Introduction

Z-Algebra:

A proposition is a statement. Propositional logic is a mathematical model that allows us to reason about the truth or falsehood of logical expressions, that are simplifications of a conditional expression. With the help of propositional calculus or logic, we represent a statement or proposition by a symbol called propositional variables. The values that can be assigned to a proposition are called truth values – either true or false. To model reasoning, an algebra is associated with the propositional logic. There are several such algebras such as BCK-algebras[25,28], BCI-algebras[27], BCH-algebras[24], Q-algebras[58], B-algebras[60], d-algebras[59] and so on. One such algebra which has been introduced recently, in the year 2017, by Chandramouleeswaran and others is that of the Z-algebras [18].

Fuzzy Sets:

The real physical world contains many problems of uncertainty, in which one cannot predict the property of "belongingness" of an object with certainty. For example, rose cannot be ascertained as a flower whose colour is rose, because we have rose flower with different colours (red, yellow,...). In order to deal with the problem of uncertainty in the real physical world, in 1965, Zadeh[73] introduced the notion of fuzzy sets. In studying the theory of fuzzy sets each element of the universe is assigned a membership grading whose value lies in [0,1]. Thus, a fuzzy set A can be characterized by a membership function $\mu_A : X \to [0,1]$ which is described by the set $\{(x, \mu_A(x)) \mid x \in X\}$.

The study of fuzzy sets and their application to mathematical concepts has reached to what is commonly called as fuzzy mathematics. Fuzzy algebra is an important branch of fuzzy mathematics. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1971 by Rosenfeld [65]. During 1985, Bhattacharya and Mukherji [13] introduced fuzzy structure in the form of fuzzy relations and fuzzy groups. In 1991, Xi [71] applied the concept of fuzzy sets to BCK-algebras.

For the general development of BCK/BCI-algebras, the ideal theory and its fuzzifications play an important role. In 1992, Meng and Xin [48] introduced the concept of implicative ideals in BCI-algebras. In 1994, Jun and Meng[32] introduced the notion of fuzzy p-ideals in BCI-algebras. In 1997, Meng et al. [49] and Mostafa [50] fuzzified the concept of implicative ideals in BCK-algebras, independently. In 1999, Jun et al.[33,38] studied the concepts of fuzzy ideals, fuzzy implicative ideals and constructed a fuzzy characteristic implicative ideal in BCK-algebras. In the same year, Khalid and Ahmad[40] introduced fuzzy H-ideals in BCI-algebras. The concept of fuzzy translations and fuzzy multiplications in BCK/ BCI-algebras have been discussed by Lee and others [44].

In 1967, the notion of fuzzy set has been extended to the notion of L-fuzzy set by Goguen [21] with the membership value $\mu_A(x) \in L$, any partial ordered algebraic structure. Further, it has been generalized by Atanassov[8] in 1986, as an intuitionistic fuzzy set, in which not only membership grading but also non-membership grading of an object is associated. Thus, an intuitionistic fuzzy set A can be described by $\{(x, \mu_A(x), \nu_A(x)) | x \in X\}$ where $\mu_A : X \to [0,1]$ is a membership function and $\nu_A : X \to [0,1]$ is a non-membership function. But in 1984 itself, Atanassov and Stoeva [11] introduced the notion of intuitionistic L-fuzzy sets. Here they have defined both membership and non-membership functions from the Universe of discourse X to the set L, where L is a complete lattice.

In 1975 itself, Zadeh [74] made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). In 1989, Atanassov and Gargov[10] introduced the notion of interval-valued intuitionistic fuzzy sets where the degree of membership and the degree of non-membership are real intervals rather than real numbers. Moreover, in 2012, Jun and others [36] introduced the concept of cubic sets, as a generalization of fuzzy sets and interval-valued fuzzy sets.

The Problem of Study:

In this work, we study the fuzzy structures on Z-algebras. In our research work, we combine the theory of fuzzy sets and the theory of Z-algebras. Thus, we have studied the notions of fuzzy Z-Subalgebras, fuzzy Z-Ideals, fuzzy H-ideals, fuzzy p-ideals, fuzzy implicative ideals, fuzzy sub-implicative ideals, fuzzy α-translations, fuzzy β-multiplications in Z-algebras. The theory has been extended to intuitionistic fuzzy Z-Subalgebras, intuitionistic fuzzy Z-ideals, intuitionistic L-fuzzy Z-Subalgebras, intuitionistic L-fuzzy Z-ideals, interval-valued fuzzy Z-Subalgebras, interval-valued fuzzy Z-ideals, interval-valued intuitionistic fuzzy Z-Subalgebras , interval-valued intuitionistic fuzzy Z-ideals, cubic Z-Subalgebras and cubic Z-ideals in Z-algebras.

Our work is divided into nine chapters.

1. Preliminaries:

In this Chapter, we have recalled some basic definitions from the theory of fuzzy sets. Also, we have recalled the definitions from algebras that arise from propositional calculi such as BCK-algebras, BCI-algebras and so on. In particular, we have presented the basic definitions on the Z-algebras.

2. Fuzzy Subalgebras of Z-Algebras:

In this Chapter , we have introduced the notions of **Fuzzy Z-Subalgebras and Fuzzy Z-Ideals in Z-Algebras** and proved some interesting results. Moreover, we have described how to deal with the Z-homomorphic image and inverse image of fuzzy Z-Subalgebras (fuzzy Z-ideals) in Z-algebras. Further, we have also proved that the Cartesian product of fuzzy Z-Subalgebras (fuzzy Z-ideals) is a fuzzy Z-Subalgebra (fuzzy Z-ideal) in Z-algebras.

We have defined the following terms.

- Let $(X,*,0)$ be a Z-algebra. A fuzzy set A in X with membership function μ_A is said to be a **fuzzy Z-Subalgebra** of a Z-algebra X if, for all x , y $\in$ X the following condition is satisfied : $\mu_A(x*y) \geq \min\{\mu_A(x), \mu_A(y)\}$.
- Let $(X,*,0)$ be a Z-algebra. A fuzzy set A in X with membership function μ_A is said to be a **fuzzy Z-ideal** of a Z-algebra X if it satisfies the following conditions: For all x, y in X,
  (i) $\mu_A(0) \geq \mu_A(x)$
  (ii) $\mu_A(x) \geq \min\{\mu_A(x*y), \mu_A(y)\}$

In this Chapter, we have proved the following interesting results.

- A fuzzy set A of a Z-algebra X is a fuzzy Z-Subalgebra if and only if for every $t \in [0,1]$, $U(\mu_A;t) = \{x \in X \mid \mu_A(x) \geq t\}$ is either empty or a Z-Subalgebra of X.

- Let X be a Z-algebra. Then given any chain of Z-Subalgebras $S_0 \subset S_1 \subset \cdots \subset S_r = X$, there exists a fuzzy Z-Subalgebra A of X whose upper level Z-Subalgebras are exactly the Z-Subalgebras of this chain.
- Let h be an Z-endomorphism of a Z-algebra $(X,*,0)$. If A be a fuzzy Z-Subalgebra of X. Then a new fuzzy set A^h in X defined by $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$ is also a fuzzy Z-Subalgebra of X.
- Let A and B be fuzzy sets in a Z-algebra X such that $A \times B$ is a fuzzy Z-ideal of $X \times X$. Then,

  (i) Either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$.

  (ii) If $\mu_A(0) \geq \mu_A(x)$ for all $x \in X$, then either $\mu_B(0) \geq \mu_A(x)$ (or) $\mu_B(0) \geq \mu_B(x)$.

  (iii) If $\mu_B(0) \geq \mu_B(x)$ for all $x \in X$, then either $\mu_A(0) \geq \mu_A(x)$ (or) $\mu_A(0) \geq \mu_B(x)$.
- Let A be a fuzzy relation on a Z-algebra X and A_B be the strongest fuzzy relation on X, where B is a fuzzy set of X. If B is a fuzzy Z-ideal of a Z-algebra X, then A_B is a fuzzy Z-ideal of X x X.

3. Classes of Fuzzy Ideals in Z-Algebras:

In this Chapter, the notions of **Fuzzy H-Ideals, Fuzzy p-Ideals, Fuzzy Implicative Ideals and Fuzzy Sub-Implicative Ideals in Z-Algebras** are introduced and proved some interesting results. Characterizations of Artinian Z-algebras and Noetherian Z-algebras via fuzzy p-ideals in Z-algebras are provided. Further, the relationship between fuzzy Z-ideal, fuzzy implicative ideal and fuzzy sub-implicative ideal of a Z-algebra are established.

We have defined the following terms.

- Let $(X,*,0)$ be a Z-algebra. A fuzzy set A in X with membership function μ_A is said to be a **fuzzy H-ideal** of a Z-algebra X if it satisfies the following conditions: For all x, y , z in X,

  (i) $\mu_A(0) \geq \mu_A(x)$

  (ii) $\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\}$

- Let $(X,*,0)$ be a Z-algebra. A fuzzy set A in X with membership function μ_A is said to be a **fuzzy p-ideal** of a Z-algebra X if it satisfies the following conditions: For all x, y, z in X,

  (i) $\mu_A(0) \geq \mu_A(x)$

  (ii) $\mu_A(x) \geq \min\{\mu_A((x*z)*(y*z)), \mu_A(y)\}$

- A Z-algebra $(X,*,0)$ is called **Noetherian** if for every ascending sequence $I_1 \subseteq I_2 \subseteq \ldots$ of p-ideals of X there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geq k$, where $\mathbb{N}$ be the set of natural numbers.

- A Z-algebra $(X,*,0)$ is called **Artinian** if for every descending sequence $I_1 \supseteq I_2 \supseteq \ldots$ of p-ideals of X there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geq k$.

- A fuzzy set A of a Z-algebra $(X,*,0)$ with membership function μ_A is called a **fuzzy implicative ideal** of X if it satisfies the following conditions:

  (i) $\mu_A(0) \geq \mu_A(x)$

  (ii) $\mu_A(x) \geq \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$, for all $x, y, z \in X$.

- A fuzzy implicative ideal A of a Z-algebra X is said to be **fuzzy characteristic** if $A^h = A$ for all $h \in Aut(X)$ where $A^h : X \to [0,1]$ is defined by $\mu_{A^h}(x) = \mu_A(h(x))$ $\forall x \in X$.

- A fuzzy set A of a Z-algebra $(X,*,0)$ with membership function μ_A is said to be a **fuzzy sub-implicative ideal** of X if it satisfies the following conditions:

  (i) $\mu_A(0) \geq \mu_A(x)$

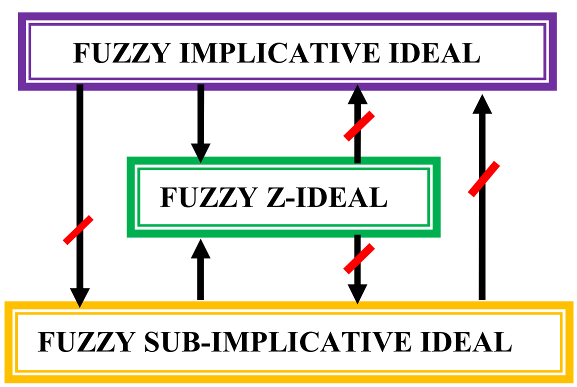
  (ii) $\mu_A(y*(y*x)) \geq \min\{\mu_A(((x*(x*y))*(y*x))*z), \mu_A(z)\}$, for all $x, y, z \in X$.

The following interesting results are obtained in this Chapter.

- A fuzzy set A of a Z-algebra $(X,*,0)$ is a fuzzy H-ideal if and only if every nonempty upper q-level subset $U(\mu_A;q)$ for $q \in Im(A)$ is an H-ideal.

- If every fuzzy p-ideal A of a Z-algebra X has only finite values, then every descending chain of p-ideals of X terminates after a finite stage.

- Let $I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots$ be a strictly ascending sequence of p-ideals in a Z-algebra X and let (t_n) be a strictly decreasing sequence in (0,1). Let A be a fuzzy set in X defined by
  $\mu_A(x) = \begin{cases} 0 & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N} \\ t_n & \text{if } x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N} \end{cases}$ where $I_0 = \phi$. Then A is a fuzzy p-ideal of X.
- Let A be a fuzzy set in a Z-algebra X with $Im(A) = \{t_0, t_1, ..., t_k\}$ where $t_0 > t_1 > t_2 > ... > t_k$. If there exists a chain of implicative ideals of X: $I_0 \subset I_1 \subset \ldots \subset I_k = X$ such that $\mu_A(I_n^*) = t_n$ where $I_n^* = I_n - I_{n-1}$, $I_{-1} = \phi$, $n = 0,1,...,k$, then A is a fuzzy implicative ideal of X.
- If A is a fuzzy characteristic implicative ideal of a Z-algebra X, then each upper level implicative ideal of A is a characteristic implicative ideal of X.
- Let X be a medial Z-algebra satisfies the condition: for all $x, y \in X$,
  $\mu_A((x*(x*y))*(y*x)) \geq \mu_A(x*(y*x))$
  Then every fuzzy sub-implicative ideal of X is a fuzzy implicative ideal of X.
- The following diagram gives the relations between fuzzy Z-ideal, fuzzy implicative ideal and fuzzy sub-implicative ideal of a Z-algebra.



4. Fuzzy α-Translations and Fuzzy β-Multiplications of Z-Algebras:

In this Chapter, we have introduced the notions of **Fuzzy α-Translations, Fuzzy Extensions and Fuzzy β-Multiplications of Fuzzy Z-Subalgebras (Fuzzy Z-Ideals) in Z-Algebras** and obtained relations between them. For any fuzzy set A in a Z-algebra $(X,*,0)$, we denote $T = 1 - \sup\{\mu_A(x) \mid x \in X\}$ unless otherwise specified. We introduced Z-homomorphism, cartesian product on fuzzy α-translations and fuzzy β-multiplications of Z-algebras and established some of their properties on the basis of fuzzy Z-Subalgebras and fuzzy Z-ideals of Z-algebras.

We have defined the following terms.

- Let A be a fuzzy set of a Z-algebra X and let $\alpha \in [0,T]$. A **fuzzy α-translation** A_α^T of A with membership function $\mu_{A_\alpha^T} : X \to [0,1]$ is defined by $\mu_{A_\alpha^T}(x) = \mu_A(x) + \alpha$, for all $x \in X$.
- When A_1 and A_2 are fuzzy sets of a Z-algebra X, A_2 is called a **fuzzy Z-Subalgebra extension** of A_1 if the following assertions are valid:

  (i) A_2 is a fuzzy extension of A_1 i.e., $\mu_{A_1}(x) \le \mu_{A_2}(x)$ for all $x \in X$.

  (ii) If A_1 is a fuzzy Z-Subalgebra of X, then A_2 is a fuzzy Z-Subalgebra of X.
- A fuzzy Z-Subalgebra extension B of a fuzzy Z-Subalgebra A in a Z-algebra X is said to be **normalized** if there exists $x_0 \in X$ such that $\mu_B(x_0) = 1$.
- Let A be a fuzzy Z-Subalgebra of a Z-algebra X. A fuzzy set B of X is called a **maximal fuzzy Z-Subalgebra extension** of A if it satisfies the following conditions:

  (i) B is a fuzzy Z-Subalgebra extension of A.

  (ii) there does not exist another fuzzy Z-Subalgebra of a Z-algebra X which is a fuzzy extension of B.
- Let A be a fuzzy set of a Z-algebra X and $\beta \in (0,1]$. A **fuzzy β-multiplication** A_β^M of A with membership function $\mu_{A_\beta^M} : X \to [0,1]$ is defined by $\mu_{A_\beta^M}(x) = \beta.\mu_A(x)$, for all $x \in X$.
- When A_1 and A_2 are fuzzy sets of a Z-algebra X, A_2 is called a **fuzzy Z-ideal extension** of A_1 if the following assertions are valid:

  (i) A_2 is a fuzzy extension of A_1

  (ii) If A_1 is a fuzzy Z-ideal of X, then A_2 is a fuzzy Z-ideal of X.

- Let $h:(X,*,0)\rightarrow(Y,*',0')$ be an Z-homomorphism of a Z-algebra X and A_{α}^{T} be a fuzzy α-translation of a fuzzy set A in Y. We define a **new fuzzy set** $(A_{\alpha}^{T})_{h}$ in X by $\mu_{(A_{\alpha}^{T})_{h}}(x)=\mu_{A_{\alpha}^{T}}(h(x))=\mu_{A}(h(x))+\alpha$, for all $x\in X$.
- Let $h:(X,*,0)\rightarrow(Y,*',0')$ be a Z-homomorphism of a Z-algebra X and A_{β}^{M} be a fuzzy β - multiplication of a fuzzy set A in Y. We define a **new fuzzy set** by $(A_{\beta}^{M})_{h}$ in X as $\mu_{(A_{\beta}^{M})_{h}}(x)=\mu_{A_{\beta}^{M}}(h(x))=\beta\cdot\mu_{A}(h(x))$, for all $x\in X$.
- Let A_{α}^{T} and B_{α}^{T} be fuzzy α-translations of fuzzy sets A and B in a Z-algebra X. The **Cartesian product** $A_{\alpha}^{T}x\,B_{\alpha}^{T}$ with membership function $\mu_{A_{\alpha}^{T}x\,B_{\alpha}^{T}}:X\,x\,X\rightarrow[0,1]$ is defined by $\mu_{A_{\alpha}^{T}\times B_{\alpha}^{T}}(x,y)=\min\{\mu_{A_{\alpha}^{T}}(x),\mu_{B_{\alpha}^{T}}(y)\}$, for all $x,y\in X$.
- Let A_{β}^{M} and B_{β}^{M} be fuzzy β-multiplications of fuzzy sets A and B in a Z-algebra X. The **Cartesian product** $A_{\beta}^{M}\times B_{\beta}^{M}$ with membership function $\mu_{A_{\beta}^{M}\times B_{\beta}^{M}}:X\times X\rightarrow[0,1]$ is defined by $\mu_{A_{\beta}^{M}\times B_{\beta}^{M}}(x,y)=\min\{\mu_{A_{\beta}^{M}}(x),\mu_{B_{\beta}^{M}}(y)\}$, for all $x,y\in X$.

The following important results are obtained in this Chapter.

- Let A be a fuzzy set of a Z-algebra X, $\alpha\in[0,T]$ and $\beta\in(0,1]$. Then every fuzzy α-translation A_{α}^{T} of A is a fuzzy Z-Subalgebra extension of the fuzzy β-multiplication A_{β}^{M} of A.
- Let A be a fuzzy Z-Subalgebra of a Z-algebra X. Then every maximal fuzzy Z-Subalgebra extension of A is normalized.
- For any fuzzy set A of a Z-algebra X, the following are equivalent:
  - (i) A is a fuzzy Z-ideal of X.
  - (ii) For all $\beta\in(0,1]$, the fuzzy β-multiplication A_{β}^{M} of A is a fuzzy Z-ideal of X.
- Let A be a fuzzy Z-ideal of a Z-algebra X and let $\alpha,\gamma\in[0,T]$. If $\alpha\geq\gamma$, then the fuzzy α- translation A_{α}^{T} of A is a fuzzy Z-ideal extension of the fuzzy γ-translation A_{γ}^{T} of A.

- Let $h:(X,*,0)\rightarrow(Y,*',0')$ be a Z-homomorphism of Z-algebras and A_{α}^{T} be a fuzzy α-translation of a fuzzy set A in Y. The pre-image of A_{α}^{T} denoted by $h^{-1}(A_{\alpha}^{T})$ is defined as $\mu_{h^{-1}(A_{\alpha}^{T})}(x)=\mu_{A_{\alpha}^{T}}(h(x))$, for all $x\in X$. If A is a fuzzy Z-Subalgebra (fuzzy Z-ideal) of Y then $h^{-1}(A_{\alpha}^{T})$ is a fuzzy Z-Subalgebra (fuzzy Z-ideal) of X.
- Let h be an Z-endomorphism of Z-algebra X. If A is a fuzzy Z-ideal of X, then $(A_{\beta}^{M})_{h}$ is also a fuzzy Z-ideal of X.
- Let A and B be fuzzy sets in a Z-algebra X such that $A_{\alpha}^{T} x B_{\alpha}^{T}$ is a fuzzy Z-ideal of $X x X$. Then either A or B is a fuzzy Z-ideal of X.
- Let A and B are fuzzy Z-Subalgebras of a Z-algebra X, then $A_{\beta}^{M}\times B_{\beta}^{M}$ is also a fuzzy Z-Subalgebra of $X\times X$.

5. Intuitionistic Fuzzy Structures in Z-Algebras:

In this Chapter, the notions of **Intuitionistic Fuzzy Z-Subalgebras and Intuitionistic Fuzzy Z-Ideals in Z-Algebras** are introduced and some of their properties are obtained. Further, some characterization theorems on these notions using the concepts of upper s-level and lower t-level subsets are also proved.

We have defined the following terms.

- An intuitionistic fuzzy set $A=(\mu_A,\nu_A)$ in a Z-algebra $(X,*,0)$ is called an **intuitionistic fuzzy Z-Subalgebra** of X if it satisfies the following conditions:
  (i) $\mu_A(x*y)\geq\min\{\mu_A(x),\mu_A(y)\}$
  (ii) $\nu_A(x*y)\leq\max\{\nu_A(x),\nu_A(y)\}$, for all $x,y\in X$.
- Let h be an Z-endomorphism of Z-algebras and A be an intuitionistic fuzzy set in a Z-algebra X. We define a **new intuitionistic fuzzy set** $A^h=(\mu_{A^h},\nu_{A^h})$ in X as $\mu_{A^h}(x)=\mu_A(h(x))$ and $\nu_{A^h}(x)=\nu_A(h(x))$ for all $x\in X$.
- An intuitionistic fuzzy set $A=(\mu_A,\nu_A)$ in a Z-algebra $(X,*,0)$ is called an **intuitionistic fuzzy Z-ideal** of X if it satisfies the following conditions:
  (i) $\mu_A(0)\geq\mu_A(x)$ and $\nu_A(0)\leq\nu_A(x)$
  (ii) $\mu_A(x)\geq\min\{\mu_A(x*y),\ \mu_A(y)\}$

(iii) $\nu_A(x) \le \max\{\nu_A(x*y),\ \nu_A(y)\}$, for all $x, y \in X$.

We have obtained the following interesting results in this Chapter.

- Any Z-Subalgebra of a Z-algebra X can be realized as both the upper s-level and lower t-level Z-Subalgebras of some intuitionistic fuzzy Z-Subalgebras of X.
- Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy Z-Subalgebra of a Z-algebra X. Then,

  (i) two upper s-level Z-Subalgebras $U(\mu_A; s_1)$ and $U(\mu_A; s_2)$ (with $s_1 < s_2$) of A are equal if and only if there is no $x \in X$ such that $s_1 \le \mu_A(x) < s_2$.

  (ii) two lower t-level Z-Subalgebras $L(\nu_A; t_1)$ and $L(\nu_A; t_2)$ (with $t_1 > t_2$) of A are equal if and only if there is no $x \in X$ such that $t_1 \ge \nu_A(x) > t_2$.
- Let h be an Z-endomorphism of Z-algebra $(X,*,0)$. If A be an intuitionistic fuzzy Z-ideal of X. Then intuitionistic fuzzy set $A^h = (\mu_{A^h}, \nu_{A^h})$ is also an intuitionistic fuzzy Z-ideal of X.
- Let A and B be two intuitionistic fuzzy Z-ideals in a Z-algebra X. Then $A \times B$ is an intuitionistic fuzzy Z-ideal of $X \times X$.

6. Intuitionistic L-Fuzzy Structures in Z-Algebras:

In this Chapter, the notions of **Intuitionistic L-Fuzzy Z-Subalgebras and Intuitionistic L-Fuzzy Z-Ideals in Z-Algebras** are introduced and some of their properties are acquired. Moreover, the csartesian product of intuitionistic L-fuzzy Z-Subalgebras (intuitionistic L-fuzzy Z-ideals) in Z-algebras is also explored.

We have defined the following terms.

- An **Intuitionistic L-fuzzy Set** $A = (\mu_A, \nu_A)$ in a Z-algebra $(X,*,0)$ is called an **Intuitionistic L-fuzzy Z-Subalgebra** of X if it satisfies the following conditions:

  (i) $\mu_A(x*y) \ge \mu_A(x) \wedge \mu_A(y)$

  (ii) $\nu_A(x*y) \le \nu_A(x) \vee \nu_A(y)$, for all $x, y \in X$.
- Let $h:(X,*,0) \to (X,*,0)$ be an Z-endomorphism of Z-algebras and A be an intuitionistic L-fuzzy set in X. We define a **new intuitionistic L-fuzzy set** $A^h = (\mu_{A^h}, \nu_{A^h})$ in X as $\mu_{A^h}(x) = \mu_A(h(x))$ and $\nu_{A^h}(x) = \nu_A(h(x))$ for all $x \in X$.

- An intuitionistic L-fuzzy set $A = (\mu_A, \nu_A)$ in a Z-algebra $(X,*,0)$ is called an **intuitionistic L-fuzzy Z-ideal** of X if it satisfies the following conditions:
  (i) $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$
  (ii) $\mu_A(x) \geq \mu_A(x * y) \wedge \mu_A(y)$
  (iii) $\nu_A(x) \leq \nu_A(x * y) \vee \nu_A(y)$, for all $x, y \in X$.

We have obtained the following interesting results in this Chapter.

- Let h be an Z-endomorphism of a Z-algebra $(X,*,0)$. If A be an intuitionistic L-fuzzy Z-Subalgebra of X. Then $A^h = (\mu_{A^h}, \nu_{A^h})$ is also an intuitionistic L-fuzzy Z-Subalgebra of X.
- Let A and B be any two intuitionistic L-fuzzy Z-Subalgebras of a Z-algebra X. Then $A \times B$ is an intuitionistic L-fuzzy Z-Subalgebra of $X \times X$.
- Let A and B be two intuitionistic L-fuzzy sets in a Z-algebra X. If $A \times B$ is an intuitionistic L-fuzzy Z-ideal of $X \times X$, the following are true.
  (i) $\mu_A(0) \geq \mu_B(y)$ and $\mu_B(0) \geq \mu_A(x)$ for all $x, y \in X$.
  (ii) $\nu_A(0) \leq \nu_B(y)$ and $\nu_B(0) \leq \nu_A(x)$ for all $x, y \in X$.
- Let A and B be two intuitionistic L-fuzzy sets in a Z-algebra X such that $A \times B$ is an intuitionistic L-fuzzy Z-ideal of $X \times X$. Then either A or B is an intuitionistic L-fuzzy Z-Ideal of X.

7. Interval-Valued Fuzzy Structures in Z-Algebras:

In this Chapter, the notions of **Interval-Valued Fuzzy Z-Subalgebras and Interval-Valued Fuzzy Z-Ideals in Z – Algebras** are introduced and some of their properties are explored. The Z-homomorphic image and inverse image of interval-valued fuzzy Z-Subalgebras and interval-valued fuzzy Z-ideals in Z-algebras are investigated. Also, the cartesian product of interval-valued fuzzy Z-Subalgebras and interval-valued fuzzy Z-ideals in Z-algebras are also discussed.

We have defined the following terms.

- Let $(X,*,0)$ be a Z-algebra. An interval-valued fuzzy set $A=\{(x,\tilde{\mu}_A(x))|x\in X\}$ in X is said to be an **interval-valued fuzzy Z-Subalgebra** of a Z-algebra X if: $\tilde{\mu}_A(x*y)\geq r\min\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$, for all $x,y\in X$.
- Let $(X,*,0)$ be a Z-algebra. An interval-valued fuzzy set $A=\{(x,\tilde{\mu}_A(x))|x\in X\}$ in X is said to be an **interval-valued fuzzy Z-ideal** of a Z-algebra X if:

  (i) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$

  (ii) $\tilde{\mu}_A(x) \geq$ rmin{ $\tilde{\mu}_A(x*y)$, $\tilde{\mu}_A(y)$}, for all $x,y\in X$.

We have proved the following results in this Chapter.

- An interval-valued fuzzy set $A=\{(x,\tilde{\mu}_A(x))\,|\,x\in X\}$ where $\tilde{\mu}_A=[\mu_A^L,\mu_A^U]$ in a Z-algebra X is an interval-valued fuzzy Z-Subalgebra of X if and only if $\mu_A^L:X\to[0,1]$ and $\mu_A^U:X\to[0,1]$ are fuzzy Z-Subalgebras of X.
- Every Z-Subalgebra of a Z-algebra X can be realized as an interval-valued $[s_1,s_2]$-level Z-Subalgebra of an interval-valued fuzzy Z-Subalgebra of X.
- Let $A=\{(x,\tilde{\mu}_A(x))\,|\,x\in X\}$ be an interval-valued fuzzy set in a Z-algebra X. Then A is an interval-valued fuzzy Z-ideal of X if and only if the nonempty set $U(\tilde{\mu}_A;[\gamma_1,\gamma_2])=\{x\in X\,|\,\tilde{\mu}_A(x)\geq[\gamma_1,\gamma_2]\}$ is an Z-ideal of X for every $[\gamma_1,\gamma_2]\in D[0,1]$.

8. Interval-Valued Intuitionistic Fuzzy Structures in Z-Algebras:

In this Chapter, the notions of **Interval-Valued Intuitionistic Fuzzy Z-Subalgebras and Interval-Valued Intuitionistic Fuzzy Z-Ideals in Z-Algebras** are introduced. Further, some of their interesting properties under Z-homomorphism and cartesian product in Z-algebras are investigated.

We have defined the following terms.

- An interval-valued intuitionistic fuzzy set $A=\{\langle x,\tilde{\mu}_A(x),\tilde{\nu}_A(x)\rangle|x\in X\}$ in a Z-algebra $(X,*,0)$ is called an **interval-valued intuitionistic fuzzy Z-Subalgebra** of X if it satisfies the following conditions:

  (i) $\tilde{\mu}_A(x*y)\geq r\min\{\tilde{\mu}_A(x),\tilde{\mu}_A(y)\}$

(ii) $\tilde{\nu}_A(x*y) \leq r\max\{\tilde{\nu}_A(x), \tilde{\nu}_A(y)\}$, for all $x, y \in X$.

- An interval-valued intuitionistic fuzzy set $A = \{\langle x, \tilde{\mu}_A(x), \tilde{\nu}_A(x)\rangle | x \in X\}$ in a Z-algebra $(X,*,0)$ is called an **interval-valued intuitionistic fuzzy Z-ideal** of X if it satisfies the following conditions:

  (i) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\nu}_A(0) \leq \tilde{\nu}_A(x)$

  (ii) $\tilde{\mu}_A(x) \geq r\min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$

  (iii) $\tilde{\nu}_A(x) \leq r\max\{\tilde{\nu}_A(x*y), \tilde{\nu}_A(y)\}$, for all $x, y \in X$.

The following important results are proved in this Chapter.

- Let $A_1 = (\tilde{\mu}_{A_1}, \tilde{\nu}_{A_1})$ and $A_2 = (\tilde{\mu}_{A_2}, \tilde{\nu}_{A_2})$ be interval-valued intuitionistic fuzzy Z-Subalgebras of a Z-algebra X. Then $A_1 \cap A_2 = (\tilde{\mu}_{A_1 \cap A_2}, \tilde{\nu}_{A_1 \cap A_2})$ is an interval-valued intuitionistic fuzzy Z-Subalgebra of X.
- Let $A = (\tilde{\mu}_A, \tilde{\nu}_A) = ([\mu_A^L, \mu_A^U], [\nu_A^L, \nu_A^U])$ be an interval-valued intuitionistic fuzzy set of a Z-algebra X then A is an interval-valued intuitionistic fuzzy Z-ideal of X $\Leftrightarrow A^L = (\mu_A^L, \nu_A^L)$ and $A^U = (\mu_A^U, \nu_A^U)$ are intuitionistic fuzzy Z-ideals of X.
- Let $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ be an interval-valued intuitionistic fuzzy set in a Z-algebra X. Then $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of X if and only if $\oplus A = (\tilde{\mu}_A, (\tilde{\mu}_A)^c)$ and $\otimes A = ((\tilde{\nu}_A)^c, \tilde{\nu}_A)$ are interval-valued intuitionistic fuzzy Z-ideals of X.
- An interval-valued intuitionistic fuzzy set $A = (\tilde{\mu}_A, \tilde{\nu}_A)$ is an interval-valued intuitionistic fuzzy Z-ideal of a Z-algebra X if and only if for all $[s_1, s_2], [t_1, t_2] \in D[0,1]$, the sets $U(\tilde{\mu}_A; [s_1, s_2])$ and $L(\tilde{\nu}_A; [t_1, t_2])$ are either empty or Z-ideals of X.

9. Cubic Structures in Z-Algebras:

In this Chapter, the notions of **Cubic Z-Subalgebras and Cubic Z-Ideals in Z-Algebras** are introduced and some of their properties are investigated. The Z-homomorphic image and inverse image of cubic Z-Subalgebras (cubic Z-ideals) in Z-algebras are investigated. Also, the cartesian product of cubic Z-Subalgebras and cubic Z-ideals in Z-algebras are also discussed.

We have defined the following terms.

- Let $(X,*,0)$ be a Z-algebra. A cubic set $A = (\widetilde{\mu}_A, \omega_A)$ in X is called a **cubic Z-Subalgebra** of X if it satisfies the following conditions:

  (i) $\widetilde{\mu}_A(x * y) \geq r\min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$

  (ii) $\omega_A(x * y) \leq \max\{\omega_A(x), \omega_A(y)\}$, for all $x, y \in X$.

- Let $(X,*,0)$ be a Z-algebra. A cubic set $A = (\widetilde{\mu}_A, \omega_A)$ in X is called a **cubic Z-ideal** of X if it satisfies the following conditions:

  (i) $\widetilde{\mu}_A(0) \geq \widetilde{\mu}_A(x)$ and $\omega_A(0) \leq \omega_A(x)$

  (ii) $\widetilde{\mu}_A(x) \geq r\min\{\widetilde{\mu}_A(x * y), \widetilde{\mu}_A(y)\}$

  (iii) $\omega_A(x) \leq \max\{\omega_A(x * y), \omega_A(y)\}$, for all $x, y \in X$.

We have obtained the following interesting results in this Chapter.

- Cubic set $A = (\widetilde{\mu}_A, \omega_A)$ of a Z-algebra X is a cubic Z-Subalgebra of X where $\widetilde{\mu}_A = [\mu_A^L, \mu_A^U]$ if and only if μ_A^L, μ_A^U and $(\omega_A)^c$ are fuzzy Z-Subalgebras of X.
- Let $h : (X,*,0) \to (Y,*',0')$ be an Z-epimorphism of Z-algebras. Let B be a cubic set of Y. If $h^{-1}(B)$ is a cubic Z-ideal of X then B is a cubic Z-ideal of Y.
- Let A and B be two cubic sets of a Z-algebra X such that $A \times B$ is a cubic Z-ideal of $X \times X$ .Then either A or B is a cubic Z-ideal of X.

Our thesis ends with a detailed **Bibliography.**