Chapter IV

Soft Hausdorff Spaces

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SOFT HAUSDORFF SPACES

Definition : 4.1

Let (X, τ) be a soft topological space and $x, y \in X$ such that $x \neq y$. (X, τ) is called **soft Hausdorff space or soft T_2 space** if there exists soft open sets F_A and G_B such that $x \tilde{\in} F_A$, $y \tilde{\in} G_B$ and $F_A \tilde{\cap} G_B = \Phi$.

Example : 4.2

Let R be the real numbers, E = R and $(F_E)_y = \{(x, (x, y) : x, y \in E$ and $x < y\}$. If $\tau = \{(F_E)_y : y \in E\} \cup \{\Phi, \tilde{E}\}$, then the pair (R, τ) is a soft topological space. Moreover (R, τ) is a soft Hausdorff space.

Definition : 4.3

Let $F_A \in S(X, E)$, $x \in X$ and $A \subseteq E$. Then $(F_A)_\Delta$ denotes the soft set over X x X for which $(F_A)_\Delta : E \to P(X \times X)$ and $(F_A)_\Delta(e) = \Delta = \{(x, x) : x \in X\}$ if $e \in A$ and $(F_A)_\Delta(e) = \Phi$ if $e \notin A$.

$(F_A)_\Delta$ is called A-diagonal soft set. If A = E, then it is called **diagonal soft set**.

Theorem : 4.4

(X, τ) is soft Hausdorff space iff the soft diagonal set $(F_A)_\Delta$ is soft closed.

Proof

Let X be a soft Hausdorff space. We must show that $(F_A)_\Delta^c$ is soft open. Suppose that $(x_1, x_2) \tilde{\in} (F_A)_\Delta^c$. Then $(x_1, x_2) \tilde{\notin} (F_A)_\Delta$ and for some $e \in E$, $(x_1, x_2) \notin (F_A)_\Delta(e)$. Thus, we have $x_1 \neq x_2$. Since X is soft Hausdorff space, there exists G_B, $H_C \in \tau$ such that $x_1 \tilde{\in} G_B$, $x_2 \tilde{\in} H_C$ and $G_B \tilde{\cap} H_C = \Phi$. So, for each $e \in E$, $x_1 \in G_B(e)$, $x_2 \in H_C(e)$ and $(G_B(e) \cap H_C(e) = \Phi$. This implies that $(x_1, x_2) \in G_B(e) \times H_C(e)$ and $(G_B(e) \times H_C(e)) \cap (F_A)_\Delta(e) = \Phi$. Hence, $(x_1, x_2) \tilde{\in} G_B \times H_C$ and $(G_B \times H_C) \tilde{\cap} (F_A)_\Delta = \Phi$.

Conversely, let $(F_A)_\Delta$ is soft closed set. Let $x, y \in X$ and $x \neq y$. Then $(x, y) \tilde{\notin} (F_A)_\Delta$. So $(x, y) \tilde{\in} (F_A)^c_\Delta$. By the definition of soft base there exists G_B and $H_C \in S(X, E)$ which is element of soft base such that $(x, y) \tilde{\in} G_B \times H_C \tilde{\subseteq} (F_A)^c_\Delta$. Hence, $x \tilde{\in} G_B$, $y \tilde{\in} H_C$, G_B, $H_C \in \tau$ and $G_B \tilde{\cap} H_C = \Phi$.

Theorem : 4.5

If (X, τ) is a soft Hausdorff space and $(\varphi, \psi) : (X, \tau) \to (Y, \tau^*)$ is injective, surjective and soft open, then (Y, τ^*) is a soft Hausdorff space.

Proof

Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since (φ, ψ) is surjective there exists $x_1, x_2 \in X$ such that $\varphi(x_1) = y_1$, $\varphi(x_2) = y_2$ and $x_1 \neq x_2$. From hypothesis (X, τ) is soft Hausdorff space, so there exists F_A, $G_B \in \tau$ such that $x_1 \tilde{\in} F_A$, $x_2 \tilde{\in} G_B$ and $F_A \tilde{\cap} G_B = \Phi$. So, for each $e \in E$, $x_1 \in F_A(e)$, $x_2 \in G_B(e)$ and $F_A(e) \cap G_B(e) = \Phi$. This implies that $\varphi(x_1) = y_1 \in \varphi(F_A(e))$, $\varphi(x_2) = y_2 \in \varphi(G_B(e))$. Hence, $y_1 \tilde{\in} (\varphi, \psi)(F_A)$, $y_2 \tilde{\in} (\varphi, \psi)(G_B)$. Since (φ, ψ) is open, then $(\varphi, \psi)(F_A)$,

$(\varphi, \psi)(G_B) \in \tau^*$ and since (φ, ψ) is injective $(\varphi, \psi)(F_A) \tilde{\cap} (\varphi, \psi)(G_B) = (\varphi, \psi)(F_A \tilde{\cap} G_B) = \Phi$. Thus, (Y, τ^*) is soft Hausdorff space.

Theorem : 4.6

The property of being a soft Hausdorff space is hereditary.

Definition : 4.7

Let (X, τ) and (Y, τ^*) be two soft topological spaces. A soft function (φ, ψ) from X to Y is called **Homeomorphism** if (φ, ψ) is one-one, onto, continuous and open.

Lemma : 4.8

Let (X, τ) and (Y, τ^*) be two soft topological spaces. Then X and Y are homeomorphic to a subspace of X x Y.

Proof

Let $(a_1, a_2) \in X \times Y$ and $(e', k') \in E \times K$ fixed. We need to show that a soft function (φ, ψ) from X to X x $\{a_2\} \subseteq X \times Y$ is a homeomorphism. Here, $\varphi : X \to X \times \{a_2\}$ and $\psi : E \to E \times \{k'\}$. φ and ψ are one-one and onto mappings, then the soft mapping (φ, ψ) is one-one and onto.

Now we show that (φ, ψ) is continuous. Let F_A be a soft set which is element of base of subspace X x $\{a_2\}$. By the definition of subspace, there exists $G_B \times H_C \in S(X \times Y, E \times K)$ open such that $F_A = (G_B \times H_C) \tilde{\cap} \tilde{\tilde{E}}_{X \times \{a_2\}}$. For $\psi(e) = (e', k')$,

$$(\varphi, \psi)^{-1} (F_A) (e', k') = (\varphi, \psi)^{-1} ((G_B \times H_C) \tilde{\cap} \tilde{\tilde{E}}_{X \times \{a_2\}}) (e', k')$$

$$= (\varphi, \psi)^{-1} (((G_B \times H_C) \tilde{\cap} \tilde{\tilde{E}}_{X \times \{a_2\}}) (\psi(e)))$$

$$= \varphi^{-1} ((G_B(e) \times H_C(k)) \cap X \times \{a_2\})$$

$$= \begin{cases} \varphi^{-1} (G_B(e) \times \{a_2\}), \text{if } a_2 \in H_C(k); \\ \Phi, \text{otherwise} \end{cases}$$

$$= \begin{cases} (G_B(e), \ a_2 \in H_C(k); \\ \Phi, \text{otherwise} \end{cases}$$

$$\text{Then } (\varphi, \psi)^{-1} (F_A) = \begin{cases} (G_B(e), a_2 \in H_C(k); \\ \Phi, \text{otherwise} \end{cases}$$

Then $(\varphi, \psi)^{-1}(F_A) = \begin{cases} (G_B, a_2 \tilde{\in} H_C; \\ \Phi, \text{otherwise.} \end{cases}$

Hence, $(\varphi, \psi)^{-1}(F_A)$ is soft open, so (φ, ψ) is soft continuous.

Now we show that (φ, ψ) is open. Let F_A be a soft open set on X. For $k \in K$,

$$(\varphi, \psi)(F_A)(k) = \begin{cases} U_{e \in \psi^{-1} \cap A}\, \varphi(F_A(e)), \psi^{-1}(k) \cap A \neq \Phi; \\ \Phi, \text{otherwise} \end{cases}$$

$$= \begin{cases} F_A(e) \times \{a_2\}, \psi^{-1}(k) \cap A \neq \Phi; \\ \Phi, \text{otherwise} \end{cases}$$

$$= \begin{cases} (F_A(e) \times Y) \cap (X \times \{a_2\}), \psi^{-1}(k) \cap A \neq \Phi; \\ \Phi, \text{otherwise} \end{cases}$$

Then $(\varphi, \psi)(F_A) = (F_A \times \tilde{E}_Y) \tilde{\cap} \tilde{E}_{X \times Y}$ is open on subspace. Consequently, the soft mapping (φ, ψ) is open.

Theorem : 4.9

X and Y are soft Hausdorff space iff X x Y is a soft Hausdorff space.

Proof

Let X and Y be soft Hausdorff spaces. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ and $(x_1, y_1) \neq (x_2, y_2)$. So we have $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume that $x_1 \neq x_2$. Since X is soft Hausdorff space there exists $F_A, G_B \in \tau$ such that $x_1 \tilde{\in} F_A$, $x_2 \tilde{\in} G_B$ and $F_A \tilde{\cap} G_B = \Phi$. Then $F_A \times \tilde{E}_Y$ and $G_B \times \tilde{E}_Y$ are soft open set on X x Y. Hence, $(x_1, y_1) \tilde{\in} F_A \times \tilde{E}_Y$, $(x_2\ y_2) \tilde{\in} G_B \times \tilde{E}_Y$ and $(F_A \times \tilde{E}_Y) \tilde{\cap} (G_B \times \tilde{E}_Y) = \Phi$.

Conversely, let X x Y be soft Hausdorff space. By the Theorem 4.6 and Lemma 4.8, it is obvious.

Theorem : 4.10

(X, τ) is a soft Hausdorff space iff $x_E = \tilde{\cap} \{F_E : F_E \in \mathcal{N}(x)$ and F_E is soft closed$\}$.

Proof

Let (X, τ) be a soft Hausdorff space. Suppose that $x_E \neq \tilde{\cap} \{F_E : F_E \in \mathcal{N}(x)$ and F_E is soft closed$\}$. Then, there exists $y \in X$ such that $x \neq y$ and $y \tilde{\in} \tilde{\cap} \{F_E : F_E \in \mathcal{N}(x)$ and F_E is soft closed$\}$. (*)

This implies that $y \in \cap F_E(e)$, for all $e \in E$. Since X is soft Hausdorff space, there exists soft open sets G_E, H_E such that $x \tilde{\in} G_E$, $y \tilde{\in} H_E$ and $G_E \tilde{\cap} H_E = \Phi$ and so $x \in G_E(e) \subset X \setminus H_E(e)$. Hence , $H_E^c \in \mathcal{N}(x)$ and H_E^c is soft closed. By the (*), we have $y \tilde{\in} H_E^c$ and so $y \tilde{\notin} H_E$. This is a contradiction and this completes the proof.

Conversely, let $x, y \in X$ with $x \neq y$. Then $y \tilde{\notin} x_E = \tilde{\cap} \{F_E : F_E \in \mathcal{N}(x)$ and F_E is soft closed$\}$. So, there exists $G_E \in \mathcal{N}(x)$ and G_E is soft closed such that $y \tilde{\notin} G_E$. This implies that $y \notin G_E(e)$ for some $e \in E$. Then $y \tilde{\in} G_E^c$ and G_E^c is soft open. Since $G_E \in \mathcal{N}(x)$ there exists $H_E \in \tau$ such that $x \tilde{\in} H_E \tilde{\subseteq} G_E$. Hence, $x \tilde{\in} H_E$, $y \tilde{\in} G_E^c$ and $H_E \tilde{\cap} G_E^c = \Phi$.

Consequently, (X, τ) is a soft Hausdorff space.

Theorem : 4.11

In soft Hausdorff space, a sequence converges to a unique point.

Proof

Suppose that (x_n) converges to x and y and $x \neq y$. Since (X, τ) is soft Hausdorff space there exists $G_B, H_C \in \tau$ such that $x \tilde{\in} G_B$, $y \tilde{\in} H_C$ and $G_B \tilde{\cap} H_C = \Phi$. This implies that for all $e \in E$, $x \in G_B(e)$, $y \in H_C(e)$ and

$G_B(e) \cap H_C(e) = \Phi$. Since x_n converges to x and G_B is soft neighborhood of x, then there exists $n_1 \in N$ such that $x_n \tilde{\in} G_B$ for all $n \geq n_1$. Since x_n converges to y and H_C is soft neighborhood of y, then there exists $n_2 \in N$ such that $x_n \tilde{\in} H_C$ for all $n \geq n_2$. Let $n_0 = \max(n_1, n_2)$, then for all $n \geq n_0$, $x_n \tilde{\in} G_B$ and $x_n \tilde{\in} H_C$. This implies that $x_n \in G_B(e)$ and $x_n \in H_C(e)$ for all $e \in E$. Then $G_B(e) \cap H_C(e) \neq \Phi$. Hence, $G_B \tilde{\cap} H_C \neq \Phi$. This is a contradiction.

Remark : 4.12

The converse of the Theorem 4.11 is not true in general. For instance, in soft topological space (R, τ^*) every sequence converges to a unique point, but this soft topological space is not Hausdorff.