CHAPTER – II

Bipolar Spherical Fuzzy Graph

2.1 Introduction

Spherical fuzzy set is more versatile than the existing fuzzy models, due to its outstanding feature of the vast space of uncertain and vagueness with the constraint $0 \le \alpha^2 + \nu^2 + \beta^2 \le 1$. A graph is a convenient way of representing the data in which the objects are vertices and their relations are edges. Many real life problems can be solved by using graphs. Based on Zadeh's fuzzy relation (1971), Kaufmann (1973) introduced the idea of fuzzy graphs. Akram (2020) examined decision making methods based on spherical fuzzy graphs. The new idea of bipolar neutrosophic cubic graphs were studied by Antony Crispin Sweety et al (2020). Here in this chapter we introduce bipolar spherical fuzzy set and bipolar spherical fuzzy graph and study the operations: symmetric difference and rejection with a description on degree and total degree of bipolar spherical fuzzy graphs with numerical example.

2.2 Bipolar Spherical Fuzzy Graph

In this section, we have defined bipolar spherical fuzzy set and bipolar spherical fuzzy graph and discussed it on a graph.

Definition 2.2.1: Let X be a non-empty set. An ***bipolar spherical fuzzy set*** (BSFS)

$$A = \left\{ < x, T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N > \mid x \in X \right\}$$

where $T_A^P, I_A^P, F_A^P : X \to [0,1]$, $T_A^N, I_A^N, F_A^N : X \to [-1,0]$ are the mappings such that $0 \le \left(\left(T_A^P\right)^2 + \left(I_A^P\right)^2 + \left(F_A^P\right)^2 \right) \le 1$ and $0 \le \left(\left(T_A^N\right)^2 + \left(I_A^N\right)^2 + \left(F_A^N\right)^2 \right) \le 1$ and T_A^P denote the positive truth membership function, I_A^P denote the positive indeterminacy membership function, F_A^P denote the positive falsity membership function, T_A^N denote the negative truth membership function, I_A^N denote the negative indeterminacy membership function, F_A^N denote the negative falsity membership function.

Definition 2.2.2: An ***bipolar spherical fuzzy graph*** (BSFG) on an underlying set V is a pair $G = (A, B)$ where A is an bipolar spherical set in V and B is an bipolar spherical relation on $V \times V$ such that

$$T_B^P(x,y) \le \min\ (T_A^P(x), T_A^P(y)), \qquad T_B^N(x,y) \ge \max\ (T_A^N(x), T_A^N(y))$$

$$I_B^P(x,y) \le \min\ (I_A^P(x), I_A^P(y)), \qquad I_B^N(x,y) \ge \max\ (I_A^N(x), I_A^N(y))$$

$$F_B^P(x,y) \le \max\ (F_A^P(x), F_A^P(y)), \qquad F_B^N(x,y) \ge \min\ (F_A^N(x), F_A^N(y))$$

where T_B^P denote the positive truth membership function, I_B^P denote the positive indeterminacy membership function, F_B^P denote the positive falsity membership function, T_B^N denote the negative truth membership function, I_B^N denote the negative indeterminacy membership function, F_B^N denote the negative falsity membership function and fulfils the following conditions: $0 \le \left(\left(T_B^P\right)^2 + \left(I_B^P\right)^2 + \left(F_B^P\right)^2 \right) \le 1$ and $0 \le \left(\left(T_B^N\right)^2 + \left(I_B^N\right)^2 + \left(F_B^N\right)^2 \right) \le 1$, where A is an bipolar spherical fuzzy vertex set and B is an bipolar spherical fuzzy edge set of G.

Definition 2.2.3: Let $G=(A,B)$ be an BSFG defined on $G^{*}=(V,E)$. The ***order of BSFG*** is defined by

$$O(G)=\sum_{a\in b}\{T_A^P(x), I_A^P(x), F_A^P(x), T_A^N(x), I_A^N(x), F_A^N(x)\}$$

and the ***degree of a vertex x of G*** is defined by

$$\deg(x)=\sum_{xy\in E}\{T_B^P(xy), I_B^P(xy), F_B^P(xy), T_B^N(xy), I_B^N(xy), F_B^N(xy)\}.$$

Definition 2.2.4

Let $G=(A,B)$ be an BSFG defined on $G^{*}=(V,E)$. The ***total degree of a vertex x of G*** is defined by

$$t\deg(x)=\sum_{xy\in E}\left\{\begin{matrix} T_B^P(xy)+T_A^P(x), I_B^P(xy)+I_A^P(x), F_B^P(xy)+F_A^P(x), \\ T_B^N(xy)+T_A^N(x), I_B^N(xy)+I_A^N(x), F_B^N(xy)+F_A^N(x)\end{matrix}\right\}.$$

Theorem 2.2.5: An bipolar spherical fuzzy graph is the generalization of Pythagorean fuzzy graph, bipolar Pythagorean fuzzy graph and spherical fuzzy graph.

Proof: Assume $G=(X,Y)$ be an bipolar spherical fuzzy graph.

By fixing the positive indeterminacy-membership and negative degree membership such truth-membership, indeterminacy-membership, falsity-membership values of vertex set and edge set equals to zero bring down the bipolar spherical fuzzy graph to Pythagorean fuzzy graph.

By fixing the positive indeterminacy-membership and negative indeterminacy-membership values of vertex set and edge set equals to zero bring down the bipolar spherical fuzzy graph to bipolar Pythagorean fuzzy graph.

By fixing the negative degree membership such truth-membership, indeterminacy-membership, falsity-membership values of vertex set and edge set equals to zero bring down the bipolar spherical fuzzy graph to spherical fuzzy graph.

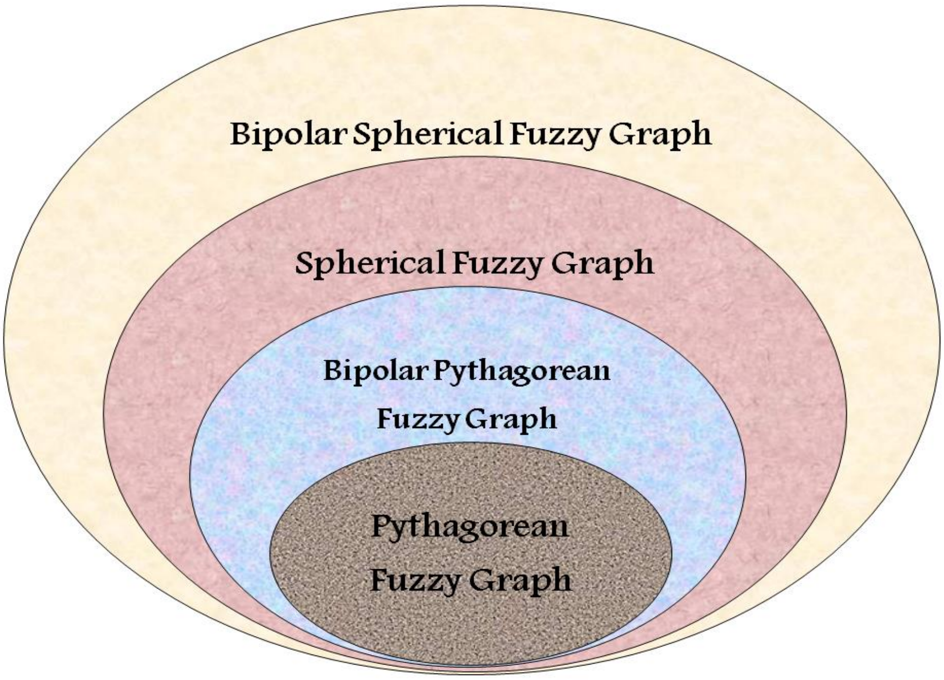

Fig. 2.1 Schematic representation of Bipolar Spherical Fuzzy Graph

2.3 Symmetric difference of BSFG

In this section, we have studied the operation: symmetric difference with a description on degree and total degree of bipolar spherical fuzzy graphs with numerical example.

Definition 2.3.1: Let $A_1 = (T^P_{A_1}, I^P_{A_1}, F^P_{A_1}, T^N_{A_1}, I^N_{A_1}, F^N_{A_1})$ and $A_2 = (T^P_{A_2}, I^P_{A_2}, F^P_{A_2}, T^N_{A_2}, I^N_{A_2}, F^N_{A_2})$ be bipolar spherical sets defined on V_1 and V_2, and let $B_1 = (T^P_{B_1}, I^P_{B_1}, F^P_{B_1}, T^N_{B_1}, I^N_{B_1}, F^N_{B_1})$ and $B_2 = (T^P_{B_2}, I^P_{B_2}, F^P_{B_2}, T^N_{B_2}, I^N_{B_2}, F^N_{B_2})$ be bipolar spherical sets defined on E_1 and E_2, respectively. Then, we denote the symmetric difference of two BSFGs G_1 and G_2 of the graphs G_1^* and G_2^* by $G_1 \oplus G_2 = (A_1 \oplus A_2, B_1 \oplus B_2)$ and defined as follows:

1. $\forall (x_1, x_2) \in V$

$$(T_{A_1}^P \oplus T_{A_2}^P)(x_1, x_2) = \min\,(T_{A_1}^P(x_1), T_{A_2}^P(x_2))$$
$$(I_{A_1}^P \oplus I_{A_2}^P)(x_1, x_2) = \min\,(I_{A_1}^P(x_1), I_{A_2}^P(x_2))$$
$$(F_{A_1}^P \oplus F_{A_2}^P)(x_1, x_2) = \max\,(F_{A_1}^P(x_1), F_{A_2}^P(x_2))$$

$$(T_{A_1}^N \oplus T_{A_2}^N)(x_1, x_2) = \max\,(T_{A_1}^N(x_1), T_{A_2}^N(x_2))$$
$$(I_{A_1}^N \oplus I_{A_2}^N)(x_1, x_2) = \max\,(I_{A_1}^N(x_1), I_{A_2}^N(x_2))$$
$$(F_{A_1}^N \oplus F_{A_2}^N)(x_1, x_2) = \min\,(F_{A_1}^N(x_1), F_{A_2}^N(x_2))$$

2. $\forall\ x \in V_1\ and\ x_2y_2 \in E_2$

$$(T_{B_1}^P \oplus T_{B_2}^P)((x, x_2)(x, y_2)) = \min\,(T_{A_1}^P(x), T_{B_1}^P(x_2y_2))$$
$$(I_{B_1}^P \oplus I_{B_2}^P)((x, x_2)(x, y_2)) = \min\,(I_{A_1}^P(x), I_{B_1}^P(x_2y_2))$$
$$(F_{B_1}^P \oplus F_{B_2}^P)((x, x_2)(x, y_2)) = \max\,(F_{A_1}^P(x), F_{B_1}^P(x_2y_2))$$

$$(T_{B_1}^N \oplus T_{B_2}^N)((x, x_2)(x, y_2)) = \min\,(T_{A_1}^N(x), T_{B_1}^N(x_2y_2))$$
$$(I_{B_1}^N \oplus I_{B_2}^N)((x, x_2)(x, y_2)) = \min\,(I_{A_1}^N(x), I_{B_1}^N(x_2y_2))$$
$$(F_{B_1}^N \oplus F_{B_2}^N)((x, x_2)(x, y_2)) = \max\,(F_{A_1}^N(x), F_{B_1}^N(x_2y_2))$$

3. $\forall\ z \in V_2\ and\ x_1y_1 \in E_1$

$$(T_{B_1}^P \oplus T_{B_2}^P)((x_1, z)(y_1, z)) = \min\,(T_{B_1}^P(x_1y_1), T_{B_2}^P(z))$$
$$(I_{B_1}^P \oplus I_{B_2}^P)((x_1, z)(y_1, z)) = \min\,(I_{B_1}^P(x_1y_1), I_{B_2}^P(z))$$
$$(F_{B_1}^P \oplus F_{B_2}^P)((x_1, z)(y_1, z)) = \max\,(F_{B_1}^P(x_1y_1), F_{B_2}^P(z))$$

$$(T_{B_1}^N \oplus T_{B_2}^N)((x_1, z)(y_1, z)) = \min\,(T_{B_1}^N(x_1y_1), T_{B_2}^N(z))$$
$$(I_{B_1}^N \oplus I_{B_2}^N)((x_1, z)(y_1, z)) = \min\,(I_{B_1}^N(x_1y_1), I_{B_2}^N(z))$$
$$(F_{B_1}^N \oplus F_{B_2}^N)((x_1, z)(y_1, z)) = \max\,(F_{B_1}^N(x_1y_1), F_{B_2}^N(z))$$

4. $\forall\ x_1y_1 \notin E_1, x_2y_2 \in E_2$

$$(T^P_{B_1} \oplus T^P_{B_2})((x_1, x_2)(y_1, y_2)) = \min\,(T^P_{A_1}(x_1), T^P_{A_1}(y_1), T^P_{B_2}(x_2 y_2))$$
$$(I^P_{B_1} \oplus I^P_{B_2})((x_1, x_2)(y_1, y_2)) = \min\,(I^P_{A_1}(x_1), I^P_{A_1}(y_1), I^P_{B_2}(x_2 y_2))$$
$$(F^P_{B_1} \oplus F^P_{B_2})((x_1, x_2)(y_1, y_2)) = \max\,(F^P_{A_1}(x_1), F^P_{A_1}(y_1), F^P_{B_2}(x_2 y_2))$$
$$(T^N_{B_1} \oplus T^N_{B_2})((x_1, x_2)(y_1, y_2)) = \max\,(T^N_{A_1}(x_1), T^N_{A_1}(y_1), T^N_{B_2}(x_2 y_2))$$
$$(I^N_{B_1} \oplus I^N_{B_2})((x_1, x_2)(y_1, y_2)) = \max\,(I^N_{A_1}(x_1), I^N_{A_1}(y_1), I^N_{B_2}(x_2 y_2))$$
$$(F^N_{B_1} \oplus F^N_{B_2})((x_1, x_2)(y_1, y_2)) = \min\,(F^N_{A_1}(x_1), F^N_{A_1}(y_1), F^N_{B_2}(x_2 y_2))$$

Example 2.3.2: Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two BSFGs where $V_1 = (a,b)$ and $V_2 = (c,d)$. Suppose R_1 and R_2 be the BSFS representations of E_1 and E_2 defined as follows:

$$R_1 = \begin{Bmatrix} a, (0.5, 0.6, 0.4, -0.7, -0.3, -0.6) \\ b, (0.4, 0.9, 0.4, -0.5, -0.7, -0.8) \end{Bmatrix}$$

$$S_1 = \{ab, (0.4, 0.6, 0.4, -0.5, -0.3, -0.8)\}$$

$$R_1 = \begin{Bmatrix} c, (0.8, 0.3, 0.5, -0.6, -0.7, -0.9) \\ d, (0.4, 0.6, 0.2, -0.5, -0.9, -0.8) \end{Bmatrix}$$

$$S_2 = \{cd, (0.4, 0.3, 0.5, -0.5, -0.7, -0.9)\}$$

$$R_1 \oplus R_2 = \left\langle \begin{matrix} (a,c), 0.5, 0.3, 0.5, -0.6, -0.3, -0.9 \\ (a,d), 0.4, 0.6, 0.4, -0.5, -0.3, -0.8 \\ (b,c), 0.4, 0.3, 0.5, -0.5, -0.7, -0.9 \\ (b,d), 0.4, 0.6, 0.4, -0.5, -0.7, -0.8 \end{matrix} \right\rangle$$

$$S_1 \oplus S_2 = \left\langle \begin{matrix} (a,c)(a,d), 0.4, 0.3, 0.5, -0.5, -0.3, -0.9 \\ (a,c)(b,c), 0.4, 0.3, 0.5, -0.5, -0.3, -0.9 \\ (a,c)(b,d), 0.4, 0.3, 0.5, -0.5, -0.3, -0.9 \\ (a,d)(b,c), 0.4, 0.3, 0.5, -0.5, -0.3, -0.9 \\ (a,d)(b,d), 0.4, 0.6, 0.4, -0.5, -0.3, -0.8 \\ (b,c)(b,d), 0.4, 0.3, 0.5, -0.5, -0.7, -0.9 \end{matrix} \right\rangle$$

G_1 G_2

Fig. 2.2 Bipolar spherical fuzzy graphs G_1 and G_2

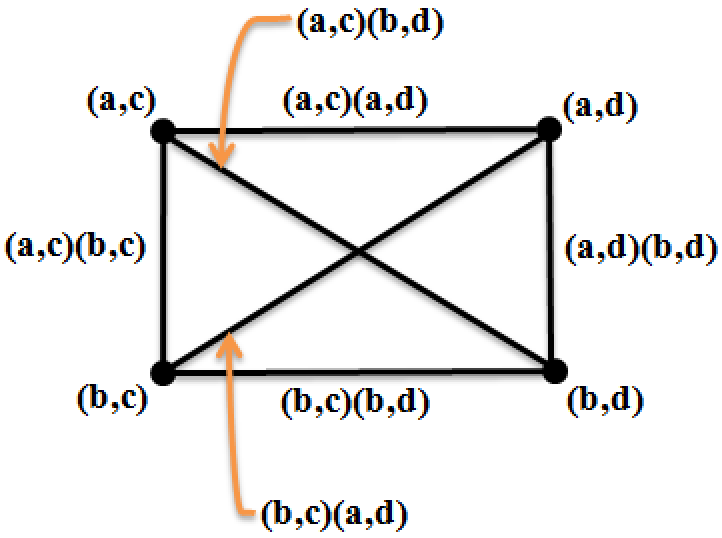

Fig. 2.3 $G_1 \oplus G_2$

Theorem 2.3.3: If G_1 and G_2 are BSFGs, then $G_1 \oplus G_2$ is a BSFG.

Proof: Let $x \in V$ and $x_2 y_2 \in E_2$. Then we have

$$\begin{aligned}(T_{B_1}^P \oplus T_{B_2}^P)((x, x_2)(x, y_2)) &= \min\{T_{A_1}^P(x), T_{B_2}^P(x_2 y_2)\} \\ &\leq \min\{T_{A_1}^P(x), \min(T_{A_2}^P(x_2), T_{A_2}^P(y_2))\} \\ &= \min\{\min(T_{A_1}^P(x), T_{A_2}^P(x_2)), \min(T_{A_1}^P(x), T_{A_2}^P(y_2))\} \\ &= \min\{((T_{A_1}^P \oplus T_{A_2}^P)(x, x_2)), ((T_{A_1}^P \oplus T_{A_2}^P)(x, y_2))\}\end{aligned}$$

$$\begin{aligned}(T_{B_1}^N \oplus T_{B_2}^N)((x, x_2)(x, y_2)) &= \max\{T_{A_1}^N(x), T_{B_2}^N(x_2 y_2)\} \\ &\leq \max\{T_{A_1}^N(x), \max(T_{A_2}^N(x_2), T_{A_2}^N(y_2))\} \\ &= \max\{\max(T_{A_1}^N(x), T_{A_2}^N(x_2)), \max(T_{A_1}^N(x), T_{A_2}^N(y_2))\} \\ &= \max\{((T_{A_1}^N \oplus T_{A_2}^N)(x, x_2)), ((T_{A_1}^N \oplus T_{A_2}^N)(x, y_2))\}\end{aligned}$$

$$(I_{B_1}^P \oplus I_{B_2}^P)((x,x_2)(x,y_2)) = \min\{ I_{A_1}^P(x), I_{B_2}^P(x_2y_2)\}$$
$$\leq \min\{ I_{A_1}^P(x), \min(I_{A_2}^P(x_2), I_{A_2}^P(y_2))\}$$
$$= \min\{\min(I_{A_1}^P(x), I_{A_2}^P(x_2)), \min(I_{A_1}^P(x), I_{A_2}^P(y_2))\}$$
$$= \min\{((I_{A_1}^P \oplus I_{A_2}^P)(x,x_2)), ((I_{A_1}^P \oplus I_{A_2}^P)(x,y_2))\}$$

$$(I_{B_1}^N \oplus I_{B_2}^N)((x,x_2)(x,y_2)) = \max\{ I_{A_1}^N(x), I_{B_2}^N(x_2y_2)\}$$
$$\leq \max\{ I_{A_1}^N(x), \max(I_{A_2}^N(x_2), I_{A_2}^N(y_2))\}$$
$$= \max\{\max(I_{A_1}^N(x), I_{A_2}^N(x_2)), \max(I_{A_1}^N(x), I_{A_2}^N(y_2))\}$$
$$= \max\{((I_{A_1}^N \oplus I_{A_2}^N)(x,x_2)), ((I_{A_1}^N \oplus I_{A_2}^N)(x,y_2))\}$$

$$(F_{B_1}^P \oplus F_{B_2}^P)((x,x_2)(x,y_2)) = \max\{ F_{A_1}^P(x), F_{B_2}^P(x_2y_2)\}$$
$$\leq \max\{ F_{A_1}^P(x), \max(F_{A_2}^P(x_2), F_{A_2}^P(y_2))\}$$
$$= \max\{\max(F_{A_1}^P(x), F_{A_2}^P(x_2)), \max(F_{A_1}^P(x), F_{A_2}^P(y_2))\}$$
$$= \max\{((F_{A_1}^P \oplus F_{A_2}^P)(x,x_2)), ((F_{A_1}^P \oplus F_{A_2}^P)(x,y_2))\}$$

$$(F_{B_1}^N \oplus F_{B_2}^N)((x,x_2)(x,y_2)) = \min\{ F_{A_1}^N(x), F_{B_2}^N(x_2y_2)\}$$
$$\leq \min\{ F_{A_1}^N(x), \min(F_{A_2}^N(x_2), F_{A_2}^N(y_2))\}$$
$$= \min\{\min(F_{A_1}^N(x), F_{A_2}^N(x_2)), \min(F_{A_1}^N(x), F_{A_2}^N(y_2))\}$$
$$= \min\{((F_{A_1}^N \oplus F_{A_2}^N)(x,x_2)), ((F_{A_1}^N \oplus F_{A_2}^N)(x,y_2))\}$$

Let $z \in V_2$ and $x_1y_1 \in E_1$. Then we have

$$(T_{B_1}^P \oplus T_{B_2}^P)((x_1,z)(y_1,z)) = \min\{ T_{B_1}^P(x_1y_1), T_{A_2}^P(z)\}$$
$$\leq \min\{\min(T_{A_1}^P(x_1), T_{A_1}^P(y_1)), T_{A_2}^P(z)\}$$
$$= \min\{\min(T_{A_1}^P(x_1), T_{A_2}^P(z)), \min(T_{A_1}^P(y_1), T_{A_2}^P(z))\}$$
$$= \min\{((T_{A_1}^P \oplus T_{A_2}^P)(x_1,z)), ((T_{A_1}^P \oplus T_{A_2}^P)(y_1,z))\}$$

$$(T_{B_1}^N \oplus T_{B_2}^N)((x_1,z)(y_1,z)) = \max\{ T_{B_1}^N(x_1y_1), T_{A_2}^N(z)\}$$
$$\leq \max\{\max(T_{A_1}^N(x_1), T_{A_1}^N(y_1)), T_{A_2}^N(z)\}$$
$$= \max\{\max(T_{A_1}^N(x_1), T_{A_2}^N(z)), \max(T_{A_1}^N(y_1), T_{A_2}^N(z))\}$$
$$= \max\{((T_{A_1}^N \oplus T_{A_2}^N)(x_1,z)), ((T_{A_1}^N \oplus T_{A_2}^N)(y_1,z))\}$$

$$(I_{B_1}^P \oplus I_{B_2}^P)((x_1,z)(y_1,z)) = \min\{ I_{B_1}^P(x_1y_1), I_{A_2}^P(z)\}$$
$$\leq \min\{\min(I_{A_1}^P(x_1), I_{A_1}^P(y_1)), I_{A_2}^P(z)\}$$
$$= \min\{\min(I_{A_1}^P(x_1), I_{A_2}^P(z)), \min(I_{A_1}^P(y_1), I_{A_2}^P(z))\}$$
$$= \min\{((I_{A_1}^P \oplus I_{A_2}^P)(x_1,z)), ((I_{A_1}^P \oplus I_{A_2}^P)(y_1,z))\}$$

$$(I_{B_1}^N \oplus I_{B_2}^N)((x_1,z)(y_1,z)) = \max\{ I_{B_1}^N(x_1y_1), I_{A_2}^N(z)\}$$
$$\leq \max\{\max(I_{A_1}^N(x_1), I_{A_1}^N(y_1)), I_{A_2}^N(z)\}$$
$$= \max\{\max(I_{A_1}^N(x_1), I_{A_2}^N(z)), \max(I_{A_1}^N(y_1), I_{A_2}^N(z))\}$$
$$= \max\{((I_{A_1}^N \oplus I_{A_2}^N)(x_1,z)), ((I_{A_1}^N \oplus I_{A_2}^N)(y_1,z))\}$$

$$(F_{B_1}^P \oplus F_{B_2}^P)((x_1,z)(y_1,z)) = \max\{ F_{B_1}^P(x_1y_1), F_{A_2}^P(z)\}$$
$$\leq \max\{\max(F_{A_1}^P(x_1), F_{A_1}^P(y_1)), F_{A_2}^P(z)\}$$
$$= \max\{\max(F_{A_1}^P(x_1), F_{A_2}^P(z)), \max(F_{A_1}^P(y_1), F_{A_2}^P(z))\}$$
$$= \max\{((F_{A_1}^P \oplus F_{A_2}^P)(x_1,z)), ((F_{A_1}^P \oplus F_{A_2}^P)(y_1,z))\}$$

$$(F_{B_1}^N \oplus F_{B_2}^N)((x_1,z)(y_1,z)) = \min\{ F_{B_1}^N(x_1y_1), F_{A_2}^N(z)\}$$
$$\leq \min\{\min(F_{A_1}^N(x_1), F_{A_1}^N(y_1)), F_{A_2}^N(z)\}$$
$$= \min\{\min(F_{A_1}^N(x_1), F_{A_2}^N(z)), \min(F_{A_1}^N(y_1), F_{A_2}^N(z))\}$$
$$= \min\{((F_{A_1}^N \oplus F_{A_2}^N)(x_1,z)), ((F_{A_1}^N \oplus F_{A_2}^N)(y_1,z))\}$$

Let $x_1y_1 \notin E_1, x_2y_2 \in E_2$. Then we have

$$(T_{B_1}^P \oplus T_{B_2}^P)((x_1,x_2)(y_1,y_2)) = \min\{ T_{A_1}^P(x_1), T_{A_1}^P(y_1), T_{B_2}^P(x_2y_2)\}$$
$$\leq \min\{ T_{A_1}^P(x_1), T_{A_1}^P(y_1), \min(T_{A_2}^P(x_2), T_{A_2}^P(y_2))\}$$
$$= \min\{\min(T_{A_1}^P(x_1), T_{A_2}^P(x_2)), \min(T_{A_1}^P(y_1), T_{A_2}^P(y_2))\}$$
$$= \min\{((T_{A_1}^P \oplus T_{A_2}^P)(x_1,x_2)), ((T_{A_1}^P \oplus T_{A_2}^P)(y_1,y_2))\}$$

$$(T_{B_1}^N \oplus T_{B_2}^N)((x_1,x_2)(y_1,y_2)) = \min\{ T_{A_1}^N(x_1), T_{A_1}^N(y_1), T_{B_2}^N(x_2y_2)\}$$
$$\leq \min\{ T_{A_1}^N(x_1), T_{A_1}^N(y_1), \min(T_{A_2}^N(x_2), T_{A_2}^N(y_2))\}$$
$$= \min\{\min(T_{A_1}^N(x_1), T_{A_2}^N(x_2)), \min(T_{A_1}^N(y_1), T_{A_2}^N(y_2))\}$$
$$= \min\{((T_{A_1}^N \oplus T_{A_2}^N)(x_1,x_2)), ((T_{A_1}^N \oplus T_{A_2}^N)(y_1,y_2))\}$$

$$(I_{B_1}^P \oplus I_{B_2}^P)((x_1,x_2)(y_1,y_2)) = \min\{ I_{A_1}^P(x_1), I_{A_1}^P(y_1), I_{B_2}^P(x_2y_2)\}$$
$$\leq \min\{ I_{A_1}^P(x_1), I_{A_1}^P(y_1), \min(I_{A_2}^P(x_2), I_{A_2}^P(y_2))\}$$
$$= \min\{\min(I_{A_1}^P(x_1), I_{A_2}^P(x_2)), \min(I_{A_1}^P(y_1), I_{A_2}^P(y_2))\}$$
$$= \min\{((I_{A_1}^P \oplus I_{A_2}^P)(x_1,x_2)), ((I_{A_1}^P \oplus I_{A_2}^P)(y_1,y_2))\}$$

$$(I_{B_1}^N \oplus I_{B_2}^N)((x_1,x_2)(y_1,y_2)) = \max\{ I_{A_1}^N(x_1), I_{A_1}^N(y_1), I_{B_2}^N(x_2y_2)\}$$
$$\leq \max\{ I_{A_1}^N(x_1), I_{A_1}^N(y_1), \max(I_{A_2}^N(x_2), I_{A_2}^N(y_2))\}$$
$$= \max\{\max(I_{A_1}^N(x_1), I_{A_2}^N(x_2)), \max(I_{A_1}^N(y_1), I_{A_2}^N(y_2))\}$$
$$= \max\{((I_{A_1}^N \oplus I_{A_2}^N)(x_1,x_2)), ((I_{A_1}^N \oplus I_{A_2}^N)(y_1,y_2))\}$$

$$(F_{B_1}^P \oplus F_{B_2}^P)((x_1,x_2)(y_1,y_2)) = \max\{ F_{A_1}^P(x_1), F_{A_1}^P(y_1), F_{B_2}^P(x_2y_2)\}$$
$$\leq \max\{ F_{A_1}^P(x_1), F_{A_1}^P(y_1), \max(F_{A_2}^P(x_2), F_{A_2}^P(y_2))\}$$
$$= \max\{\max(F_{A_1}^P(x_1), F_{A_2}^P(x_2)), \max(F_{A_1}^P(y_1), F_{A_2}^P(y_2))\}$$
$$= \max\{((F_{A_1}^P \oplus F_{A_2}^P)(x_1,x_2)), ((F_{A_1}^P \oplus F_{A_2}^P)(y_1,y_2))\}$$

$$(F_{B_1}^N \oplus F_{B_2}^N)((x_1,x_2)(y_1,y_2)) = \min\{ F_{A_1}^N(x_1), F_{A_1}^N(y_1), F_{B_2}^N(x_2y_2)\}$$
$$\leq \min\{ F_{A_1}^N(x_1), F_{A_1}^N(y_1), \min(F_{A_2}^N(x_2), F_{A_2}^N(y_2))\}$$
$$= \min\{\min(F_{A_1}^N(x_1), F_{A_2}^N(x_2)), \min(F_{A_1}^N(y_1), F_{A_2}^N(y_2))\}$$
$$= \min\{((F_{A_1}^N \oplus F_{A_2}^N)(x_1,x_2)), ((F_{A_1}^N \oplus F_{A_2}^N)(y_1,y_2))\}$$

Let $x_1y_1 \in E_1, x_2y_2 \notin E_2$. Then we have

$$(T_{B_1}^P \oplus T_{B_2}^P)((x_1,x_2)(y_1,y_2)) = \min\{ T_{A_2}^P(x_2), T_{A_2}^P(y_2), T_{B_1}^P(x_1y_1)\}$$
$$\leq \min\{ T_{A_2}^P(x_2), T_{A_2}^P(y_2), \min(T_{A_1}^P(x_1), T_{A_1}^P(y_1))\}$$
$$= \min\{\min(T_{A_1}^P(x_1), T_{A_2}^P(x_2)), \min(T_{A_1}^P(y_1), T_{A_2}^P(y_2))\}$$
$$= \min\{((T_{A_1}^P \oplus T_{A_2}^P)(x_1,x_2)), ((T_{A_1}^P \oplus T_{A_2}^P)(y_1,y_2))\}$$

$$(T_{B_1}^N \oplus T_{B_2}^N)((x_1,x_2)(y_1,y_2)) = \max\{ T_{A_2}^N(x_2), T_{A_2}^N(y_2), T_{B_1}^N(x_1y_1)\}$$
$$\leq \max\{ T_{A_2}^N(x_2), T_{A_2}^N(y_2), \max(T_{A_1}^N(x_1), T_{A_1}^N(y_1))\}$$
$$= \max\{\max(T_{A_1}^N(x_1), T_{A_2}^N(x_2)), \max(T_{A_1}^N(y_1), T_{A_2}^N(y_2))\}$$
$$= \max\{((T_{A_1}^N \oplus T_{A_2}^N)(x_1,x_2)), ((T_{A_1}^N \oplus T_{A_2}^N)(y_1,y_2))\}$$

$$(I^P_{B_1} \oplus I^P_{B_2})((x_1,x_2)(y_1,y_2)) = \min\{ I^P_{A_2}(x_2), I^P_{A_2}(y_2), I^P_{B_1}(x_1y_1)\}$$
$$\leq \min\{ I^P_{A_2}(x_2), I^P_{A_2}(y_2), \min(I^P_{A_1}(x_1), I^P_{A_1}(y_1))\}$$
$$= \min\{\min(I^P_{A_1}(x_1), I^P_{A_2}(x_2)), \min(I^P_{A_1}(y_1), I^P_{A_2}(y_2))\}$$
$$= \min\{((I^P_{A_1} \oplus I^P_{A_2})(x_1,x_2)), ((I^P_{A_1} \oplus I^P_{A_2})(y_1,y_2))\}$$

$$(I^N_{B_1} \oplus I^N_{B_2})((x_1,x_2)(y_1,y_2)) = \max\{ I^N_{A_2}(x_2), I^N_{A_2}(y_2), I^N_{B_1}(x_1y_1)\}$$
$$\leq \max\{ I^N_{A_2}(x_2), I^N_{A_2}(y_2), \max(I^N_{A_1}(x_1), I^N_{A_1}(y_1))\}$$
$$= \max\{\max(I^N_{A_1}(x_1), I^N_{A_2}(x_2)), \max(I^N_{A_1}(y_1), I^N_{A_2}(y_2))\}$$
$$= \max\{((I^N_{A_1} \oplus I^N_{A_2})(x_1,x_2)), ((I^N_{A_1} \oplus I^N_{A_2})(y_1,y_2))\}$$

$$(F^P_{B_1} \oplus F^P_{B_2})((x_1,x_2)(y_1,y_2)) = \max\{ F^P_{A_2}(x_2), F^P_{A_2}(y_2), F^P_{B_1}(x_1y_1)\}$$
$$\leq \max\{ F^P_{A_2}(x_2), F^P_{A_2}(y_2), \max(F^P_{A_1}(x_1), F^P_{A_1}(y_1))\}$$
$$= \max\{\max(F^P_{A_1}(x_1), F^P_{A_2}(x_2)), \max(F^P_{A_1}(y_1), F^P_{A_2}(y_2))\}$$
$$= \max\{((F^P_{A_1} \oplus F^P_{A_2})(x_1,x_2)), ((F^P_{A_1} \oplus F^P_{A_2})(y_1,y_2))\}$$

$$(F^N_{B_1} \oplus F^N_{B_2})((x_1,x_2)(y_1,y_2)) = \min\{ F^N_{A_2}(x_2), F^N_{A_2}(y_2), F^N_{B_1}(x_1y_1)\}$$
$$\leq \min\{ F^N_{A_2}(x_2), F^N_{A_2}(y_2), \min(F^N_{A_1}(x_1), F^N_{A_1}(y_1))\}$$
$$= \min\{\min(F^N_{A_1}(x_1), F^N_{A_2}(x_2)), \min(F^N_{A_1}(y_1), F^N_{A_2}(y_2))\}$$
$$= \min\{((F^N_{A_1} \oplus F^N_{A_2})(x_1,x_2)), ((F^N_{A_1} \oplus F^N_{A_2})(y_1,y_2))\}$$

Hence $G_1 \oplus G_2$ is an BSFG.

Definition 2.3.4: Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two BSFGs. Then for any vertex, $(x_1, x_2) \in V_1 \times V_2$.

$$\deg(T^P_{G_1} \oplus T^P_{G_2})(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1 \oplus E_2} (T^P_{B_1} \oplus T^P_{B_2})(x_1,x_2)(y_1,y_2)$$
$$= \sum_{x_1=y_1, x_2y_2\in E_2} \min\{ T^P_{A_1}(x_1), T^P_{B_2}(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \min\{ T^P_{A_2}(x_2), T^P_{B_1}(x_1y_1)\}$$
$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \min\{ T^P_{B_1}(x_1y_1), T^P_{A_2}(x_2), T^P_{A_2}(y_2)\}$$
$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \min\{ T^P_{B_2}(x_2y_2), T^P_{A_1}(x_1), T^P_{A_1}(y_1)\}$$

$$\deg(T_{G_1}^N \oplus T_{G_2}^N)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (T_{B_1}^N \oplus T_{B_2}^N)(x_1,x_2)(y_1,y_2)$$
$$= \sum_{x_1=y_1, x_2y_2\in E_2} \max\{T_{A_1}^N(x_1), T_{B_2}^N(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \max\{T_{A_2}^N(x_2), T_{B_1}^N(x_1y_1)\}$$
$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \max\{T_{B_1}^N(x_1y_1), T_{A_2}^N(x_2), T_{A_2}^N(y_2)\}$$
$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \max\{T_{B_2}^N(x_2y_2), T_{A_1}^N(x_1), T_{A_1}^N(y_1)\}$$

$$\deg(I_{G_1}^P \oplus I_{G_2}^P)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (I_{B_1}^P \oplus I_{B_2}^P)(x_1,x_2)(y_1,y_2)$$
$$= \sum_{x_1=y_1, x_2y_2\in E_2} \min\{I_{A_1}^P(x_1), I_{B_2}^P(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \min\{I_{A_2}^P(x_2), I_{B_1}^P(x_1y_1)\}$$
$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \min\{I_{B_1}^P(x_1y_1), I_{A_2}^P(x_2), I_{A_2}^P(y_2)\}$$
$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \min\{I_{B_2}^P(x_2y_2), I_{A_1}^P(x_1), I_{A_1}^P(y_1)\}$$

$$\deg(I_{G_1}^N \oplus I_{G_2}^N)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (I_{B_1}^N \oplus I_{B_2}^N)(x_1,x_2)(y_1,y_2)$$
$$= \sum_{x_1=y_1, x_2y_2\in E_2} \max\{I_{A_1}^N(x_1), I_{B_2}^N(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \max\{I_{A_2}^N(x_2), I_{B_1}^N(x_1y_1)\}$$
$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \max\{I_{B_1}^N(x_1y_1), I_{A_2}^N(x_2), I_{A_2}^N(y_2)\}$$
$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \max\{I_{B_2}^N(x_2y_2), I_{A_1}^N(x_1), I_{A_1}^N(y_1)\}$$

$$\deg(F_{G_1}^P \oplus F_{G_2}^P)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (F_{B_1}^P \oplus F_{B_2}^P)(x_1,x_2)(y_1,y_2)$$
$$= \sum_{x_1=y_1, x_2y_2\in E_2} \max\{F_{A_1}^P(x_1), F_{B_2}^P(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \max\{F_{A_2}^P(x_2), F_{B_1}^P(x_1y_1)\}$$
$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \max\{F_{B_1}^P(x_1y_1), F_{A_2}^P(x_2), F_{A_2}^P(y_2)\}$$
$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \max\{F_{B_2}^P(x_2y_2), F_{A_1}^P(x_1), F_{A_1}^P(y_1)\}$$

$$\deg(F_{G_1}^N \oplus F_{G_2}^N)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (F_{B_1}^N \oplus F_{B_2}^N)(x_1,x_2)(y_1,y_2)$$
$$= \sum_{x_1=y_1, x_2y_2\in E_2} \min\{ F_{A_1}^N(x_1), F_{B_2}^N(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \min\{ F_{A_2}^N(x_2), F_{B_1}^N(x_1y_1)\}$$
$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \min\{ F_{B_1}^N(x_1y_1), F_{A_2}^N(x_2), F_{A_2}^N(y_2)\}$$
$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \min\{ F_{B_2}^N(x_2y_2), F_{A_1}^N(x_1), F_{A_1}^N(y_1)\}$$

Definition 2.3.5: Let $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ be two BSFGs. Then for any vertex, $(x_1,x_2)\in V_1\times V_2$.

$$t\deg(T_{G_1}^P \oplus T_{G_2}^P)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (T_{B_1}^P \oplus T_{B_2}^P)(x_1,x_2)(y_1,y_2) + (T_{A_1}^P \oplus T_{A_2}^P)(x_1,x_2)$$
$$= \sum_{x_1=y_1, x_2y_2\in E_2} \min\{ T_{A_1}^P(x_1), T_{B_2}^P(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \min\{ T_{A_2}^P(x_2), T_{B_1}^P(x_1y_1)\}$$
$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \min\{ T_{B_1}^P(x_1y_1), T_{A_2}^P(x_2), T_{A_2}^P(y_2)\}$$
$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \min\{ T_{B_2}^P(x_2y_2), T_{A_1}^P(x_1), T_{A_1}^P(y_1)\}$$
$$+ \min\{(T_{A_1}^P(x_1) \oplus T_{A_2}^P(x_2)\}$$

$$t\deg(T_{G_1}^N \oplus T_{G_2}^N)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (T_{B_1}^N \oplus T_{B_2}^N)(x_1,x_2)(y_1,y_2) + (T_{A_1}^N \oplus T_{A_2}^N)(x_1,x_2)$$
$$= \sum_{x_1=y_1, x_2y_2\in E_2} \max\{ T_{A_1}^N(x_1), T_{B_2}^N(x_2y_2)\}$$
$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \max\{ T_{A_2}^N(x_2), T_{B_1}^N(x_1y_1)\}$$
$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \max\{ T_{B_1}^N(x_1y_1), T_{A_2}^N(x_2), T_{A_2}^N(y_2)\}$$
$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \max\{ T_{B_2}^N(x_2y_2), T_{A_1}^N(x_1), T_{A_1}^N(y_1)\}$$
$$+ \min\{(T_{A_1}^N(x_1) \oplus T_{A_2}^N(x_2)$$

$$t\deg(I_{G_1}^P \oplus I_{G_2}^P)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (I_{B_1}^P \oplus I_{B_2}^P)(x_1,x_2)(y_1,y_2) + (I_{A_1}^P \oplus I_{A_2}^P)(x_1,x_2)$$

$$= \sum_{x_1=y_1, x_2y_2\in E_2} \min\{ I_{A_1}^P(x_1), I_{B_2}^P(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \min\{ I_{A_2}^P(x_2), I_{B_1}^P(x_1y_1)\}$$

$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \min\{ I_{B_1}^P(x_1y_1), I_{A_2}^P(x_2), I_{A_2}^P(y_2)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \min\{ I_{B_2}^P(x_2y_2), I_{A_1}^P(x_1), I_{A_1}^P(y_1)\}$$

$$+ \min\{(I_{A_1}^P(x_1) \oplus I_{A_2}^P(x_2)$$

$$t\deg(I_{G_1}^N \oplus I_{G_2}^N)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (I_{B_1}^N \oplus I_{B_2}^N)(x_1,x_2)(y_1,y_2) + (I_{A_1}^N \oplus I_{A_2}^N)(x_1,x_2)$$

$$= \sum_{x_1=y_1, x_2y_2\in E_2} \max\{ I_{A_1}^N(x_1), I_{B_2}^N(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \max\{ I_{A_2}^N(x_2), I_{B_1}^N(x_1y_1)\}$$

$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \max\{ I_{B_1}^N(x_1y_1), I_{A_2}^N(x_2), I_{A_2}^N(y_2)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \max\{ I_{B_2}^N(x_2y_2), I_{A_1}^N(x_1), I_{A_1}^N(y_1)\}$$

$$+ \min\{(I_{A_1}^N(x_1) \oplus I_{A_2}^N(x_2)$$

$$t\deg(F_{G_1}^P \oplus F_{G_2}^P)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (F_{B_1}^P \oplus F_{B_2}^P)(x_1,x_2)(y_1,y_2) + (F_{A_1}^P \oplus F_{A_2}^P)(x_1,x_2)$$

$$= \sum_{x_1=y_1, x_2y_2\in E_2} \max\{ F_{A_1}^P(x_1), F_{B_2}^P(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \max\{ F_{A_2}^P(x_2), F_{B_1}^P(x_1y_1)\}$$

$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \max\{ F_{B_1}^P(x_1y_1), F_{A_2}^P(x_2), F_{A_2}^P(y_2)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \max\{ F_{B_2}^P(x_2y_2), F_{A_1}^P(x_1), F_{A_1}^P(y_1)\}$$

$$+ \min\{(F_{A_1}^P(x_1) \oplus F_{A_2}^P(x_2)$$

$$t\deg(F_{G_1}^N \oplus F_{G_2}^N)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1\oplus E_2} (F_{B_1}^N \oplus F_{B_2}^N)(x_1,x_2)(y_1,y_2) + (F_{A_1}^N \oplus F_{A_2}^N)(x_1,x_2)$$

$$= \sum_{x_1=y_1, x_2y_2\in E_2} \min\{ F_{A_1}^N(x_1), F_{B_2}^N(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\in E_1} \min\{ F_{A_2}^N(x_2), F_{B_1}^N(x_1y_1)\}$$

$$+ \sum_{x_1y_1\in E_1, x_2y_2\notin E_2} \min\{ F_{B_1}^N(x_1y_1), F_{A_2}^N(x_2), F_{A_2}^N(y_2)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\in E_2} \min\{ F_{B_2}^N(x_2y_2), F_{A_1}^N(x_1), F_{A_1}^N(y_1)\}$$

$$+ \min\{(F_{A_1}^N(x_1) \oplus F_{A_2}^N(x_2)$$

2.4 Rejection of BSFG

In this section, we have discussed the operation: rejection with a description on degree and total degree of bipolar spherical fuzzy graphs with numerical example.

Definition 2.4.1: Let $A_1 = (T_{A_1}^P, I_{A_1}^P, F_{A_1}^P, T_{A_1}^N, I_{A_1}^N, F_{A_1}^N)$ and $A_2 = (T_{A_2}^P, I_{A_2}^P, F_{A_2}^P, T_{A_2}^N, I_{A_2}^N, F_{A_2}^N)$ be bipolar spherical sets defined on V_1 and V_2, and let $B_1 = (T_{B_1}^P, I_{B_1}^P, F_{B_1}^P, T_{B_1}^N, I_{B_1}^N, F_{B_1}^N)$ and $B_2 = (T_{B_2}^P, I_{B_2}^P, F_{B_2}^P, T_{B_2}^N, I_{B_2}^N, F_{B_2}^N)$ be bipolar spherical sets defined on E_1 and E_2, respectively. Then, we denote the rejection of two BSFGs G_1 and G_2 of the graphs G_1^* and G_2^* by $G_1 | G_2 = (A_1 | A_2, B_1 | B_2)$ and defined as follows:

1. $\forall (x_1,x_2) \in V \times V$

$$(T_{A_1}^P | T_{A_2}^P)(x_1,x_2) = \min\,(T_{A_1}^P(x_1), T_{A_2}^P(x_2))$$
$$(I_{A_1}^P | I_{A_2}^P)(x_1,x_2) = \min\,(I_{A_1}^P(x_1), I_{A_2}^P(x_2))$$
$$(F_{A_1}^P | F_{A_2}^P)(x_1,x_2) = \max\,(F_{A_1}^P(x_1), F_{A_2}^P(x_2))$$

$$(T_{A_1}^N | T_{A_2}^N)(x_1,x_2) = \max\,(T_{A_1}^N(x_1), T_{A_2}^N(x_2))$$
$$(I_{A_1}^N | I_{A_2}^N)(x_1,x_2) = \max\,(I_{A_1}^N(x_1), I_{A_2}^N(x_2))$$
$$(F_{A_1}^N | F_{A_2}^N)(x_1,x_2) = \min\,(F_{A_1}^N(x_1), F_{A_2}^N(x_2))$$

2. $\forall\, x \in V_1$ *and* $x_2 y_2 \in E_2$

$$(T^P_{B_1} | T^P_{B_2})((x, x_2)(x, y_2)) = \min\,(T^P_{A_1}(x), T^P_{A_2}(x_2), T^P_{A_2}(y_2))$$
$$(I^P_{B_1} | I^P_{B_2})((x, x_2)(x, y_2)) = \min\,(I^P_{A_1}(x), I^P_{A_2}(x_2), I^P_{A_2}(y_2))$$
$$(F^P_{B_1} | F^P_{B_2})((x, x_2)(x, y_2)) = \max\,(F^P_{A_1}(x), F^P_{A_2}(x_2), F^P_{A_2}(y_2))$$

$$(T^N_{B_1} | T^N_{B_2})((x, x_2)(x, y_2)) = \max\,(T^N_{A_1}(x), T^N_{A_2}(x_2), T^N_{A_2}(y_2))$$
$$(I^N_{B_1} | I^N_{B_2})((x, x_2)(x, y_2)) = \max\,(I^N_{A_1}(x), I^N_{A_2}(x_2), I^N_{A_2}(y_2))$$
$$(F^N_{B_1} | F^N_{B_2})((x, x_2)(x, y_2)) = \min\,(F^N_{A_1}(x), F^N_{A_2}(x_2), F^N_{A_2}(y_2))$$

3. $\forall\, z \in V_2$ *and* $x_1 y_1 \notin E_1$

$$(T^P_{B_1} | T^P_{B_2})((x_1, z)(y_1, z)) = \min\,(T^P_{B_1}(x_1), T^P_{B_1}(y_1), T^P_{A_2}(z))$$
$$(I^P_{B_1} | I^P_{B_2})((x_1, z)(y_1, z)) = \min\,(I^P_{B_1}(x_1), I^P_{B_1}(y_1), I^P_{A_2}(z))$$
$$(F^P_{B_1} | F^P_{B_2})((x_1, z)(y_1, z)) = \max\,(F^P_{B_1}(x_1), F^P_{B_1}(y_1), F^P_{A_2}(z))$$

$$(T^N_{B_1} | T^N_{B_2})((x_1, z)(y_1, z)) = \max\,(T^N_{B_1}(x_1), T^N_{B_1}(y_1), T^N_{A_2}(z))$$
$$(I^N_{B_1} | I^N_{B_2})((x_1, z)(y_1, z)) = \max\,(I^N_{B_1}(x_1), I^N_{B_1}(y_1), I^N_{A_2}(z))$$
$$(F^N_{B_1} | F^N_{B_2})((x_1, z)(y_1, z)) = \min\,(F^N_{B_1}(x_1), F^N_{B_1}(y_1), F^N_{A_2}(z))$$

4. $\forall\, x_1 y_1 \notin E_1, x_2 y_2 \notin E_2$

$$(T^P_{B_1} | T^P_{B_2})((x_1, x_2)(y_1, y_2)) = \min\,(T^P_{A_1}(x_1), T^P_{A_1}(y_1), T^P_{A_2}(x_2), T^P_{A_2}(y_2))$$
$$(I^P_{B_1} | I^P_{B_2})((x_1, x_2)(y_1, y_2)) = \min\,(I^P_{A_1}(x_1), I^P_{A_1}(y_1), I^P_{A_2}(x_2), I^P_{A_2}(y_2))$$
$$(F^P_{B_1} | F^P_{B_2})((x_1, x_2)(y_1, y_2)) = \max\,(F^P_{A_1}(x_1), F^P_{A_1}(y_1), F^P_{A_2}(x_2), F^P_{A_2}(y_2))$$

$$(T^N_{B_1} | T^N_{B_2})((x_1, x_2)(y_1, y_2)) = \max\,(T^N_{A_1}(x_1), T^N_{A_1}(y_1), T^N_{A_2}(x_2), T^N_{A_2}(y_2))$$
$$(I^N_{B_1} | I^N_{B_2})((x_1, x_2)(y_1, y_2)) = \max\,(I^N_{A_1}(x_1), I^N_{A_1}(y_1), I^N_{A_2}(x_2), I^N_{A_2}(y_2))$$
$$(F^N_{B_1} | F^N_{B_2})((x_1, x_2)(y_1, y_2)) = \min\,(F^N_{A_1}(x_1), F^N_{A_1}(y_1), F^N_{A_2}(x_2), F^N_{A_2}(y_2))$$

Theorem 2.4.2: If G_1 and G_2 are BSFGs, then $G_1 | G_2$ is a BSFG.

Proof: Let $x \in V_1$ and $x_2 y_2 \notin E_2$. Then we have

$$
\begin{aligned}
(T_{B_1}^P \mid T_{B_2}^P)((x,x_2)(x,y_2)) &= \min\{ T_{A_1}^P(x), T_{B_2}^P(x_2y_2)\} \\
&\le \min\{ T_{A_1}^P(x), \min(T_{A_2}^P(x_2), T_{A_2}^P(y_2))\} \\
&= \min\{\min(T_{A_1}^P(x), T_{A_2}^P(x_2)), \min(T_{A_1}^P(x), T_{A_2}^P(y_2))\} \\
&= \min\{((T_{A_1}^P \mid T_{A_2}^P)(x, x_2)), ((T_{A_1}^P \mid T_{A_2}^P)(x, y_2))\}
\end{aligned}
$$

$$
\begin{aligned}
(T_{B_1}^N \mid T_{B_2}^N)((x,x_2)(x,y_2)) &= \max\{ T_{A_1}^N(x), T_{B_2}^N(x_2y_2)\} \\
&\le \max\{ T_{A_1}^N(x), \max(T_{A_2}^N(x_2), T_{A_2}^N(y_2))\} \\
&= \max\{\max(T_{A_1}^N(x), T_{A_2}^N(x_2)), \max(T_{A_1}^N(x), T_{A_2}^N(y_2))\} \\
&= \max\{((T_{A_1}^N \mid T_{A_2}^N)(x, x_2)), ((T_{A_1}^N \mid T_{A_2}^N)(x, y_2))\}
\end{aligned}
$$

$$
\begin{aligned}
(I_{B_1}^P \mid I_{B_2}^P)((x,x_2)(x,y_2)) &= \min\{ I_{A_1}^P(x), I_{B_2}^P(x_2y_2)\} \\
&\le \min\{ I_{A_1}^P(x), \min(I_{A_2}^P(x_2), I_{A_2}^P(y_2))\} \\
&= \min\{\min(I_{A_1}^P(x), I_{A_2}^P(x_2)), \min(I_{A_1}^P(x), I_{A_2}^P(y_2))\} \\
&= \min\{((I_{A_1}^P \mid I_{A_2}^P)(x, x_2)), ((I_{A_1}^P \mid I_{A_2}^P)(x, y_2))\}
\end{aligned}
$$

$$
\begin{aligned}
(I_{B_1}^N \mid I_{B_2}^N)((x,x_2)(x,y_2)) &= \max\{ I_{A_1}^N(x), I_{B_2}^N(x_2y_2)\} \\
&\le \max\{ I_{A_1}^N(x), \max(I_{A_2}^N(x_2), I_{A_2}^N(y_2))\} \\
&= \max\{\max(I_{A_1}^N(x), I_{A_2}^N(x_2)), \max(I_{A_1}^N(x), I_{A_2}^N(y_2))\} \\
&= \max\{((I_{A_1}^N \mid I_{A_2}^N)(x, x_2)), ((I_{A_1}^N \mid I_{A_2}^N)(x, y_2))\}
\end{aligned}
$$

$$
\begin{aligned}
(F_{B_1}^P \mid F_{B_2}^P)((x,x_2)(x,y_2)) &= \max\{ F_{A_1}^P(x), F_{B_2}^P(x_2y_2)\} \\
&\le \max\{ F_{A_1}^P(x), \max(F_{A_2}^P(x_2), F_{A_2}^P(y_2))\} \\
&= \max\{\max(F_{A_1}^P(x), F_{A_2}^P(x_2)), \max(F_{A_1}^P(x), F_{A_2}^P(y_2))\} \\
&= \max\{((F_{A_1}^P \mid F_{A_2}^P)(x, x_2)), ((F_{A_1}^P \mid F_{A_2}^P)(x, y_2))\}
\end{aligned}
$$

$$
\begin{aligned}
(F_{B_1}^N \mid F_{B_2}^N)((x,x_2)(x,y_2)) &= \min\{ F_{A_1}^N(x), F_{B_2}^N(x_2y_2)\} \\
&\le \min\{ F_{A_1}^N(x), \min(F_{A_2}^N(x_2), F_{A_2}^N(y_2))\} \\
&= \min\{\min(F_{A_1}^N(x), F_{A_2}^N(x_2)), \min(F_{A_1}^N(x), F_{A_2}^N(y_2))\} \\
&= \min\{((F_{A_1}^N \mid F_{A_2}^N)(x, x_2)), ((F_{A_1}^N \mid F_{A_2}^N)(x, y_2))\}
\end{aligned}
$$

Let $z \in V_2$ and $x_1 y_1 \notin E_1$. Then we have

$$\begin{aligned}(T_{B_1}^P \mid T_{B_2}^P)((x_1, z)(y_1, z)) &= \min\{ T_{B_1}^P(x_1 y_1), T_{A_2}^P(z)\} \\ &\le \min\{\min(T_{A_1}^P(x_1), T_{A_1}^P(y_1)), T_{A_2}^P(z)\} \\ &= \min\{\min(T_{A_1}^P(x_1), T_{A_2}^P(z)), \min(T_{A_1}^P(y_1), T_{A_2}^P(z))\} \\ &= \min\{((T_{A_1}^P \mid T_{A_2}^P)(x_1, z)), ((T_{A_1}^P \mid T_{A_2}^P)(y_1, z))\}\end{aligned}$$

$$\begin{aligned}(T_{B_1}^N \mid T_{B_2}^N)((x_1, z)(y_1, z)) &= \max\{ T_{B_1}^N(x_1 y_1), T_{A_2}^N(z)\} \\ &\le \max\{\max(T_{A_1}^N(x_1), T_{A_1}^N(y_1)), T_{A_2}^N(z)\} \\ &= \max\{\max(T_{A_1}^N(x_1), T_{A_2}^N(z)), \max(T_{A_1}^N(y_1), T_{A_2}^N(z))\} \\ &= \max\{((T_{A_1}^N \mid T_{A_2}^N)(x_1, z)), ((T_{A_1}^N \mid T_{A_2}^N)(y_1, z))\}\end{aligned}$$

$$\begin{aligned}(I_{B_1}^P \mid I_{B_2}^P)((x_1, z)(y_1, z)) &= \min\{ I_{B_1}^P(x_1 y_1), I_{A_2}^P(z)\} \\ &\le \min\{\min(I_{A_1}^P(x_1), I_{A_1}^P(y_1)), I_{A_2}^P(z)\} \\ &= \min\{\min(I_{A_1}^P(x_1), I_{A_2}^P(z)), \min(I_{A_1}^P(y_1), I_{A_2}^P(z))\} \\ &= \min\{((I_{A_1}^P \mid I_{A_2}^P)(x_1, z)), ((I_{A_1}^P \mid I_{A_2}^P)(y_1, z))\}\end{aligned}$$

$$\begin{aligned}(I_{B_1}^N \mid I_{B_2}^N)((x_1, z)(y_1, z)) &= \max\{ I_{B_1}^N(x_1 y_1), I_{A_2}^N(z)\} \\ &\le \max\{\max(I_{A_1}^N(x_1), I_{A_1}^N(y_1)), I_{A_2}^N(z)\} \\ &= \max\{\max(I_{A_1}^N(x_1), I_{A_2}^N(z)), \max(I_{A_1}^N(y_1), I_{A_2}^N(z))\} \\ &= \max\{((I_{A_1}^N \mid I_{A_2}^N)(x_1, z)), ((I_{A_1}^N \mid I_{A_2}^N)(y_1, z))\}\end{aligned}$$

$$\begin{aligned}(F_{B_1}^P \mid F_{B_2}^P)((x_1, z)(y_1, z)) &= \max\{ F_{B_1}^P(x_1 y_1), F_{A_2}^P(z)\} \\ &\le \max\{\max(F_{A_1}^P(x_1), F_{A_1}^P(y_1)), F_{A_2}^P(z)\} \\ &= \max\{\max(F_{A_1}^P(x_1), F_{A_2}^P(z)), \max(F_{A_1}^P(y_1), F_{A_2}^P(z))\} \\ &= \max\{((F_{A_1}^P \mid F_{A_2}^P)(x_1, z)), ((F_{A_1}^P \mid F_{A_2}^P)(y_1, z))\}\end{aligned}$$

$$\begin{aligned}(F_{B_1}^N \mid F_{B_2}^N)((x_1, z)(y_1, z)) &= \min\{ F_{B_1}^N(x_1 y_1), F_{A_2}^N(z)\} \\ &\le \min\{\min(F_{A_1}^N(x_1), F_{A_1}^N(y_1)), F_{A_2}^N(z)\} \\ &= \min\{\min(F_{A_1}^N(x_1), F_{A_2}^N(z)), \min(F_{A_1}^N(y_1), F_{A_2}^N(z))\} \\ &= \min\{((F_{A_1}^N \mid F_{A_2}^N)(x_1, z)), ((F_{A_1}^N \mid F_{A_2}^N)(y_1, z))\}\end{aligned}$$

Let $x_1y_1 \notin E_1, x_2y_2 \notin E_2$. Then we have

$$(T_{B_1}^P \mid T_{B_2}^P)((x_1,x_2)(y_1,y_2)) = \min\{T_{A_1}^P(x_1), T_{A_1}^P(y_1), T_{B_2}^P(x_2y_2)\}$$
$$\leq \min\{T_{A_1}^P(x_1), T_{A_1}^P(y_1), \min(T_{A_2}^P(x_2), T_{A_2}^P(y_2))\}$$
$$= \min\{\min(T_{A_1}^P(x_1), T_{A_2}^P(x_2)), \min(T_{A_1}^P(y_1), T_{A_2}^P(y_2))\}$$
$$= \min\{((T_{A_1}^P \mid T_{A_2}^P)(x_1,x_2)), ((T_{A_1}^P \mid T_{A_2}^P)(y_1,y_2))\}$$

$$(T_{B_1}^N \mid T_{B_2}^N)((x_1,x_2)(y_1,y_2)) = \min\{T_{A_1}^N(x_1), T_{A_1}^N(y_1), T_{B_2}^N(x_2y_2)\}$$
$$\leq \min\{T_{A_1}^N(x_1), T_{A_1}^N(y_1), \min(T_{A_2}^N(x_2), T_{A_2}^N(y_2))\}$$
$$= \min\{\min(T_{A_1}^N(x_1), T_{A_2}^N(x_2)), \min(T_{A_1}^N(y_1), T_{A_2}^N(y_2))\}$$
$$= \min\{((T_{A_1}^N \mid T_{A_2}^N)(x_1,x_2)), ((T_{A_1}^N \mid T_{A_2}^N)(y_1,y_2))\}$$

$$(I_{B_1}^P \mid I_{B_2}^P)((x_1,x_2)(y_1,y_2)) = \min\{I_{A_1}^P(x_1), I_{A_1}^P(y_1), I_{B_2}^P(x_2y_2)\}$$
$$\leq \min\{I_{A_1}^P(x_1), I_{A_1}^P(y_1), \min(I_{A_2}^P(x_2), I_{A_2}^P(y_2))\}$$
$$= \min\{\min(I_{A_1}^P(x_1), I_{A_2}^P(x_2)), \min(I_{A_1}^P(y_1), I_{A_2}^P(y_2))\}$$
$$= \min\{((I_{A_1}^P \mid I_{A_2}^P)(x_1,x_2)), ((I_{A_1}^P \mid I_{A_2}^P)(y_1,y_2))\}$$

$$(I_{B_1}^N \mid I_{B_2}^N)((x_1,x_2)(y_1,y_2)) = \max\{I_{A_1}^N(x_1), I_{A_1}^N(y_1), I_{B_2}^N(x_2y_2)\}$$
$$\leq \max\{I_{A_1}^N(x_1), I_{A_1}^N(y_1), \max(I_{A_2}^N(x_2), I_{A_2}^N(y_2))\}$$
$$= \max\{\max(I_{A_1}^N(x_1), I_{A_2}^N(x_2)), \max(I_{A_1}^N(y_1), I_{A_2}^N(y_2))\}$$
$$= \max\{((I_{A_1}^N \mid I_{A_2}^N)(x_1,x_2)), ((I_{A_1}^N \mid I_{A_2}^N)(y_1,y_2))\}$$

$$(F_{B_1}^P \mid F_{B_2}^P)((x_1,x_2)(y_1,y_2)) = \max\{F_{A_1}^P(x_1), F_{A_1}^P(y_1), F_{B_2}^P(x_2y_2)\}$$
$$\leq \max\{F_{A_1}^P(x_1), F_{A_1}^P(y_1), \max(F_{A_2}^P(x_2), F_{A_2}^P(y_2))\}$$
$$= \max\{\max(F_{A_1}^P(x_1), F_{A_2}^P(x_2)), \max(F_{A_1}^P(y_1), F_{A_2}^P(y_2))\}$$
$$= \max\{((F_{A_1}^P \mid F_{A_2}^P)(x_1,x_2)), ((F_{A_1}^P \mid F_{A_2}^P)(y_1,y_2))\}$$

$$(F_{B_1}^N \mid F_{B_2}^N)((x_1,x_2)(y_1,y_2)) = \min\{F_{A_1}^N(x_1), F_{A_1}^N(y_1), F_{B_2}^N(x_2y_2)\}$$
$$\leq \min\{F_{A_1}^N(x_1), F_{A_1}^N(y_1), \min(F_{A_2}^N(x_2), F_{A_2}^N(y_2))\}$$
$$= \min\{\min(F_{A_1}^N(x_1), F_{A_2}^N(x_2)), \min(F_{A_1}^N(y_1), F_{A_2}^N(y_2))\}$$
$$= \min\{((F_{A_1}^N \mid F_{A_2}^N)(x_1,x_2)), ((F_{A_1}^N \mid F_{A_2}^N)(y_1,y_2))\}$$

Hence $G_1 \mid G_2$ is an BSFG.

Definition 2.4.3: Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two BSFGs. Then for any vertex, $(x_1, x_2) \in V_1 \times V_2$.

$$\deg(T_{G_1}^P \mid T_{G_2}^P)(x_1, x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (T_{B_1}^P \mid T_{B_2}^P)(x_1, x_2)(y_1, y_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \min\{ T_{A_1}^P(x_1), T_{B_2}^P(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \min\{ T_{A_2}^P(x_2), T_{B_1}^P(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \min\{ T_{B_1}^P(x_1y_1), T_{B_2}^P(x_2y_2)\}$$

$$\deg(T_{G_1}^N \mid T_{G_2}^N)(x_1, x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (T_{B_1}^N \mid T_{B_2}^N)(x_1, x_2)(y_1, y_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \max\{ T_{A_1}^N(x_1), T_{B_2}^N(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \max\{ T_{A_2}^N(x_2), T_{B_1}^N(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \max\{ T_{B_1}^N(x_1y_1), T_{B_2}^N(x_2y_2)\}$$

$$\deg(I_{G_1}^P \mid I_{G_2}^P)(x_1, x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (I_{B_1}^P \mid I_{B_2}^P)(x_1, x_2)(y_1, y_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \min\{ I_{A_1}^P(x_1), I_{B_2}^P(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \min\{ I_{A_2}^P(x_2), I_{B_1}^P(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \min\{ I_{B_1}^P(x_1y_1), I_{B_2}^P(x_2y_2)\}$$

$$\deg(I_{G_1}^N \mid I_{G_2}^N)(x_1, x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (I_{B_1}^N \mid I_{B_2}^N)(x_1, x_2)(y_1, y_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \max\{ I_{A_1}^N(x_1), I_{B_2}^N(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \max\{ I_{A_2}^N(x_2), I_{B_1}^N(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \max\{ I_{B_1}^N(x_1y_1), I_{B_2}^N(x_2y_2)\}$$

$$\deg(F_{G_1}^P \mid F_{G_2}^P)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (F_{B_1}^P \mid F_{B_2}^P)(x_1,x_2)(y_1,y_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \max\{F_{A_1}^P(x_1), F_{B_2}^P(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \max\{F_{A_2}^P(x_2), F_{B_1}^P(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \max\{F_{B_1}^P(x_1y_1), F_{B_2}^P(x_2y_2)\}$$

$$\deg(F_{G_1}^N \mid F_{G_2}^N)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (F_{B_1}^N \mid F_{B_2}^N)(x_1,x_2)(y_1,y_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \min\{F_{A_1}^N(x_1), F_{B_2}^N(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \min\{F_{A_2}^N(x_2), F_{B_1}^N(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \min\{F_{B_1}^N(x_1y_1), F_{B_2}^N(x_2y_2)\}$$

Definition 2.4.4: Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two BSFGs. Then for any vertex, $(x_1, x_2) \in V_1 \times V_2$.

$$t\deg(T_{G_1}^P \mid T_{G_2}^P)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (T_{B_1}^P \mid T_{B_2}^P)(x_1,x_2)(y_1,y_2) + (T_{A_1}^P \mid T_{A_2}^P)(x_1,x_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \min\{T_{A_1}^P(x_1), T_{B_2}^P(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \min\{T_{A_2}^P(x_2), T_{B_1}^P(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \min\{T_{B_1}^P(x_1y_1), T_{B_2}^P(x_2y_2)\} + \min\{(T_{A_1}^P(x_1), T_{A_2}^P(x_2)\}$$

$$t\deg(T_{G_1}^N \mid T_{G_2}^N)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (T_{B_1}^N \mid T_{B_2}^N)(x_1,x_2)(y_1,y_2) + (T_{A_1}^N \mid T_{A_2}^N)(x_1,x_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \max\{T_{A_1}^N(x_1), T_{B_2}^N(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \max\{T_{A_2}^N(x_2), T_{B_1}^N(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \max\{T_{B_1}^N(x_1y_1), T_{B_2}^N(x_2y_2)\} + \max\{(T_{A_1}^N(x_1), T_{A_2}^N(x_2)\}$$

$$t\deg(I_{G_1}^P \mid I_{G_2}^P)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (I_{B_1}^P \mid I_{B_2}^P)(x_1,x_2)(y_1,y_2) + (I_{A_1}^P \mid I_{A_2}^P)(x_1,x_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \min\{ I_{A_1}^P(x_1), I_{B_2}^P(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \min\{ I_{A_2}^P(x_2), I_{B_1}^P(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \min\{ I_{B_1}^P(x_1y_1), I_{B_2}^P(x_2y_2)\} + \min\{(I_{A_1}^P(x_1), I_{A_2}^P(x_2)\}$$

$$t\deg(I_{G_1}^N \mid I_{G_2}^N)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (I_{B_1}^N \mid I_{B_2}^N)(x_1,x_2)(y_1,y_2) + (I_{A_1}^N \mid I_{A_2}^N)(x_1,x_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \max\{ I_{A_1}^N(x_1), I_{B_2}^N(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \max\{ I_{A_2}^N(x_2), I_{B_1}^N(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \max\{ I_{B_1}^N(x_1y_1), I_{B_2}^N(x_2y_2)\} + \max\{(I_{A_1}^N(x_1), I_{A_2}^N(x_2)\}$$

$$t\deg(F_{G_1}^P \mid F_{G_2}^P)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (F_{B_1}^P \mid F_{B_2}^P)(x_1,x_2)(y_1,y_2) + (F_{A_1}^P \mid F_{A_2}^P)(x_1,x_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \max\{ F_{A_1}^P(x_1), F_{B_2}^P(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \max\{ F_{A_2}^P(x_2), F_{B_1}^P(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \max\{ F_{B_1}^P(x_1y_1), F_{B_2}^P(x_2y_2)\} + \max\{(F_{A_1}^P(x_1), F_{A_2}^P(x_2)\}$$

$$t\deg(F_{G_1}^N \mid F_{G_2}^N)(x_1,x_2) = \sum_{(x_1,x_2)(y_1,y_2)\in E_1|E_2} (F_{B_1}^N \mid F_{B_2}^N)(x_1,x_2)(y_1,y_2) + (F_{A_1}^N \mid F_{A_2}^N)(x_1,x_2)$$

$$= \sum_{x_1=y_1, x_2y_2\notin E_2} \min\{ F_{A_1}^N(x_1), F_{B_2}^N(x_2y_2)\}$$

$$+ \sum_{x_2=y_2, x_1y_1\notin E_1} \min\{ F_{A_2}^N(x_2), F_{B_1}^N(x_1y_1)\}$$

$$+ \sum_{x_1y_1\notin E_1, x_2y_2\notin E_2} \min\{ F_{B_1}^N(x_1y_1), F_{B_2}^N(x_2y_2)\} + \min\{(F_{A_1}^N(x_1), F_{A_2}^N(x_2)\}$$