MULTI-CHANNEL MARKOVIAN QUEUES

By

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CONTENTS

SYNOPSIS

In Introduction, the definitionof Queueing Theory, their basic characteristics and the notations are given.

Chapter I is devoted to study the paper "OPERATING STRATEGIES FOR A MULTI-VEHICLE TRANSPORTATION SYSTEM" by S.H. SIM and J.G.C. TEMPLETON. In this chapter, a multi-vehicle transportation systen with one source terminal is considered. We assume passenger arrivals to be described by a Poisson process. The dispatching policy is of the form:

Dispatch a vehicle if the passenger queue is atleast as large as some control limit α. It is assume that vehicles have sufficient capacity that a passenger can always board the next departing vehicle. A minimum average cost criterion is used to determine the firm's fleet size and dispatching strategy. This is a generalization of the results of Weiss (1979) for a single-vehicle transportation system.

Chapter II is dealt with the "COMPUTATIONAL PROCEDURES FOR STEADY STATE CHARACTERISTICS OF UN-SCHEDULED MULTI-CARRIER SHUTTLE SYSTEM" by S.H. SIM and J.G.C. TEMPLETON. The transportation system considered in this paper has a number of vehicles with no capacity constraint, which take passengers from a source terminal to various destinations and return to the terminal. The trip times are considered to be independent and identically distributed random variables with a common exponential distribution passengers arrive at the terminal in accordance with a Poisson process. The system is operated under the following policy: When a vehicle is available and there are atleast α passengers waiting for service, then a vehicle is dispatched immediately. The passenger queue length and waiting time distributions are obtained under steady state conditions. System performance measures such as average passenger, passenger queue length and waiting time are then derived. A minimum average cost criterion is then used to determine the optimal fleet size and dispatching policy. This is a generalization of the results of Weiss (1979) for a single-vehicle system.

At the end of this dissertation, a list of Bibliography used for reference is added.

"Mathematics - in a strict sense - is the abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations".

INTRODUCTION

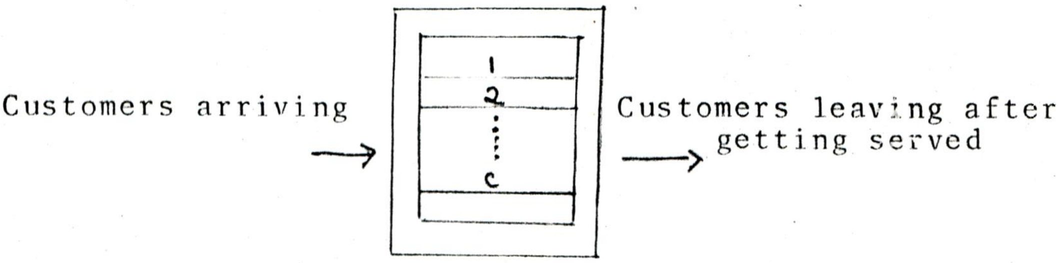
INTRODUCTION

The study of queueing system has increasingly occupied the attention of researchers, since the annoyance of having to wait in line is becoming more and more prevalent in our increasingly congested and urbanised society. A queueing system can be described as customers arriving for service and waiting in case of it is not immediate and if having waiting for service, leaving the system after getting served.

The study of queues is mainly applied in the field of business (banks, super markets, booking storage), industries (serving of automatic machines production lines storage), engineering (telephones, communication networks, electronic computers), transportation (air-ports, harbours, railways, traffic operation in cities, postal services) and in common places like elevators, restaurants, barber shops etc. These are concerned with the design and planning of service facilities to meet a randomly fluctuating demand for service, so that congestion is minimised and economic balance between the cost of service and the cost associated with waiting for the service is maintained.

Queueing Systems and their Basic Characteristics:

A system consisting of a servicing facility a process of arrival of customers who wish to be served by the facility and the process of service is called a queueing system. Graphically, a queueing system can be described as follows:



Arrival Pattern of Customers:

The arrival pattern is measured in terms of the mean arrival rate or the mean inter-arrival time. If the arrivals are strictly according to schedule, queues can be avoided, but in practice this is not the case and in most situations arrivals are controlled by factors external to the system.

Arrivals may occur in batches or one at a time. When more than one arrival can enter the system simultaneously, the input is said to occur in bulk or batches. In the bulk arrival situation, the time between successive arrivals and the number of customers in a batch are probabilistic.

If the queue is too long, a customer may decide not to enter the system upon his arrival and he is said to have balked.

On the other hand, a customer may enter the queue but after sometime he may lose patience and decide to leave and in this case, he is said to have reneged. When there are two or more parallel waiting lines, customers may switch over from one to another and it is called jockeying for position.

If an arrival patterno does not change with time, it is called a stationary arrival pattern, otherwise it is called nonstationary.

Service Mechanism:

The service mechanism describes the manner, in which service is rendered. Service may be deterministic or probabilistic. Random variable representations of these characteristics are essential.

Customers may be served singly or in batches. There are many situations where a batch of customers is served by a single server.

In many queueing situations, the number of servers may depend on queue length. As soon as the number of customers exceeds a preassinged number, an additional server may be employed. When the queue size becomes less than or equal to the preassinged number the additional server may be withdrawn after the completion of service on hand.

System Capacity:

Some of the queueing processes admit the physical limitation to the amount of waiting room so that when the queue reaches a certain length no further customers are allowed to enter until space becomes available by a service completion. In some cases, the capacity can be considered to be infinite.

The finite queueing systems can be viewed as having forced balking where a customer is forced to balk if he arrives at a time when the queue size is at its maximum limit.

Service Channels:

Queueing system may have several service channels toprovide service. These service channels may be arranged in parallel or in series or as a more complex combination of both, depending on the design of the system's service mechanism.

In parallel channels, a number of channels provide identical service facilities so that several customers may be served simultaneously. In series channels, a customer must pass through successfully all the ordered channels before service is completed.

A queueing system is called a single server model when the system has only one server and when the system has a number of parallel servers it is known as multisever model.

Queue Discipline:

Queue disciplien is the rule according to which customers are selected for service when a queue is formed. The most common disicpline is 'First in First Out (FIFO) rule under which the customers are served in the strict order of their arrivals. Another queue discipline is "Last in First Out"(LIFO). According to this rule, the last arrival in the system is served first. In some cases, the arrivals are served randomly irrespective of their arrivals in the system "Priority" discipline allows priority in service to some customers in relation to other customers waiting in the queue.

Notation

A queueing process is usually represented by the notation A/B/C/X/Y where A is the interarrival time distribution and B is the service time distribution. "C" is the number of parallel servers or sevice channels, X is the system capacity and "Y" is the queue discipline. This notation is called Kendall Notation.

π_j = Steady state probability that j passengers are waiting at the terminal $j \leq 0$.

P_m = Steady state probability that m vehicles are available at the terminal immediately after a dispatch m = 0,1,2,.....N-1.

c = Cost of dispatching a vehicle.

w = Waiting cost per passenger per unit time.

D(N) = Cost per unit time associated with the fleet size N.

fw(t) = Probability density function (pdf) of W, the waiting time of an arriving passenger in the queue.

$\Psi(N, \alpha)$ = Long-run average cost per unit time associated with the dispatching policy α and a fleet size N.

Methods for Solving Queueing Models:

Queueing mmodels can be classified into Markovian Queueing models and non-Markovian queueing models.

Markovian Queueing Modells:

Queueing models with inter-arrival time of customers and service time exponentially distributed are called Markovian queueing models.

Markovian queueing models can be analysed by the following methods:

a) the difference-differential equation method.

b) the Matrix-geometric algorithmic method.

Non-Markovian Queueing Models:

In practice, there are models that do not rely on strict Markov assumptions. Queuing models having the inter-arrival times and/or service times which are not exponentially distributed are called non-Markovian queueing models.

Development of Queueing Theory:

The queueing theory has its origin in 1909, when the Danish Mathematician A.K. Erlang published his paper relating to the study of congestion in telephone traffic, namely "The theory of probabilities and telephone conversations". Being the pioneer researcher, Erlang is called the father of queueing theory. Felix Follaezele did some further pioneering work in the early 1930's. These early works in queueing theory picked up momentum slowly and now intensive research is going on in this field.

Thus queueing theory has proved to be a very useful tool and we anticipate that its use will continue to grow as recognition of the many guises of queueing theorem.

"I find both a special pleasure and constraint in describing the progress of Mathematics because it has been part of so much speculation; as ladder for mystical as well as rational thoughts in the intellectual ascent of man".

-Jacob Bronowski

CHAPTER - I

CHAPTER - I

OPERATING STRATEGIES FOR A MULTI-VEHICLE TRANSPORTATION SYSTEM

In this chapter we are interested in a transportation system in which dispatching of vehicles is carried out from a source terminal. There are N vehicles in the system. Each vehicle takes passengers from the source terminal to various destinations and returns after some random trip time to make another trip. The travel times of successive trips are assumed to be independent, identically distributed exponential random variables with mean $1/\mu$. Passengers arrive at the source terminal according to a poisson process with rate λ.

Vehicle dispatching decisions are made at the following two types of decision epoch:-

i) Upon the arrival of a vehicle at the source terminal

ii) At every passenger arrival, if there is at least one waiting vehicle at the source terminal.

At eacch of the above review points, one has the option of holding or dispatching a vehicle immediately. The dispatching strategy is as follows:-

Dispatch a vehicle if the passenger queue exceeds or equals a contorl limit α . If no vehicle is available, the passenngers will continue to wait.

It is also assumed that vehicles have sufficient capacity that a passenger can always board the next departing vehicle.

The performance measures for the above system have been obtained by Sim and Templeton[5] using a two-dimensional state space formulation. The above queue-dependent dispatching policy has beer shown by Deeb and Serfozo[2] to exhibit optimal properties for the one-vehicle system. Weiss[7] obtained the "optimal' queue limit α for the one-vehicle system by minimizing the long-run average cost of waiting and service charges.

In this paper, the distribution of number of passengers waiting at the terminal is obtained by using a one-dimensionnal state-space formulation. A minimum average cost criterion is then used to determine the firm's fleet size and dispatching strategy. This cost consists of:-

i) dispatching cost

ii) passenger waiting costs

iii) cost associated with the fleet size.

Section 2: PASSENGER QUEUE LENGTH DISTRIBUTION

Define:-

π_j = Steady state probability that j passengers are waiting at the terminal $j \geq 0$.

P_m = Steady state probability that m vehicles are available at the terminal immediately after a dispatch. m = 0,1,2,.....,N-1.

β = Steady state probability that passengers have arrived at the terminal before a vehicle arrival and that there are no vehicles at the terminal after the last dispatch.

$$= \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} P_o$$

The steady state probability equations describing the passenger queue are:-

$$\lambda \pi_j = \lambda \pi_{j-1} \quad , 1 \leq j \leq \alpha - 1 \qquad .. 1.2.1$$

$$\lambda \pi_o = N\mu \sum_{k=\alpha}^{\infty} \pi_k + \lambda \pi_{\alpha-1}(1 - \beta) \qquad .. 1.2.2$$

$$(\lambda + N\mu)\pi_\alpha = \lambda \pi_{\alpha-1} \beta \qquad .. 1.2.3$$

$$(\lambda + N\mu)\pi_j = \lambda \pi_{j-1}, \quad j \geq \alpha + 1 \qquad .. 1.2.4$$

From equation 1.2.1:

$$\pi_j = \pi_o , \quad 1 \leq j \leq \alpha - 1 \qquad .. 1.2.5$$

From equation 1.2.2

$$\lambda \pi_0 = N\mu \sum_{K=\alpha}^{\infty} \pi_K + \lambda \pi_{\alpha-1}(1-\beta)$$

$$\therefore N\mu \sum_{K=\alpha}^{\infty} \pi_K = \lambda \pi_0 - \lambda \pi_{\alpha-1}(1-\beta)$$

$$\sum_{K=\alpha}^{\infty} \pi_K = \frac{1}{N\mu}(\lambda \pi_0 - \lambda \pi_{\alpha-1}[1-\beta])$$

Using equation 1.2.5

$$\sum_{K=\alpha}^{\infty} \pi_K = \frac{1}{N\mu}[\lambda \pi_0 - \lambda \pi_0 (1-\beta)]$$

$$= \frac{\lambda \pi_0}{N\mu} . \beta$$

But we know $\beta = \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} P_0$

$$\therefore \sum_{K=\alpha}^{\infty} \pi_K = \frac{\lambda}{N\mu}\left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} \pi_0 P_0 \qquad .. 1.2.6$$

From equation 1.2.3

$$(\lambda + N\mu)\pi_\alpha = \lambda \pi_{\alpha-1} \beta$$

$$\pi_\alpha = \frac{\lambda \pi_{\alpha-1} \beta}{\lambda + N\mu}$$

$$= \frac{\lambda}{\lambda + N\mu}\left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} \pi_{\alpha-1} P_0$$

Use equation 1.2.5

$$\pi_\alpha = \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha+1} P_0 \pi_0 \qquad .. 1.2.7$$

From equation 1.2.4

$$(\lambda + N\mu)\pi_j = \lambda \pi_{j-1}, \quad j \geq \alpha + 1$$

$$\pi_j = \frac{\lambda \pi_{j-1}}{(\lambda + N\mu)}$$

Using equation 1.2.7

$$\pi_j = \left(\frac{\lambda}{\lambda + N\mu}\right)\left(\frac{\lambda}{\lambda + N\mu}\right)^j P_o \pi_o$$

$$= \left(\frac{\lambda}{\lambda + N\mu}\right)^{j+1} P_o \pi_o \qquad \text{.. 1.2.8}$$

But $\sum_{j=o}^{\infty} \pi_j = 1$

$$\sum_{j=o}^{\infty} \pi_j = \sum_{j=o}^{\alpha-1} \pi_j + \sum_{j=\alpha}^{\infty} \pi_j = 1$$

But we know $\pi_j = \pi_o,\ 1 \leq j \leq \alpha - 1$ and from equation 1.2.6 we get

$$\sum_{j=o}^{\alpha-1} \pi_o + \sum_{j=\alpha}^{\infty} \pi_j = \pi_o + \frac{\lambda}{N\mu}\left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} P_o \pi_c = 1$$

$$\Rightarrow \pi_o \left[\alpha + \frac{\lambda}{N\mu}\left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} P_o\right] = 1$$

$$\Rightarrow \pi_o = \left[\alpha + \frac{\lambda}{N\mu}\left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} P_o\right]^{-1}$$

Hence, the steady state probabilities π_j can be expressed in terms of α, λ, μ and P_o

$$\pi_j = \begin{cases} \left[\alpha + \frac{\lambda}{N\mu}\left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} P_o\right]^{-1}, & 0 \leq j \leq c-1 \\ \left(\frac{\lambda}{\lambda + N\mu}\right)^{j+1} \left[\alpha + \frac{\lambda}{N\mu}\left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} P_o\right]^{-1} P_o, & j \geq \alpha \end{cases}$$

.. 1.2.9

These results are in agreement with those obtained by Sim and Templeton[5] using a two-dimensional state space formulation.

The average passenger queue size is given by

$$E(Q) = \sum_{j=1}^{\infty} j\,\pi_j = \sum_{j=1}^{\alpha} j\,\pi_j + \sum_{j=\alpha}^{\infty} j\,\pi_j$$

Using equation 1.2.9

$$E(Q) = \sum_{j=1}^{\alpha-1} j\left[\alpha + \frac{\lambda}{N\mu}\left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha} P_o\right]^{-1} +$$

$$\sum_{j=\alpha}^{\infty} j\left[\alpha + \frac{\lambda}{N\mu}\left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha} P_o\right]^{-1}\left(\frac{\lambda}{\lambda+N\mu}\right)^{j+1} P_o$$

$$= \sum_{j=1}^{\alpha-1} j\,\pi_o + \sum_{j=\alpha}^{\infty} j\,\pi_o\left(\frac{\lambda}{\lambda+N\mu}\right)^{j+1} P_o$$

$$= \pi_o[1+2+3+\ldots\ldots\ldots+\alpha-1] + \pi_o P_o\left[\alpha\left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha+1}\right.$$

$$\left.+(\alpha+1)\left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha+2} + \ldots\ldots\ldots\right]$$

$$= \pi_o\frac{\alpha(\alpha-1)}{2} + \pi_o P_o\left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha+1}\left[\alpha+(\alpha+1)\right.$$

$$\left.\left(\frac{\lambda}{\lambda+N\mu}\right) + (\alpha+2)\left(\frac{\lambda}{\lambda\alpha+N\mu}\right)^2 + \ldots\ldots\ldots\right]$$

$$= \pi_o\frac{\alpha(\alpha-1)}{2} + \pi_o P_o\left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha+1}\left[\alpha + \alpha\left(\frac{\lambda}{\lambda+N\mu}\right)\right.$$

$$\left.+\left(\frac{\lambda}{\lambda+N\mu}\right)^2+\ldots\ldots\right] + \left[\frac{\lambda}{\lambda+N\mu} + 2\left(\frac{\lambda}{\lambda+N\mu}\right)^2\right.$$

$$\left.\ldots\ldots\ldots\right]$$

$$= \pi_o \frac{\alpha(\alpha - 1)}{2} + \pi_o P_o \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha+1}$$

$$\left(1 - \frac{\lambda}{\lambda + N\mu}\right)^{-1} + \frac{\lambda}{\lambda + N\mu}\left(1 - \frac{\lambda}{\lambda + N\mu}\right)^{-2}$$

$$= \pi_o \frac{\alpha(\alpha - 1)}{2} + \pi_o P_o \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha+1}$$

$$\alpha\left(\frac{\lambda + N\mu - \lambda}{\lambda + N\mu}\right)^{\alpha-1} + \frac{\lambda}{\lambda + N\mu}\left(\frac{\lambda + N\mu - \lambda}{\lambda + N\mu}\right)^{\alpha-1}$$

$$= \pi_o \frac{\alpha(\alpha - 1)}{2} + \pi_o P_o \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha+1} \alpha \left(\frac{N\mu}{\lambda + N\mu}\right)^{\alpha-1} +$$

$$\frac{\lambda}{\lambda + N\mu}\left(\frac{N\mu}{\lambda + N\mu}\right)^{\alpha-2}$$

$$= \pi_o \frac{\alpha(\alpha - 1)}{2} + \pi_o P_o \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha+1} \alpha \cdot \frac{\lambda + N\mu}{N\mu}$$

$$+ \pi_o P_o \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha+1} \frac{\lambda}{\lambda + N\mu}\left(\frac{\lambda + N\mu}{N\mu}\right)^2$$

$$= \pi_o \frac{\alpha(\alpha - 1)}{2} + \pi_o P_o \frac{\lambda^{\alpha+1}}{(\lambda + N\mu)^{\alpha}} \frac{1}{N\mu}$$

$$+ \pi_o P_o \frac{\lambda^{\alpha+2}}{(N\mu)^2} \cdot \frac{1}{(\lambda + N\mu)^{\alpha}}$$

$$= \pi_o \alpha \frac{(\alpha - 1)}{2} + \alpha \pi_o P_o \frac{\lambda}{N\mu}\left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha}$$

$$+ \pi_o P_o \left(\frac{\lambda}{N\mu}\right)^2 \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha}$$

$$= \pi_o \frac{\alpha(\alpha-1)}{2} + \pi_o P_o\left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha}\left(\frac{\lambda}{N\mu}\right)\left[\alpha+\frac{\lambda}{N\mu}\right]$$

$$\therefore\ E(Q) = \left\{\frac{\alpha(\alpha-1)}{2} + \frac{\lambda}{N\mu}\left[\alpha+\frac{\lambda}{N\mu}\right]\left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha} P_o\right\}\pi_o \qquad \text{.. 1.2.10}$$

Also, the average passenger waiting time in the queue is given by

$$E(W) = \frac{E(Q)}{\lambda}$$

Section:3 COST FUNCTION AND OPTIMIZATION

Define the following cost parameters:-

i) C = Cost of dispatching a vehicle. This includes fuel consumption and vehicle depreciation.

ii) w = Waiting cost per passenger per unit time.

iii) $D(N)$ = Cost per unit time associated with the fleet size N. This includes maintenance and insurance costs of the fleet as well as associated administrative and labour costs. It is assumed that D(.) is monotonically increasing and $D(0)=0$.

The long-run average cost per unit time associated with the dispatching strategy α and a fleet size N is given by

$$\psi(N,\alpha) = D(N) + \sum_{n=0}^{\alpha} w\, j\pi_j + \frac{c}{E(H)} \qquad .. \ 1.3.1$$

where $E(H)$ is the expected time between dispatches.

$$E(H) = \frac{\alpha}{\lambda} + \frac{1}{N\mu}\left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} P_0$$

Using equation 1.2.9

Since $\pi_j = [\alpha + \frac{\lambda}{N\mu}(\frac{\lambda}{\lambda + N\mu})^{\alpha} P_o]^{-1}$, $0 \leq j \leq \alpha - 1$

$= \pi_o$

Multiplying E(H) by λ,

$$\lambda E(H) = \lambda\, [\frac{\alpha}{\lambda} + \frac{1}{N\mu}(\frac{\lambda}{\lambda + N\mu})^{\alpha} P_o]$$

$$= \alpha + \frac{\lambda}{N\mu}(\frac{\lambda}{\lambda + N\mu})^{\alpha} Po$$

$$E(H) = \pi_o^{\alpha - 1}$$

$$\therefore \quad E(H) = (\lambda\, \pi_o)^{-1} \qquad .. 1.3.2$$

Using equation 1.3.1 and 1.3.2, we obtain,

$$\Psi(N, \alpha) = D(N) + w[\pi_o \frac{\alpha(\alpha - 1)}{2} + \pi_o P_o (\frac{\lambda}{\lambda + N\mu})^{\alpha}$$

$$\frac{\lambda}{N\mu}(\alpha + \frac{\lambda}{N\mu})] + c\, \lambda \pi_o \qquad .. 1.3.3$$

where $\pi_o = [\alpha + \frac{\lambda}{N\mu}(\frac{\lambda}{\lambda + N\mu})^{\alpha} P_o]^{-1}$

An optional dispatching policy is found by minimizing $\Psi(N, \alpha)$ with respect to N and α .

"I cherish both faces of Mathematics, the pure as a beautiful retreat from reality and the applied as an ardent hoper for life".

CHAPTER-II

CHAPTER - II

COMPUTATIONAL PROCEDURES FOR STEADY STATE CHARACTERISTICS OF UNSCHEDULED MULTI-CARRIER SHUTTLE SYSTEMS

In this chapter we study a transportation system with one source terminal and a fleet of N vehicles, each of maximum capacity b. Each vehicle takes passengers from the terminal to various destinations and returns after some random trip time to make anothe trip. The travel times of successive trips are independent and identically distributed random having mean trip time $1/\mu$. After a vehicle returns to the terminal, one has the option holding it or dispatching it immediately. Passengers arrive individually in accordance with a Poisson process having rate λ. The system is operated under the following policy:

When a vehicle is available and there are atleast α passengers waiting to be served, then a vehicle is dispatched. The above system may be denoted by $M/M(\alpha, b)/N$.

The $M/M(\alpha, b)/N$ system has been studied by several authors and algorithmic solutions have been given, but no complete analytic solution is known. We consider the limiting case $b \to \infty$, and assume that vehicles have sufficient capacity that an arriving passenger can always board the next departing vehicle. In this limiting case, we give analytical expressions for the following steady state characteristics of the system at a random epoch:

i) Passenger queue length distribution

ii) Passenger waiting time distribution

iii) Average passenger queue length

iv) The r-th moment of passenger waiting time in the queue.

The above quantities are obtained as functions of P_o, the steady state probability that there is no vehicle at the terminal immediately after dispatch.

A minimum average total cost criterion is then used to determine the optimal fleet size and dispatching policy. The total cost consists of:

i) dispatching cost

ii) passenger waiting costs

iii) cost associated with the fleet size.

SECTION 2

2.1 One-Vehicle Systems:

One-vehicle transportation systems with service in batches of atleast 'α' and at most 'b' customers have been studied, mostly by traditional methods, in a large number of papers.

Kosten[20,21] was perhaps the first researcher to consider one-vehicle systems with dispatching policy of the forum considered in this paper. He studied the $M/D(\alpha, \infty)/1$ system in his doctoral thesis in 1942. Most of the research on one-vehicle systems has been carried out during the last fifteen years.

Neuts[26] studied the $M/G(\alpha, b)/1$ queue using the semi-Markov process technique and obtained the transient state distribution of the number of customers in the system. Borthakur and Medhi[10] derived the queue length distribution for the $M^{(x)}/G(\alpha, b)/1$ queue using the supplementary variable technique. Borthakur[12] has also studied the busy period distribution for the $M^{(x)}/G(\alpha, b)/1$ queue and presented a detailed solution for the $M/M(\alpha, b)/1$ queue. In addition, Medhi[23] derived the steady state waiting time distribution in the $M/M(\alpha, b)/1$ queue. Steady state results were also obtained by Easton and Chaudry[17] for the $E_K/M(\alpha, b)/1$ queue.

All the above studies of single finite-capacity vehicle systems use methods which require the application of Rouche's theorem. Algorthimic methods, not using Rouche's theorem, were used by Neuts[28] to analyse one-vehicle systems. Neuts[27] developed an algorithmic approach for the analysis of he M/G(α ',b)/1 queue. His approach involves only real arithmetic and avoids the calculation of complex roots based on Rouche's theorem.

Thus, we see that one-vehicle systems have been extensively studied in the literature. It is interesting to note that the dispatching strategy for our one-vehicle transportation system possesses optimal properties. This was shown by Deb and Serfozo[15] and by Weiss[7].

2.2 Multi-Vehicle Systems:

Multi-vehicle transportation systems with queue-dependent dispatching policies have been studied by Osuna and Newell[31], Asgharzadeh and Newell[11], Ghare[19], Medhi and Borthakur[25], Medhi[22], Neuts and Nadarajan[29] and Sim and Templeton[5]. Note that two of these papers [11, 31] assume that passenger inter-arrival times are constant while the other papers assume poisson arrivals.

Given that the travel times of successive trips are inndependent and identically distributed random variables with a known distribution(not necessarily an exponential distribution) and that the vehicles have no capacity constraints, Osuna and Newell[31] determined the optimal vehicle dispatching strategy by formulating the problem as a dynamic programming model with the objective of minimizing the long term average wait per passenger at the source terminal. The dynamic programming model was solved by direct anlytical methods for N=1. For N=2, only crude approximations were obtained. A more thorough analysis of this two-vehicle system was subsequently carried out by Newell [30] using approximation methods.

By assuming that vehicles have exponentially distributed trip times, Asgharzadeh and Newell[1] were able to extend Osuna and Newell's results. They prescribed a dispatching strategy of vehicles was kept at the terminal in orderr to regulate the pattern oof departures. Dispatching deccisions were made depending on the number of vehicles being held at the terminal and on the time since the last dispatch. That is, if there are m vehicles waiting at the terminal, dispatch a vechile if the time since the last dispatch exceeds or equals T_m where

$m = 1,2,\ldots\ldots N$ and $T_1, T_2. T_3\ldots\ldots T_N > 0$. However, for $N >> 1$, Asgharzadeh and Newell were able to obtain an approximate analytical expression for the optimal dispatch rate as a function of the number of vehicles at the terminal.

It is interesting to note that since the passenger interarrival times are constant, the optimal dispatching strategy of Asgharzadeh and Newell can be written as follows:

If there are m vehicles waiting at the terminal, dispatch a vehicle if the passenger queue exceeds or equals a_m where $m = 1,\ldots\ldots,N$ and $\alpha > a_1 \geq a_2 \geq a_3 \geq \ldots\ldots\ldots\ldots \geq a_N > 0$. Hence, the transportation system of Asgharzadeh and Newell is a varient of the D/M(α , α)/N model.

Ghare[19] was able to obtain an explicit expression forr the steady state distribution of the number of passengers in the M/M(1,b)/N queue. Further steady state results including the waiting time distribution in the queue were obtained by Cromie and Chaudry[14]. The GI/M(1,b)/N system was studied by shyu[33,34] using the imbedded Markov Chain method. No numerical results were given. Roes[32] has also discussed the GI/M(1,b)/N queue using the theory of derived Markov CChains which is due to Cohen[13]. Using the semi-Markov process technique, Love[22] developed the steady state solution for the F_k/M(b.b)/N queue.

Medhi and Borthakur[25] derived the queue length distribution for the M/M(α ,b)/2 system. Medhi[24] was able to study the waiting time distribution in the M/M(α ,b)/N queue under steady state conditions and obtained numerical results for one-vehicle and two-vehicle systems. The M/M(α ,b)/N queue appears to be analytically interactable for N > 2, but algorithmic solutions have been given by Neuts and Nadarajan[29] and by the present authors [36].

Assuming that vehicles have sufficient capacity that an arriving passenger can always board the next departing vehicle, we are able to perform a more detailed and complete analysis of the multi-vehicle transportation system considered by Medhi [24], Neuts and Nadarajan[29] and Sim and Templeton [36].

Section 3: DISTRIBUTION OF VEHICLES AT THE TERMINAL IMMEDIATELY AFTER DISPATCH

The future behaviour of the system depends only on the state vector.

$$\tilde{x} = \langle m, j \rangle \quad (m = 0,\ j \geq 0) \text{ or } (1 \leq m \leq N,\ 0 \leq j < \alpha)$$

where m = number of vehicles at the source terminal.

j = number of passenges waiting at the source terminal.

Since the trip times of the vehicceles are assumed to be exponentially distributed, the number of vehicles available at the source terminal immediately after a dispatch can be modelled as a Markov chain with state space $\{< m, 0 > 1, m = 0, 1, 2, \ldots\ldots. N-1\}$.

Let $P_n(k)$ be the probability of having n vehicles at the source terminal immediately after the k-th dispatch and let $q_{m,n}(k)$ be the conditional probability of having n vehicles after the k-th dispatch given hat we had m vehicles after the previous dispatch. Then

$$\sum_{n=0}^{N-1} P_n(k) = 1 \quad \text{and} \quad \sum_{n=o}^{N-1} q_{m,n}(k) = 1$$

where $m = 0,1,2,\ldots\ldots,N-1$ and $k = 1,2,3,\ldots\ldots$

Under the Markovian condition stated above, we have

$$q_{m,n}(k) = q_{m,n} \qquad k = 1,2,\ldots\ldots\ldots$$

Assuming that the system will be in equilibrium after operating for a sufficient length of time, $\lim_{k} P_n(k) = P_n$. The equilibrium distribution satisfies the equation

$$\underline{P}\,\underset{\sim}{Q} = \underline{P}$$

where $Q = (q_{m,n})$; m and $n = 0,1,2,\ldots\ldots N-1$ is a stochastic matrix and $P = (P_0, P_1, P_2, \ldots\ldots P_{N-1})$ satisfies $\sum_{n=0}^{N-1} P_n = 1$.

The transition probabilities $q_{m,n}$ are obtained as follows:

The trip times are assumed to be exponentially distributed with probability distribution given by $F(t) = 1-e^{-\mu t}$. If there are m vehicles in the terminal and no dispatch occurs during a time interval ti, then for each of the (N-m) vehicles enroute, there is a probability $(1-e^{-\mu t})$ that it will return within the time interval t and $e^{-\mu t}$ that it will not. Hence, the probability $q_{m,m+\ell}(t)$ of exactly ℓ vehicle arrivals in the interval t, given that m vehicles were in the terminal at the beginning of the interval and no dispatch occurred during the interval, is the binomial probability of ℓ successes and $N-m-\ell$ failures when the probability of a failure is equal to $e^{-\mu t}$. That is

$$\tilde{q}_{m,m+\ell}(t) = \begin{bmatrix} N-m \\ \ell \end{bmatrix} (e^{-\mu t})^{N-m-\ell} (1-e^{-\mu t})$$

The probability density function of the time interval required for the arrival of α passengers is given by the gamma density

$$f(\lambda,\alpha;t) = \frac{\lambda(\lambda t)^{\alpha-1}}{(\alpha-1)!} e^{-\mu t}$$

For the case $(1 < m < n-1,\ m-1 < n < N-1)$ or $(m=0,\ 1 < n < N-1)$, the transition probabilities are given by:

$$q_{m,n} = \int_0^{\infty} \tilde{q}_{m,n+1}(t) f(\lambda,\alpha;t)\,dt$$

$$= \int_0^{\infty} \begin{bmatrix} N-m \\ n-m+1 \end{bmatrix} (1-e^{-\mu t})^{n-m+1} e^{-(\lambda+(N-n-1)\mu)t} \frac{\lambda(\lambda t)^{\alpha-1}}{(\alpha-1)!}\,dt$$

$$= \begin{bmatrix} N-m \\ n-m+1 \end{bmatrix} \int_0^{\infty} \Big[1-(n-m+1)C_1(e^{-\mu t})^{n-m} + (n-m+1)C_2(e^{-\mu t}) - \dots\dots\dots\dots + (-1)^m (e^{-\mu t})^{n-m+1} \Big] e^{-[\lambda+(N-n-1)\mu]t} \frac{\lambda(\lambda t)^{\alpha-1}}{(\alpha-1)!}\,dt$$

$$= \frac{\lambda^{\alpha}}{(\alpha-1)!} \begin{bmatrix} N-m \\ n-m+1 \end{bmatrix} \int_0^{\infty} \Big\{ e^{-[\lambda + (N-n-1)\mu]t}$$

$$- (n-m+1)\, e^{[-\mu(n-m) - \lambda - (N-n-1)\mu]t}$$

$$+ \frac{(n-m+1)(n-m)}{2}\, e^{[-\mu(n-m-1) - \lambda(N-n-1)\mu]t}$$

$$- \ldots\ldots\ldots + (-1)^m e^{[\mu(n-m+1 - \lambda - (N-n-1)\mu]t} \Big\}$$

$$t^{\alpha-1}\, dt$$

But we know $\int_0^{\infty} e^{-a\varkappa} \varkappa^{\alpha}\, d\varkappa = \frac{\alpha!}{a^{\alpha+1}}$

$$\therefore q_{m,n} = \frac{\lambda^{\alpha}}{(\alpha-1)!} \begin{bmatrix} N-m \\ n-m+1 \end{bmatrix} \Big\{ \frac{(\alpha-1)!}{[\lambda + (N-n-1)\mu]^{\alpha}} -$$

$$\frac{(n-m+1)(\alpha-1)!}{(\lambda + N\mu - m\mu - \mu)} + \frac{(n-m+1)(n-m)}{2} \cdot$$

$$\frac{(\alpha-1)!}{(\lambda + N\mu - m\mu - 2\mu)^{\alpha}}$$

$$(-1)^{n-m+1} \frac{(\alpha-1)1}{(\lambda + N - (m+1)\mu]^{\alpha}} \Big\}$$

$\therefore$ m = 0

$1 \leq n \leq N-1$

$$= \begin{bmatrix} N-m \\ n-m+1 \end{bmatrix} \Big\{ \frac{\lambda^{\alpha}}{[\lambda + (N-n-1)\mu]^{\alpha}} +$$

$$(-1)^1 \frac{(n-m+1)\lambda^{\alpha}}{[\lambda + (N-n)\mu]^{\alpha}}$$

$$+ (-1)^2 \frac{(n-m+1)(n-m)}{2} \frac{\lambda^{\alpha}}{[\lambda + (N-n+1)\mu]^{\alpha}}$$

$$+ \ldots\ldots\ldots + (-1)^{n-m+1} \frac{\lambda\alpha}{[\lambda + (N-m+1)\mu]^{\alpha}} \Big\}$$

$$\therefore \quad q_{m,n} = \begin{bmatrix} N-m \\ n-m+1 \end{bmatrix} \sum_{r=0}^{n-m+1} (-1)^r \begin{bmatrix} n-m+1 \\ r \end{bmatrix} \left[\frac{\lambda}{\lambda + (N-n+r-1)\mu} \right]^{\alpha}$$

The transition probability $q_{0,0}$ is obtained as follows:

$$q_{0,0} = P_{rob} \{ \alpha \text{ or more passengers arrive before the first vehicle arrival/N vehicles are out} \}$$

$$+ \sum_{r=0}^{\alpha-1} \text{Prob} \{ \text{exactly } r \text{ passengers arrive before the first vehicle arrival and atleast } \alpha - r \text{ passengers arrive between first and second vehicle arrivals/N vehicles are out} \}$$

$$= \sum_{r=\alpha}^{\infty} \left[\frac{\lambda}{\lambda + N\mu}\right]^r \frac{N\mu}{\lambda + N\mu} + \sum_{r=0}^{\alpha-1} \left[\frac{\lambda}{\lambda + N\mu}\right]^r \frac{\lambda}{\lambda + N\mu} \left[\frac{\lambda}{\lambda + (N-1)\mu}\right]^{\alpha - r}$$

$$= \frac{N\mu}{\lambda + N\mu} \sum_{r=0}^{\alpha-1} \left[\frac{\lambda}{\lambda + N\mu}\right]^r \left[\frac{\lambda}{\lambda + (N-1)\mu}\right]^{\alpha - r} + \sum_{r=\alpha}^{\infty} \left[\frac{\lambda}{\lambda + N\mu}\right]^r$$

$$= \frac{N\mu}{\lambda + N\mu} \left\{ \left\{ \left[\frac{\lambda}{\lambda + (N-1)\mu}\right]^{\alpha} + \left[\frac{\lambda}{\lambda + (N-1)\mu}\right]^{\alpha-1} * \left[\frac{\lambda}{\lambda + N\mu}\right] + \cdots \right.\right.$$

$$\left. + \left[\frac{\lambda}{\lambda + N\mu}\right]^{\alpha-1} \left[\frac{\lambda}{\lambda + (N-1)\mu}\right]^{1} \right\} + \left\{ \left[\frac{\lambda}{\lambda + N\mu}\right]^{\alpha} + \left[\frac{\lambda}{\lambda + N\mu}\right]^{\alpha+1} + \cdots\cdots\cdots \right\}\Bigg\}$$

$$= N \left[\frac{\lambda}{\lambda + (N-1)\mu}\right]^{\alpha} - (N-1)\left[\frac{\lambda}{\lambda 1 + N\mu}\right]^{\alpha}$$

Since we do not dispatch more than one vehicle at a time,

$$q_{m+\ell, m} = 0 \cdot \qquad \ell = 2, 3, \ldots\ldots\ldots, N-m-1$$
$$m = 0, 1, 2, \ldots\ldots\ldots, N-2$$

Define $\omega_m = \dfrac{\lambda}{\lambda + m\mu}$, $m = 0, 1, 2, \ldots\ldots\ldots N$

Then we have,

$$q_{m,n} = \begin{cases} N\omega^{\alpha}_{N-1} - (N-1)\omega^{\alpha}_{N} & (m=0, n=0) \\ \begin{bmatrix} N-m \\ n-m+1 \end{bmatrix} \sum_{r=0}^{n-m+1} (-1)^r \begin{bmatrix} n-m+1 \\ r \end{bmatrix} \omega^{\alpha}_{N+r-n-1} & (1 \le m \le N-1,\ 1 \le m \le N-1) \text{ or } (m = 0,\ 1 \le n \le N-1) \\ 0 & \text{otherwise} \end{cases} \quad \ldots 2.3.1$$

Section 4: PASSENGER QUEUE LENGTH DISTRIBUTION

Let $P(m,j)$ be the steady state probability that there are m vehicles available at the terminal and j passengers are present in the queue. We shall assume that vehciles have sufficient capacity that a passenger can always board the next departing vehicle. Hence $P(m,j)$ is defined only when $(m=0,\ j\geq 0)$ or $(1\leq m\leq N,\ 0\leq j<\alpha)$.

Following Medhi and Borthakur[25], the steady state probability equations are

$$(\lambda + N\mu)\,P(0,0) = \lambda\,P(1,\alpha-1) + N\sum_{k=\alpha}^{\alpha} P(0,k) \qquad 2.4.1$$

$$(\lambda + N\mu)\,P(0,j) = \lambda\,P(o,j-1),\ j\geq 1 \qquad \ldots\ 2.4.2$$

$$(\lambda + N\mu)\,P(m,j) = \lambda\,P(m+1,\alpha-1) + (N-m+1)\,P(m-1,0)$$

$$1\leq m\leq N-1 \qquad \ldots\ 2.4.3$$

$$\lambda\,P(N,0) = \mu\,P(N-1,0) \qquad \ldots\ 2.4.4$$

$$[\lambda + (N-m)\mu]\,P(m,j) = \lambda\,P(m,j-1) + (N-m+1)\,\mu\,P(m-1,j)$$

$$1\leq m\leq N,\ 1\leq j\leq \alpha-1 \qquad \ldots 2.4.5$$

$$\sum_{j=o}^{\alpha} P(0,j) + \sum_{m=1}^{N}\sum_{j=0}^{\alpha-1} P(m,j) = 1 \qquad \ldots\ 2.4.6$$

Using equations 2.4.2 and 2.4.5

$$\sum_{m=0}^{N} P(m,j) = \sum_{m=0}^{N} P(m,j-1) = \sum_{m=0}^{N} P(m,o)$$

$$1\leq j\leq \alpha-1 \qquad \ldots\ 2.4.7$$

From equation 2.4.2

$$P(0,j) = \frac{\lambda}{\lambda + N\mu}\,P(0,j-1) = w_N^j\,P(0,0) \qquad \ldots\ 2.4.8$$

Define π_j = Steady state probability that j passengers are waiting the terminal

$$\pi_j = \begin{cases} \sum_{m=0}^{N} P(m,0) & 0 \leq j \leq \alpha - 1 \\ P(0,j) & j \geq \alpha \end{cases} \quad \ldots 2.4.9$$

Therefore, since $\sum_{j=0}^{\infty} \pi_j = 1$,

$$\sum_{j=0}^{\infty} \pi_j = \sum_{j=0}^{\alpha-1} \pi_j + \sum_{j=\alpha}^{\infty} \pi_j = 1$$

$$\Rightarrow \sum_{j=0}^{\alpha-1} \sum_{m=0}^{N} P(m,0) + \sum_{j=\alpha}^{\infty} P(0,j) = 1$$

$$\Rightarrow \alpha \pi_0 + \sum_{j=\alpha}^{\infty} P(0,j) = 1 \quad \ldots 2.4.10$$

Substituting equation2.4.8 into equation 2.4.10

$$\alpha \pi_0 + \sum_{j=\alpha}^{\infty} w_N^j P(0,0) = 1$$

$$\alpha \pi_0 + \sum_{j=\alpha}^{\infty} \left(\frac{\lambda}{\lambda + N\mu}\right)^j P(0,0) = 1$$

$$\alpha \pi_0 + \left[\left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} + \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha+1} + \ldots\ldots\right] P(0,0) = 1$$

$$\alpha \pi_0 + \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} \left[1 + \frac{\lambda}{\lambda + N\mu} + \left(\frac{\lambda}{\lambda + N\mu}\right)^2 + \ldots\ldots\right] p(0,0) = 1$$

$$\alpha \pi_0 + \left(\frac{\lambda}{\lambda + N\mu}\right)^{\alpha} \left(1 - \frac{\lambda}{\lambda + N\mu}\right)^{-1} P(0,0) = 1$$

$$\alpha\pi_o + \left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha}\left(\frac{\lambda+N\mu-\lambda}{\lambda+N\mu}\right)^{-1} P(0,0) = 1$$

$$\alpha\pi_o + \left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha}\left(\frac{N\mu}{\lambda+N\mu}\right)^{-1} \frac{\lambda}{N\mu}\frac{N\mu}{\lambda} P(0,0)=1$$

$$\alpha\pi_o + \left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha}\left(\frac{\lambda}{\lambda+N\mu}\right)^{-1} \frac{\lambda}{N\mu}\frac{N\mu}{\lambda} P(0,0)=1$$

$$\alpha\pi_o + \left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha}\left(\frac{\lambda}{\lambda+N\mu}\right)^{-1} \frac{\lambda}{N\mu} P(0,0) = 1$$

$$\alpha\pi_o + \left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha-1} \frac{\lambda}{N\mu} P(0,0) = 1$$

$$\alpha\pi_o + \frac{\lambda}{N\mu} w_N^{\alpha-1} P(0,0) = 1 \qquad ..2.4.11$$

where P(0,0) is determined as follows.

By regarding the states 'Terminal empty' and 'Terminal occupied' as defining an alternating renewall process and applying Feller[18], we obtain

$$P(0,0) = \frac{E[\text{Time during which system is empty between two successive dispatches}]}{E[\text{Time between two successive dispatches}]} \qquad ..2.4.12$$

E[Time during which system is empty between two successive dispatches]

= E[Time to first arrival/system was empty just after last dispatch] P(system was empty just after last dispatch)

$$= \frac{1}{\lambda+N\mu} P_o \qquad ..\ 2.4.13$$

E[Time between successive dispatches]

= P_o E[Inter-departure time/system was empty after last departure] + $(1-P_o)$ E[Inter-departure time/ system was not empty after last deparrture]

$$= P_o[\frac{\alpha}{\lambda} + (\frac{\lambda}{\lambda+N\mu})^{\alpha} \frac{1}{N\mu} + (1-P_o)\frac{\alpha}{\lambda}$$

$$= \frac{\alpha}{\lambda} P_o + (\frac{\lambda}{\lambda+N\mu}) \frac{1}{N\mu} + \frac{\alpha}{\lambda} - \frac{\alpha}{\lambda} P_o$$

$$= \frac{1}{\lambda} [\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} P_o] \qquad ..2.4.14$$

Therefore using 2.4.12, 2.4.13 and 2.4.14

$$P(0,0) = \frac{\frac{1}{\lambda+N\mu} P_o}{\frac{1}{\lambda}[\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} P_o]}$$

$$= \frac{\lambda}{\lambda+N\mu} P_o [\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} P_o]^{-1}.$$

$$= w_N[\alpha + w_N^{\alpha} \frac{\lambda}{N\mu} P_o]^{-1} P_o \qquad .. 2.4.15$$

Using equations 2.4.15, 2.4.11 and 2.4.8 equation 2.4.9 becomes,

Using 2.4.15 and 2.4.8

$$P(0,j) = w_N^j P(0,0)$$

$$= w_N^j \; w_N [\alpha + w_N^{\alpha} \frac{\lambda}{N\mu} P_o]^{-1} P_o$$

$$= w_N^{j+1} [\alpha + w_N^{\alpha} \frac{\lambda}{N\mu} P_o]^{-1} P_o$$

Using 2.15 and 2.4.11 we get

From 2.4.11

$$\alpha \pi_o + \frac{\lambda}{N\mu} w_N^{\alpha -1} P(0,0) = 1$$

$$\alpha \pi_o + \frac{\lambda}{N\mu} w_N^{\alpha -1} w_N [\alpha + w_N^{\alpha} \frac{\lambda}{N\mu} P_o]^{-1} P_o = 1$$

$$\alpha \pi_o + \frac{\lambda}{N\mu} w_N^{\alpha} \pi_o P_o = 1$$

$$\pi_o [\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} P_o] = 1$$

$$\pi_o = [\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} P_o]^{-1}$$

Therefore 2.4.9 becomes

$$\pi_j = \begin{cases} [\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} P_o]^{-1} & 0 \leq j \leq \alpha -1 \\ w_N^{j+1} [\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} P_o]^{-1} P_o & j \geq \alpha \end{cases} \qquad 2.4.16$$

The average passenger queue size is given by

$$E(Q) = \sum_{j=1}^{\infty} j \pi_j$$

Using equation 2.416 we get

$$E(Q) = \sum_{j=1}^{\alpha - 1} j \pi_j + \sum_{j=\alpha}^{\infty} j \pi_j$$

$$= \sum_{j=1}^{\alpha - 1} j [\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} P_o]^{-1} + \sum_{j=\alpha}^{\infty} j w_N^{j+1} [\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} P_o]^{-1} P_o$$

$$PUt\ [\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} P_o]^{-1} = \pi_o$$

$$= \sum_{j=1}^{\alpha-1} j\pi_o + \sum_{j=\alpha}^{\infty} j\, w_N^{j+1} \pi_o P_o$$

$$= \pi_o \sum_{j=1}^{\alpha-1} j + \pi_o P_o \sum_{j=\alpha}^{\infty} j\, w_N^{j+1}$$

$$= \pi_o [1+2+\ldots\ldots\ldots\ldots+\alpha-1] + \pi_o P_o [\alpha\, w_N^{\alpha+1} +$$

$$(\alpha+1)\alpha\, w_N^{\alpha+2} + \ldots\ldots\ldots\ldots]$$

$$= \pi_o \frac{\alpha(\alpha-1)}{2} + \pi_o P_o [(\alpha\, w_N^{\alpha+1} + \alpha\, w_N^{\alpha+2}$$

$$+ \ldots\ldots\ldots\ldots] + w_N^{\alpha+2} + 2w_N^{\alpha+3} + \ldots\ldots\ldots)]$$

$$= \pi_o \frac{\alpha(\alpha-1)}{2} + \pi_o P_o [\alpha\, w_N^{\alpha+1} (1+w_N + w_N^2$$

$$+ \ldots\ldots\ldots\ldots) + w_N^{\alpha+1} (w_N + 2w_N^2 + \ldots\ldots\ldots]$$

$$= \frac{\alpha(\alpha-1)\pi_o}{2} + \pi_o P_o [\alpha (1-w_N)^{-1} +$$

$$w_N (1-w_N)^{-2}]$$

$$= \frac{\alpha(\alpha-1)\pi_o}{2} + \pi_o P_o (\frac{\lambda}{\lambda+N\mu})^{\alpha+1} [\alpha (1 - \frac{\lambda}{\lambda+N\mu})^{-1}$$

$$+ \frac{\lambda}{\lambda+N\mu} (1 - \frac{\lambda}{\lambda+N\mu})^{-2}]$$

$$= \frac{\alpha(\alpha-1)\pi_o}{2} + \pi_o P_o (\frac{\lambda}{\lambda+N\mu})^{\alpha+1} [\alpha (\frac{\lambda+N\mu-\lambda}{\lambda+N\mu})^{-1}$$

$$+ \frac{\lambda}{\lambda+N\mu} (\frac{\lambda+N\mu-\lambda}{\lambda+N\mu})^{-2}]$$

$$= \frac{\alpha(\alpha-1)\pi_o}{2} + \pi_o P_o \left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha+1} \left\{ \left[\left(\frac{\lambda N\mu}{\lambda+N\mu}\right)^{-1} + \frac{\lambda}{\lambda+N\mu}\left(\frac{N\mu}{\lambda+N\mu}\right)^{-2} \right] \right\}$$

$$= \frac{\alpha(\alpha-1)\pi_o}{2} + \pi_o P_o \left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha+1} \left\{ \alpha\left(\frac{N\mu}{\lambda N\mu}\right)^{-1} + \left(\frac{\lambda}{\lambda+N\mu}\right)\left(\frac{N\mu}{\lambda+N\mu}\right)^{-2} \right\}$$

$$= \frac{\alpha(\alpha-1)\pi_o}{2} + \pi_o P_o \frac{(\lambda)^{\alpha+1}}{(\lambda+N\mu)^{\alpha}} \left\{ \alpha\left(\frac{\lambda+N\mu}{+N\mu}\right) + \pi_o P_o \left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha+1} \left(\frac{\lambda}{\lambda+N\mu}\right)\left(\frac{\lambda+N\mu}{N\mu}\right)^2 \right\}$$

$$= \frac{\alpha(\alpha-1)\pi_o}{2} + \pi_o P_o \alpha \frac{\lambda^{\alpha+1}}{(\lambda+N\mu)^{\alpha}} \frac{1}{N\mu} + \pi_o P_o \frac{\lambda^{\alpha+2}}{(\lambda+N\mu)^{\alpha}} \cdot \frac{1}{(N\mu)^2}$$

$$= \pi_o \alpha \frac{(\alpha-1)}{2} + \pi_o P_o \alpha \frac{\lambda}{N\mu}\left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha} + \pi_o P_o \left(\frac{\lambda}{N\mu}\right)^2 \left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha}$$

$$= \frac{\pi_o \alpha(\alpha-1)}{2} + \pi_o P_o \left(\frac{\lambda}{\lambda+N\mu}\right)^{\alpha} \frac{\lambda}{N\mu}\left[\alpha + \frac{\lambda}{N\mu}\right]$$

$$E(Q) = \left[\frac{\alpha(\alpha-1)}{2} + \frac{\lambda}{N_\mu}\left(\alpha + \frac{\lambda}{N_\mu}\right) w_N^{\alpha} P_o\right] \pi_o \qquad \text{..2.4.17}$$

Hence $E(Q) = \alpha + \frac{\lambda}{N\mu} - \alpha\left(\frac{\alpha+1}{2} + \frac{\lambda}{N\mu}\right)\pi_o$ ..2.4.17

Section : 5 PASSENGER WAITING TIME DISTRIBUTION

The waiting time distribution of an arriving passenger in the queue is determined using the same approach as Medhi[23,24]. Define:

i) $f_w(t)$ = Probability density function (pdf) of w, the waiting tine of an arriving passenger in the queue.

ii) $f(v,k,t)$ = Probability density function (pdf) of a gamma variate with parameters v,k

$$= \frac{v(vt)^{k-1} e^{-vt}}{(k-1)!}$$

iii) F(v,k;t)

$$= \int^{t} f(v,k;u)\,du$$

$$= 1 - S(k,vt) \text{ where } S(k,vt) =$$

$$\sum_{r=0}^{k-1} \frac{(vt)^r}{r!} e^{-vt}$$

An arriving passenger finds the system in one of the following classes of states:

i) $(m, \alpha-1)$. $m = 1,2,\ldots,N$

ii) (m,j) $1 \le m \le N,\ 0 \le j \le \alpha-2$

iii) $(0,j)$ $0 \le j \le \alpha-2$

iv) $(0,j)$ $j \ge \alpha-1$

In case (i), an ariving passenger does not wait. The probability of zero delay is thus

$$\sum_{m=1}^{N} P(m, \alpha-1)$$

In case (ii), an arriving passenger has to wait for the arrival of $(\alpha-1-j)$ passengers and a vehicle. This duration may be denoted by the random variable z_j given by

z_j = max [Gamma variable with parameters λ, $\alpha-1-j$; Exponential variate with parameter $N\mu$] $0 \le j \le \alpha-1$

The pdf of z_j is given by

$$f_j(t) = f(\lambda, \alpha-1-j;\ t)\ F(N\mu, 1; t) + f(N\mu, 1; t)\ F(\lambda, \alpha-1-j; t)$$

$$= f(\lambda, \alpha-1-j;\ t)[1-S(1, N\mu t)] + f(N\mu, 1; t)\ [1 - S(\alpha-1-k, \lambda t)]$$

In case (iv), an arriving passenger has to wait for the arrival of a vehicle. The time for a vehicle arrival has an exponential distribution with parameter $N\mu$.

From the above, it follows that

$$f_w(t) = \sum_{m=1}^{N} \sum_{j=0}^{\alpha-2} P(m,j)\, f(\lambda, \alpha-1-j;t) + \sum_{j=0}^{\alpha-2} P(0,j)\, f_j(t) + \sum_{j=\alpha-1}^{\infty} P(0,j)\, f(N\mu,1;t)$$

$$= \sum_{j=0}^{\alpha-2} [\pi_j - P(0,j)]\, f(\lambda,\alpha-1-j;t) + \sum_{j=0}^{\alpha-2} P(0,j) f_j(t) + \sum_{j=\alpha-1}^{\infty} P(0,j) f(N\mu,1;t) \quad 2.4.18$$

From equations 2.4.8, 2.4.15, and 2.4.16

Equation 2.4.8 $\Rightarrow P(o,j) = w_N^j\, P(0,0)$

Equation 2.4.15 $\Rightarrow P(0,0) = w_N [\alpha + w_n^\alpha \frac{\lambda}{N\mu} P_o]^{-1} P_o$

Equation 2.4.16 $\Rightarrow \pi_j = [\alpha + \frac{\lambda}{N\mu} w_N^\alpha P_o]^{-1}, \quad 0 \le j \le \alpha-1$

$= \pi_o$

Put equation 2.4.16 in equation 2.4.15, we get

$$P(0,j) = w_N^j\, w_N\, \pi_o\, P_o = w_N^{j+1}\, \pi_o\, P_o, \quad j \geq 0 \quad .. \ 2.4.19$$

Using equations 2.4.16 and 2.4.19, 2.4.18 becomes

$$f_w(t) = \sum_{j=0}^{\alpha-2} [\pi_o - w_N^{j+1} P_o \pi_o]\, f(\lambda,\alpha-1-j;t) + \sum_{j=0}^{\alpha-2} w_N^{j+1} P_o \pi_o\, f_j(t) + \sum_{j=\alpha-1}^{\infty} w_N^{j+1} P_o \pi_o\, f(N\mu,1;t) \quad ...2.4.20$$

$$= \sum_{j=0}^{\alpha-2} [\pi_o - w_N^{j+1} P_o \pi_o] f(\lambda, \alpha-1-j;t) +$$

$$\sum_{j=0}^{\alpha-2} w_N^{j+1} P_o \pi_o [f(\lambda, \alpha-1-j;t)(1-S(1, N\mu t)]$$

$$+ f(N\mu, 1;t)(1-S(\alpha-1-j, \lambda t)]$$

$$+ \sum_{j=\alpha-1}^{\infty} w_N^{j+1} P_o \pi_o f(N\mu, 1;t)$$

$$= \sum_{j=0}^{\alpha-2} [\pi_o - w_N^{j+1} P_o \pi_o] f(\lambda, \alpha-1-j;t)$$

$$+ \sum_{j=0}^{\alpha-2} w_N^{j+1} P_o \pi_o f(\lambda, \alpha-1-j;t)(1-S[1, N\mu t])$$

$$+ \sum_{j=0}^{\alpha-2} w_N^{j+1} P_o \pi_o f(N\mu, 1;t)$$

$$[1 - S(\alpha-1-j, \lambda t)]$$

$$+ \sum_{j=\alpha-1}^{\infty} w_N^{j+1} P_o \pi_o f(N\mu, 1;t)$$

$$= \sum_{j=0}^{\alpha-2} [\pi_o - w_N^{j+1} P_o \pi_o]$$

$$\frac{(\lambda t)^{\alpha-1-j} e^{-\lambda t}}{(\alpha-1-j)!}$$

$$+ \sum_{j=0}^{\alpha-2} w_N^{j+1} P_o \pi_o \frac{(\lambda t)^{\alpha-1-j} e^{-\lambda t}}{(\alpha-1-j)!}$$

$$(1 - \sum_{r=0}^{1} \frac{(N\mu t)^r}{r!} e^{-N\mu t})$$

$$+ \sum_{j=0}^{\alpha-2} w_N^{j+1} P_o \pi_o \frac{N\mu (N\mu t) e^{-N\mu t}}{1!}$$

$$\left[1 - \sum_{r=o}^{\alpha-1-j} \frac{(\lambda t)^r}{r!} e^{-\lambda t}\right]$$

$$+ \sum_{j=\alpha-1}^{\infty} w_N^{j+1} P_o \Pi_o \frac{N^{\mu} (N \mu t)^1 e^{-N\mu t}}{1!}$$

$$f_w(t) = \Pi_o [P_o e^{-N\mu t} + S(\alpha-1, \lambda t)(1-P_o e^{-N\mu t})]$$

Let U = Probability that an arriving passenger will not have to wait.

and $f_T(t)$ = Probability density function of the (conditional) waiting time in the queue of only those passengers who have to wait.

Hence, using equations 2.4.8, 2.4.9, and 2.4.15 we find

$$U = \sum_{m=1}^{N} P(m, \alpha-1)$$

$$= \pi_o - P(0, \alpha-1)$$

$$= \pi_o - w_N^{\alpha} P_o \pi_o$$

$$U = \pi_o (1 - w_N^{\alpha} P_o) \qquad ..\ 2.4.21$$

$$\text{and} \quad f_T(t) = f_W(t) / 1-U \qquad ..2.4.22$$

The r-th moment of passenger waiting time in the queue is given by

$$E(W^r) = \lambda \pi_o P_o \int_0^{\infty} t^r e^{-N\mu t} dt + \lambda \pi_o \int_0^{\infty} t^r S(\alpha-1, \lambda t)$$
$$(1-P_o e^{-N\mu t}) dt \qquad (A)$$

Consider the first term of (A)

$$\lambda \pi_o P_o \int_0^{\infty} t^r e^{-n\mu t}\, dt = \lambda \pi_o P_o \frac{r!}{(N\mu)^{r+1}}$$

$$= \lambda \pi_o P_o \frac{r!}{(N\mu)^{r+1}} \cdot \frac{\lambda^r}{\lambda^r}$$

$$= \lambda^{-r} \pi_o P_o\, r! \left(\frac{\lambda}{N\mu}\right)^{r+1}$$

Consider the second term of (A)

$$\lambda \pi_o \left\{ \int_0^{\infty} t^r S(\alpha-1, \lambda t)\,(1 - P_o e^{-N\mu t})dt \right\}$$

$$= \lambda \pi_o \left\{ \int_0^{\infty} t^r \sum_{r=0}^{\alpha-2} \frac{(\lambda t)^r}{r!} e^{-\lambda t} (1 - P_o e^{-N\mu t})dt \right\}$$

$$= \lambda \pi_o \left\{ \int_0^{\infty} t^r \sum_{r=0}^{\alpha-2} \frac{(\lambda t)^r}{r!} e^{-\lambda t}\, dt - \int_0^{\infty} t^r \sum_{r=0}^{\alpha-2} \frac{(\lambda t)^r}{r!} e^{-\lambda t} P_o e^{-N\mu t}\, dt \right\}$$

$$= \lambda \pi_o \left\{ \int_0^{\infty} t^r [1 + \frac{\lambda t}{r!} + \frac{(\lambda t)^2}{2!} + \ldots + \frac{(\lambda t)^{\alpha-2}}{(\alpha-2)!}] e^{-\lambda t}\, dt \right\}$$

$$- P_o \left\{ \int_0^{\infty} t^r [1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \ldots + \frac{(\lambda t)^{\alpha-2}}{(\alpha-2)!}] e^{-(\lambda \pm N\mu)t}\, dt \right\}$$

$$- \pi_0 \{ \int_0^{\infty} t^r e^{-\lambda t} dt + \int_0^{\infty} \frac{\lambda t}{1!} e^{-\lambda t} dt + \dots\dots$$

$$+ \int_0^{\infty} \frac{(\lambda t)^{\alpha-2}}{(\alpha-2)!} e^{-\lambda t} dt$$

$$- P_0 [\int_0^{\infty} \lambda t^r e^{-(\lambda + N\mu)t} dt +$$

$$\int_0^{\infty} \frac{\lambda t}{1!} e^{-(\lambda + N\mu)t} dt + \dots\dots\dots +$$

$$\int_0^{\infty} \frac{(\lambda t)^{\alpha-2}}{(\alpha-2)!} e^{-(\lambda + N\mu)t} dt] \}$$

$$= \lambda \pi_0 \{ [\frac{r!}{\lambda^{r+1}} + \frac{\lambda}{1!} \frac{(r+1)!}{\lambda^{r+2}} + \dots\dots\dots\dots +$$

$$\frac{\lambda^{\alpha-2}}{(\alpha-2)!} \frac{(r+\alpha-2)!}{\lambda^{r+\alpha-2}}]$$

$$- P_0 [\frac{r!}{(\lambda + N\mu)^{r+1}} + \frac{\lambda}{1!} \frac{(r+1)!}{(\lambda + N\mu)^{r+2}} +$$

$$+ \dots\dots\dots \frac{\lambda^{\alpha-2}}{(\alpha-2)!} \cdot \frac{(r+\alpha-2)!}{(\lambda + N\mu)^{r+\alpha-2}}]$$

$$= \frac{\lambda \pi o \, r!}{\lambda^{r+1}} [1 + \frac{r+1}{1!} + \frac{(r+2)(r+1)}{2!} + \dots\dots +$$

$$\frac{(r+\alpha-2)(r+\alpha-1)\dots.2.1}{(\alpha-2)!}]$$

$$- \frac{\lambda \pi o \, r! \, P_o}{(\lambda + N\mu)^{r+1}} [1 + \frac{r+1}{\lambda + N\mu} + 2 \frac{(r+2(r+1)}{2!(\lambda + N\mu)^2}$$

$$+ \dots\dots\dots + \frac{(r+\alpha-2)\dots.2.1}{(\alpha-2)!} \frac{\lambda^{\alpha-2}}{(\lambda + N\mu)^{r+\alpha-2}}]$$

$$= \lambda^{-r} \pi_o r! \begin{bmatrix} r+\alpha-1 \\ \alpha-2 \end{bmatrix} - \lambda^{-r} \pi_o r! P_o$$

$$[\beta(\frac{\lambda}{\lambda+N\mu})^{r+1} + (\frac{\lambda}{\lambda+N\mu})^{r+2} \frac{r+1}{1!} +$$

$$\ldots\ldots + (\frac{\lambda}{\lambda+N\mu})^{r+\alpha-2} \frac{(r+\alpha-2)\ldots\ldots 2.1}{(\alpha-2)!}$$

$$= \lambda^{-r} \pi_o r! \begin{bmatrix} r+\alpha-1 \\ \alpha-2 \end{bmatrix} - \lambda^{r} \pi_o r! P_o$$

$$\sum_{k=0}^{\alpha-2} \begin{bmatrix} r+k \\ k \end{bmatrix} w_N^{r+k+1}$$

$$= \lambda^{-r} \pi_o r! \begin{bmatrix} r+\alpha-1 \\ \alpha-2 \end{bmatrix} - P_o \sum_{k=0}^{\alpha-2} \begin{bmatrix} r+k \\ k \end{bmatrix}$$

$$w_N^{r+k+1}$$

Therefore

$$E(W^r) = \pi_o r! \lambda^{-r} \beta (\frac{\lambda}{N\mu})^{r+1} P_o \begin{bmatrix} r+\alpha-1 \\ \alpha-1 \end{bmatrix} - P_o$$

$$\sum_{k=0}^{\alpha-2} \begin{bmatrix} r+k \\ k \end{bmatrix} w_N^{k+r+1} \qquad ..2.4.23$$

But $\sum_{k=0}^{\alpha-2} \begin{bmatrix} r+k \\ k \end{bmatrix} w_n^k = (1-w_N)^{-r-1} - w_N^{\alpha-1} \begin{bmatrix} r+\alpha-1 \\ \alpha-1 \end{bmatrix}_2$

$$F_1(r+\alpha, 1, \alpha; w_N)$$

$$= (1-w_N)^{-r-1} - w_N^{\alpha-2} \sum_{n=0}^{r} \begin{bmatrix} r+\alpha-1 \\ r-n \end{bmatrix}$$

$$(\frac{\lambda}{N\mu})^{n+1}$$

Hence equation 2.4.23 becomes,

$$E(W^r) = \pi_o\, r!\, \lambda^{-r} \left\{ \begin{bmatrix} r+\alpha-1 \\ \alpha-2 \end{bmatrix} + P_o\, w_N^{r+\alpha-1} \sum_{n=0}^{r} \begin{bmatrix} r+\alpha-1 \\ r-n \end{bmatrix} \left(\frac{\lambda}{N\mu}\right)^{n+1} \right\} \quad ..2.4.24$$

Using equation 2.4.24, the r-th moment of passenger waiting time in the queue can be easily evaluated. Setting r = 0 in equation 2.2.24 and using equations 2.4.16 and 2.4.21, one finds

$$\int_0^{\infty} f_w(t)\,dt = \pi_o\, 1.1 \binom{\alpha-1}{\alpha-2} + P_o\, w_N^{\alpha-1} \sum_{n=0} \binom{\alpha-1}{n} \left(\frac{\lambda}{N\mu}\right)^{n+1}$$

$$= \pi_o [(\alpha-1) + P_o\, W_N^{\alpha-1} (\frac{\lambda}{N\mu})]$$

$$= \pi_o [(\alpha-1) + P_o (\frac{\lambda}{\lambda+N\mu})^{\alpha-1} (\frac{\lambda}{N\mu})]$$

$$= \pi_o [(\alpha-1) + P_o (\frac{\lambda}{\lambda+N\mu}) (\frac{\lambda+N\mu}{N\mu})]$$

$$= \pi_o [(\alpha-1) + P_o\, w_N^{\alpha} (\frac{\lambda+N\mu}{N\mu})]$$

$$\int_0^{\infty} f_w(t)\,dt = \pi_o [(\alpha-1) + P_o\, w_N^{\alpha} (\frac{\lambda+N\mu}{N\mu}) = 1-U \quad ..2.4.25$$

Setting r=1 in equation 2.2.24

$$E(W) = \pi_o\, 1!\, \lambda^{-1} \left\{ \binom{\alpha}{\alpha-2} + P_o\, w_N^{\alpha} \sum_{n=0}^{1} \binom{\alpha}{1-n} \left(\frac{\lambda}{N\mu}\right)^{n-1} \right\}$$

$$= \pi_o\, \lambda^{-1} \left\{ \frac{\alpha(\alpha-1)}{2} + P_o w_N^{\alpha} \left[\binom{\alpha}{1} \left(\frac{\lambda}{N\mu}\right) + \binom{\alpha}{0} \left(\frac{\lambda}{N\mu}\right)^2 \right] \right\}$$

$$= \pi_o\, \lambda^{-1} \left\{ \frac{\alpha(\alpha-1)}{2} + P_o w_N^{\alpha} \left(\alpha \frac{\lambda}{N\mu} + \left(\frac{\lambda}{N\mu}\right)^2\right) \right\}$$

$$= \pi_o \lambda^{-1} \left\{ \frac{\alpha (\alpha - 1)}{2} + P_o w_N^{\alpha} \frac{\lambda}{N\mu} (\alpha + \frac{\lambda}{N \mu}) \right\}$$

$$E(W) = \pi_o \left\{ \frac{\alpha (\alpha - 1)}{2 \lambda} + \frac{P_o w_N^{\alpha}}{N \mu} (\alpha + \frac{\lambda}{N \mu}) \right\}$$

..2.4.26

This is the average waiting time per passenger in the queue. The above result can also be obtained from equation 2.4.17 by using Little's formula $E(Q) = \lambda E(W)$. Setting $r=2$ in equation 2.4.24, we obtain,

$$E(W^2) = \pi_o 2! \lambda^{-2} \left\{ \binom{\alpha+1}{\alpha-1} + P_o w_N^{\alpha+1} \sum_{n=0}^{2} \binom{\alpha+1}{2-n} \left(\frac{\lambda}{N \mu}\right)^{n+1} \right\}$$

$$= \pi_o 2! \lambda^{-2} \left\{ \frac{(\alpha+1) \alpha (\alpha - 1)}{3} + P_o w_N^{\alpha+1} \left[\binom{\alpha+1}{2} \frac{\lambda}{N\mu} + \binom{\alpha+1}{1} \left(\frac{\lambda}{N\mu}\right)^2 + \binom{\alpha+1}{2} \left(\frac{\lambda}{N\mu}\right)^3 \right] \right\}$$

$$= \pi_o 2 \lambda^{-2} \binom{\alpha+1}{3} + P_o w_N^{\alpha+1} \frac{\lambda}{N\mu} \left(\frac{\alpha (\alpha + 1)}{2} + (\alpha + 1) \frac{\lambda}{N\mu} + \frac{\alpha (\alpha + 1)}{2} \left(\frac{\lambda}{N\mu}\right)^2 \right.$$

..2.4.27

From equations 2.4.27 and 2.4.28, it is easy to compute the varience of waiting time for given N, α, λ, μ and known P_o.

Section : 6 SPECIAL CASES:

6.1 One-Vehicle System:

For N=1, the formulation and solution of the problem is simple enough be carried out manually

Using equations 2.1.1 and 2.1.2 we get

$$P_o = 1 \text{ and } P_1 = 0$$

From equation 2.4.18, since $P_o = 1$ and $w_1 = \frac{\lambda}{\lambda + \mu}$

$$\pi_j = \begin{cases} [\alpha + \frac{\lambda}{\mu} w_1^{\alpha}]^{-1} & 0 \le j \le \alpha - 1 \\ w_1^{j+1} [\alpha + \frac{\lambda}{\mu} w_1^{\alpha}]^{-1} & j > \alpha \end{cases}$$

Using equation 2.4.17 we get

$$E(Q) = \alpha + \frac{\lambda}{\mu} - \alpha \left(\frac{\alpha + 1}{2} + \frac{\lambda}{\mu}\right) \left(\alpha + \frac{\lambda}{\mu} w_1^{\alpha}\right)$$

..2.4.28

Using Little's formula, the expected waiting time in the queue is given by

$$E(W) = \frac{E(Q)}{\lambda}$$

$$= \frac{1}{\lambda}\left[\alpha + \frac{\lambda}{\mu} - \alpha \left(\frac{\alpha + 1}{2} + \frac{\lambda}{\mu}\right)\right]\left[\alpha + \frac{\lambda}{\mu} w_1^{\alpha}\right]^{-1}$$

$$= \frac{\alpha}{\mu} + \frac{1}{\mu} - \alpha \left[\frac{\alpha + 1}{2\lambda} + \frac{1}{\mu}\right] \left[\alpha + \frac{\lambda}{\mu} w_1^{\alpha}\right]^{-1}$$

..2.4.29

The above performance measures for the one-vehicle system can also be derived from Medhi [23].

6.2 Two-Vehicle System

Although the two-vehicle system is more complicated than the one-vehicle system, the problem is still manageable. Using equations 2.1.1 and 2.1.2 as before we obtain

$$(P_o, P_1)\begin{vmatrix} q_{0,0} & q_{0,1} \\ q_{1,0} & q_{1,1} \end{vmatrix} = (P_o, P_1)$$

where $q_{1,1} = 1 - w_1$, $q_{1,0} = w_1$, $q_{0,1} = 1 - 2\, w_1^{\alpha} + w_2^{\alpha}$

$q_{0,0} = 2w_1^{\alpha} - w_2^{\alpha}$

Solving $P_o q_{0,0} + P_1 q_{1,0} = P_o$

Put $P_1 = 1$

$$P_o q_{0,0} + q_{1,0} = P_o$$

$$P_o\,(q_{0,0} - 1) = -q_{1,0}$$

$$p_o = \frac{q_{1,0}}{1 - q_{0,0}} = \frac{w_1^{\alpha}}{1 - w_1^{\alpha} + w_2^{\alpha}}$$

Substituting the value of P_o into equation 2.4.16 we get

$$\pi_j = [\,\alpha + \frac{\lambda}{2\mu}\, w_2^{\alpha}\, \frac{w_1^{\alpha}}{1 - w_1^{\alpha} + w_2^{\alpha}}\,]^{-1}$$

$0 \leq j \leq \alpha - 1$ and $N=2$

$$= \left[\frac{(1-w_1^\alpha + w_2^\alpha)\alpha + \frac{\lambda}{2\mu} w_2^\alpha \; w_1^\alpha}{1 - w_1^\alpha + w_2^\alpha} \right]^{-1}$$

Put $\frac{\lambda}{2\mu}$ $= R$,

$$= (1-w_1^\alpha + w_2^\alpha) | (\alpha \; (1-w_1^\alpha + w_2^\alpha) + RW_2^\alpha \; w_1^\alpha)$$

and π_j $= w_2^{j+1} \left[\alpha + \frac{\lambda}{2\mu} \frac{w_2^\alpha \; w_1^\alpha}{(1-w_1^\alpha + w_2^\alpha)} \right]^{-1}$

$$\frac{w_1^\alpha}{(1 - w_1^\alpha + w_2)^\alpha}, \; j \geq \alpha$$

$$= \frac{w_2^{j+1} \; w_1^\alpha}{1-w_1^\alpha + w_2^\alpha} \left[\frac{\alpha \; (1-w_1^\alpha + w_2^\alpha) + R(W_1^\alpha \; w_2^\alpha)}{1 - w_1^\alpha + w_2^\alpha} \right]^{-1}$$

$$= w_1^\alpha \; w_2^{j+1} \; | (\alpha (1-w_1^\alpha + w_2^\alpha) + Rw_1^\alpha \; w_2^\alpha)$$

Therefore,

$$\pi_j = \begin{cases} (1-w_1^\alpha + w_2^\alpha) | (\alpha (1-w_1^\alpha + w_2^\alpha) + RW_1^\alpha \; w_2^\alpha) & 0 \leq j \leq \alpha-1 \\ w_1^\alpha \; w_2^{j+1} \; | (\alpha \; (1-w_1^\alpha + w_2^\alpha) + Rw_1^\alpha \; w_2^\alpha) & j \geq \alpha \end{cases}$$

where $R = \frac{\lambda}{2\mu}$

Using equation 2.4.17, the average passenger queue size is

$$E(Q) = \alpha + R - \alpha \left(\frac{\alpha+1}{2} + R \right)(1-w_1^\alpha + w_2^\alpha) | \; (\alpha (1-w_1^\alpha + w_2^\alpha) + Rw_1^\alpha \; w_2^\alpha)$$

..2.4.30

Using Little's formula, the average passenger waiting time in the queue is given by

$$E(W) = \frac{E(Q)}{\lambda}$$

$$= \frac{\alpha}{\lambda} + \frac{1}{2\mu} - \alpha\left(\frac{\alpha+1}{2\lambda} + \frac{1}{2\mu}\right)(1 - w_1^{\alpha} + w_2^{\alpha})\,/$$

$$[\alpha(1 - w_1^{\alpha} + w_2^{\alpha}) + R w_1^{\alpha} w_2^{\alpha}]$$

..2.4.31

6.3 **N-Vehicle System ($N \geq 3$)**

The system of equation 2.1.1 is solved vecursively to obtain P_m ($m = 0,1,2......N-1$).

π_j ($j = 0,1,2,.....$) can be easily calculated once P_o is known similarly, the average passenger queue size and waiting time in the queue can be easily computed.

We know from equation 2.4.16 that

$$\pi_o = \frac{1}{\alpha}\left[1 + \frac{\lambda}{N\mu}\,\frac{w_N^{\alpha} P_o}{\alpha}\right]^{-1}$$

Since $P_o < 1 \lesssim \alpha$ and $\frac{\lambda}{N}$ $w_N^{\alpha} \to 0$ and $N \to \alpha$,

$$\lim_{N \to \alpha} \pi_o = \frac{1}{\alpha}$$

Section : 7. COST FUNCTION AND OPTIMIZATION

Define:

c = Cost of dispatching a vehicle. This includes fuel consumption and vehicle depreciation.

w = Waiting cost per passenger per unit time.

$D(N)$ = Cost per unit time associated with the fleet size N. This includes maintenance and insurance costs of the fleet as well as associated administrative and labour costs. It is assumed that $D(.)$ is monotonically increasing and $D(0) = 0$.

$\Psi(N, \alpha)$ = long-run average cost per unit time associated with the dispatching policy α and a fleet size N.

$$\Psi(N^*, \alpha^*) = \min_{N, \alpha} \Psi(N, \alpha)$$

$k(N, \alpha)$ = long-run average cost per passenger associated with dispatching policy α and a fleet size N.

$$k(N^*, \alpha^*) = \min_{N, \alpha} k(N, \alpha)$$

$E(H)$ = Expected time between successive dispatches.

From these definitions, we see that

$$\Psi(N, \alpha) = D(N) + \sum_{j=0}^{\alpha} w \pi_j \, j + C / E(H) \quad \text{..2.4.32}$$

From equations 2.4.14 and 2.4.16 we obtain

$$\begin{aligned} E(H) &= \frac{1}{\lambda}\left[\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} p_o\right] \\ &= \frac{1}{\lambda} \pi_o^{-1} \quad \text{where } \pi_o^{-1} = \alpha + \frac{\lambda}{N\mu} w_N^{\alpha} p_o \\ &= (\lambda \pi_o)^{-1} \end{aligned}$$

Substituting equations 2.4.33 and 2.4.17 into equations 2.4.32,

$$\Psi(N, \alpha) = D(N) + w\left[\alpha + \frac{\lambda}{N\mu} \alpha \left(\frac{\alpha+1}{2} + \frac{\lambda}{N\mu}\right) \pi_o\right] + c \lambda \pi_o \quad \text{..2.4.34}$$

where $\pi_o = \left[\alpha + \frac{\lambda}{N\mu} w_N^{\alpha} p_o\right]^{-1}$

It may be noted that

$$k(N, \alpha) = \Psi(N, \alpha)/\lambda \quad \text{..2.4.35}$$

The numerical results in Table show that as the passenger arrival rate λ increases, the minimum long-run average cost per passenger $k(N^*, \alpha^*)$ decreases.

Table : **Optimal Policy and Performance Measures**

λ	$\Psi(N^*, \alpha^*)$	Operrating Policies		Performance Measures					$k(N^*, \alpha^*)$
		N^*	α^*	$E(Q)$	$E(W)$	π_o	P_o	$E(H)$	
10	118.08	3	7	3.14	0.31	0.1401	0.2601	0.71	11.81
20	130.29	4	10	4.64	0.23	0.0987	0.1647	0.51	6.51
30	139.69	5	12	5.60	0.19	0.0827	0.0983	0.40	4.65
40	147.58	5	15	7.18	0.18	0.0659	0.1240	0.38	3.69
50	154.27	6	16	7.60	0.15	0.0621	0.0721	0.32	3.09

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