Some Interesting Results From Number Theory

By

M. Anuradha

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER EDUCATION FOR WOMEN - DEEMED UNIVERSITY, COIMBATORE - 641 043

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

MAY - 1999

SOME INTERESTING RESULTS FROM NUMBER THEORY

By

M. ANURADHA

A Dissertation submitted to the Avinashilingam Institute for
Home Science and Higher Education for Women,
Deemed University, Coimbatore – 641 043.

In Partial fulfilment of the requirements
for the degree of

MASTER OF SCIENCE IN MATHEMATICS

May 1999

K. N. Meenakshi
Signature of the Head
Of the Department
26.5.99

K. N. Meenakshi
Signature of the Guide

Acknowledgement

ACKNOWLEDGEMENT

The investigator is grateful to ***Dr. (Tmt.) RAJAMMAL P. DEVADAS****, M.A., M.Sc., Ph.D. (Ohio State), D.Sc. (Madras), Hon. D.H.L. (Oregon State), Hon. D.H.L. (Ohio State), Hon. D.Sc., (C.A. Agri., Kanpur), Hon. D.Sc. (University of Vlster, Northern Ireland), Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for providing the opportunity to conduct her investigation in this esteemed university.*

The author expresses her gratitude to ***Dr. Mrs. LAKSHMI SANTA RAJAGOPAL****, M.S., (Tennessee), Ph.D. (Madras), Vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for providing facilities to carry out the study.*

The author extends her deep sense of gratitude to ***Dr. Mrs. SAROJA PRABHAKARAN,*** *M.A., Diploma in Education (Madras), Ph.D. (Mother Teresa), Registrar and* ***Dr. (Mrs.) S. SIVAKAMASUNDARI,*** *M.Sc. (Annamalai), M.Phil, Ph.D. (Madras), Dean, Faculty of Science, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for facilitating the conduct of the study.*

*The author wishes to express her sincere gratitude to **Dr. (Miss.) K.N. MEENAKSHI,** M.Sc., Ph.D., Professor and Head of the Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Deemed University, Coimbatore, for her constant benevolent help, invaluable advice and illuminating guidance. She gratefully acknowledges her persistent efforts and affectionate endeavours.*

*The author would like to extend her thanks to other members of the **STAFF** for their ever ready help extended to her.*

*The author is greatly indebted to her loving **PARENTS**, and family members for their encouragement moral support in carrying out the work.*

Finally, the author wishes to express her heartfelt gratitude to all the people who helped her bringing out this work successfully.

Contents

CONTENTS

Introduction

INTRODUCTION

Number theory is considered to be the queen of all branches of mathematics. Some of the celebrated names of mathematicians who have done research in number theory are SRINIVASA RAMANUJAM, P. FERMAT, EUCLID, MINKOWSKI, GOLDBACK, P. BACKMANN, R.D. CARMICHAEL, DIOPHANTINE, DIRICHLET, L.E. DICKSON, G.H.HARDY, KRONECKER, LAGRANGE, MERSENNE.

The aim of this thesis is to discuss some interesting results from number theory.

In the first chapter, we discuss the paper "Some interesting results on a new combinatorial arithmetic function" by Soumendra Bera. Here the author has introduced the function $\overline{\left|\begin{matrix} r \\ t \end{matrix}\right|}$ defined by $\overline{\left|\begin{matrix} r \\ t \end{matrix}\right|}$ = sum of the products of the natural numbers taken t at a time out of first r natural numbers where $2 \leq t \leq r-1$. Using this he has introduced the function

$$f_n^{(a)} = \overline{\left|\begin{matrix} a+n-1 \\ 1 \end{matrix}\right|} f_{n-1}^{(a)} - \overline{\left|\begin{matrix} a+n-1 \\ 2 \end{matrix}\right|} f_{n-2}^{(a)} + \overline{\left|\begin{matrix} a+n-1 \\ 3 \end{matrix}\right|} f_{n-3}^{(a)} - \cdots$$

$$+ (-1)^{n-2} \overline{\left|\begin{matrix} a+n-1 \\ n-1 \end{matrix}\right|} + (-1)^{n-1} \overline{\left|\begin{matrix} a+n-1 \\ n \end{matrix}\right|}$$

He has obtained the following results.

Theorem – 1 : $\overline{\left|\begin{matrix} r \\ t \end{matrix}\right|} = \overline{\left|\begin{matrix} r-1 \\ t \end{matrix}\right|} + r\,\overline{\left|\begin{matrix} r-1 \\ t-1 \end{matrix}\right|}$

Theorem – 2 : $\overline{\left|\begin{matrix} n \\ 1 \end{matrix}\right|} - \overline{\left|\begin{matrix} n \\ 2 \end{matrix}\right|} + \overline{\left|\begin{matrix} n \\ 3 \end{matrix}\right|} - \ldots + (-1)^{n-1}\overline{\left|\begin{matrix} n \\ n \end{matrix}\right|} = 1$

Theorem – 3 : $f_n^{(1)} = 1$

Theorem – 4 : $f_n^{(0)} = 0$

Theorem – 5 : $f_{n+1}^{(a)} - f_{n+1}^{(a-1)} = a\, f_n^{(a)}$ where a and n are both positive integers

Theorem – 6 : $f_n^{(a)} = \frac{a^{a+n-1}}{0!(a-1)!} - \frac{(a-1)^{a+n-1}}{1!(a-2)!} + \frac{(a-2)^{a-1}}{2!(a-3)!} - \cdots (-1)^{a-1}\frac{1}{(a-1)!0!}$

In chapter 2 we discuss the following 3 papers.

(i) "On a paper of Andre Schinzel" by D. SURYANARAYANA and N.VENKATESWARA RAO

(ii) "Solutions of a Mordell Diophantine equation" by WAH KEUNG CHAN

(iii) "A report on primes of the form k and on factors of Fermat numbers" by ROBINSON.

In the first paper the author gives a method of solving and finding integral solutions of the equations.

$$\frac{3}{2n+1} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

In the second paper the author has discussed the integral solutions of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} + \frac{1}{wxyz} = 0$$

He has obtained the following results.

(1) For each N there exist infinite number of solutions with $N = w + z$

(2) For each integer m there exists a finite number of solutions with $w = 2m$

In the 3rd paper the author has given a report on primes of the form $k.2^n + 1$.

He has constructed a list of primes of the form $k.2^n + 1$.

In the 3rd chapter, we discuss the following 2 papers.

(1) "Pick's theorem revisited by DALE E. VARBERG

(2) "The determination of Kaprekar convergence and loop convergence of all three digit – numbers" by KLAUS E. ELDRIDGE and SEOK SAGONG.

In the first paper the following Pick's theorem is proved.

The area of a simple lattice polygon s is given by

$$A(s) = i + \tfrac{1}{2} b - 1$$

$$= r - \tfrac{1}{2} b - 1$$

where i = the number of interior lattice points

b = the boundary of lattice points

r = the total number of lattice point of s.

In the second paper the Kaprekar transformation k is defined on the set of n digit numbers to the base r.

The transformation k permutes the digits of x to obtain the largest and smallest number using the same n digits and then substracts the smallest from the largest.

Recursively, we define,

$$K^n(x) = \begin{cases} x & \text{for } n = 0 \\ k(k^{n-1}(x)) & \text{for } n = 1, 2, 3, \cdots \end{cases}$$

For example if x = 4 5 6 in base 10, then $k^0(456) = 456$,

$$k^1(456) = k(456) = 654 - 456 = 198,$$

$$k^2(456) = k(k^1(456)) = k(198) = 931 - 189 = 792$$

and so on.

Let s(3,r) denote the set of all three digit numbers exclusive of those digit sequences with all digits equal.

Let x be in s(3,r). If k(x) = x, then x is said to be a fixed point of k.

For example, take 495 in s(3,r). Then k(495) = 495. If x is a fixed point in s(3,r) such that $k^m(s(3,r)) = \{x\}$ for some positive integer m, then x is called the Kaprekar constant of s(3,r). The smallest such integer m is called the degree of Kaprekar convergence of s(3,r).

Let x be a fixed point of k in s(3,r). If y is in s(3,r) and $k^n(y) = x$ for some positive integer n then y is said to converge to x under k and the smallest such integer n is called the degree of convergence y to x.

If, in particular, x is Kaprekar constant, then y is said to be Kaprekar convergent and the degree of convergence of y to x is called the degree of Kaprekar convergence of y.

If, for some integer m, $k^m(x) = k^{m+i}(x)$ $k^m(x) \neq k^{m+j}(x)$ for all j such that $1 \leq j < i$ then $k^m(x)$ is said to generate a loop of length i and each of $k^{m+j}(x)$, j = 0, 1, ……, I-1 is called a loop element. For example, consider 363 in base 7. Then 363 → 264 → 363 form a loop of length 2 where 264 is a loop element.

The main result proved in this paper is the following :

a) The set s(3,r) has the Kaprekar constant $(\frac{r-2}{2}$,r-1,r/2) iff r is even

b) If x is in s(3.r), where r is even, is not the Kaprekar constant, then its degree of convergence is given by;

(i) $2 - I + r/2$ if $I < r/2$

(ii) $1 + I + r/2$ if $I \geq r/2$

Here I is the difference between the greatest and the smallest digits of x.

Review of Literature

REVIEW OF LITERATURE

Some of the important topics in the study of number theory are :

(1) Distribution of primes

(2) Diophantine equation

(3) Partition of numbers

(4) Study of arithmetic functions $\mu(n)$, $\sigma(n)$, $d(n)$ and $R(n)$

(5) The representation of a number as a sum of two, four and five squares.

(6) Geometry of numbers.

Since number theory is an age old branch of mathematics, the origin of the study of these topics is very difficult to find. A beautiful introduction to the theory of numbers can be found in the classical work of G.H.Hardy and E.M.Wright published in 1938. This book contains an ocean of information on the topics mentioned above.

There are many good books in number theory. To mention a few we have,

1. An introduciton to the theory of numbers by Iran Niven and Herbert S. Zucherman [7].
2. Introduction to analysis number theory by A.M. Apostol [2]
3. A selection of problems in the thoery of nos by Waclaw Sierpinski [15].

On ancient Indian Mathematics, we have the following important books.

1. Sari Bharathi Krishna Tirthaji Maharaja, Vedic Methematics, Motilal Banarji Das, New Delhi (1978).

2. Sri Bharathi, Krishna Tirthaji Maharaja, Vedic Metaphysics, Motilal Banerji Das, New Delhi (1978).

3. B.B. Datta and A.N. Singh, History of Hindu Mathematics, Parts I and II, Motilal Banarji Das, (1935), (1938).

4. Krishna Ganaka Bija Pallavam, a Commentory on Bija Ganita of Bhaskara

5. T.A. Saraswati Amma; Geometry in Ancient and Medieval India, Motilal Banarji Das, (1979).

6. C.N. Srinivasiengar; History of ancient Indian Mathematics. The World Press Private Ltd., Calcutta (1967).

7. T.S. Banumathy, A modern introduction to Ancient Indian Mathematics, (1992), Wiley Eastern Ltd.

Chapter 1

CHAPTER I

In this chapter, we shall discuss the article "SOME INTERESTING RESULTS ON A NEW COMBINATORIAL ARITHMETIC FUNCTION" by SOUMENDRA DERA.

SECTION : 1

DEFINITION : 1.1.1

$\overline{\left|\begin{matrix} r \\ t \end{matrix}\right|}$ means sum of the products of the natural numbers taken t at a time out of first r natural numbers where $2 \leq t \leq r\text{-}1$.

EXAMPLE :

$$\overline{\left|\begin{matrix} 4 \\ 1 \end{matrix}\right|} = 1 + 2 + 3 + 4$$

$$\overline{\left|\begin{matrix} 4 \\ 2 \end{matrix}\right|} = 1.2 + 1.3 + 1.4 + 2.3 + 2.4 + 3.4$$

$$\overline{\left|\begin{matrix} 4 \\ 3 \end{matrix}\right|} = 1.2.3 + 1.2.4 + 1.3.4 + 2.3.4$$

and $\overline{\left|\begin{matrix} 4 \\ 4 \end{matrix}\right|} = 1 . 2 . 3 . 4$

THEOREM : 1.1.2

$$\overline{\left|\begin{matrix} r \\ t \end{matrix}\right|} = \overline{\left|\begin{matrix} r-1 \\ t \end{matrix}\right|} + r\,\overline{\left|\begin{matrix} r-1 \\ t-1 \end{matrix}\right|}$$

PROOF :

$\overline{\left|\begin{matrix} r \\ t \end{matrix}\right|}$ is the sum of $\binom{r}{t}$ products we shall divide the products into two parts. One part contains the products which have the common factor r. From all these products, we get the new products where total number of different natural numbers is (r-1) and number of factors in each product is (t-1). In other words, these new products are obtained taken (t-1) at a time out of (r-1) successive natural numbers starting with 1. Then sum of the products of the first part is clearly $r\overline{\left|\begin{matrix} r-1 \\ t-1 \end{matrix}\right|}$. The second part contains the remaining products each of which has t factors as it is, but total number of different natural numbers in this case is (r-1) due to absence of the highest natural number r. So sum of products of the second part is $\overline{\left|\begin{matrix} r-1 \\ t \end{matrix}\right|}$.

$$\text{Then } \overline{\left|\begin{matrix} r \\ t \end{matrix}\right|} = \text{sum of two parts}$$

$$= \overline{\left|\begin{matrix} r-1 \\ t \end{matrix}\right|} + r\overline{\left|\begin{matrix} r-1 \\ t-1 \end{matrix}\right|}$$

and hence we have the theorem 1.

EXAMPLE :

$$\overline{\left|\begin{matrix} 4 \\ 3 \end{matrix}\right|} = 1.2.3 + 1.2.4 + 1.3.4 + 2.3.4$$

$$= 1.2.3 + 4(1.2 + 1.3 + 2.3)$$

$$= \overline{\left|\begin{matrix}3\\3\end{matrix}\right|} + 4\overline{\left|\begin{matrix}3\\2\end{matrix}\right|}$$

THEOREM : 1.1.3

$$\overline{\left|\begin{matrix}n\\1\end{matrix}\right|} - \overline{\left|\begin{matrix}n\\2\end{matrix}\right|} + \overline{\left|\begin{matrix}n\\3\end{matrix}\right|} - \ldots + (-1)^{n-1}\overline{\left|\begin{matrix}n\\n\end{matrix}\right|} = 1$$

PROOF :

Proof is by mathematical induction. Let the above theorem be true for n = r, r being some positive integer. That is we suppose that,

$$\overline{\left|\begin{matrix}r\\1\end{matrix}\right|} - \overline{\left|\begin{matrix}r\\2\end{matrix}\right|} + \overline{\left|\begin{matrix}r\\3\end{matrix}\right|} - \ldots + (-1)^{r-1}\overline{\left|\begin{matrix}r\\r\end{matrix}\right|} = 1$$

Then $\overline{\left|\begin{matrix}r+1\\1\end{matrix}\right|} - \overline{\left|\begin{matrix}r+1\\2\end{matrix}\right|} + \overline{\left|\begin{matrix}r+1\\3\end{matrix}\right|} - \ldots + (-1)^{r-1}\overline{\left|\begin{matrix}r+1\\r\end{matrix}\right|} + (-1)^{r}\overline{\left|\begin{matrix}r+1\\r+1\end{matrix}\right|}$

$$= \{\overline{\left|\begin{matrix}r\\1\end{matrix}\right|} + (r+1)\} - \{\overline{\left|\begin{matrix}r\\2\end{matrix}\right|} + (r+1)\overline{\left|\begin{matrix}r\\1\end{matrix}\right|}\} + \{\overline{\left|\begin{matrix}r\\3\end{matrix}\right|} + (r+1)\overline{\left|\begin{matrix}r\\2\end{matrix}\right|}\} - \ldots$$

$$+ (-1)^{r-1}\{\overline{\left|\begin{matrix}r\\r\end{matrix}\right|} + (r+1)\overline{\left|\begin{matrix}r\\r-1\end{matrix}\right|}\} + (-1)^{r}(r+1)\overline{\left|\begin{matrix}r\\r\end{matrix}\right|}$$

$$= \{\overline{\left|\begin{matrix}r\\1\end{matrix}\right|} - \overline{\left|\begin{matrix}r\\2\end{matrix}\right|} + \overline{\left|\begin{matrix}r\\3\end{matrix}\right|} - \ldots(-1)^{r-1}\overline{\left|\begin{matrix}r\\r\end{matrix}\right|}\} \qquad \text{(by theorem 1)}$$

$$+ (r+1) - (r+1)\{\overline{\left|\begin{matrix}r\\1\end{matrix}\right|} - \overline{\left|\begin{matrix}r\\2\end{matrix}\right|} + \ldots(-1)^{r-1}\overline{\left|\begin{matrix}r\\r\end{matrix}\right|}\}$$

$$= \quad 1 + (r+1) - (r+1),\ 1 = 1$$

Above relation shows that the theorem is true for n = r+1, if the theorem is true. For n = 1, also when n = 1, we have $\overline{\left|\begin{matrix}1\\1\end{matrix}\right|} = 1$ and when n = 2, we have

$$\overline{\left|\begin{matrix}2\\1\end{matrix}\right|} = \overline{\left|\begin{matrix}2\\2\end{matrix}\right|} = (1+2) - (1\,.\,2) = 1.$$

Hence the theorem is true for all positive integers.

SECTION : 2

DEFINITION : 1.2.1

$$f_1^{(a)} = \overline{\left|\begin{matrix}a\\1\end{matrix}\right|};\quad f_2^{(a)} = \overline{\left|\begin{matrix}a+1\\1\end{matrix}\right|} f_1^{(a)} - \overline{\left|\begin{matrix}a+1\\2\end{matrix}\right|}$$

In general, $f_n^{(a)}$ is defined in terms of $f_{n-1}^{(a)}$, $f_{n-2}^{(a)}$ and so on.

$$f_n^{(a)} = \overline{\left|\begin{matrix}a+n-1\\1\end{matrix}\right|} f_{n-1}^{(a)} - \overline{\left|\begin{matrix}a+n-1\\2\end{matrix}\right|} f_{n-2}^{(a)} + \cdots$$

$$+ (-1)^{n-2} \overline{\left|\begin{matrix}a+n-1\\n-1\end{matrix}\right|} f_1^{(a)} + (-1)^{n-1} \overline{\left|\begin{matrix}a+n-1\\n\end{matrix}\right|}$$

In this equation, $f_n^{(a)}$, $f_{n-1}^{(a)}$, $f_1^{(a)}$ and $\overline{\left|\begin{matrix}a+n-1\\1\end{matrix}\right|}$, $\overline{\left|\begin{matrix}a+n-1\\2\end{matrix}\right|}$, ... , $\overline{\left|\begin{matrix}a+n-1\\n\end{matrix}\right|}$ are two groups of successive values of dependent variables each of a and n being any positive integer.

EXAMPLES :

When n = 3, we have

$$f_3^{(a)} = \overline{\left|\begin{matrix} a+2 \\ 1 \end{matrix}\right|} f_2^{(a)} - \overline{\left|\begin{matrix} a+2 \\ 2 \end{matrix}\right|} f_1^{(a)} + \cdots \overline{\left|\begin{matrix} a+2 \\ 3 \end{matrix}\right|}$$

$$= \overline{\left|\begin{matrix} a+2 \\ 1 \end{matrix}\right|} \left[\overline{\left|\begin{matrix} a+1 \\ 1 \end{matrix}\right|} \; \overline{\left|\begin{matrix} a \\ 1 \end{matrix}\right|} - \overline{\left|\begin{matrix} a+1 \\ 2 \end{matrix}\right|} \right]$$

$$= \overline{\left|\begin{matrix} a+2 \\ 2 \end{matrix}\right|} \; \overline{\left|\begin{matrix} a \\ 1 \end{matrix}\right|} + \overline{\left|\begin{matrix} a+2 \\ 3 \end{matrix}\right|}$$

When a = 0, we have

$$f_n^{(0)} = \overline{\left|\begin{matrix} n-1 \\ 1 \end{matrix}\right|} f_{n-1}^{(0)} - \overline{\left|\begin{matrix} n-1 \\ 2 \end{matrix}\right|} f_{n-2}^{(0)} + \cdots$$

$$+ (-1)^{n-2} \overline{\left|\begin{matrix} n-1 \\ n-1 \end{matrix}\right|} f_1^{(0)} + (-1)^{n-1} \overline{\left|\begin{matrix} n-1 \\ n \end{matrix}\right|}$$

THEOREM : 1.2.2

$$f_n^{(1)} = 1$$

PROOF

When n = 1, we have $f_1^{(1)} = \overline{\left|\begin{matrix} 1 \\ 1 \end{matrix}\right|} = 1$ and theorem obviously holds for n = 1.

When n = 2, we have

$$f_2^{(1)} = \overline{\left|\begin{matrix} 2 \\ 1 \end{matrix}\right|} f_1^{(1)} - \overline{\left|\begin{matrix} 2 \\ 2 \end{matrix}\right|}$$

$$= \overline{\left|\begin{matrix}2\\1\end{matrix}\right|}.1 - \overline{\left|\begin{matrix}2\\2\end{matrix}\right|}$$

$$= 1$$

Hence the theorem holds for n = 2 whenever it holds for n = 1.

To complete the proof, we now assume that the theorem holds for n = 1, 2, 3, ..., r. Thus by assumption we write

$$f_r^{(1)} = f_{r-1}^{(1)} = f_{r-2}^{(1)} = \dots = f_1^{(1)} = 1$$

Also by the initial equation we have

$$f_r^{(1)} = \overline{\left|\begin{matrix}r\\1\end{matrix}\right|}\, f_{r-1}^{(1)} - \overline{\left|\begin{matrix}r\\2\end{matrix}\right|}\, f_{r-2}^{(1)} + \dots + (-1)^{r-2}\overline{\left|\begin{matrix}r\\r-1\end{matrix}\right|}\, f_r^{(1)} + (-1)^{r-1}\overline{\left|\begin{matrix}r\\r\end{matrix}\right|}$$

Then $f_{r+1}^{(1)} = \overline{\left|\begin{matrix}r+1\\1\end{matrix}\right|}\, f_r^{(1)} - \overline{\left|\begin{matrix}r+1\\2\end{matrix}\right|}\, f_{r-1}^{(1)} + \overline{\left|\begin{matrix}r+1\\3\end{matrix}\right|}\, f_{r-2}^{(1)} - \dots + (-1)^{r-1}\overline{\left|\begin{matrix}r+1\\r\end{matrix}\right|}\, f_1^{(1)} + (-1)^r\overline{\left|\begin{matrix}r+1\\r+1\end{matrix}\right|}$

$$= \overline{\left|\begin{matrix}r+1\\1\end{matrix}\right|}\,.\,1 - \overline{\left|\begin{matrix}r+1\\2\end{matrix}\right|}\,.\,1 + (-1)^{r-1}\overline{\left|\begin{matrix}r+1\\r\end{matrix}\right|}.1 + (-1)^r\overline{\left|\begin{matrix}r+1\\r+1\end{matrix}\right|}$$

$$= 1$$

The above relation shows that the theorem holds for n = r +1, whenever it holds for n = 1, 2 … r. But we already proved that the theorem holds for n = 1 and for n = 2. Hence we have theorem.

For every natural number n by mathematical induction.

THEOREM : 1.2.3

$$f_n^{(0)} = 0$$

PROOF :

For n = 1, we have $f_1^{(0)} = \left|\begin{matrix}0\\1\end{matrix}\right| = 0 \left[\left|\begin{matrix}r\\t\end{matrix}\right| = 0\ r < t\right]$. For n = 2, we have

$f_2^{(0)} = \left|\begin{matrix}1\\1\end{matrix}\right| f_1^{(0)} - \left|\begin{matrix}1\\2\end{matrix}\right| = 1.0 - 0 = 0$. Thoerem thus holds for n = 1 and for n = 2.

Now we assume theorem to be true for n = 1, 2, … , r.

Thus by assumption we have

$$f_r^{(0)} = f_{r-1}^{(0)} = f_{r-2}^{(0)} = f_1^{(0)} = 0 \text{ where}$$

$$f_r^{(0)} = \left|\begin{matrix}r-1\\1\end{matrix}\right| f_{r-1}^{(0)} - \left|\begin{matrix}r-1\\2\end{matrix}\right| f_{r-2}^{(0)} + \dots + (-1)^{r-1} \left|\begin{matrix}r\\r-1\end{matrix}\right|$$

$$\text{Then } f_{r+1}^{(0)} = \left|\begin{matrix}r\\1\end{matrix}\right| f_r^{(0)} - \left|\begin{matrix}r\\2\end{matrix}\right| f_{r-1}^{(0)} + \dots + (-1)^{r-1} \left|\begin{matrix}r\\r\end{matrix}\right| f_r^{(0)} + (-1)^r \left|\begin{matrix}r\\r+1\end{matrix}\right|$$

$$= 0 + 0 + 0 \dots = 0$$

This shows that the theorem holds for n = r + 1 whenever it holds for n = 1, 2, .. r. Hence by induction, the required theorem holds for all positive integers.

THEOREM : 1.2.4

$$f_{n+1}^{(a)} - f_{n+1}^{(a-1)} = a\, f_n^{(a)} \text{ where a and n are both positive integers}$$

PROOF :

When n = 1, we first notice that

$$f_2^{(a)} = \overline{\left|\begin{matrix}a+1\\1\end{matrix}\right|}\, f_1^{(a)} - \overline{\left|\begin{matrix}a+1\\2\end{matrix}\right|} \quad \text{a being any positive integer.}$$

Then $f_2^{(a-1)} = \overline{\left|\begin{matrix}a\\1\end{matrix}\right|}\, f_1^{(a-1)} - \overline{\left|\begin{matrix}a\\2\end{matrix}\right|}$

$$= \overline{\left|\begin{matrix}a\\1\end{matrix}\right|}\left(f_1^{(a)} - a\right) - \overline{\left|\begin{matrix}a\\2\end{matrix}\right|}$$

$$= \overline{\left|\begin{matrix}a\\1\end{matrix}\right|}\, f_1^{(a)} - a\overline{\left|\begin{matrix}a\\1\end{matrix}\right|} - \overline{\left|\begin{matrix}a\\2\end{matrix}\right|}$$

$$f_2^{(a)} - f_2^{(a-1)} = \left[\overline{\left|\begin{matrix}a+1\\1\end{matrix}\right|} - \overline{\left|\begin{matrix}a\\1\end{matrix}\right|}\right] f_1^{(a)} \left[\overline{\left|\begin{matrix}a+1\\2\end{matrix}\right|} - a\overline{\left|\begin{matrix}a\\1\end{matrix}\right|} - \overline{\left|\begin{matrix}a\\2\end{matrix}\right|}\right]$$

$$= (a+1)\; f_1^{(a)} - 1\overline{\left|\begin{matrix}a\\1\end{matrix}\right|}$$

$$= (a+1)\; f_1^{(a)} - f_1^{(a)}$$

$$= a\; f_1^{(a)}$$

Above relation shows that the theorem holds for n = 1, while a is any positive integer. To complete the proof, we know assume that theorem hold good, when $n = 1, 2, 3 \ldots r$ and $a = p$. p being a fixed positive integer. Thus by assumption, we write

$$f_2^{(p)} - f_2^{(p-1)} = p\, f_1^{(p)} \qquad \text{(A1.1)}$$

$$f_3^{(p)} - f_3^{(p-1)} = p\, f_2^{(p)} \qquad \text{(A1.2)}$$

$$f_4^{(p)} - f_4^{(p-1)} = p\, f_3^{(p)} \qquad \text{(A1.3)}$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$f_r^{(p)} - f_r^{(p-1)} = p\, f_{r-1}^{(p)} \qquad \text{(A1.r-1)}$$

and $$f_{r+1}^{(p)} - f_{r+1}^{(p-1)} = p\, f_r^{(p)} \qquad \text{(A1.r)}$$

Again by initial equation, we have,

$$f_{r+2}^{(p)} = \left|\begin{array}{c}p+r+1\\1\end{array}\right| f_{r+1}^{(p)} - \left|\begin{array}{c}p+r+1\\2\end{array}\right| f_r^{(p)} + \left|\begin{array}{c}p+r+1\\3\end{array}\right| f_{r-1}^{(p)} - \cdots$$

$$+(-1)^r \left|\begin{array}{c}p+r+1\\r+1\end{array}\right| f_1^{(p)} + (-1)^{r+1} \left|\begin{array}{c}p+r+1\\r+2\end{array}\right|$$

Then, $$f_{r+2}^{(p)} = \left|\begin{array}{c}p+r\\1\end{array}\right| f_{r+1}^{(p-1)} - \left|\begin{array}{c}p+r\\2\end{array}\right| f_r^{(p-1)} + \left|\begin{array}{c}p+r\\3\end{array}\right| f_{r-1}^{(p-1)} - \cdots$$

$$+(-1)^r \left|\begin{array}{c}p+r\\r+1\end{array}\right| f_1^{(p-1)} + (-1)^{r+1} \left|\begin{array}{c}p+r\\r+2\end{array}\right|$$

$$= \binom{p+r}{1}\left[f_{r+1}^{(p-1)} - pf_{r}^{(p)}\right] - \binom{p+r}{2}\left[f_{r}^{(p)} - f_{r-1}^{(p)}\right] + \binom{p+r}{3}\left[f_{r-1}^{(p-1)} - pf_{r-2}^{(p)}\right] - \cdots$$

$$+ (-1)^{r}\binom{p+r}{r+1}$$

$$\left[f_{1}^{(p)} - p\right] + (-1)^{r+1}\binom{p+r}{r+2} \qquad \text{(by A1.r, A1.r-1, A1.r-2)}$$

$$\therefore \quad f_{r+2}^{(p)} = f_{r+2}^{(p-1)} = \left[\binom{p+r+1}{1} - \binom{p+r}{1}\right]f_{r+1}^{(p)} - \left[\binom{p+r+1}{2} - p\binom{p+r}{1} - \binom{p+r}{2}\right]f_{r}^{(p)}$$

$$\left[\binom{p+r+1}{3} - p\binom{p+r}{2} - \binom{p+r}{3}\right]f_{r-1}^{(p)} + \cdots$$

$$+ (-1)^{r+1}\left[\binom{p+r+1}{r+2} - p\binom{p+r}{r+1} - \binom{p+r}{r+2}\right]$$

$$= (p+r+1)\ f_{r+1}^{(p)} - (r+1)\binom{p+r}{1} f_{r}^{(p)} + (r+1)\binom{p+r}{2} f_{r-1}^{(p)} - \cdots +$$

$$(-1)^{r}(r+1)\binom{p+r}{r} f_{1}^{(p)} + (-1)^{r+1}\binom{p+r}{r+1}$$

Clearly similar relation will be obtained if we consider any fixed positive integral values of a, while n = 1, 2, 3 … r. Thus the above relation shows that while a is any positive integer, the theorem holds for n = r +1, if it holds for n = 1, 2, … r. But it is already proved that the theorem holds for n = 1, where a is any positive integer. Therefore, we have the thoerem 5 for all positive integral value of a and n following mathematical induction. This completes the proof.

THEOREM : 1.2.5

$f_0^{(a)} = 1$, a being any positive integer.

PROOF :

We find the occurrence of the symbol $f_0^{(a)}$, when we put n = 0 in the theorem. Thus, we get,

$$a\, f_0^{(a)} = f_1^{(a)} - f_1^{(a-1)} = \begin{vmatrix} a \\ 1 \end{vmatrix} - \begin{vmatrix} a-1 \\ 1 \end{vmatrix} = a.$$

$\therefore f_0^{(a)} = 1$ a being any positive integer and hence we have the theorem.

Using the properties of $f_n^{(a)}$ we can prove the following identities.

$$(1) \sum \frac{a_1 a^{a+n-2}}{0!(a-1)!} - \sum_a \frac{a(a-1)^{a+n-2}}{1!(a-2)!} + \sum_a \frac{a(a-2)^{a+n-2}}{2!(a-2)!} - \cdots + (-1) \sum_a \frac{a}{(a-1)!0!}$$

$$= \frac{a^{a+n-1}}{0!(a-1)!} - \frac{(a-1)^{a+n-1}}{1!(a-2)!} + \cdots + (-1)^{a-1} \frac{1}{(a-1)!0!}$$

(Numbers of terms in each side = a)

$$(2)\ 1 - \frac{(a+1)}{1!} + \frac{(a+2)a}{2!} - \frac{(a+3)a^2}{3!} + \ldots + \frac{(-1)^n a^{n-1}}{n!} = (-1)^n \frac{a^n}{n!}$$

Chapter II

CHAPTER II

SECTION 1

In this section we shall discuss the paper "ON A PAPER OF ANDRE SCHINZEL" by D. SURYANARAYANA and N. VENKATESWARA RAO.

In this paper the author has obtained simple methods of finding integral solutions to the equation

$$\frac{3}{2n+1} = 1/x + 1/y + 1/z \quad n > 1 \qquad \text{I}$$

Before discussing the solution we note that any odd integer > 3 can be expressed as either $\sum_{i=0}^{m} 2^i$ $m \geq 2$ (or) $2^{r+2}K + \sum_{i=0}^{r} 2^i$ $r \geq 0$ and $K \geq 1$

We find solutions to I under different cases.

CASE (i)

$$2n + 1 = \sum_{i=0}^{m} 2^i \ . \ m \geq 2$$

SUB CASE (i)

If m is even then

$$\frac{3}{2n+1} = \frac{3}{2^{m+1}-1}$$

$$= \frac{3}{(2^{m+1}+1)} + \frac{3}{(2^{2m+1}+1)} + \frac{3.3}{(2^{2m+2}-1)(2^{2m+1}-1)}$$

EXAMPLE

Take m = 2

$$\frac{3}{2^{m+1}-1} = \frac{3}{2^{2+1}-1} = \frac{3}{8-1} = \frac{3}{7}$$

$$\frac{3}{(2^{m+1}+1)} + \frac{3}{(2^{2m+1}+1)} + \frac{3.3}{(2^{2m+2}-1)(2^{2m+1}-1)} = \frac{3}{9} + \frac{3}{33} + \frac{1}{7x33}$$

$$= \frac{3 \text{ x } 11 \text{ x } 7 + 9 \text{ x } 7 + 3}{3 \text{ x } 33 \text{ x } 7}$$

$$= \frac{42 \text{ x } 7 \text{ x } 3}{3 \text{ x } 33 \text{ x } 7} = \frac{294 + 3}{3 \text{ x } 33 \text{ x } 7} = \frac{297}{3 \text{ x } 33 \text{ x } 7}$$

$$= \frac{9 \text{ x } 33}{3 \text{ x } 33 \text{ x } 7} = \frac{3}{7}$$

SUB CASE (ii)

m odd we have

$$\frac{3}{2n+1} = 3\left(\sum_{i=0}^{m} 2^i\right)^{-1}$$

$$= 3\ (1 + 2)^{-1} \left\{\sum_{i=0}^{(m-1)/2} (2^2)^i\right\}^{-1} = \frac{1}{4t+1} \text{ say } t \geq 1$$

To prove this consider the following example

$$1 + 2 + 2^2 + \ldots.\ 2^9 = 1 \text{ x } (1+2^2 + 2^4 + 2^6 + 2^8) + 2 \text{ x } (1 + 2^2 + 2^4 + 2^6 + 2^8)$$

$$= (1 + 2)\ (1 + 2^2 + 2^4 + 2^6 + 2^8)$$

When $t = 3P$

$$\frac{1}{4t+1} = \frac{1}{12P+1}$$

In this case the solution is given by $x = 12P + 3$, $y = 3(24P^2 + 8P + 1)$,

$z = (12P + 1)(12P + 3)(24P^2 + 8P + 1)$

when $t = 3P + 1$

$$\frac{1}{4t+1} = \frac{1}{12P+5}$$

In this case the solution is given by

$x = 12P + 7$

$y = (12P + 7)(6P + 3)$

$z = (12P + 5)(12P + 7)(6P + 3)$

When $t = 3P + 2$

$$\frac{1}{4t+1} = \frac{1}{12P+9}$$

In this case the solution is given by

$x = (12P + 11)$

$y = 3(24P^2 + 40P + 17)$

$$z = (12P + 9)\,(12P + 11)\,(24P^2 + 40P + 17)$$

CASE (2)

Suppose $\quad 2n + 1 = 2^{r+2}K + \sum_{i=0}^{r} 2^i,\ r \geq 0,\ K \geq 1$

$$= 2^{r+2}K + (2^{r+1} - 1)$$

For r = 0 clearly $\dfrac{3}{2n+1} = \dfrac{3}{4K+1}$

$$= \frac{1}{(2K+1)} + \frac{1}{(4K+1)} + \frac{1}{(2K+1)(4K+1)}$$

CASE (2) SUB CASE (1)

Let now r be even we shall distinguish the cases.

CASE (2) SUB CASE (1,1)

Let $K = 3q$

Let $S = 2^r q + (2^{r-1} - 2)/3$

$$\frac{3}{2n+1} = 3\left\{2^{r+2}K + (2^{r-1} - 1)\right\}^{-1}$$

$$= 3\left\{12(2^r q + (2^{r-1} - 2)/3) + 7\right\}^{-1}$$

$$= \frac{3}{12S+7} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \quad \text{where}$$

$x = 4S + 3$

$y = (24S^2 + 32S + 11)$

$z = (4S + 3)(12S + 7)(24S^2 + 32S + 11)$

CASE (2) SUBCASE (1,2)

Let $K = 3q + 1$

$$\frac{3}{2n+1} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \quad \text{where}$$

$$x = (2^{r+1} + 1)(2S + 1)$$

$$y = \frac{(2^{r+1} + 1)(2q + 1)}{3} \quad \text{and}$$

$$z = (2^{r+1} + 1)(2q + 1)(2n + 1)$$

CASE (2) SUBCASE (1,3)

Finally if $K = 3q + 2$

$$\frac{3}{2n+1} = 3\left\{2^{r+2}.3q + (2^{r+3} + 2^{r+1} - 1)\right\}^{-1}$$

$$= \frac{1}{4t+1}, \quad t = 2^r q + (2^{r+3} + 2^{r+1} - 1)/3$$

and so can be expressed in the form

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

CASE (2) SUBCASE (2)

Let now r be odd. Again we distinguish the cases $K = 3q$, $3q + 1$ and $3q + 2$.

CASE (2) SUBCASE (2,1)

If K = 3q we have

$$\frac{3}{2n+1} = 3\left\{2^{r+2}.3q + (2^{r+1} - 1)\right\}^{-1} = \frac{1}{4t+1}$$

where $t = 2^r q + (2^{r-1} - 1)/3$

CASE (2) SUBCASE (2,2)

If K = 3q +1 then

$$\frac{3}{2n+1} = 3\left\{2^{r+2}.3q + (2^{r+2} + 2^{r+1} - 1)\right\}^{-1}$$

$$= \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

where $x = 2^{r+2}q + (2^{r+3} - 1)/3$

$y = \{3(2q + 1)(2^{r+2} q + (2^{r+3} - 1)/3\}$

$z = 3(2q + 1)\{2^{r+2} q + (2^{r+3} - 1)/3\}(2n + 1)$

CASE (2) SUBCASE (2,3)

If K = 3q + 2 then

$$\frac{3}{2n+1} = 3\left\{2^{r+2}.3q + (2^{r+3} + 2^{r+1} - 1)\right\}^{-1} = \frac{1}{12t+7}$$

where $t = 2r \,.\, q + (2^{r+1} + 2^{r-1} - 2)/3$

which can be expressed in the required form.

PROBLEM II

Now we discuss the solvability of

$$\frac{2}{2n+1} = \frac{1}{x} + \frac{1}{y}$$

for any fixed integer n > 1 in distinct odd positive integers and specify the solutions whenever it is solvable.

For n even, $\frac{2}{(2n+1)} = \frac{1}{(n+1)} + \frac{1}{(n+1)(2n+1)}$

For n odd, $(2\gamma + 1)$, $\frac{2}{(2n+1)} = \frac{2}{(4\gamma+3)}$

If $4\gamma + 3$ is composite we have a prime $(4K + 3)$ dividing $(4\gamma + 3)$ so that we can write

$$(4\gamma + 3) = (4K + 3)(4l + 1)$$

where $l \geq 1$ and then

$$\frac{2}{2n+1} = \frac{1}{(2l+1)(4K+3)} + \frac{1}{(2l+1)(2n+1)}$$

EXAMPLE

Take $\gamma = 3$ $4\gamma + 3 = 15$

$$2/15 = 1/3.3 + 1/3.15$$

$$= \frac{5+1}{45} = \frac{6}{45} = \frac{2}{15}$$

If $4\gamma + 3$ is a prime, suppose there exist distinct odd positive integers x, y such that

$$\frac{2}{2n+1} = \frac{1}{x} + \frac{1}{y} \qquad n > 1$$

Since $2x - (4\gamma + 3)$ divides $(4\gamma + 3)x$ and

$2y - (4\gamma + 3)$ divides $(4\gamma + 3)y$ and

$2x - (4\gamma + 3) > 1$, $2y - (4\gamma + 3) > 1$ as

x and y are odd positive integers, it follows that $(4\gamma + 3)$ divides x and y respectively. This cannot happen unless $x = y = 4\gamma+3$, which is a contradiction. Hence the equation is not solvable in distinct odd positive integers if $(4\gamma + 3)$ is a prime.

For example we cannot write

$\frac{2}{11} = \frac{1}{x} + \frac{1}{y}$ where x and y are distinct integers.

SECTION 2

In this section we shall discuss the paper "SOLUTIONS OF MORDELL DIOPHANTINE EQUATION" by WAH KEUNG CHAN.

The problem of finding integral solutions of the equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} + \frac{1}{wxyz} = 0 \qquad \text{— I}$$

was posed by MORDELL 1974

WAH KEUNG CHAN has discussed the equation and found a method of solving this equation.

Equation I can be rewritten as

$$(w + z)\, xy + wz\, (x + y) = -1, \quad wxyz \neq 0 \qquad \text{— II}$$

First we prove that

PROPOSITION : 1

Exactly one of w, x, y, z is even.

PROOF

Suppose two or more of w, x, y, z be even L.H.S. of II is even and therefore this is not possible.

Suppose all the four members w, x, y, z are odd then also L.H.S. of II is even. Therefore this is not possible.

Exactly one must be even.

Assume w is even.

Let $N = w + z$

We can rewrite equation II as

$$Nxy + w(N - w)(x + y) = -1 \qquad \text{— III}$$

Consider the equation III of the form

$$Ax + By = -1$$

$$Ax = -1 - By$$

Therefore $x = \frac{-1 - By}{A}$

Therefore equation III have solutions of the form x + y = M where M is any integer.

$$(xy, x + y) \in \left\{\left(\frac{-w(N-w)M-1}{N}, M\right) M \in Z\right\}$$

Since $x + y = M$ we have $y = M - x$ then

$$xy = (M - x)\, x = \frac{-w(N-w)M-1}{N}$$

$$\Rightarrow Mw^2 - 1 \equiv 0 \text{ (Mod N)}$$

$$Nx^2 - NMx - w(N - w)\, M - 1 = 0$$

$$x = \frac{NM \pm \sqrt{N^2M^2 - 4N[-w(N-w)M-1]}}{2N}$$

$$= \frac{NM \pm \sqrt{N^2M^2 - 4WNM(N-w) + 4N}}{2N}$$

In order for x to be a rational number we want

$$N^2M^2 + 4wNM(N - w) + 4N = z^2$$

$$\text{L.H.S.} = N^2M^2 + 4wNM(N - w) + 4N$$

$$= N^2M^2 + 2N[2w(N - w)]M + 4N$$

$$= \quad N^2M^2 + 2N[2w(N-w)]M + 4w^2(N-w)^2$$

$$- 4w^2(N-w)^2 + 4N$$

$$z^2 \quad = \quad (NM + 2w(N-w))^2 - 4\,([w(N-w)])^2 - N)$$

Let $T = NM + 2w(N-w)$ then

$$T^2 - z^2 = 4([w(N-w)]^2 - N)$$

$$(T+z)\,(T-z) = 2^2\,([w(N-w)]^2 - N)$$

Let $G = [w(N-w)]^2 - N$. then we must find its factorization.

PROPOSITION : 2

Given w and N III has solution iff G can be written as $G = ab$ where

$$a \equiv b \pmod{N} \equiv -\,w^2 \pmod{N}$$

PROOF

Now let $2^2G = a'b'$ where $|a'| \geq |b'|$ then

$$\left.\begin{matrix} T + z = a' \\ T - z = b' \end{matrix}\right\} \text{give} \quad \begin{matrix} z = \dfrac{a' - b'}{2} \\ T = \dfrac{a' + b'}{2} \end{matrix}$$

Since $z, T \in z$ we have $a' \equiv b' \pmod{2}$

Furthermore, since $4\,|a'\,b'$ then $2|a'$ or b' gives

$a' = 2a$ and $b' = 2b$ Thus

$G = ab$ and $z = a - b$, $T = a + b$

Looking at T we have

$$NM + 2w(N - w) = a + b$$

$$NM = a + b - 2w(N - w)$$

Considering again our equation in x we have

$$x = \frac{NM \pm (a+b)}{2N}$$

$$= \frac{a + b - 2w(N - w) \pm (a - b)}{2N}$$

$$= \frac{a - w(N - w)}{N} \text{ or } \frac{b - w(N - w)}{N}$$

$y = M - x$

$$= 1/N \ \left[a + b - 2w(N - w) - \begin{Bmatrix} a \\ b \end{Bmatrix} + w(N - w)\right]$$

$$= \frac{b - w(N - w)}{N} \text{ or } \frac{a - w(N - w)}{N}$$

Thus we can let

$$x = \frac{a - w(N - w)}{N} \qquad y = \frac{b - w(N - w)}{N}$$

$$x = \frac{a - wz}{w + z} \qquad y = \frac{b - wz}{w + z}$$

For $x = \dfrac{a - w(N - w)}{N} \in z$ and $y = \dfrac{b - w(N - w)}{N} \in z$

we must have

$a + w^2 \equiv 0 \pmod{N}$ and $b + w^2 \equiv 0 \pmod{N}$

This satisfies our equation for G since

$$ab = [w(N - w)]^2 - N = w^2(N^2 - 2Nw + w^2) - N$$

gives us

$$ab \equiv w^4 \pmod{N}$$

PROPOSITION : 3

For each N there exist infinite number of solutions to equation III.

PROOF

We will construct an infinite number of solutions for each N.

Consider $w \equiv 1 \pmod{N}$ then $w^2 \equiv 1 \pmod{N}$

and $w^4 \equiv 1 \pmod{N}$ since $ab \equiv w^4 \pmod{N} \Rightarrow ab \equiv 1 \pmod{N}$

Take $b = -1$ and $a = -G \equiv -1 \pmod{N} \equiv N - 1 \pmod{N}$ then $a + w^2 \equiv 0 \pmod{N}$ and $b + w^2 \equiv \pmod{N}$. Hence x, y are integers and it follows that there are an infinite number of solutions for each N.

PROPOSITION : 4

For each $m \in z$ there exists a solution to II with $w = 2m$

PROOF

Let $N = 2M - 1$ then by proposition 3, II has a solution.

PROPOSITION : 5

For each integer m, when $w = 2m$ there exists only a finite number of solutions to (II).

PROOF

First we fix the value of $w = 2m$. Now suppose that there exists an infinite of solutions to II with w fixed. Then there are an infinite number of values $N = w - z$ which yield solutions. This is because for each N, the number of solution depend on the factors of

$G = [w(N - w)]^2 - N$ of which there are only a finite number. Now attached to each w there are really three N's, N_1, N_2 and N_3, where $N_2 = w + x$ and $N_3 = w + y$ and $N = w + z$. Here without loss of generality we let $|N| \geq |N_2| \geq |N_3|$.

Since N is arbitrary, we can choose N with its absolute value greater than that of w. (ie) $|N| > |w|$.

We now show that there is a bound on N_3, the smallest of these N's we have

$$N_2 = w + z = \frac{a - w(N - w)}{N} + = \frac{a + w^2}{N}$$

and similarly

$$N_3 = \frac{b + w^2}{N}$$

Since $ab = w^2(N - w)^2 - N$ we can choose a and b such that as a crude estimate

$$|b| < |wN| + w^2 + |N|$$

$$|N_3| < 2|w| + 1$$

Thus for each solution we find the smallest N to be bounded dependent only on the value of w. This contradicts our hypothesis that there are an infinite number of N. To find the solutions we need only look at these finite number of values of N.

Let $\eta(w)$ be the number of solutions for each even w.

The author has proved that a lower bound for $\eta(w)$ is given by the expression

$$\mu(w) = \tau(w^2(w-1)^2 - 1) + \tau(w^2(w+1)^2 + 1)$$

Here τ is the number of divisions of G.

The author has constructed the table for $\eta(w)$ and $\mu(w)$.

w	-32	-30	-28	-26	-24	-22	-20	-18	-16	-14	-12	-10	-8	-6	-4	-2
η(w)	49	21	25	31	15	49	14	24	32	30	23	15	13	10	14	5
μ(w)	24	16	12	16	12	28	8	18	18	18	20	10	6	8	12	5
w	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
η(w)	4	7	10	29	15	15	20	15	24	26	29	25	57	31	23	32
μ(w)	4	6	8	24	10	10	10	8	16	10	8	20	36	16	18	12

SECTION : 3

In this section, we shall discuss the paper "A REPORT ON PRIMES OF THE FORM $K.2^N+1$ AND ON FACTORS OF FERMAT NUMBERS" by RAPHAEL M. ROBINSON.

Fermat numbers are of the form $F_k. 2^{2^k}+1$ where k = 0, 1, 2, ...

Famous mathematician of the 17th century P. Fermat conjucted that all these numbers are prime. This is true. For k = 0, 1, 2, 3, 4 but L. Euler in 1732 showed that the number.

$F_5 = 2^{2^5}+1$ = 4,294,967,297 having 10 digits is composite divisible by 641.

In 1963, 38 composite numbers f_k were known. These are F_k for k = 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 23, 26, 36, 38, 39, 55, 58, 63, 73, 77, 81, 117, 125, 144, 150, 207, 226, 228, 260, 267, 268, 284, 316, 452, 1945.

The author has constructed a table of list of primes of the form $k.2^n+1$.

LIST OF PRIMES OF THE FORM $k.2^n + 1$

K	*n*
1	0, 1, 2, 4, 8, 16
3	1, 2, 5, 6, 8, 12, 18, 30, 36, 41, 66, 189, 201, 209, 276, 353, 408, 438, 534
5	1, 3, 7, 13, 15, 25, 39, 55, 75, 85, 127, 1947
7	2, 4, 6, 14, 20, 26, 50, 52, 92, 120, 174, 180, 190, 290, 320, 390, 432, 616, 830
9	1, 2, 3, 6, 7, 11, 14, 17, 33, 42, 43, 63, 65, 67, 81, 134, 162, 206, 211, 366
11	1, 3, 5, 7, 19, 21, 43, 81, 125, 127, 209, 211
13	2, 8, 10, 20, 28, 82, 188, 308, 316
15	1, 2, 4, 9, 10, 12, 27, 37, 38, 44, 48, 78, 112, 168, 229, 297, 339
17	3, 15, 27, 51, 147, 243, 267, 347, 471
19	6, 10, 46, 366
21	1, 4, 5, 7, 9, 12, 16, 17, 41, 124, 128, 129, 187, 209, 276, 313, 397
23	1, 9, 13, 29, 41, 49, 69, 73, 341, 381, 389
25	2, 4, 6, 10, 20, 22, 52, 64, 78, 184, 232, 268, 340, 448
27	2, 4, 7, 16, 19, 20, 22, 26, 40, 44, 46, 47, 50, 56, 59, 64, 92, 175, 215, 275, 407, 455
29	1, 3, 5, 11, 27, 43, 57, 75, 77, 93, 103, 143, 185, 231, 245, 391
31	8, 60, 68, 140, 210, 416

33	1, 6, 13, 81, 21, 22, 25, 28, 66, 93, 118, 289, 412, 453
35	1, 3, 7, 9, 13, 15, 31, 45, 47, 49, 55, 147, 245, 327, 355
37	2, 4, 8, 10, 12, 16, 22, 26, 68, 82, 84, 106, 110, 166, 236, 254, 286, 290
39	1, 2, 3, 5, 7, 10, 11, 13, 14, 18, 21, 22, 31, 42, 67, 70, 71, 73, 251, 370, 375, 389, 407
41	1, 11, 19, 215, 289, 379
43	2, 6, 12, 18, 26, 32, 94, 98, 104, 144, 158, 252
45	2, 9, 12, 14, 23, 24, 29, 60, 189, 200, 333, 372, 443, 464
47	
49	2, 6, 10, 30, 42, 54, 66, 118, 390
51	1, 3, 7, 9, 13, 17, 25, 43, 53, 83, 89, 92, 119, 175, 187, 257, 263, 267, 321, 333
53	1, 5, 17, 21, 61, 85, 93, 105, 133, 485
55	4, 8, 16, 22, 32, 94, 220, 244, 262, 286, 344, 356, 392
57	2, 3, 7, 8, 10, 16, 18, 19, 40, 48, 55, 90, 96, 98, 190, 398, 456, 502
59	5, 11, 27, 35, 291
61	4, 12, 48, 88, 168
63	1, 4, 5, 9, 10, 14, 17, 18, 21, 25, 37, 38, 44, 60, 65, 94, 133, 153, 228, 280, 314, 326, 334, 340, 410, 429
65	1, 3, 5, 11, 17, 21, 29, 47, 85, 93, 129, 151, 205, 239, 257, 271, 307, 351, 397, 479
67	2, 6, 14, 20, 44, 66, 74, 102, 134, 214, 236, 288, 342, 354, 382, 454, 470

69	1, 2, 10, 14, 19, 26, 50, 55, 145
71	3, 5, 9, 19, 23, 27, 57, 59, 65, 119, 299, 417
73	2, 6, 14, 24, 30, 32, 42, 44, 60, 110, 212, 230
75	1, 3, 4, 6, 7, 10, 12, 34, 43, 51, 57, 60, 63, 67, 87, 102, 163, 222, 247, 312, 397, 430
77	3, 7, 19, 23, 95, 287, 483
79	2, 10, 46, 206
81	1, 4, 5, 7, 12, 15, 16, 21, 25, 27, 32, 35, 36, 39, 48, 89, 104, 121, 125, 148, 152, 267, 271, 277, 296, 324, 344, 396, 421, 436, 447
83	1, 5, 157, 181, 233, 273
85	4, 6, 10, 30, 34, 36, 38, 74, 88, 94, 148, 200
87	2, 6, 8, 18, 26, 56, 78, 86, 104, 134, 207
89	2, 7, 9, 21, 37, 61
91	8, 168, 260
93	2, 4, 6, 10, 12, 30, 42, 44, 52, 70, 76, 108, 122, 164, 170, 226, 298, 398
95	1, 3, 5, 7, 13, 17, 21, 53, 57, 61, 83, 89, 111, 167, 175, 237
97	2, 4, 14, 20, 40, 266, 400
99	1, 2, 5, 6, 10, 11, 22, 31, 33, 34, 41, 42, 53, 58, 65, 82, 126, 143, 162, 170, 186, 189, 206, 211, 270, 319, 369, 410, 433

The author has given interesting survey of different method used to find primes of the form $k.2^n+1$ and factors of Fermat number.

This table was constructed during 1956 – 1957 using the high speed computer located on the Los Angles campus of the university of California.

In 1878, Proth constructed the following test for primeness $N = k.\ 2^n+1$, where $0 < k < 2^n$ suppose $(a/N) = -1$. This test was found to be useful in the search for primes.

The only cases to which the test does not apply are those n which are very small. But no test is needed when n is small.

The author has proved the following theorem.

THEOREM : 2.3.1

If $n = K.2^n + 1$ is prime, where k is odd $0 < k < 100$ and $0 < n < 512$ then the smallest positive quadratic non residue of N does not exceed 23. The smallest non residue is 23 in just 3 cases.

$N = 39.\ 2^{12} + 1 \qquad 33.2^{28} + 1, \quad 57.2^{90} + 1$

The number of the form $N = n.2^n + 1$ are called Cullen numbers. They were studied by Cunningham and Woodall who stated that for just 47 of these numbers with $n \leq 1000$ is the smallest factor greater than 1000. They gave a factor in three of these cases. The remaining 44 numbers were tested on the SWAC and of these only one turned out to be prime, namely $N = 141.\ 2^{141} + 1$.

This is the only Cullen prime with $2 \leqq n \leqq 1000$.

We now give a brief survey of tables of primes of the form $k.2^n + 1$ or of factors of numbers of this form.

A short list of primes of the form $k.2^n + 1$ was given by Seelhoff in 1886. Cunningham lists a number of primes of the form $k.2^n + 1$ not exceeding 10^8 together with certain quadratic partitions. Kraitchik [5 pp 12 – 13] gives the smallest factor of each number $N = k.2^n + 1$ with k odd $1 < k < 100$ and 2. $10^8 < N < 10^{12}$. In Kraitchik [6 pp. 222 – 232]. The range is extended to $0 < k < 1000$ and $10^8 < N < 10^{12}$ and a list of primes form the table appears on pp. 233 – 235.

2. *FACTORS OF FERMAT NUMBERS*

If the number N turned out to be a prime, then it was also tested to find whether it is a factor of any Fermat number $F_m = 2^{2m} + 1$. Since for $m \geq 2$, every prime factor p of F_m satisfies $p \equiv 1 \pmod{2^{m+2}}$. We needed to try the numbers $N = k.2^n + 1$ as a factor of F_m only for $m \leq n-1$. The routine for making this test was very similar to that used for testing N for primeness.

A list of all known prime factors of composite Fermat numbers appear in table together with the date found or published and the discoverer.

FACTORS $k.2^n + 1$ OF FERMAT NUMBERS F_m

k	*n*	*m*	*Date*	*Discoverer*
5	7	5	1732	Euler
52347	7	5	1732	Euler
1071	8	6	1880	Landry
262814145745	8	6	1880	Landry, Le Lasseur
37	16	9	1903	Western
11131	12	10	1953	Selfride (SWAL)
39	13	11	1899	Cunningham
119	13	11	1899	Cunningham
7	14	12	1877	Perrouchine, Lucas
397	16	12	1903	Western
973	16	12	1903	Western
579	21	15	1925	Kiaitchik
1575	19	16	1953	Selfridge (SWAC)
13	20	18	1903	Western
5	25	23	1878	Perrouchine
5	39	36	1886	Seelhoff
3	41	38	1903	Cullen, Cunningham, Western

21	41	39	1956	Robinson (SWAL)
29	57	55	1956	Robinson (SWAL)
95	61	58	1957	Robinson (SWAL)
9	67	63	1956	Robinson (SWAL)
5	75	73	1906	Morelead
125	79	77	1957	Robinson, Selfridge (SWAL)
271	84	81	1957	Robinson, Selfridge (SWAL)
7	120	117	1956	Robinson (SWAL)
5	127	125	1956	Robinson (SWAL)
17	147	144	1956	Robinson (SWAL)
1575	157	150	1956	Robinson (SWAL)
3	209	207	1956	Robinson (SWAL)
15	229	226	1956	Robinson (SWAL)
29	231	228	1956	Robinson (SWAL)
403	252	250	1957	Robinson, Selfridge (SWAL)
177	271	267	1957	Robinson, Selfridge (SWAL)
21	276	268	1956	Robinson (SWAL)
7	290	284	1956	Robinson (SWAL)

7	320	316	1956	Robinson (SWAL)
27	455	452	1956	Robinson (SWAL)
5	1947	1945	1957	Robinson (SWAL)

THEOREM : 2.3.2

A prime of the form $N = 3.2^n + 1$ where n is even, cannot be a factor of any fermat number.

Chapter III

CHAPTER III

This chapter deals with the papers "PICK'S THEOREM REVISITED" by DALE E. VERBURG and "THE DETERMINATION OF KAPREKAR CONVERGENCE and LOOP CONVERGENCE OF ALL THREE DIGIT NUMBERS" by KLAUS E. ELDRIDGE and SEOK SAGONG and "CONWAY'S RATS and other REVERSALS" edited by RICHARD GUY.

SECTION I

In this paper entitled PICK'S THEOREM REVISITED the author DALE E. VARBERG deals with Pick's theorem and its generalization using simple lattice polygon.

DEFINITION : 3.1.1

A polygon is simple if its boundary is a simple closed curve.

PICK'S THEOREM : 3.1.2

Pick's theorem is one of the vital part of elementary mathematics. Because it gives us an interesting conclusion.

STATEMENT

The area of simple lattice polygon S is given by

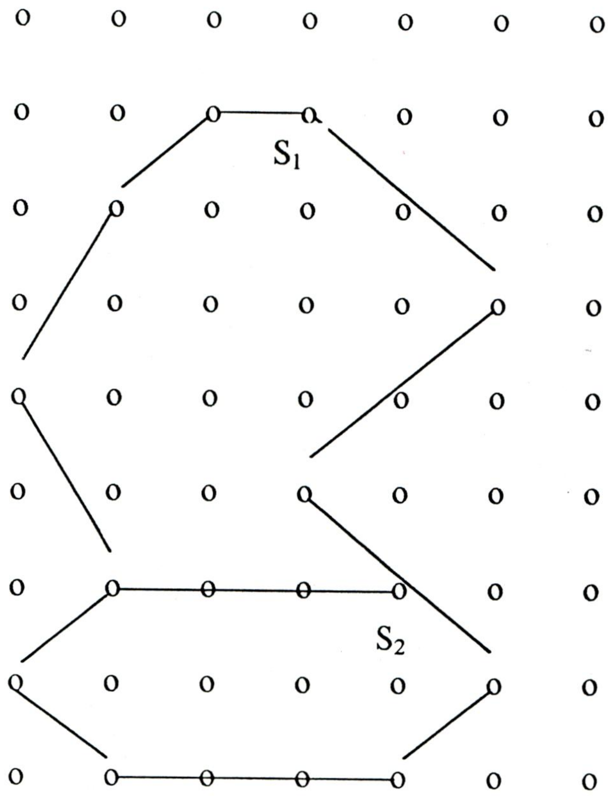
$$A(S) = i + \tfrac{1}{2} b - 1$$

$$= v - \tfrac{1}{2} b - 1$$

where i – the number of interior lattice points b – the number of boundary lattice points

and v – the total number of lattice point of S.

The truth of the above theorem is easily verified in the following example.



In S_1,

i – number of interior lattice points = 11

b – boundary lattice points = 12

v – Total number of lattice points = 23

In S_2,

i = 4

$b = 10$

$v = 14$

$S = S_1 \cup S_2$

$i = 17, \quad b = 16, \quad v = 33$

$$\text{Area of } S_1 = A(S_1) = i + \tfrac{1}{2} b - 1$$
$$= 11 + \tfrac{1}{2} \times 12 - 1$$
$$= 11 + 6 - 1$$
$$= 16$$

$$\text{Area of } S_2 = A(S_2) = i + \tfrac{1}{2} b - 1$$
$$= 4 + 10/2 - 1$$
$$= 4 + 5 - 1$$
$$= 8$$

$A(S_1) + A(S_2)$

The proof of Pick's theorem is direct and it is easy to prove by induction, because a lattice polygon can be decomposed as a union of lattice triangles.

Before giving the proof, let us define the following.

With each vertex P_k we associate weight $W_k = \frac{\theta_k}{2\pi}$, where θ_k measures the "visibility" angle with which P_k can see into S.

Thus, W_k = 1 at an interior lattice point

W_k = ½ at a boundary lattice point that is not a vertex

W_k = ¼ at a right angled corner point

Let $W(S) = \sum_{P_k \in S} W_k$ and

$A(S)$ = area of S

LEMMA : 3.1.3

$$W(S) = A(S)$$

PROOF:

We prove W is additive.

(i.e.) if $S = S_1 \cup S_2$ where S_1 and S_2 are lattice polygons, then $W(S) = W(S_1) + W(S_2)$

This is equivalent to the following statement (i.e.) : The sum of the visibility angles in S_1 and S_2 at a common lattice point P is equal to the visibility angle in S at the point P.

Now consider a lattice rectangle with sides parallel to the lattice points as shown in the following fig : 2.

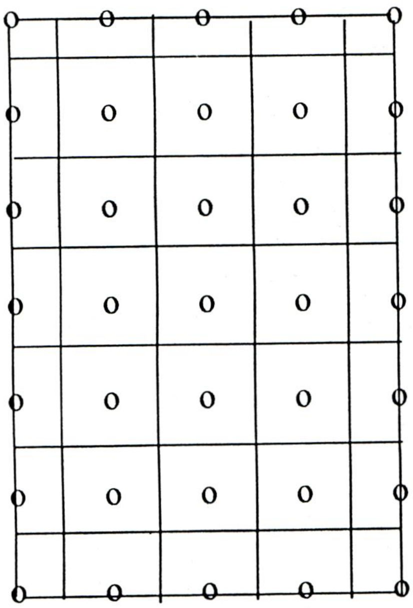


Fig 2

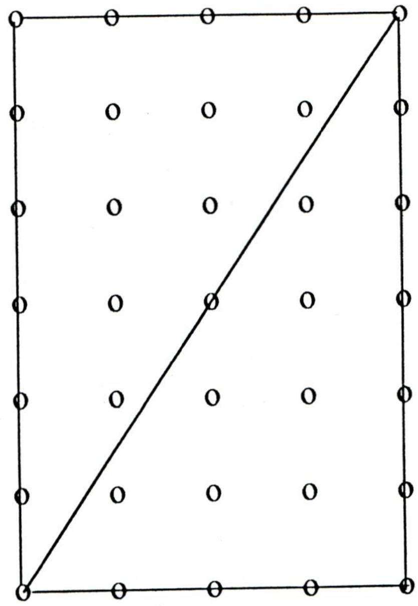


Fig 3

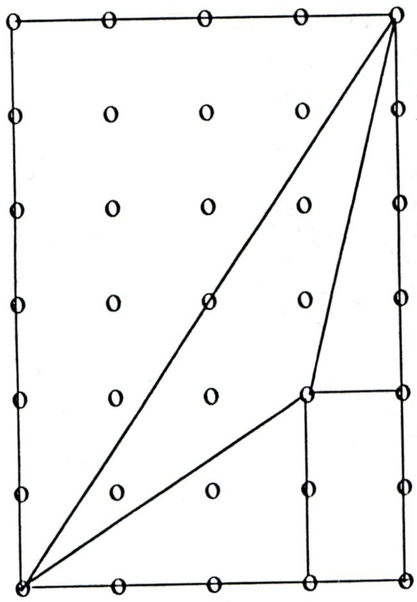


Fig 4

In this

$$W(S) = \sum_{P_k \in S} W_k$$

$$= \sum_{P_k \in I} W_k + \sum_{P_k \in B} W_k + \sum_{\substack{P_k = right.angled. \\ corner.point}} W_k$$

I = interior o

f S

B = boundary of S

Number of interior points = 15

Number of boundary points = 16

Right angled corner points = 4

W(S) = 15 + 16/2 + 4/4

= 15 + 8 + 1

= 24

A(S) = i + ½ b – 1

= 15 + 20/2 – 1

= 25 – 1 = 24

A(S) = W(S)

Next consider a lattice triangle with legs parallel to the lattice (Fig 3).

In this case we get the result by dividing the above result by 2.

Now consider an arbitrary lattice triangle (Fig.4). In this case we get the result by using the shared additivity of W and A. By decomposing an arbitrary lattice polygon as an union of lattice triangles and by applying the additivity of W, we get the result.

PICK'S THEOREM

For a simple lattice polygon

$$A(S) = i + \tfrac{1}{2} b - 1$$

$$= v - \tfrac{1}{2} b - 1$$

PROOF

Consider a simple polygon S with c interior vertex angles. Then the angle sum of this polygon is (c – 2).

For example,

A triangle has angle sum π.

A quadrilateral has angle sum 2π etc.

If b is the number of boundary points, then the sum of all the visibility angles at points P_k along the boundary of S is $(b-2)\pi$.

Therefore if I and B denote the interior and boundary of S then,

$$A(S) = W(S) = \sum_{P_k \in I} W_k + \sum_{P_k \in B} W_k$$

Since at an interior lattice point

$W_k = 1$

$$A(S) = W(S) = i + \frac{(b-2)}{2\pi}\pi$$

$$= i + \tfrac{1}{2} b - 1$$

i – The number of interior lattice points

b – The number of boundary lattice points

$$\therefore A(S) = i + \tfrac{1}{2} b - 1$$

= Area of a simple polygon S

SECTION 2

In this section we shall define and discuss the problem of Kaprekar convergence.

DEFINITION : 3.2.1

We denote by S(n,r) the set of all positive integers expressible in n-digit sequences in base r exclusive of these digit sequences with all digits equal.

EXAMPLE

S(3,10) is the set of all decimal integers between 001 and 998 excluding 111, 222, ... 888.

A "three digit number" means an element of S(3,r) for an appropriate base r.

DEFINITION : 3.2.2

Let k : S(n,r) → S(n,r) be that transformation which permutes the n digits of a number in S(n,r) to obtain the largest and the smallest number expressible using the same n digits and then subtracts the smallest from the largest k is the Kaprekar transformation.

Recursively we define,

$$k^n(x) = \begin{cases} x & \text{for } n = 0 \\ k \mid k^{n-1}(x) & \text{for } n = 1,2,3\ldots \end{cases}$$

EXAMPLE

(i) Consider 456 in base 10

When n = 0 $K^0(456) = 456$

When n = 1 $K^1(456) = K(K^0(456) = K(456) = 198$

(ie)

$$\begin{array}{r} 654 \\ \underline{456} \\ 198 \end{array}$$

when n = 2 $K^2(456) = K(K^1(456) = K(198) = 792$

(ie)

$$\begin{array}{r} 981 \\ \underline{189} \\ \underline{792} \end{array}$$

and so on.

(ii) Consider 456 in base 7

When n = 0 $K^0(456) = 456$

When n = 1 $K^1(456) = K(456) = 165$

(ie) 654

456

165

when n = 2 $K^2(456) = K(K^1(456) = K(165) = 462$

(ie) 651

156

---

462 and so on.

DEFINITION : 3.2.3

Let $x \in S(n,r)$. If $K(x) = x$ then x is said to be fixed point of K.

EXAMPLE

495 in base 10 is a fixed point of K since $K(495) = 495$

(ie) 954

459

495

DEFINITION : 3.2.4

If $x \in S(n,r)$ is a fixed point and $K^m[S(n,r)] = \{x\}$ for some positive integer m, then x if exists, is called the Kaprekar constant of S(n,r) and the smallest such integer m is called the degree of Kaprekar convergence of S(n,r).

DEFINITION : 3.2.5

Let x be a fixed point of K in S(n,r). If $y \in S(n,r)$ and $K^n(y) = x$ for some positive integer n then y is said to converge to x under K and the smallest such integer n is called the degree of convergence of y to K.

If, in particular, x is the Kaprekar constant then y is said to be Kaprekar convergent and n^{th} degree of convergence of y to x is called the degree of Kaprekar convergence of y.

NOTE

The fixed points may not be unique.

FOR EXAMPLE

Both 1001 and 0111 are fixed point in S(4,2)

DEFINITION : 3.2.6

If for some integer m, $K^m(x) = K^{m+i}(x)$ and $K^m(x) \neq K^{m+j}(x)$ for all j such that $1 \leq j \leq i$, then $K^m(x)$ is said to generate a loop of length i and each of $K^{m+j}(x)$, j = 0,1,....i-1 is called a loop element.

(i) Consider 264 in base 7

6 4 2	6 3 3	6 4 2
2 4 6	3 3 6	2 4 6
3 6 3	2 6 4	3 6 3

Therefore, 363 → 264 → 363 form a loop of length 2 and 264 is a loop element.

(ii) Consider 7. 19 19 12 in base 20

19 19 12 7	12 12 8 6	15 14 6 3
7 12 19 19	6 8 12 12	3 6 14 15
12 6 12 8	6 3 15 14	12 7 11 8

12 11 8 7	16 15 5 2	14 9 9 6
7 8 11 12	2 5 15 16	6 9 9 14
5 2 16 15	14 9 9 6	7 19 19 12

Therefore 7 <u>19</u> <u>19</u> 12 → <u>12</u> 6 <u>12</u> 8 → 6 3 <u>15</u> <u>14</u> → <u>12</u> 7 <u>11</u> 8 → 5 2 <u>16</u> <u>15</u> → <u>14</u> 9 9 6 → 7 <u>19</u> <u>19</u> 12 form a loop of length 6.

RESULT : 3.2.7

For a three digit number x in base r, the result of a single Kaprekar transformation is determined by the difference between the largest and smallest digits, If of x and the base r. The first digit of the result is I – 1, the second digit r – 1 and the third digit r – 1.

PROOF

Let x = abc be a three digit number in base r where the digits are such that $c \leq b \leq a$ then I = a – c.

Consider

a	b	c
c	b	a
a-1-c	r+b-1-b	r+c-a

(ii)

a	b	c
c	b	a
I-1	r-1	r-1

Therefore K(abc) = (I-1, r-1, r-1)

REMARK : 3.2.8

. When r is even, consider the number $x = (\frac{r-2}{2}, r-1, r/2)$ in S(3,r).

Then x is a fixed point of K.

To find K(x) consider

$r-1 \qquad r/2 \qquad \frac{r-2}{2}$

$$
\begin{array}{ccc}
\frac{r-2}{2} & r/2 & r-1 \\
\hline
\frac{r-2}{2} & r-1 & r/2
\end{array}
$$

Therefore $K(x) = (\frac{r-2}{2}, r-1, r/2) = x$

It will be shown in the course of this chapter every three digit number converges to this number when r is even.

REMARK : 3.2.9

When r is odd, we can show that the two numbers

$(\frac{r-3}{2}, r-1, \frac{r+1}{2})$ and $(\frac{r-1}{2}, r-1, \frac{r+1}{2})$ are

loop elements.

For example, consider 264 in base 11.

Then

$$
\begin{array}{ccc}
6 \; 4 \; 2 \\
2 \; 4 \; 6 \\
\hline
3 \; \underline{10} \; 7
\end{array}
\qquad
\begin{array}{c}
10 \; 7 \; 3 \\
3 \; 7 \; 10 \\
\hline
6 \; \underline{10} \; 4
\end{array}
\qquad
\begin{array}{c}
10 \; 6 \; 4 \\
4 \; 6 \; 10 \\
\hline
5 \; \underline{10} \; 5
\end{array}
\qquad
\begin{array}{c}
10 \; 5 \; 5 \\
5 \; 5 \; 10 \\
\hline
4 \; \underline{10} \; 6
\end{array}
\qquad
\begin{array}{c}
10 \; 6 \; 4 \\
4 \; 6 \; 10 \\
\hline
5 \; \underline{10} \; 5
\end{array}
$$

Hence we see that the two numbers (4,10,6) and (5,10,5) in base 11 are loop elements.

If $K^n(x)$ that is, $m(n,1,x) \leq m(n,2;x) \leq m(n,3;x)$ with at least one strict inequality.

EXAMPLES

(i) Consider x = 792 in S(3,10)

Then d(0,1;792) = 7 while m(0,1;792) = 2, the smallest digit

d(0,3;792) = 2 while m(0;3;792) = 9 the largest digit

d(0,2;792) = 9 while m(0;2;792) = 7 the middle digit

(ii) Consider K'(792) = K(792) = 693 we have

d(1,7;792) = 6 while m(1,1;792) = 3 ;

d(1,2;792) = 9 while m(1,2;792) = 6 ;

d(1,3;792) = 3 while m(1,3;792) = 9 and so on.

With the above notations we can write the image of x in S(3,r) after n applications of K as follows.

$K^n(x) = \{d(n,1;x), d(n,2;x), d(n,3;x)\}$ or equivalently as

$$d(n,1;x)r^2 + d(n,2;x) + d(n,3;x)$$

We shall start with proving the fundamental results.

LEMMA : 3.2.11

Let x be in S(3,r), then for every n > 1 we have

$$d(n,1;x) = m(n-1,\ 3;x) - m(n-1,\ 1;x) - 1 \qquad (1)$$

$d(n,2;x) = r - 1$ (2)

$d(n,3;x) = r + m(n-1, 1;x) - m(n-1, 3; x)$ (3)

DEFINITION : 3.2.12

Let $x \in S(3,r)$, we denote the positive difference between the largest and smallest digits of $K^n(x)$ by $I(n;x)$

(ii) $I(n;x) = m(n,3;x) - m(n,1;x)$ for every $n \geq 0$

NOTE We write I for $I(0;x)$

REMARK : 3.2.13

If x and y are in S(3,r) then for each $n \geq 1$ $K^n(x) = K^n(y)$ if and only if $I(n-1;x) = I(n-1;y)$

After the initial application of K the middle digit, being r-1, remains always the largest. So we investigate the behaviour of the first and third digits under K, which are, in some order, the smallest and middle sized digits.

DEFINITION : 3.2.14

For $x \in S(3,r)$ we define $J(n;x)$ to be the (non negative) difference between the first and third digits of $K^n(x)$ for all $n \geq 0$.

(ii) $J(n;x) = m(n,2,x) - m(n,1;x)$ for $n \geq 0$

Then for $n \geq 1$

$$J(n;x) = m(n,2;x) - m(n,1;x)$$

$$= |d(n,1;x) - d(n,3;x)|$$

If we write (1) and (3) using I(n-1;x) for m(n-1, 3;x) – m(n-1, 1;x) then

$J(n;x) = |r - 2I(n-1;x) + 1|$. Thus for $n \geq 1$

$$J(n;x) = \begin{cases} r - 2I(n-1;x) + 1 \text{ if } I(n-1,x) \leq \dfrac{r+1}{2} \\ 2I(n-1;x) - r - 1 \text{ if } I(n-1;x) \geq \dfrac{r+1}{2} \end{cases}$$

LEMMA : 3.2.15

Let x be in S(3,r). Then for all $n \geq 1$, J(n;x) is odd if and only if the base r is even.

LEMMA : 3.2.16

Let x be in S(3,r). If $J(1;x) \geq 2$ then $J(1+j;x) = J(1;x) - 2i$ for all $i \leq [J(1;x)/2]$

THEOREM : 3.2.17

(A) The set S(3,r) has the Kaprekar constant ((r-2)/2, r-1, r/2) if and only if r is even.

(B) If x in S(3,r) where r is even, is not the Kaprekar constant, then its degree of Kaprekar convergence is given by

(i) $2 - I + r/2$ if $I < r/2$

(ii) $1 + I - r/2$ if $I \geq r/2$

Two important consequences of the above theorem are as follows.

(i) The degree of Kaprekar convergence of S(3,r), where r is even, is I + r/2 for $r \geq 4$ and 1 if $r = 2$

(ii) The degree of loop convergence of S(3,r) where r is odd, is (r+1)/2 for $r \geq 5$ and 1 if $r = 3$.

Finally algorithm for $K^n(x)$

$K^n(x) = m(n-1, 2, x-1, r-1, m(n-1, 1 ; x) +1)$

for $n \geq 2$

FOR EXAMPLE

Consider 787 in base 10

We obtain the successive images 787 → 099 → 891 → 792 → 693 → 594 → 495 → 495 → .....

Consider 787 in base 15. Then its successive images are

787 → 0 <u>14</u> <u>14</u> → <u>13</u> <u>14</u> 1 → <u>12</u> <u>14</u> 2 → <u>11</u> <u>14</u> 3 → <u>10</u> <u>14</u> 4 → 9 <u>14</u> 5 → 8 <u>14</u> 6 → 7 <u>14</u> 7 → 6 <u>14</u> 8 → 7 <u>14</u> 7 → 6 <u>14</u> 8 → ....

It is interesting to note that the degree of convergence of 787 is equal to the degree of convergence of the set S(3,10) which is equal to 6.

Summary and Conclusion

SUMMARY AND CONCLUSION

In this thesis we have made an attempt to discuss some interesting results from Number Theory. These results are contained in the following ~~results~~. articles.

(1) Some interesting Results on a New Combinatorial Arithmetic Function by Soumendara Bera.

(2) On a paper of Andre Schinzel by D.Suryanarayana and N.Venkateswara Rao.

(3) Solutions of a Mordell Diophantine Equation by Wah Keung Chan.

(4) A report on primes of the form K . $2^n + 1$ and on factors of Fermat numbers by R.M.Robinson.

(5) Pick's Theorem Revisited by Dale E. Varberg.

(6) The Determination of Kaprekar convergence and loop convergence of all Three-Digit Numbers by Kalus E. Eldridge and Seok Sagong.

We discuss the first paper in chapter I we study the 2nd, 3rd, 4th paper in chapter 2 in the last chapter we discuss the articles on Pick's theorem and Kaprekar convergence.

Bibliography

BIBLIOGRAPHY

1. TOM. M. APOSTAL — "Introduction to Analytic Number Theory" – Springer International Student Edition, 1980

2. T.S. BANU MURTHY — "A modern introduction to Ancient Indian Mathematics", 1992, Wiley Eastern Limited, New Delhi

3. T.S. BANU MURTHY — "A modern introduction to Ancient Indian Mathematics", (1992), Wiley Eastern Ltd.

4. Sri. BHARATHI KRISHNA TIRTHAJI MAHARAJA — "Vedic Mathematics", Motilal Banarasi Das, New Delhi (1978)

5. Sri. BHARATHI KRISHNA TIRTHAJI MAHARAJA — "Vedic Meta Physics", Motilal Banarasi Das, New Delhi (1978)

6. B.B. DATTA and A.N. SINGH — "History of Hindu Mathematics, Parts I and II", Motilal Banarasi Das, (1935) (1938)

7. L.E. DICKSON — "History of Theory of Numbers" – Vol. 2 Chelsea Publishing Company, New York, 1952

8. KLAUS E. ELDRIDGE — "The Determination of Kaprekar convergence and Loop convergence of All Three-digit Numbers" – Amer – Math. Monthly, Vol. 95, No.2, Feb. 1988 (105 – 112).

9. G.H. HARDY and E.M.WRIGHT — "An Introduction to the Theory of Numbers", Fourth Edition, 1975, The English Language Book Society and Oxford University Press

10. IVAN NIVEN and HERBERT S. ZUCKERMAN — "An Introduction to the Theory of Numbers", Third Edition, 1972 Wiley Eastern Limited.

11. KRISHNA GANAKA BIJA PALLAVAM — "A Commentary on Bija Ganita of Bhaskara"

12. R.M. ROBINSON — "A report on primes of the form $K \cdot 2^n + 1$ and on factors of Fermat numbers", Journals of the Indian Mathematical Soc. Oct. 1958.

13. T.A. SARASWATHIAMMA — "Geometry in Ancient and Medieval India", Motilal Banarasi Das (1979)

14. SOUMENDRA BERA — "History of Ancient Indian Mathematics" The World Press Private Ltd., Calcutta (1967)

15. C.N. SRINIVASIENGAR — "Some Interesting Results on a New Combinatorial Arithmetic Function", Indian J. Pure Appl. Math., 29 (5) : 541 – 556, May 1998

16. D.SURYANARAYANA and N.VENKATESWARA RAO — "On a paper of Andre Schinzel", The Journal of the Indian Mathematical Society, Vol. XXIX No. 4, Oct – Dec. 1965

17. DALE E. VARBERG — "Pick's Theorem Revisited", American Mathematical Monthly, October 1985 (584 – 587)

18. WALLAW SIERPINSKI — "A Selection of Problems in the Theory of Numbers" – PWN – Polish Scientific Publishers, Wargzawa – 1964

19. WAK KEUNG CHAN "Solutions of a Modern Diophantine Equation", J. Ramanujam Math. Soc., 6, Nos. 1 & 2, 1991 pp. 129 – 140.