is a square TFNM then | A | = | A′ |.

(iii) adj A = (adj A)′.

Proof

(i) We have to prove co-factor of the elements $\tilde{A}_{i(j\oplus 1)}$ and $\tilde{A}_{(i\oplus \overline{n-1})j}$ for all i, j ∈ n are same.

Since A is circulant then by Remark 4.5 $\tilde{A}_{i(j\oplus 1)} = \tilde{A}_{(i\oplus \overline{n-1})j}$ and so the minor of $\tilde{A}_{i(j\oplus 1)}$ and $\tilde{A}_{(i\oplus \overline{n-1})j}$ will be same.

Now, co-factor of $\tilde{A}_{i(j\oplus 1)} = (-1)^{i+(j\oplus 1)} \sum_{\sigma\in S_n} \operatorname{Sgn}\sigma \prod_{\substack{k=1 \\ k\neq i \\ k\neq j\oplus 1}}^{n} \tilde{A}_{k\,\sigma(k)}$ and

$$\tilde{A}_{(i\oplus \overline{n-1})j} = (-1)^{(i\oplus \overline{n-1})+j} \sum_{\sigma\in S_n} \operatorname{Sgn}\sigma \prod_{\substack{k=1 \\ k\neq i \\ k\neq (i\oplus \overline{n-1})}}^{n} \tilde{A}_{k\,\sigma(k)}$$

It is obvious that for fixed n, the sign of $(-1)^{i+(j\oplus 1)}$ and $(-1)^{(i\oplus \overline{n-1})+j}$ is same for all $i, j \in \{1, 2, \ldots, n\}$.

So, the co-factor of $\tilde{A}_{i(j\oplus 1)}$ and $\tilde{A}_{(i\oplus \overline{n-1})j}$ are same.

Hence adj A is also circulant TFNM.

(ii) Let $A = (\tilde{A}_{ij})_{n\times n}$ be a square TFNM and $A' = B = (\tilde{B}_{ij})_{n\times n}$. Then

$$|B| = \sum_{\sigma\in S_n} \operatorname{Sgn}\sigma\ \tilde{B}_{1\sigma(1)}, \tilde{B}_{2\sigma(2)}, \ldots, \tilde{B}_{n\sigma(n)}$$

$$= \sum_{\sigma\in S_n} \operatorname{Sgn}\sigma\ \tilde{A}_{1\sigma(1)}, \tilde{A}_{2\sigma(2)}, \ldots, \tilde{A}_{n\sigma(n)}$$

Let ϕ be a permutation of $\{1, 2, \ldots, n\}$ such that $\phi\,\sigma = I$, the identity permutation. Then $\phi = \sigma^{-1}$. Since σ runs over the whole set of permutations, ϕ also runs over the same set of permutation. Let $\sigma(i) = j$ then $i = \sigma^{-1}(j)$ and $a_{\sigma(i)i} = a_{j\,\phi(j)}$ for all i, j. Therefore,

$$|B| = \sum_{\sigma\in S_n} \operatorname{Sgn}\sigma\ \langle a^{\ell}_{\sigma(1)1}, \tilde{a}_{\sigma(1)1}, a^{u}_{\sigma(1)1}\rangle \langle a^{\ell}_{\sigma(2)2}, \tilde{a}_{\sigma(2)2}, a^{u}_{\sigma(2)2}\rangle \ldots \langle a^{\ell}_{\sigma(n)n}, \tilde{a}_{\sigma(n)n}, a^{u}_{\sigma(n)n}\rangle$$

$$= \sum_{\phi\in S_n} \operatorname{Sgn}\phi\ \langle a^{\ell}_{1\phi(1)}, \tilde{a}_{1\phi(1)}, a^{u}_{1\phi(1)}\rangle \langle a^{\ell}_{2\phi(2)}, \tilde{a}_{2\phi(2)}, a^{u}_{2\phi(2)}\rangle \ldots \langle a^{\ell}_{n\phi(n)}, \tilde{a}_{n\phi(n)}, a^{u}_{n\phi(n)}\rangle$$

$$= |A|$$

Hence, $|A| = |A'|$

(iii) Similar to (i) and (ii).