Chapter 6

CHAPTER – VI

ANALYSIS OF $M^X/M/1$ AND M / M(a, b) /1 QUEUEING SYSTEM UNDER WORKING VACATIONS

Introduction:

In classical vacation queues the server may not be fully available for a period of time or may stop the primary service and perform other supplementary jobs. For the past three decades there had been considerable attention paid to the analysis of queueing system with server vacations because of their applications in modeling the computer networks, communication network and manufacturing service system. In 2002, Servi and Finn , introduced a class of semi vacation policies, in which servers work at a lower rate rather than completely stopping primary service during vacation. Such a vacation is called working vacation (WV). Part of service ability keeps the system operating in a lower speed during a vacation. If service speed degenerates into zero in a working vacation, the working vacation queue becomes a classical vacation queueing model. Therefore the working vacation queue is a generalization of the classical vacation queue and the analysis of these kinds of models is more complicated than the previous work.

The M/M/1 queueing system with working vacation has been analysed by Servi and Finn (2002), Liu et al., (2007) , Zhang and Xu (2008) and Tian *et al.,* (2008b) Their work is motivated and illustrated, by the analysis of a WDM (Wavelength Division Multiplexing) optical access network using multiple wave length which can be reconfigured. Wu and Takagi (2006) later extended Servi and Finn's M/M/1/WV to a M/G/1/WV model. They assumed that both the service times during a regular service period and during working vacations are generally distributed. They derived the distribution for the number of customers in the system and the sojourn time for an arbitrary customer in the state. Baba (2005) has considered a GI/M/1/WV queueing system in which the inter arrival times form an independent identically distributed sequence of random variables having a general distribution function. The steady state distributions for the number of customers in the system both at arrival and arbitrary epochs are derived using Embedded Markov Chain technique.

The available research on working vacation mainly concentrated on single arrival and single service queueing systems. In the following chapters we analyze some bulk arrival and bulk service queueing models with working vacations.

In sections (6.1) and (6.2) of this chapter, a batch arrival $M^X/G/1$ queueing system is analysed under exponentially distributed multiple and single working vacations. The steady state results of an M / M(a,b) / 1 queueing system with multiple working vacation are analyzed in section (6.3).

SECTION: 6.1

M^X/M/1 QUEUE WITH MULTIPLE WORKING VACATIONS (MWV)

6.1.1: The Mathematical Analysis of the Queueing Model (MWV)

I Model description:

Consider a batch arrival $M^X/M/1$ queue in which, the arrival stream forms a Poisson process and the actual number of customers in any arriving module is a random variable X, which may take on any positive integral value k (< ∞) with probability g_k. If λ_k is the arrival rate of a Poisson process of batches of size k then $g_k = \frac{\lambda_k}{\lambda}$, k = 1, 2, 3, . . . where λ is the composite arrival rate of all batches equal to $\sum_{i=1}^{\infty} \lambda_i$. This total process, which arises from the overlap of the set of Poisson processes with rates $\{\lambda_k / k = 1, 2, \ldots\}$ is a compound Poisson process. Let X(z), E(X) and $E(X^2)$ denote the PGF , first and second moments of random variable X.

Then $X(z) = \sum_{k=1}^{\infty} g_k z^k$; $E(X^k) = \sum_{n=1}^{\infty} g_n n^k$

The server serves the customers one at time with exponential service rate μ in a regular busy period. Whenever the system becomes empty at service completion instant, the server starts a working vacation during which the service is done at a lower rate. The vacation duration V follows an exponential distribution with parameter η. During working vacations, arriving customers are served with exponential service rate μ_v ($\leq \mu$).

When a vacation terminates and the server finds the system is empty, then he begins another working vacation. On the other hand, if the server finds the system is not empty at the vacation termination instant, then he switches to a regular service period. The distributions of the service times during regular busy period and working vacation period are both exponential but with different rates μ and μ_v respectively. It is assumed that, inter arrival times, service time and working vacation times are mutually independent of each other.

This model is denoted by **M^x /M/1 /MWV**

II System Size Distribution:

Let $N_s(t)$ denote the number of customers in the system at time t and

$$J(t) = \begin{cases} 0 & \text{the system is in a working vacation period at time t} \\ 1 & \text{the system is in regular busy period at time t} \end{cases}$$

Then $\{N_s(t), J(t)\}$ is a Markov process.

Let $P_n(t) = Pr\{N_s(t) = n ; J(t) = 1\}$, $n \geq 1$

and $Q_n(t) = Pr\{N_s(t) = n ; J(t) = 0\}$, $n \geq 0$

denote the system size probabilities at time t.

Assuming the steady state system size probabilities $P_n = \lim_{t \to \infty} P_n(t)$ and $Q_n = \lim_{t \to \infty} Q_n(t)$ exist, the steady state equations satisfied by P_n's and Q_n's are given by

$$\lambda Q_0 = \mu_v Q_1 + \mu P_1 \quad (6.1.1)$$

$$(\lambda + \eta + \mu_v) Q_n = \lambda \sum_{k=1}^{n} Q_{n-k} g_k + \mu_v Q_{n+1}, \quad n \geq 1 \quad (6.1.2)$$

$$(\lambda + \mu) P_1 = \mu P_2 + \eta Q_1 \quad (6.1.3)$$

$$(\lambda + \mu)P_n = \lambda \sum_{k=1}^{n-1} P_{n-k} g_k + \mu P_{n+1} + \eta Q_n \quad n \geq 2 \quad (6.1.4)$$

To obtain the steady state distribution of the model, the partial PGFs,

$$Q(z) = \sum_{n=0}^{\infty} Q_n z^n$$

$$P(z) = \sum_{n=1}^{\infty} P_n z^n \text{ are defined.}$$

Multiplying equation (6.1.2) by z^n, summing over $n \geq 1$ and then adding with (6.1.1) we have

$$(\lambda + \eta + \mu_v) Q(z) = (\eta + \mu_v) Q_0 + \frac{\mu_v}{z} (Q(z) - Q_0) + \mu P_1 + \lambda X(z)Q(z),$$

since, $\sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} Q_{n-k} g_k = Q(z) X(z)$

(i.e.) $(\lambda z (1 - X(z)) + \mu_v (z - 1) + \eta z) Q(z) = \mu_v (z - 1) Q_0 + Pz$ (6.1.5)

where $P = \eta Q_0 + \mu P_1$.

The equation , $h(z) = \lambda z (1 - X(z)) + \mu_v(z - 1) + \eta z$

$= (\lambda + \eta + \mu_v) z - \lambda z X(z) - \mu_v$ (6.1.6)

has a unique root z_1, with $|z_1| < 1$, by Rouche's theorem.

For $f(z) = (\lambda + \eta + \mu_v) z$ and $g(z) = -\lambda z X(z) - \mu_v$ satisfy the condition $|g(z)| < |f(z)|$ on $|z| = 1$ and hence the equations $f(z) = 0$ and

$f(z) + g(z) = h(z)=0$ have the same number of roots inside the circle $| z | = 1$. Since $f(z)$ is linear, the root is real and simple and $h(0) = -\mu_v$ and $h(1) = \eta$ are of opposite sign.

Thus evaluating P at $z = z_1$ from equation (6.1.5),

$$P = \frac{\mu_v (1-z_1) Q_0}{z_1} \tag{6.1.7}$$

With this, the partial PGF Q(z) is given by

$$Q(z) = \frac{\mu_v (z-z_1) Q_0}{z_1 (\lambda z (1-X(z)) + \mu_v (z-1) + \eta z)} \tag{6.1.8}$$

Similarly equations (6.1.3) and (6.1.4) imply,

$$[\lambda z (1 - X(z)) + \mu (z-1)] P(z) = \eta z Q(z) - Pz$$

substituting the value of P from (6.1.7) and Q(z) from (6.1.8), P(z) is given by,

$$(\lambda z (1 - X(z)) + \mu (z-1)) P(z) =$$

$$\left[\frac{\mu_v \eta z (z-z_1)}{z_1 (\lambda z (1-X(z)) + \eta z + \mu_v (z-1))} - \frac{\mu_v z (1-z_1)}{z_1} \right] Q_0$$

On simplification,

$$P(z) = \frac{\lambda z \mu_v Q_0}{z_1} \left[\frac{(z-1) z_1 (X(z_1)-1) - (z_1-1) z (X(z)-1)}{(\lambda z (1-X(z)) + \eta z + \mu_v (z-1))(\lambda z (1-X(z)) + \mu (z-1))} \right] \tag{6.1.9}$$

Thus the total PGF $P_{wv}^{M}(z) = P(z) + Q(z)$ is given by

$$P_{wv}^{M}(z) = \frac{\mu_v (z-1) Q_0}{z_1 (\lambda z (1-X(z)) + \mu (z-1))} \left(\frac{\mu (z-z_1) + \lambda z z_1 (X(z_1) - X(z))}{\lambda z (1-X(z)) + \mu_v (z-1) + \eta z} \right) \tag{6.1.10}$$

The normalizing condition $P_{wv}^{M}(1) = 1$ implies

$$Q_0 = \frac{\eta z_1 \mu (1-\rho)}{\mu_v (\lambda z_1 (X(z_1)-1) + \mu (1-z_1))}$$

$$\text{where} \quad \rho = \frac{\lambda}{\mu} E(X) \tag{6.1.11}$$

By substituting for Q_0 in equation (6.1.10), the PGF is given by

$$P_{wv}^{M}(z) = \frac{\mu(1-\rho)(z-1)}{(\lambda z(1-X(z))+\mu(z-1))} \frac{\left[\dfrac{\mu(z-z_1)+\lambda z z_1(X(z_1)-X(z))}{\lambda z(1-X(z))+\mu_v(z-1)+\eta z}\right]}{\left[\dfrac{\mu(1-z_1)+\lambda z_1(X(z_1)-1)}{\eta}\right]}$$

$$\text{(i.e)}\ \mathbf{P_{wv}^{M}(z)} = P_{M^x/M/1}(z) \frac{\left(\dfrac{\mu(z-z_1)+\lambda z z_1(X(z_1)-X(z))}{\lambda z(1-X(z))+\mu_v(z-1)+\eta z}\right)}{\left(\dfrac{\mu(1-z_1)+\lambda z_1(X(z_1)-1)}{\eta}\right)} \quad (6.1.12)$$

III Decomposition property

Thus the PGF of the M^X / M / 1 multiple working vacation model is decomposed into the product of two random variables one is the PGF of classical M^X/M/1 queueing model and the other is PGF of the additional queue length. This justifies the **Decomposition property.**

6.1.2: Performance Measures

(i) The probability that the server is on vacation P_v is given by

$$P_v = \underset{z \to 1}{\text{lt}}\ Q(z) = \frac{\mu_v(1-z_1)Q_0}{\eta z_1} = \frac{\mu(1-\rho)(1-z_1)}{\mu(1-z_1)+\lambda z_1(X(z_1)-1)} \quad (6.1.13)$$

(ii) The probability that the server is on regular busy period (P_{busy}) is given by

$$P_{busy} = \underset{z \to 1}{\text{lt}}\ P(z) = \frac{\lambda \mu_v}{\eta z_1}\left(\frac{z_1(X(z_1)-1)+E(X)(1-z_1)}{\mu(1-\rho)}\right) Q_0$$

$$= \frac{\lambda(z_1(X(z_1)-1)+(1-z_1)E(X))}{(\mu(1-z_1)+\lambda z_1(X(z_1)-1))} \quad (6.1.14)$$

(iii) The expected system size when the server is on vacation (L_v) is,

$$L_v = \frac{d}{dz}(Q(z))_{z=1} = \frac{\lambda \mu_v Q_0}{\eta^2 z_1}(E(X)(1-z_1)+z_1(X(z_1)-1)) \quad (6.1.15)$$

(iv) Thus the expected system size of the model is given by

$$L = \frac{d}{dz}\left(P_{wv}^{M}(z)\right)_{z=1}$$

$$= \frac{\lambda\,(E(X)+E(X^2))}{2\,\mu\,(1-\rho)} + \frac{\lambda\,E(X) - \mu_v}{\eta} + \frac{z_1\,(\mu - \lambda\,E(X))}{(\mu\,(1-z_1) + \lambda\,z_1\,(X(z_1)-1))} \qquad (6.1.16)$$

6.1.3: Particular Case : M/M/1 Multiple Working Vacation Model

The results of M/M/1 multiple working vacation can be obtained by taking $X(z) = z$ in the corresponding equations.

Let $z_1 < 1$ and $z_2 > 1$ be the two roots of the equation,

$$\lambda\,z\,(1-z) + \mu_v\,(z-1) + \eta\,z = 0$$

Then $\lambda\,z\,(1-z) + \mu_v\,(z-1) + \eta\,z = -\lambda\,(z-z_1)\,(z-z_2)$

and the relation $z_2 = \frac{\mu_v}{\lambda\,z_1} = \frac{1}{r_v}$ (6.1.17)

implies $\lambda\,z\,(1-z) + \mu_v\,(z-1) + \eta\,z = -\lambda\,(z-z_1)\,(z - \frac{1}{r_v})$ (6.1.18)

then $Q(z) = \frac{Q_0}{(1-r_v\,z)}$ from equation (6.1.8)

$$= \sum_{n=0}^{\infty} (r_v)^n\, z^n\, Q_0$$

and $P(z) = \frac{\rho\,z\,(1-z_1)\,Q_0}{(1-\rho\,z)\,(1-r_v z)}$ from equation (6.1.9)

$$= \frac{\eta\,r_v\,Q_0}{\mu\,(\rho - r_v)\,(1-r_v)}\left(\frac{1}{1-\rho z} - \frac{1}{1-r_v z}\right)$$

$$= \frac{\eta r_v}{\mu\,(\rho - r_v)\,(1-r_v)} \sum_{n=0}^{\infty} (\rho^n - r_v^{\,n})\, z^n\, Q_0$$

and $Q_0 = \left((1-\rho) + \dfrac{\eta r_v}{\mu(1-r_v)}\right)^{-1}(1-\rho)(1-r_v)$

also $L = \dfrac{\rho}{(1-\rho)} + \left(1 - \dfrac{r_v \mu_v}{\mu}\right)^{-1}\left(1 - \dfrac{\mu_v}{\mu}\right)\left(\dfrac{r_v}{1-r_v}\right)$

follow from equations 6.1.11 and 6.1.16 where $\rho = \dfrac{\lambda}{\mu}$ and these results coincide with the results of (Liu *et al.,* 2007).

SECTION: 6.2

M^X/ M/1 QUEUE WITH SINGLE WORKING VACATION (SWV)

6.2.1: The Mathematical Analysis of SWV

I Model Description:

In this section we consider single working vacation into a classical M^X/M/1 queue with batch arrival rate λ and service rate μ in regular busy period. The server begins a working vacation of random length V at the instant where the queue becomes empty and vacation duration V follows an exponential distribution with parameter η. During the working vacation time the arriving customers are served at rate μ_v. When the vacation ends, if there are customers in the queue the server changes service rate from μ_v to μ and a regular busy period starts. Otherwise, the server enters an idle period, and a new regular busy period starts when arrival occurs. In this model also working vacation is an operation period with a lower service rate (μ_v). It is assumed that the inter arrival times of batches and service times (during regular busy and working vacation) and vacation time are all exponentially distributed with parameters λ, μ, μ_v and η respectively and mutually independent of each other. In addition, the service is done one at a time.

This model is also denoted by **M^x /M/1 /SWV**

II System Size Distribution

In this case the system is studied with $(N_S(t), J(t))$ as a Markov process, where $N_S(t)$ denotes the number of customers in the system at time t and $J(t) = -1, 0, 1$ according as the server is idle, on working vacation and regular busy state in the system.

Then defining,

$$Q_n^S(t) = \Pr\{N_S(t) = n, J(t) = 0\}\ ; \quad n \geq 0$$

$$P_n^S(t) = \Pr\{N_S(t) = n, J(t) = 1\}\ ; \quad n \geq 1$$

$$P_0^S(t) = \Pr\{N_S(t) = n, J(t) = -1\} \text{ and}$$

assuming the steady state probabilities

$P_n^S = \underset{t \to \infty}{\text{lt}} P_n^S(t)$; $Q_n^S = \underset{t \to \infty}{\text{lt}} Q_n^S(t)$ and $P_0 = \underset{t \to \infty}{\text{lt}} P_0^S(t)$ exist,

the steady state equations satisfied by the steady state probabilities are :

$$(\eta + \lambda)\, Q_0^S = \mu_v\, Q_1^S + \mu\, P_1^S \qquad (6.2.1)$$

$$(\lambda + \eta + \mu_v)\, Q_n^S = \lambda \sum_{k=1}^{n} Q_{n-k}^S\, g_k + \mu_v\, Q_{n+1}^S \qquad n \geq 1 \qquad (6.2.2)$$

$$(\lambda + \mu)\, P_1^S = \mu\, P_2^S + \eta\, Q_1^S + \lambda\, P_0^S\, g_1 \qquad (6.2.3)$$

$$(\lambda + \mu)\, P_n^S = \lambda \sum_{k=1}^{n-1} P_{n-k}^S\, g_k + \mu\, P_{n+1}^S + \eta\, Q_n^S + \lambda\, P_0^S\, g_n \quad n \geq 2 \qquad (6.2.4)$$

$$\lambda\, P_0^S = \eta\, Q_0^S \qquad (6.2.5)$$

Let $Q^S(z) = \sum_{n=0}^{\infty} Q_n^S z^n$, $P^S(z) = \sum_{n=1}^{\infty} P_n^S z^n$ denote the partial PGFs, then

$P_{wv}^S(z) = P^S(z) + Q^S(z) + P_0^S$ gives the total PGF of the model.

By using equations (6.2.1) and (6.2.2) the PGF when the server is on vacation is given by

$$(\lambda z\,(1 - X(z)) + \eta\, z + \mu_v\,(z - 1))\, Q^S(z) = \mu_v\,(z - 1)\, Q_0^S + \mu P_1^S\, z \qquad (6.2.6)$$

Since the equation $\lambda z(1 - X(z)) + \eta z + \mu_v(z - 1) = 0$ has unique root $z_1 \in (0, 1)$, (provided in section 6.1) P_1^S is evaluated at $z = z_1$ and given by

$$P_1^S = \frac{\mu_v (1 - z_1) Q_0^S}{\mu z_1}$$

Thus $$Q^S(z) = \frac{\mu_v (z - z_1) Q_0}{z_1 (\lambda z (1 - X(z)) + \eta z + \mu_v (z - 1))} \tag{6.2.7}$$

Similarly equations (6.2.3) to (6.2.5) give the PGF $P^S(z)$ corresponding to regular busy period namely,

$$(\lambda z (1 - X(z)) + \mu (z - 1)) P^S(z) = -(\mu P_1^S + \eta Q_0^S) z + \eta z Q^S(z) + \lambda z X(z) P_0^S$$

Substituting for $Q^S(z)$, $P^S(z)$ is given by

$$P^S(z) = \frac{Q_0^S z}{(\lambda z (1 - X(z)) + \mu (z - 1))} \left(\frac{\lambda \mu_v}{z_1} \left(\frac{z_1 (z - 1)(X(z_1) - 1) - z(z_1 - 1)(X(z) - 1)}{\lambda z (1 - X(z)) + \eta z + \mu_v (z - 1)} \right) - \eta (1 - X(z)) \right) \tag{6.2.8}$$

Thus the total PGF $P_{wv}^S(z) = P^S(z) + Q^S(z) + P_0^S$ gives

$$P_{wv}^S(z) = \frac{(z - 1) Q_0}{(\lambda z (1 - X(z)) + \mu (z - 1))} \left(\frac{\mu_v}{z_1} \left(\frac{\mu (z - z_1) + \lambda z z_1 (X(z_1) - X(z))}{\lambda z (1 - X(z)) + \eta z + \mu_v (z - 1)} \right) + \frac{\eta \mu}{\lambda} \right) \tag{6.2.9}$$

Evaluating Q_0^S at $z = 1$, $P_{wv}^S(1) = 1$ implies

$$Q_0^S = \frac{\mu (1 - \rho)}{\left(\frac{\mu_v}{\eta z_1} (\mu (1 - z_1) + \lambda z_1 (X(z_1) - 1)) + \frac{\eta \mu}{\lambda} \right)} \tag{6.2.10}$$

By substituting Q_0^S in (6.2.9) we get,

$$P_{wv}^S(z) = \frac{\mu (1 - \rho)(z - 1)}{(\lambda z (1 - X(z)) + \mu (z - 1))} \left[\frac{\frac{\mu_v}{z_1} \left(\frac{\mu (z - z_1) + \lambda z z_1 (X(z_1) - X(z))}{\lambda z (1 - X(z)) + \eta z + \mu_v (z - 1)} \right) + \frac{\eta \mu}{\lambda}}{\frac{\mu_v}{z_1} \left(\frac{\mu (1 - z_1) + \lambda z_1 (X(z_1) - 1)}{\eta} \right) + \frac{\eta \mu}{\lambda}} \right]$$

which is the product of two PGF of which one is

$$P_{M^X/M/1}(z) = \frac{\mu(1-\rho)(z-1)}{(\lambda z(1-X(z))+\mu(z-1))}$$ and the other is

$$Q_d(z) = \left[\frac{\dfrac{\mu_v}{z_1}\left(\dfrac{\mu(z-z_1)+\lambda z z_1(X(z_1)-X(z))}{\lambda z(1-X(z)+\eta z+\mu_v(z-1)}\right)+\dfrac{\eta\mu}{\lambda}}{\dfrac{\mu_v}{z_1}\left(\dfrac{\mu(1-z_1)+\lambda z_1(X(z_1)-1)}{\eta}\right)+\dfrac{\eta\mu}{\lambda}}\right]$$

6.2.2: Performance Measures

Let P_v^S, P_{busy}^S and P_{idle}^S denote that the server is on working vacation, regular busy and idle in the system then

(i) $$P_v^S = \lim_{z\to 1} Q^S(z) = \frac{\mu_v(1-z_1)Q_0}{\eta z_1}$$

(ii) $$P_{busy}^S = \lim_{z\to 1} P^S(z) = \left[\eta E(X)+\frac{\lambda\mu_v}{\eta z_1}(z_1(X(z_1)-1)-(z_1-1)E(X))\right]\frac{Q_0^S}{\mu(1-\rho)}$$

(iii) $$P_{idle}^S = \lim_{z\to 1} P_0^S = \frac{\eta}{\lambda}Q_0^S$$

Further the expected system size of the Single working vacation model is given by $L^S = \frac{d}{dz}(P(z))_{z=1}$ implies,

$$L^S = \lambda\frac{(E(X)+E(X^2))}{2\mu(1-\rho)}+\frac{\lambda E(X)-\mu_v}{\eta}+\frac{\mu z_1\left((1-\rho)+\dfrac{\eta}{\lambda}(1-\dfrac{\lambda E(X)}{\mu_v})\right)}{\left(\mu(1-z_1)+\lambda z_1(X(z_1)-1)+\dfrac{\eta^2\mu z_1}{\lambda\mu_v}\right)}$$

6.2.3: Particular Case : M/M/1 Single Working Vacation Model

The system size distribution of M/M/1 single working vacation can be obtained by taking $X(z) = z$. By proceeding as in the previous section, the partial PGFs of M/M/1 single working vacation models are obtained respectively from equations (6.2.7) to (6.2.9). Then,

$$Q^S(z) = \sum_{r=0}^{\infty} r_V{}^n z^n \, Q_0^S$$

$$P^S(z) = \left(\frac{\lambda z (1 - z_1)}{\mu (1 - rz)(1 - \rho z)} + \frac{\eta z}{\mu (1 - \rho z)} \right) Q_0^S$$

$$\text{(i.e)} \; P^S(z) = \left[\frac{\eta r_V}{\mu (\rho - r_V)(1 - r_V)} \sum_{n=0}^{\infty} (\rho^n - r_V{}^n) z^n + \frac{\eta z}{\mu} \sum_{n=0}^{\infty} (\rho z)^n \right] Q_0^S$$

$$\text{and} \quad P_0^S = \frac{\eta}{\lambda} Q_0^S$$

$$\text{where} \quad Q_0^S = \frac{(1 - r_V)(1 - \rho)}{(1 - \rho) + \dfrac{\eta r_V}{\mu (1 - r_V)} + \dfrac{\eta (1 - r_V)}{\lambda}}$$

and the expected system size is given by

$$L_S = \frac{\rho}{(1 - \rho)} + \left(1 - \frac{\mu_V}{\mu} \right) \frac{r_V}{(1 - r_V)} \left((1 - \frac{r_V \mu_V}{\mu}) + \frac{\eta}{\lambda} (1 - r_V) \right)^{-1}$$

where $\left(\frac{1}{r_V} \right)$ is the root of $(\lambda z (1 - z) + \eta z + \mu_V (z - 1)) = 0$ as in (6.1.18).

These results coincide with the corresponding results of M/M/1 Single working vacation model (Tian *et al.,* 2008b).

SECTION : 6.3

M / M(a, b)/1 QUEUEING SYSTEM WITH MULTIPLE WORKING VACATIONS

In queueing models, there are situations particularly in transportation system, where the service provided is such that a group of customers can be served simultaneously. Examples include shuttle-bus service, freight trains, express elevators, tour guides and batch servicing in manufacturing process. The theory of batch queues originated with the work of Bailey (1954). He considered a queue with Poisson arrival and fixed size service. Numerous authors (Jaishwal 1960 & Madil and Choudhury 1986) have investigated a variety of extensions of the basic model. Neuts (1967) has introduced the most general bulk service rule in considering a queueing system with Poisson arrival and general service time distribution.

Most of the general bulk service queueing models with servers vacation have been analysed by many authors using matrix geometric method. Some of the notable works for matrix geometric method can be seen in Neuts (1981). Analytic solution of bulk service queueing models can be found in (Madhil and Choudhury 1986 and Medhi 1984). But these works do not include vacations. In general it is difficult to obtain a closed form solution for bulk service queueing models with servers vacation. Afthab Begum (1996) has obtained analytic solutions for M/M(a,b)/1 queue, E_k/M(a,b)/1 queue and GI/M(a,b)/1 queue with servers single and multiple vacation in her Ph.D. thesis and presented the steady state results.

Since Servi and Finn's (2002) pioneer work on M/M/1 working vacation queueing systems, the research on working vacation queues has considerable development. In sections (6.1) and (6.2), a bulk arrival queueing system is analysed under single and multiple working vacations and PGFs of the systems are presented in a closed form so that various performance measures can be calculated directly. In this section (6.3), a general bulk service queueing system M/M(a,b)/1 with server's multiple working vacation is analysed.

The steady state queue size probabilities are obtained by solving the difference equations and the results of Liu *et al.,* (2007) are obtained as particular case of the model. Numerical examples are presented and the graphical representations of the effects of regular service rate (μ), vacation service (μ_v) and vacation parameters (η) on the expected queue length are also presented.

This model is denoted by **M/M(a,b)/1/MWV.**

6.3.1: Mathematical Analysis of M/M(a,b)/1/MWV

I Model Description

In this model it is assumed that the arrival process is Poisson with parameter λ. The server processes the customers in batches according to the General Bulk Service Rule (GBSR) introduced by Neutus (1967).

According to this rule, the server starts service only when a minimum of '**a**' customers are present in the system. If the server after a service completion finds **a** (or) more but at most **b** customers present in the system, then he takes them all in a batch, and if he finds more than **b**, then he takes in the batch the first **b**-customers for service, while others wait. Thus each batch for service contains a minimum of '**a**' units and a maximum of '**b**' units. This rule is called general bulk service rule (GBSR). The service time of batches of size s ($a \leq s \leq b$) is assumed to be independent identically distributed random variable with exponential distribution of parameter μ.

Whenever the server completes a service and finds less than '**a**' customers in the queue he begins a vacation which is an exponentially distributed random variable V with parameter η. After completing a vacation, if the system length is still less than '**a**' he takes another vacation and the vacations are continued until the server finds at least '**a**' customers in the queue (i.e.,) multiple vacation is adopted. Suppose during the vacation if the queue size becomes at least '**a**' the server starts his service under the GBSR with service rate μ_v which is different from the regular service rate μ. When the vacation ends he switches his service rate from μ_v to μ. When the server changes his service mode from μ_v rate to μ, the size of the service batch that being served remains unchanged . When the server is working, the size of the batch in service is x with $a \leq x \leq b$, and the service rates are independent of the size of the batch in service.

The above queueing model is denoted by **M/M(a,b)/1/MWV** and the steady state results including the probability distribution of the queue size and the expected number of units in the queue are obtained.

II Steady State System Size Equations

Let $N_Q(t)$ = the number of customers in the queue at time t and J(t) = 0, 1 or 2 according as the server is idle on vacation, busy on vacation or in regular busy state respectively.

Let
$$R_n(t) = Pr\{N_Q(t) = n, J(t) = 0\} \qquad 0 \le n \le a-1$$
$$Q_n(t) = Pr\{N_Q(t) = n, J(t) = 1\} \qquad n \ge 0$$
$$P_n(t) = Pr\{N_Q(t) = n, J(t) = 2\} \qquad n \ge 0$$

when J(t) = 0, the queue size and the system size are same.
when J(t) = 1 (or) 2, the number of customers in the system is the sum of the number of customers in queue and the size of the service batch containing $a \le x \le b$ customers.

By assuming the steady state probabilities

$$Q_n = \lim_{t\to\infty} Q_n(t) \;;\; R_n = \lim_{t\to\infty} R_n(t) \;;\; P_n = \lim_{t\to\infty} P_n(t) \quad \text{exist,}$$

the Chapman Kolmogrov equations satisfied by them in the steady state are given by

$$\lambda R_0 = \mu P_0 + \mu_v Q_0 \qquad (6.3.1)$$

$$\lambda R_n = \lambda R_{n-1} + \mu P_n + \mu_v Q_n \qquad 1 \le n \le a-1 \qquad (6.3.2)$$

$$(\lambda + \eta + \mu_v) Q_1 = \lambda R_{a-1} + \mu_v \sum_{n=a}^{b} Q_n \qquad (6.3.3)$$

$$(\lambda + \eta + \mu_v) Q_n = \lambda Q_{n-1} + \mu_v Q_{n+b} \qquad n \ge 1 \qquad (6.3.4)$$

$$(\lambda + \mu) P_0 = \mu \sum_{n=a}^{b} P_n + \eta Q_0 \qquad (6.3.5)$$

$$(\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+b} + \eta Q_n \qquad n \ge 1 \qquad (6.3.6)$$

III Steady State Solution

To solve the steady state equations, the forward shifting operator E on P_n and Q_n are introduced as

$$E(P_n) = P_{n+1} \;;\; E(Q_n) = Q_{n+1} \qquad \text{for } (n \ge 0)$$

Thus the equation (6.2.4) gives homogeneous difference equation,

$$(\mu_v E^{b+1} - (\lambda + \mu_v + \eta)E + \lambda)\, Q_n = 0 \qquad n \geq 0 \qquad (6.3.7)$$

The characteristic equation of the difference equation is given by

$$h(z) = \mu_v z^{b+1} - (\lambda + \mu_v + \eta)\, z + \lambda = 0$$

By taking $f(z) = (\lambda + \mu_v + \eta)\, z$; $g(z) = \mu_v z^{b+1} + \lambda$ it is found that $|g(z)| < |f(z)|$ on $|z| = 1$ and hence by Rouche's theorem h(z) has only one root r_v inside the contour $|z| = 1$. Further the root is real, since $h(0) = \lambda$ and $h(1) = -\eta$ are of opposite sign and hence the solution of the homogeneous difference equation (6.3.7) is given by

$$Q_n = (r_v^n)\, Q_0 \qquad n \geq 0 \qquad (6.3.8)$$

Next, the equation (6.3.6) can be written as

$$(\mu E^{b+1} - (\lambda + \mu)E + \lambda)\, P_n = -\eta\, Q_{n+1} \qquad n \geq 0$$

$$= -\eta\, r_v^{n+1} Q_0 \qquad n \geq 0 \qquad (6.3.9)$$

Again by Rouche's theorem, the equation $\mu z^{b+1} - (\lambda + \mu)\, z + \lambda = 0$ has unique root r with $|r| < 1$ provided $\frac{\lambda}{b\mu} < 1$ (Medhi 1981).

Hence the solution of the non-homogeneous difference equation (6.3.9) is given by,

$$P_n = \left(A r^n - \frac{\eta\, r_v^{n+1}}{\mu\, r_v^{b+1} - (\lambda + \mu)\, r_v + \lambda} \right) Q_0$$

$$= (A\, r^n + B\, r_v^n)\, Q_0 \qquad (6.3.10)$$

$$\text{where } B = \frac{\eta\, r_v}{\lambda (r_v - 1) + \mu\, r_v (1 - r_v^b)} \quad \text{if } r_v \neq r \qquad (6.3.11)$$

The expression for R_n $(0 \leq n \leq a-1)$ is obtained by adding equations (6.3.1) and (6.3.2) over 0 to n.

$$\text{(i.e)}\ \lambda \sum_{k=0}^{n} R_n = \lambda \sum_{k=0}^{n-1} R_k + \mu \sum_{k=0}^{n} P_k + \mu_v \sum_{k=0}^{n} Q_k$$

$$\lambda R_n = \mu \sum_{k=0}^{n} P_k + \mu_v \sum_{k=0}^{n} Q_k$$

Substituting for Q_n and P_n from equations (6.3.8) and (6.3.10) it is found that

$$R_n \quad = \quad \left[\frac{\mu}{\lambda} \left(\frac{A(1-r^{n+1})}{(1-r)} + \frac{B(1-r_v^{n+1})}{(1-r_v)} \right) + \frac{\mu_v}{\lambda} \frac{(1-r_v^{n+1})}{(1-r_v)} \right] Q_0$$

Hence the steady state queue size probabilities are expressed in terms of the unknowns A and Q_0.

Now to calculate A, the equation (6.3.5)

$$(\lambda + \mu) P_0 = \mu \sum_{n=a}^{b} P_n + \eta Q_0 \quad \text{is considered.}$$

Substituting for P_n from equation (6.3.10) it is found that

$$A \left(\lambda + \mu - \frac{\mu (r^a - r^{b+1})}{(1-r)} \right) = \left(\eta - B \left((\lambda + \mu) - \mu \frac{(r^a - r^{b+1})}{(1-r_v)} \right) \right) \tag{6.3.12}$$

which can be simplified as

$$\frac{A \mu (1-r^a)}{(1-r)} = \frac{\eta}{(1-r_v)} - \frac{B \mu (1-r_v^a)}{(1-r_v)} \tag{6.3.13}$$

The equation (6.3.3) is also verified.

Hence the steady state queue size probabilities of the model are expressed in terms of Q_0 and are given by

$$Q_n = r_v^n Q_0 \qquad n \geq 0 \tag{6.3.14}$$

$$P_n = (A r^n + B r_v^n) Q_0 \qquad n \geq 0 \tag{6.3.15}$$

where $$A = \frac{(1-r)}{\mu (1-r^a)} \left[\frac{\eta}{(1-r_v)} - \frac{B \mu (1-r_v^a)}{(1-r_v)} \right]$$

with $$B = \frac{\eta r_v}{\mu r_v (1-r_v^b) + \lambda (r_v - 1)}$$ and

$$R_n = \left[\frac{\mu}{\lambda} \left(\frac{A(1-r^{n+1})}{(1-r)} + \frac{B(1-r_v^{n+1})}{(1-r_v)} \right) + \frac{\mu_v}{\lambda} \frac{(1-r_v^{n+1})}{(1-r_v)} \right] Q_0 \qquad 0 \leq n \leq a-1 \tag{6.3.16}$$

and the expression for Q_0 can be calculated by using the normalizing condition $\sum_{n=0}^{\infty} Q_n + \sum_{n=0}^{\infty} P_n + \sum_{n=0}^{a-1} R_n = 1$.

(i.e.) $Q_0^{-1} = F(r_v, \mu_v) + AF(r, \mu) + BF(r_v, \mu)$

where $F(x, y) = \dfrac{1}{(1-x)}\left(1+\dfrac{y}{\lambda}\left(a-\dfrac{x(1-x^a)}{(1-x)}\right)\right)$

6.3.2: Performance Measures

I Mean Queue Length

The expected queue length (L_q) of the model is calculated, as

$$L_q = \sum_{n=1}^{\infty} n(Q_n + P_n) + \sum_{n=1}^{a-1} nR_n \qquad (6.3.17)$$

Substituting the values of Q_n, P_n and R_n from (6.3.14) to (6.3.16), Lq is simplified as

$$L_q = A\,H(r, \mu) + B\,H(r_v, \mu) + H(r_v, \mu_v)$$

where $H(x, y) = \dfrac{x}{(1-x)^2} + \dfrac{y}{\lambda(1-x)}\left\{\dfrac{a(a-1)}{2} + \dfrac{a\,x^{a+1}(1-x) - x^2(1-x^a)}{(1-x)^2}\right\}$

II Other Performance Measures

If P_v, P_{busy} and P_{idle} respectively denote the probability that the server is busy in vacation state , is regular busy and is idle in vacation state then

i) $P_v = \sum_{n=0}^{\infty} Q_n = \dfrac{Q_0}{(1-r_v)}$

ii) $P_{busy} = \sum_{n=0}^{\infty} P_n = \sum_{n=1}^{\infty} (A\,r^n + B\,r_v^n)\,Q_0 = \left(\dfrac{A}{(1-r)} + \dfrac{B}{(1-r_v)}\right)Q_0$

iii) $P_{idle} = \sum_{n=0}^{a-1} R_n$

6.3.3: Particular Cases

Case 1: M/M/1 multiple working vacation model.

The steady state queue size probabilities of M/M/1 multiple working vacation (Liu *et al.*, (2007)) queueing model are deduced.

The equations (6.3.14) to (6.3.16) at a = b = 1 imply

$$Q_n = r_v^n Q_0 \qquad n \geq 0$$

$$P_n = \frac{B}{r_v}(r_v^{n+1} - r^{n+1})Q_0 . \qquad n \geq 0$$

and $$R_0 = (Q_0 / r_v)$$

where $r = \frac{\lambda}{\mu} = \rho$, $A = -\frac{B\rho}{r_v}$ and $B = \frac{\eta r_v}{\mu(1-r_v)(r_v - \rho)}$

These give the **queue** size probabilities of M/M/1 multiple working vacation queueing model (Liu *et al.*,2007).

Case 2: M/M(a,b)/1 classical multiple vacation model

The results of the classical multiple vacation (Afthab Begum 1996) queuing model M / M (a, b) / 1 can be derived by letting $\mu_v = 0$ in the multiple working vacation model. When $\mu_v = 0$, $r_v = \frac{\lambda}{\lambda + \eta}$

with this $$Q_n = r_v^n Q_0 \qquad n \geq 0$$

$$P_n = (A r^n + B r_v^n) Q_0 \qquad n \geq 0$$

and $$R_n = \frac{\mu}{\lambda}(A h_n(r) + B h_n(r_v)) Q_0$$

where $$B = \frac{\eta}{\mu(1-r_v^b) - \eta} \quad \text{and} \quad \frac{A(1-r^a)}{(1-r)} = \frac{\lambda+\eta}{\mu} - \frac{B(1-r_v^a)}{(1-r_v)}$$

Further $$Q_0^{-1} = A F(r, \mu) + B F(r_v, \mu) + \frac{1}{(1-r_v)}$$

where $$F(x, y) = \frac{1}{(1-x)}\left(1 + \frac{y}{\lambda}\left(a - \frac{x(1-x^a)}{(1-x)}\right)\right)$$

and the mean queue length is given by

$$L_q = A H(r, \mu) + B H(r_v, \mu) + \frac{r_v}{(1-r_v)^2}$$

where $$H(x, y) = \frac{x}{(1-x)^2} + \frac{y}{\lambda(1-x)}\left\{\frac{a(a-1)}{2} + \frac{a x^{a+1}(1-x) - x^2(1-x^a)}{(1-x)^2}\right\}$$

6.4 Numerical Analysis

In this section numerical results are obtained to study the effects of various parameters namely, mean vacation time ($1/\eta$), service rate during working vacation (μ_v), regular service rate (μ) and the mean batch size ($E(X)$) on the expected system size of batch arrival multiple working vacation model (L_{mwv}), single working vacation model (L_{swv}) and the bulk service model ($L_{(a,b)}^{mwv}$).The effect of the parameters on the system size probabilities and on the expected system size when the server is in different states are also analysed. For the numerical computation, the batch size X is assumed to follow the geometric distribution with probabilities $Pr(X=k) = (1-p)\, p^{k-1}$ and mean $E(X) = 1/(1-p)$. z_1, r_v and r respectively denote the roots of characteristic equations, $\lambda z(1 - X(z)) + \eta z + \mu_v (z - 1) = 0$,

$$\mu_v z^{b+1} - (\lambda + \mu_v + \eta) z + \lambda = 0;$$

$$\mu z^{b+1} - (\lambda + \mu) z + \lambda = 0$$ that lie inside interval (0,1).

In classical vacation models, since the service is stopped completely during vacation, the system size increases notably as the mean vacation time increases. But in working vacation models, since the service is done with a smaller rate μ_v ($< \mu$) during vacation, the vacation parameter η has less effect on the system size. The effects of η and μ_v on the expected system size under two situations ((i.e.,) ρ =.3 and ρ=.9) are presented in tables (6.1a) and (6.1b). The graphical representation for the case ρ = .3 can be seen in figures (6.1 a to c) for all the three models.

The table values and the figures show that as μ_v (service rate during vacation) or η increases the mean system (queue) size decreases. Also the table values show that the effect of η on L is very much notable for smaller values of η. It is interesting to note that as μ_v approaches 0, the system or queue size of the working vacation models approaches the system or queue size of the corresponding classical vacation models. Further the mean system size of working vacation models and classical non-vacation models coincide when $\mu_v = \mu$.

Table (6.1a) Mean system size of working vacation and classical vacation models with respect to η and μ_v
$(p, \lambda, \mu) = (.75, 0.0675, 0.9)$ $\rho = .3$ $(\lambda, \mu, a, b) = (4.05, 0.9, 5, 15)$.

$\rho = (\lambda E(X) / \mu)$ $\rho = (\lambda / b\mu)$

μ_v	η	$M^X/M/1/MV$		$M^X/M/1/SV$		z_1	M/M(5,15)/1		r =.8268
		L_{mwv}	L_{mv}	L_{swv}	L_{sv}		$L^{mwv}_{(5,15)}$	$L^{mv}_{(5,15)}$	r_v
	.01	28.260	28.714	27.753	28.208	0.0614	391.780	400.271	0.9974
.005	.05	7.0441	7.1143	5.7578	5.821	0.0411	75.6210	76.9760	0.9875
	.5	2.2505	2.2543	1.7854	1.786	0.0087	7.9736	8.0241	0.8892
	.01	24.264	28.714	23.751	28.208	.4278	327.570	400.271	0.9969
.05	.05	6.4405	7.1143	5.2276	5.821	.3112	64.0852	76.9760	0.9854
	.5	2.2165	2.2543	1.7804	1.786	.0816	7.5449	8.0241	0.8819
	.01	20.0686	28.714	19.5572	28.208	0.6373	255.893	400.271	0.9962
.1	.05	5.8256	7.1143	4.7014	5.821	0.4891	51.894	76.9760	0.9821
	.5	2.1799	2.2543	1.7751	1.786	0.1505	7.1174	8.0241	0.8734
	.01	3.8405	28.714	3.6360	28.208	0.9636	9.6618	400.271	0.9143
.5	.05	2.8251	7.1143	2.4066	5.821	0.8689	8.2392	76.9760	0.9008
	.5	1.9216	2.2543	1.7400	1.786	0.4738	5.0136	8.0241	0.8051
	.01	1.7143	28.714	1.7143	28.208	0.9848	4.0681	400.271	0.8248
.9	.05	1.7143	7.1143	1.7143	5.821	0.9325	4.0681	76.9760	0.8171
	.5	1.7143	2.2543	1.7143	1.786	0.6216	4.0681	8.0241	0.7446

mwv (swv)-multiple (single) working vacations. mv(sv) – classical multiple (single) vacation

Table (6.1b) Mean system size of working vacation and classical vacation models with respect to η and μ_v

(p, λ , μ) = (.75, 0.2025, 0.9) ρ = .9 (λ , μ , a, b) = (12.14 , 0.9, 5, 15)

ρ = (λ E(X) / μ) ρ = (λ / b μ)

μ_v	η	M^X /M/1 /MV		M^X /M/1 /SV		z_1	M/M(5,15)/1		r =.0.9867
		L_{mwv}	L_{mv}	L_{swv}	L_{sv}		$L^{mwv}_{(5,15)}$	$L^{mv}_{(5,15)}$	r_v
.005	.01 .05 .5	116.5024 52.1020 37.6107	117.0000 52.2000 37.6200	116.3137 51.3454 36.5836	116.8122 51.4449 36.5875	0.0231 0.0194 0.0070	1250.8322 292.0712 85.0210	1263.8726 293.6290 85.138	0.9992 0.9958 0.9603
.05	.01 .05 .5	112.0263 51.2216 37.5272	117.0000 52.2000 37.6200	111.8292 50.4521 36.5465	116.8122 51.4449 36.5875	0.1995 0.1709 0.0668	1189.1204 278.9744 84.1864	1263.8726 293.6290 85.138	0.9991 0.9956 0.9586
.1	. 01 .05 .5	107.0592 50.2471 37.4348	117.0000 52.2000 37.6200	106.8521 49.4649 36.5066	116.8122 51.4449 36.5875	0.3462 0.3004 0.1257	1114.3763 264.3677 83.2655	1263.8726 293.6290 85.138	0.9991 0.9953 0.9568
.5	.01 .05 .5	68.0000 42.6938 36.7068	117.0000 52.2000 37.6200	67.6871 41.9663 36.2248	116.8122 51.4449 36.5875	0.8333 0.7515 0.4271	519.5676 151.2693 77.1483	1263.8726 293.6290 85.138	0.9979 0.9904 0.9375
.9	.01 .05 .5	35.9999 35.9999 36.0000	117.0000 52.2000 37.6200	35.9999 35.9999 36.0000	116.8122 51.4449 36.5875	0.9558 0.8819 0.5805	73.2146 73.2147 73.2140	1263.8726 293.6290 85.138	0.9811 0.9689 0.9118

mwv (swv)-multiple (single) working vacations. mv(sv) – classical multiple (single) vacation.

Mean System Size Vs η and μ_v for $\rho = .3$

Fig. (6.1a)

$M^X/M/1/MWV$ model

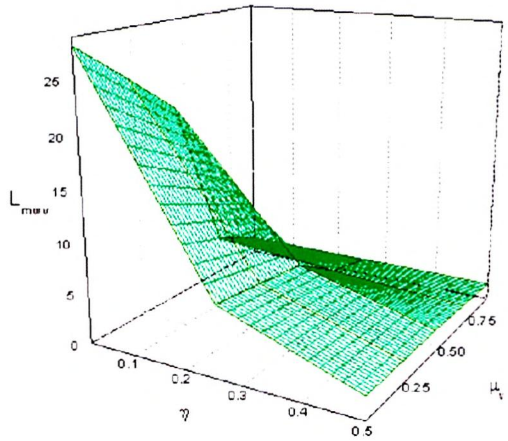


Fig. (6.1b)

$M^X/M/1/SWV$ model

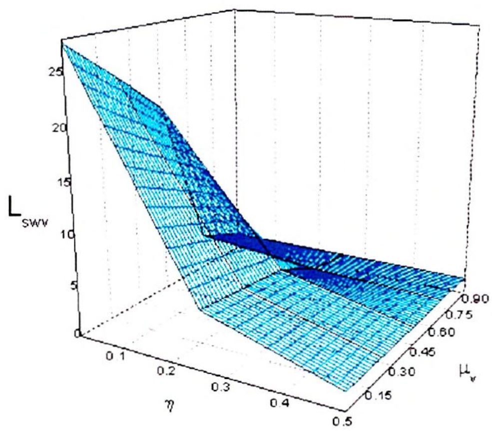
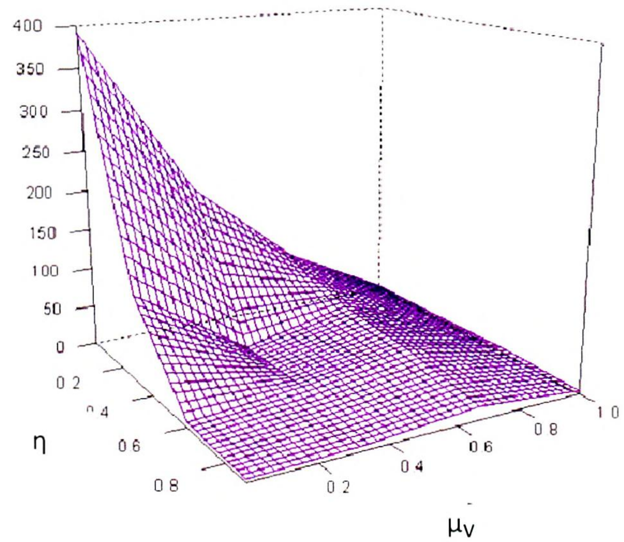


Fig. (6.1c)

M/M(a,b)/1/MWV model



In table (6.2), the values of the expected system size for all the three models are presented for different arrival rates λ and the service rates μ_v (in vacation) for two different regular service rates μ. It is noted from the table (6.2) and graphs (6.2a and b) that L increases with λ and decreases as μ_v increases.

Table (6.2): System size with respect to λ and μ_v for μ = 2.5 and 5

The parameters are (η, p) = (.01, .75) ; (η, a, b) = (.01, 5, 15)

λ	μ_v	.05	.2	.4	.6	.8
.2	μ=2.5	77.0572 76.8644 126.5050	62.7841 62.5828 13.6400	44.5729 44.3627 4.4680	28.4174 28.2066 2.9666	16.5035 16.3097 2.4611
	μ = 5	75.9759 75.7848 126.4706	61.8323 61.6384 13.6400	43.8228 43.6275 4.4652	27.8895 27.6990 2.9648	16.1722 15.9975 2.4598
.3	μ=2.5	118.7813 118.6501 224.2580	104.1225 103.9842 44.7085	84.8357 84.6873 8.2167	66.0785 65.9198 4.5237	48.3768 48.3097 3.3051
	μ = 5	116.3919 116.2629 224.2431	101.8667 101.7337 44.7125	82.7963 82.6594 8.2133	64.3068 64.1669 4.5204	46.9271 46.7851 3.3023
.4	μ=2.5	162.1572 162.0579 323.2525	147.3271 147.2221 117.2524	127.6551 127.5414 15.035	108.1684 108.0449 6.8488	89.0030 88.8689 4.5532
	μ = 5	156.9689 156.8703 323.2911	142.2739 142.1729 117.2898	122.8237 122.7191 15.0348	103.6187 103.5106 6.8449	84.8123 84.7008 4.5491

L_{mwv} L_{swv} $L^{mwv}_{(5,15)}$

Mean System Size Vs λ and μ_v when $\mu = 2.5$

Fig. (6.2a) $M^x/M/1/MWV$ model

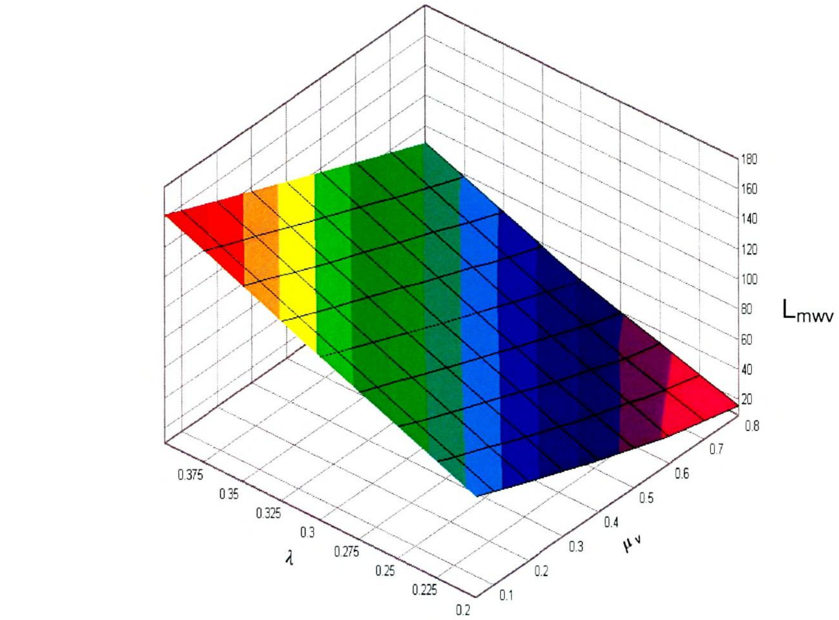
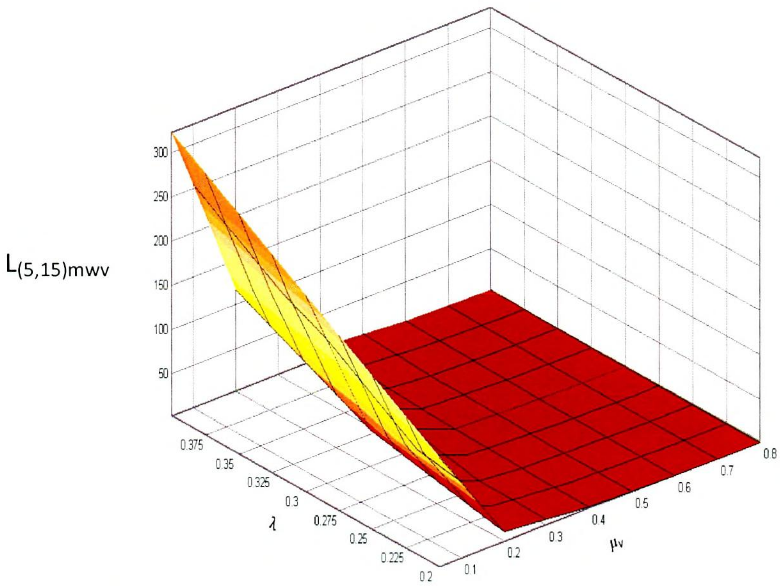


Fig. (6.2b) M /M (5,15)/1/MWV model



The system size probabilities P_v, $P_{busy,}$ P_I and the expected system size L_v, L_{busy} corresponding to different states of the server are presented in table (6.3) . The table values show the effect of the parameters on the performance measures. The parameters chosen are $(p , \eta , \lambda , \mu_v , \mu) = (.75, .05 , .3, 1, 2)$

Table (6.3.) : System measures with respect to system parameters

parameters		P_v	P_I	$P_{busy,}$	L_v	L_{busy}
η	.01	0.719 0.714	- 0.002	0.281 0.284	22.510 22.359	11.627 11.589
	.05	0.620 0.577	- 0.028	0.380 0.395	6.077 5.653	7.519 7.413
	.1	0.571 0.480	- 0.064	0.429 0.456	3.429 2.880	6.857 6.720
λ	.1	0.947 0.672	- 0.232	0.053 0.096	1.702 1.209	0.360 0.546
	.3	0.620 0.577	- 0.028	0.380 0.395	6.077 5.653	7.519 7.413
	.4	0.333 0.320	- 0.008	0.667 0.672	5.333 5.120	24.000 23.680
μ_v	.6	0.523 0.500	- 0.018	0.477 0.482	7.629 7.287	11.722 11.466
	.7	0.547 0.520	- 0.020	0.453 0.460	7.247 6.883	10.529 10.302
	.8	0.571 0.539	- 0.022	0.429 0.438	6.857 6.472	9.429 9.236
μ	1.5	0.380 0.348	- 0.017	0.620 0.635	3.721 3.408	18.481 18.272
	1.75	0.529 0.489	- 0.024	0.471 0.487	5.182 4.792	10.612 10.471
	2	0.620 0.577	- 0.028	0.380 0.395	6.077 5.653	7.519 7.413

L_{mwv} (red) L_{swv} (blue)

The table values of (6.4) show that the expected system size increases with the mean batch size E(X) for different values of μ_v and η.

Table (6.4): System size with respect to various batch size

(λ, μ,)=(.1, 2) for M^x / M/1 / wv ;

(λ, μ, a, b) = (8, 2, 5, 20) for M^x / M(a,b)/1 / mwv

μ_v	.1		.5		1		1.5	
η	E(X)=4	E(X)=10	E(X)=4	E(X)=10	E(X)=4	E(X)=10	E(X)=4	E(X)=10
.01	31.95 31.58 598.63	100.52 99.59 719.89	7.67 7.41 35.67	64.71 63.69 403.43	2.48 2.37 8.09	31.27 30.41 56.77	1.41 1.37 4.73	15.98 15.56 8.40
.05	7.61 6.54 118.76	28.36 25.58 143.92	3.88 3.29 23.27	22.42 19.99 83.37	2.06 1.75 7.59	16.67 14.99 24.09	2.06 1.75 7.59	12.68 11.88 7.71
.1	4.435 3.224 58.89	19.26. 16.07 72.17	2.82 2.07 17.47	16.57 14.03 43.31	1.80 1.44 7.08	13.81 12.16 16.62	1.80 1.44 7.08	11.67 10.88 7.14
.5	1.735 1.140 11.910	11.88 10.36 15.53	1.51 1.09 7.51	11.44 10.27 11.3	1.30 1.05 5.21	10.92 10.16 7.64	1.30 1.05 5.21	10.44 10.08 5.58

L_{mwv} (red) L_{swv} (blue) $L^{mwv}_{(5,15)}$ (green)