On b* - closed sets and sb* - closed sets in topological spaces

Sowmya, N

(13PMA010)

Thesis submitted to

Avinashilingam Institute for Home Science and Higher Education for Women,

Coimbatore – 641 043

In Partial Fulfilment of the Requirements for the

Degree of Master of Science in Mathematics

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Content

CONTENT

Introduction

INTRODUCTION

"The essence of Mathematics lies in its freedom"

- Georg Cantor

Topology has a tremendous application in computer graphics, image processing, digital image and digital pictures etc. Levine [1970] introduced the notion of generalized closed sets in topological spaces and showed that compactness, countable compactness, para compactness and normality etc are all g - closed hereditary. Also he introduced a separation axiom called $T_{1/2}$ between T_1 and T_0. Recently many modifications were defined and investigated. They are applied to introduce several low separation axioms. Levine [1963] introduced the concept of semi - continuous function.

Generalized open sets play a very important role in general topology and they are now the research topics of many researchers worldwide. As a generalization of open sets, the notion of b - open sets was introduced by Andrijevic [1996]. The class of b - open sets is contained in the class of semi - pre open sets and contains all semi - open sets and pre - open sets. Moreover, it generates the same topology as the class of pre - open sets.

Bhattacharya and Lahiri [1987] introduced the notions of sg - closed sets and Arya and Nour [1990] introduced the notions of gs - closed sets in topological spaces. Veerakumar [2000] introduced ψ - closed sets in topological spaces. Veerakumar [2003] introduced the concepts of $g^{\#}$ - closed sets in topological spaces.

Extensive research on generalizing closedness was done in recent years. In continuation of the study of generalized closed sets, Al - Omari and Noorani [2009] have introduced the notion of generalized b - closed sets. Since then many authors have contributed to the study of the various concepts using the notion of generalized b - closed sets.

Muthuvel and Parimelazhagan [2012] introduced the notion of b* - closed sets and investigated some of its properties. Later the notion of b* - continuous functions and b* - open maps and b* - closed maps were introduced by Muthuvel and Parimelazhagan [2012].

Poongothai and Parimelazhagan [2012] introduced the notion of strongly b* - closed sets and investigated the relation between the associated topologies. Later on the notion of strongly b* - continuous (briefly sb* - continuous) functions and sb* - open and closed maps were also introduced by Poongothai and Parimelazhagan [2012].

The aim of this thesis is to study b* - closed sets and strongly b* - closed sets in topological spaces. The following articles are chosen for our discussion:

- b* - Closed Sets in Topological Spaces by **S. Muthuvel and R. Parimelazhagan [2012].**
- b* - Continuous Functions in Topological Spaces by **S. Muthuvel and R. Parimelazhagan [2012].**
- sb* - Closed Sets in Topological Spaces by **Poongothai and R. Parimelazhagan [2012].**
- Strongly b* - Continuous Functions in topological spaces by **Poongothai and R. Parimelazhagan [2012].**

In Chapter 1, we discussed the contributions of Muthuvel and Parimelazhagan [2012] towards the study of b* - closed sets. The Union of any family of b* - closed sets is not b* - closed. It is illustrated by means of a counter example. Regarding intersection, it is shown that the intersection of a b* - closed set and a closed set is b* - closed and also the intersection of two b* - closed sets is a b* - closed set. Moreover, some interesting results on the topology generated by b* - closed sets are discussed. The chapter is concluded by deriving some properties of b* - open sets.

In Chapter 2, b* - continuous functions in topological spaces due to Muthuvel and Parimelazhagan [2012] are studied. Some interesting properties of b* - continuous functions are discussed. Also the relationship between b* - continuous functions and other existing continuous functions are discussed.

In Chapter 3, the concept of strongly b* - closed (briefly sb* - closed) sets introduced by Poongothai and Parimelazhagan [2012] are studied. Some interesting characterizations of this set are discussed. It is shown that the intersection of a sb* - closed set and a closed set is sb* - closed set and the intersection of two sb* - closed sets is also a sb* - closed set. Regarding union, the union of two sb* - closed sets need not be a sb* - closed which is proved by a counter example. In this chapter, the relationship between sb* - closed set and other existing closed sets are discussed. Furthermore, they have introduced the notion of independency of sb* - closed sets and the following sets namely g - closed sets, αg - closed sets, sg - closed sets, semi - closed sets and $\psi g^{\#}$ - closed sets are independent of sb* - closed sets.

In Chapter 4, the notion of strongly b* - continuous (briefly sb* - continuous) functions in topological spaces by Poongothai and Parimelazhagan [2012] are discussed. Various characterization theorems and also the relationship between sb* - continuous function and other existing continuous functions are discussed. Furthermore, the notion of strongly b* - open and closed maps are also discussed. Some of its characterization theorems are also obtained.

Review of Literature

REVIEW OF LITERATURE

Levine [1970] introduced the notion of generalized closed sets. Andrijevic [1996] introduced the notion of b - open sets. Al - Omari and Noorani [2009] introduced the notion of generalized b - closed sets. The notions of b* - closed sets and b* - continuous functions were introduced by Muthuvel and Parimelazhagan [2012]. The notions of strongly b* - closed (briefly sb* - closed) sets and strongly b* - continuous functions were introduced by Poongothai and Parimelazhagan [2012].

Since the advent of these notions, several authors have contributed to the study of these concepts and several worthwhile research papers have been published. We present a brief review of literature in some of the important articles published that are related to this topic.

On b - open sets

Andrijevic, [1996]

In this article a new class of generalized open sets in topological space, called b - open sets, is introduced and studied. This class is contained in the class of semi - preopen sets and contains all semi - open sets and all pre - open sets. The class of b - open sets generates the same topology as the class of pre - open sets.

Slightly γ - continuous functions

Ekici and Caldas, [2004]

In this article the authors introduced and studied b - continuous functions in topological spaces.

Almost b - continuous functions

Rajesh, [2007]

In this article the author has introduced and characterized almost b - continuous functions by using b - open sets.

On generalized b - closed sets

Al - Omari and Noorani, [2009]

In this article, the class of generalized b - closed sets is discussed and this notion is used to consider new weak and stronger forms of continuities associated with these sets. These notions are applied to give new characterization of extremally disconnected spaces and also T_{gs} - spaces.

Properties of totally b - continuous functions

Caldas, Jafari and Rajesh, [2009]

In this article,the authors have introduced a new class of functions called totally b - continuous functions by using b - closed sets and b - open sets. Relationships between this new class and other classes of existing known functions are established.

On generalized b - closed sets and their relationships

Hussein, [2011]

In this article many relationships between some known types of generalized closed sets and b - generalized closed sets are investigated. Also some new characterizations of extremally disconnected, T_{gs} spaces and sg - submaximal spaces are obtained.

Generalizations of locally b - closed sets

Bharathi, Bhuvaneshvari and Chandramathi, [2011]

The notions of generalized locally b - closed sets (glbc sets) and generalized locally b - continuous maps (glbc maps) which are weaker forms of locally closed sets and LC continuous maps, respectively, are introduced in topological spaces. Certain results relating to them are established.

Weakly ωb - continuous functions

Mustafa, [2011]

In this article the authors have introduced a new class of functions called weakly ωb - continuous functions and investigated several properties and characterizations. Connections with other existing concepts, such as ωb - continuous and weakly b - continuous functions are also discussed.

New notions via b - open sets

Rajesh and Salleh, [2011]

In this article the authors have introduced a new class of topological spaces called b - $T_{1/2}$ space in terms of the concept of b - open sets and b - kernel and have investigated some of their fundamental properties. Also they have introduced and studied some new notions in topological spaces by utilizing b - open sets.

On Locally b - Closed, b - Pre-Open & sb - Generalized Closed Sets In Topological Spaces

Thakur C.K. Raman and Pallab Kanti Biswas, [2014]

In this article, the authors have introduced and studied locally b - closed, b - pre - open sets, sb - generalized closed sets and their connectivity with regular open sets.

Chapter – 1

CHAPTER - 1

b* - CLOSED SETS IN TOPOLOGICAL SPACES

Section - 1.1

Preliminaries

Definition : 1.1.1 [16]

A subset A of a topological space (X, τ) is called **semi - open** if A ⊆ cl(int(A)) and **semi - closed** if int(cl(A)) ⊆ A.

Definition : 1.1.2 [19]

A subset A of a topological space (X, τ) is called **pre - open** if A ⊆ int(cl(A)) and **pre - closed** set if cl(int(A)) ⊆ A.

Definition : 1.1.3 [27]

A subset A of a topological space (X, τ) is called α **- open** if A ⊆ int(cl(int(A))) and α **- closed** if cl(int(cl(A))) ⊆ A.

Definition : 1.1.4 [15]

A subset A of a topological space (X, τ) is called **generalized closed** (briefly g - closed) if cl(A) ⊆ U, whenever A ⊆ U and U is open in X.

Definition : 1.1.5 [3]

A subset A of a topological space (X, τ) is called **b - open** if A ⊆ cl(int(A)) ∪ int(cl(A)) and **b - closed** if cl(int(A)) ∩ int(cl(A)) ⊆ A.

Definition : 1.1.6 [18]

A subset A of a topological space (X, τ) is called α **- generalized closed** (briefly αg - closed) if αcl(A) ⊆ U, whenever A ⊆ U and U is open in (X, τ).

Definition : 1.1.7 [8]

A subset A of a topological space (X, τ) is called **semi - generalized closed** (briefly sg - closed) if scl(A) $\subseteq$ U, whenever and U is semi - open in (X, τ).

Definition : 1.1.8 [17]

A subset A of a topological space (X, τ) is called **generalized α - closed** (briefly gα - closed) if αcl(A) $\subseteq$ U, whenever A $\subseteq$ U and U is α - open in (X, τ).

Definition : 1.1.9 [5]

A subset A of a topological space (X, τ) is called **generalized semi - closed** (briefly gs - closed) if scl(A) $\subseteq$ U, whenever A $\subseteq$ U and U is open in (X, τ).

Definition : 1.1.10 [2]

A subset A of a topological space (X, τ) is called **generalized b - closed** (briefly gb - closed) if bcl(A) $\subseteq$ U, whenever A $\subseteq$ U and U is open in (X, τ).

Definition : 1.1.11 [44]

A subset A of a topological space (X, τ) is called **nowhere dense** if int(cl(A)) = φ.

Definition : 1.1.12 [32]

A subset A of a topological space (X, τ) is called **g * s - closed** if scl(A) $\subseteq$ U, whenever A $\subseteq$ U and U is gs - open in (X, τ).

Definition : 1.1.13 [36]

A subset A of a topological space (X, τ) is called **regular closed** if A = cl(int(A)).

Definition : 1.1.14 [26]

A subset A of a topological space (X, τ) is called **regular b - closed** (briefly rb - closed) if rcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is b - open in (X, τ).

Definition : 1.1.15 [28]

A subset A of a topological space (X, τ) is called **regular generalized closed** if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is regular open in (X, τ).

Definition : 1.1.16 [21]

A subset A of a topological Space (X, τ) is called **r*g*- closed** if rcl(A) $\subseteq$ U whenever A $\subseteq$ U, U is g - open in (X, τ).

Definition : 1.1.17 [12]

A subset A of a topological Space (X, τ) is called **r*bg*- closed** if rbcl(A) $\subseteq$ U whenever A $\subseteq$ U, U is b - open in (X, τ).

Definition : 1.1.18 [41]

A subset A of a topological space (X, τ) is called **ψ - closed** if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is sg - open set in (X, τ).

Definition : 1.1.19 [42]

A subset A of a topological space (X, τ) is called **$g^{\#}$ - closed** if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is αg - open set in (X, τ).

Definition : 1.1.20 [35]

A subset A of a topological Space (X, τ) is called **$\psi g^{\#}$ - closed** if ψcl(A) $\subseteq$ U whenever A $\subseteq$ U, U is $g^{\#}$ - open in (X, τ).

Section - 1.2

b* - Closed Sets

Definition : 1.2.1 [23]

A subset A of a topological space (X, τ) is called a b* - closed set if int(cl(A)) $\subseteq$ U, whenever A $\subseteq$ U and U is b - open in (X, τ).

Theorem : 1.2.2

A subset A of X is b* - closed if and only if int(cl(A)) - A contains no non-empty closed set in X .

Proof :

Necessity :

Suppose A be a b* - closed set. Let int(cl(A)) - A contains a closed set (say) F. That is $F \subseteq int(cl(A)) - A \Longrightarrow F \subseteq int(cl(A)) \cap A^c$

$$\Longrightarrow F \subseteq int(cl(A)) \text{ and } F \subseteq A^c \Longrightarrow A \subseteq F^c$$

Hence F^c is b - open and A is b* - closed set, we have

$$int(cl(A)) \subseteq F^c \Longrightarrow F \subseteq (int(cl(A)))^c$$

Thus we have $F \subseteq int(cl(A))$ and $F \subseteq (int(cl(A)))^c$

$$\Longrightarrow F \subseteq int(cl(A)) \cap (\, int(cl(A)))^c = \varphi.$$

Hence $F = \varphi$.

Sufficiency :

Let int(cl(A)) - A contains no non-empty closed set. Let $A \subseteq U$, where U is b - open. Suppose $int(cl(A)) \nsubseteq U$, $U^c \cap int(cl(A)) \neq \varphi$ then $U^c \cap int(cl(A))$ is a non-empty closed set of int(cl(A)) - A. This implies that int(cl(A)) - A contains a non-empty closed set which is a contradiction. Hence $int(cl(A)) \subseteq U$. Thus A is b* - closed set in X.

Corollary : 1.2.3

Let A be gb - closed set and A is b* - closed set if and only if int(cl(A)) - A is closed.

Proof :

Necessity :

Let A be gb - closed set. If A is b* - closed, then we get $int(cl(A)) - A = \varphi$

which is a closed set.

Sufficiency :

Let int(cl(A)) - A be closed. Then by **theorem 1.2.2**, int(cl(A)) - A does not contain any non-empty closed subset and since int(cl(A)) is closed subset of itself. Then we get

$$\text{int}(\text{cl}(A)) - A = \varphi \Longrightarrow \text{int}(\text{cl}(A)) = A$$

Thus A is b* - closed set in X.

Theorem : 1.2.4

Suppose that $B \subseteq A \subseteq X$, B is b* - closed set relative to A and that A is both b - open and b* - closed subset of X, then B is b* - closed set relative to X.

Proof :

Let $B \subseteq U$ and U be an open set in X. It is given that $B \subseteq A \subseteq X$. Thus

$$B \subseteq U \text{ and } B \subseteq A \Longrightarrow B \subseteq A \cap U$$

Since B is b* - closed set relative to A, $\text{int}(\text{cl}(B)) \subseteq U$ that is

$$A \cap \text{int}(\text{cl}(B)) \subseteq A \cap U \Longrightarrow A \cap \text{int}(\text{cl}(B)) \subseteq U. \text{ Thus}$$

$$[A \cap \text{int}(\text{cl}(B))] \cup [\text{int}(\text{cl}(B))]^c \subseteq U \cup [\text{int}(\text{cl}(B))]^c$$

$$\Longrightarrow A \cup [\text{int}(\text{cl}(B))]^c \subseteq U \cup [\text{int}(\text{cl}(B))]^c$$

Since A is b* - closed set in X, $\text{int}(\text{cl}(A)) \subseteq U \cup [\text{int}(\text{cl}(B))]^c$.

$$\text{Also } B \subseteq A \Longrightarrow \text{int}(\text{cl}(B)) \subseteq \text{int}(\text{cl}(A)).$$

Thus $\text{int}(\text{cl}(B)) \subseteq \text{int}(\text{cl}(A)) \subseteq U \cup [\text{int}(\text{cl}(B))]^c$. Therefore $\text{int}(\text{cl}(B)) \subseteq U$ and hence B is b* - closed set relative to X.

Theorem : 1.2.5

Let $A \subseteq Y \subseteq X$ and supposed that A is b* - closed in X then A is b* - closed relative to Y.

Proof :

Let A be a b* - closed set in X and let $A \subseteq Y \cap U$ where U is b - open set in X. Since A is b* - closed set in X, $int(cl(A)) \subseteq U$ that is $Y \cap [int(cl(A))] \subseteq Y \cap U$ where $Y \cap [int(cl(A))]$ is the interior closure of A in Y. Thus A is b* - closed set relative to Y.

Theorem : 1.2.6

A subset A of a topological space (X, τ) is b* - closed set if and only if A is a semi - closed set in X.

Proof :

Necessity :

Let A be a b* - closed set of X. Let U be an open set containing A in X. Since A is b* - closed set, $int(cl(A)) \subseteq U$ and since $A \subseteq U$, we get

$$int(cl(A)) \subseteq A \subseteq U \Longrightarrow int(cl(A)) \subseteq A$$

Hence A is a semi - closed set in X.

Sufficiency :

Let A be a semi - closed set of X. Let U be an open set containing A in X. Since A is semi - closed, $int(cl(A)) \subseteq A$ and since $A \subseteq U$, we get $int(cl(A)) \subseteq U$. Thus A is b* - closed set in X.

Theorem : 1.2.7

If A is a b* - closed set and $A \subseteq B \subseteq int(cl(A))$ then B is a b* - closed set.

Proof :

Let A be a b* - closed set, such that $A \subseteq B \subseteq int(cl(A))$. Let U be a b - open set of X such that $B \subseteq U$. Since A is b* - closed set, we have $int(cl(A)) \subseteq U$ whenever $A \subseteq U$. Since $A \subseteq B$ and $B \subseteq int(cl(A))$ then

$$int(cl(B)) \subseteq int(cl(int(cl(A)))) \subseteq int(cl(A)) \subseteq U.$$

Therefore $int(cl(B)) \subseteq U$. Thus B is b* - closed set in X.

Theorem : 1.2.8

The intersection of a b* - closed set and a closed set is a b* - closed set.

Proof :

Let A be a b* - closed set and F be a closed set. Since A is b* - closed set, int(cl(A)) ⊆ U whenever A ⊆ U, where U is a b - open set.

To show that A ∩ F is b* - closed set, it is enough to show that int(cl(A ∩ F)) ⊆ U whenever A ⊆ U, where U is b - open set. Let G = X - F then A ⊆ U ∪ G. Since G is open set, U ∪ G is b - open set and A is b* - closed set, int(cl(A)) ⊆ U ∪ G.

Now int(cl(A ∩ F)) ⊆ int(cl(A)) ∩ int(cl(F)) ⊆ int(cl(A)) ∩ F ⊆ (U ∪ G) ∩ F. This implies int(cl(A)) ∩ F ⊆ (U ∪ G) ∩ F ⊆ (U ∩ F) ∪ (G ∩ F) ⊆ (U ∩ F) ∪ φ ⊆ U. This implies that A ∩ F is b* - closed set.

Theorem : 1.2.9

If A and B are two b* - closed sets in X, then their intersection A ∩ B is b* - closed set in X.

Proof :

Let A and B are b* - closed sets in X and consider U be a b - open set in X such that A ∩ B ⊆ U. Now int(cl(A ∩ B)) ⊆ int(cl(A)) ∩ int(cl(B)) ⊆ U. Hence A ∩ B is b* - closed set in X.

Remark : 1.2.10

The Union of two b* - closed sets need not be a b* - closed set.

Example : 1.2.11

Let X = {a, b, c} with topology τ = {φ, {a}, {b}, {a, b}, X}. Then the sets { φ, {a}, {b}, {c}, {b, c}, {c, a}, X} are b* - closed sets. The union of the sets {a} and {b} is {a, b} is not a b* - closed set in X.

Theorem : 1.2.12

If a subset A of a topological space (X, τ) is both open and b* - closed then it is closed set in X.

Proof :

Let A be a subset of X which is both open and b* - closed set in X. We know that int(cl(A)) $\subseteq$ cl(A) $\subseteq$ A, hence A is closed set in X.

Remark : 1.2.13

The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.14

Let X = {a, b, c} with topology τ = {φ, {a}, {b}, {a, b}, X}. Then the sets {φ, {a}, {b}, {c}, {b, c}, {c, a}, X} are b* - closed sets whereas the sets {φ, {b, c}, {c, a}, {c}, X} are closed sets. The sets {a} and {b} are open and b* - closed sets but not closed sets in X.

Theorem : 1.2.15

If a subset A of a topological space (X, τ) is closed then it is b* - closed set in X.

Proof :

Let A be closed set of X. Let U be an open set containing A in X. Since A is closed, cl(A) $\subseteq$ A $\subseteq$ U. We know that int(cl(A)) $\subseteq$ cl(A), we get int(cl(A)) $\subseteq$ U. Thus A is a b* - closed set in X.

Remark : 1.2.16

The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.17

Let X = {a, b, c} with topology τ = {φ, {a}, {b}, {a, b}, X}. Then the sets {φ, {a}, {b}, {c}, {b, c}, {c, a}, X} are b* - closed sets whereas the sets {φ, {c}, {b, c}, {a, c}, X} are closed sets. The sets {a} and {b} are b* - closed sets but not closed sets in X.

Theorem : 1.2.18

If a subset A of a topological space (X, τ) is b* - closed then it is b - closed set in X.

Proof :

Let A be a b* - closed set in X. Let U be an open set containing A in X. Since A is b* - closed, $int(cl(A)) \subseteq U$. Taking complement on both sides we get, $U \subseteq cl(int(A))$.

Therefore $int(cl(A)) \subseteq U \subseteq cl(int(A))$.

Therefore $int(cl(A)) \cap cl(int(A)) \subseteq U$. As $A \subseteq U$, we get

$int(cl(A)) \cap cl(int(A)) \subseteq A$.

Hence A is b* - closed set in X.

Remark : 1.2.19

The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.20

Let X = {a, b, c} with topology τ = {φ, {b}, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}, X} are b - closed sets whereas the sets {φ, {b}, {a, c}, X} are b* - closed sets. The sets {a}, {c}, {a, b} and {b, c} are b - closed sets but not b* - closed sets in X.

Theorem : 1.2.21

If a subset A of a topological space (X, τ) is pre - closed then it is b* - closed set in X.

Proof :

Suppose A is pre - closed set in X. Let U be an open set containing A in X. Since A is pre - closed, cl(int(A)) = int(cl(A)). Hence we get

$$cl(int(A)) \subseteq int(cl(A)) \subseteq A \subseteq U$$

Thus A is b* - closed set in X.

Remark : 1.2.22

The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.23

Let X = {a, b, c} with topology τ = {φ, {a}, {b}, {a, b}, X}. Then the sets {φ, {a}, {b}, {c}, {b, c}, {c, a}, X} are b* - closed sets whereas the sets {φ, {c}, {b, c}, {a, c}, X} are pre - closed sets. The sets {a} and {b} are b* - closed sets but not pre - closed sets in X.

Theorem : 1.2.24

If a subset A of a topological space (X, τ) is b* - closed set then it is $\psi g^{\#}$ - closed set in X.

Proof :

Let A be a b* - closed set of X. Let U be an open set containing A in X. Since A is b*- closed, int(cl(A)) ⊆ U and since int(cl(A)) ⊆ ψcl(A), we get ψcl(A) ⊆ U. Hence A is a $\psi g^{\#}$ - closed set in X.

Remark : 1.2.25

The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.26

Let X = {a, b, c} with topology τ = {φ, {a, b}, X}. Then the sets {φ, {c}, {b, c}, {c, a}, X} are $\psi g^{\#}$ - closed sets whereas the sets {φ, {c}, X} are b* - closed sets. The sets {b, c} and {c, a} are $\psi g^{\#}$ - closed sets but not b* - closed sets in X.

Theorem : 1.2.27

If a subset A of a topological space (X, τ) is b* - closed set then it is g*s - closed set in X.

Proof :

Let A be b* - closed subset of X. Let U be an open set containing A in X. Since A is b* - closed, int(cl(A)) ⊆ U. We know that int(cl(A)) ⊆ scl(A). Hence we get scl(A) ⊆ U. Hence A is a g*s - closed set in X.

Remark : 1.2.28

The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.29

Let X = {a, b, c} with topology τ = {φ, {a, b}, X}. Then the sets {φ, {c}, {b, c}, {c, a}, X} are g*s - closed sets whereas the sets {φ, {c}, X} are b* - closed sets. The sets {b, c} and {c, a} are g*s - closed sets but not b* - closed sets in X.

Theorem : 1.2.30

If a subset A of a topological space (X, τ) is b* - closed set then it is gs - closed set in X.

Proof :

Let A be a b* - closed set of X. Let U be an open set containing A in X. Since A is b* - closed, int(cl(A)) ⊆ U and since int(cl(A)) ⊆ scl(A), we get scl(A) ⊆ U. Thus A is gs - closed set in X.

Remark : 1.2.31

The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.32

Let X = {a, b, c} with topology τ = {φ, {a, c}, X}. Then the sets {φ, {b}, {a, b}, {b, c}, X} are gs - closed sets whereas the sets {φ, {b}, X} are b* - closed sets. The sets {a, b} and {b, c} are gs - closed sets but not b* - closed sets in X.

Theorem : 1.2.33

If a subset A of a topological space (X, τ) is b* - closed set then it is gb - closed set in X.

Proof :

Let A be a b* - closed set of X. Let U be an open set containing A in X. Since A is b* - closed, int(cl(A)) ⊆ U and since int(cl(A)) ⊆ bcl(A), we get bcl(A) ⊆ U. Hence A is gb - closed set in X.

Remark : 1.2.34

The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.35

Let X = {a, b, c} with topology τ = {φ, {a}, {a, c}, X}. Then the sets {φ, {b}, {c}, {a, b}, {b, c}, X} are gb - closed sets whereas the sets {φ, {b}, {c}, {b, c}, X} are b* - closed sets. The set {a, b} is gb - closed set but not b* - closed set in X.

Theorem : 1.2.36

If a subset A of a topological space (X, τ) is r*bg* - closed set then it is b* - closed set in X.

Proof :

Let A be a r*bg* - closed set of X. Let U be an open set containing A in X. Since A is r*bg* - closed, rbcl (A) $\subseteq$ U and since int(cl(A)) $\subseteq$ rbcl(A), we get int(cl(A)) $\subseteq$ U. Hence A is b* - closed set in X.

Remark : 1.2.37

The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.38

Let X = {a, b, c} with topology τ = {φ, {a}, {a, c}, X}. Then the sets {φ, {b}, {c}, {b, c}, X} are b* - closed sets whereas the sets {φ, {b, c}, X} are r*bg* - closed sets. The sets {b} and {c} are b* - closed sets but not r*bg* - closed sets in X.

Theorem : 1.2.39

If a subset A of a topological space (X, τ) is nowhere dense then it is b* - closed set in X.

Proof :

Suppose a subset A is nowhere dense then int(cl(A)) = φ. It is obvious that A $\subseteq$ cl(A) and also int(A) $\subseteq$ int(cl(A)). Since A is nowhere dense,

$$\text{int}(A) = \varphi \Longrightarrow \text{int}(\text{cl}(A)) = \varphi$$

Thus A is b* - closed set in X.

Remark : 1.2.40

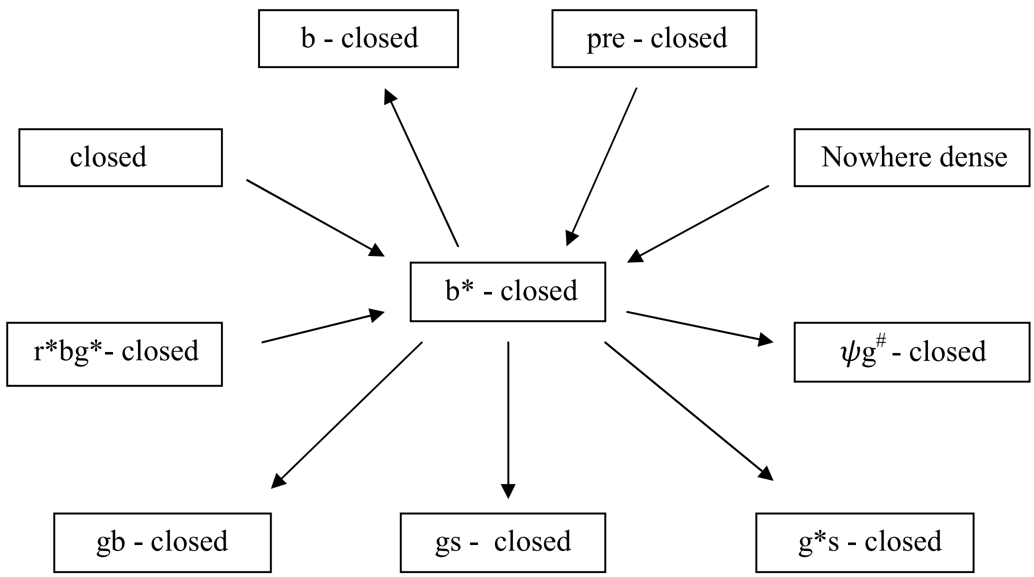
The converse of the above theorem need not be true as seen from the following example.

Example : 1.2.41

Let X = {a, b, c} with topology τ = {φ, {a}, {b}, {a, b}, X}. Then the sets {φ, {a}, {b}, {c}, {b, c }, {c, a}, X} are b* - closed sets whereas the sets {φ, {c}} are nowhere dense sets. The sets {a}, {b}, {b, c}, {c, a}, X are b* - closed sets but not nowhere dense sets in X.

Remark : 1.2.42

The following diagram illustrates the relationship between b* - closed sets and other closed sets.



where A → B indicates A implies B, but not conversely.

Remark : 1.2.43

The following examples show that b* - closed set is independent of g - closed set.

Example : 1.2.44

Let X = {a, b, c} with topology τ = {φ, {a}, {b}, {a, b}, X}. Then the sets {φ, {a}, {b}, {c}, {b, c}, {c, a}, X} are b* - closed sets whereas the sets {φ, {c}, {b, c}, {c, a}, X} are g - closed sets. The sets {a} and {b} are b* - closed sets but not g - closed sets in X.

Example : 1.2.45

Let X = {a, b, c} with topology τ = {φ, {a, b}, X}. Then the sets {φ, {c}, {b, c}, {c, a}, X} are g - closed sets whereas the sets {φ, {c}, X} are b* - closed sets. The sets {b, c} and {c, a} are g - closed sets but not b* - closed sets in X.

Remark : 1.2.46

The following examples show that b* - closed set is independent of gα - closed set.

Example : 1.2.47

Let X = {a, b, c} with topology τ = {φ, {a}, {b}, {a, b}, X}. Then the sets {φ, {a}, {b}, {c}, {b, c}, {c, a}, X} are b* - closed sets whereas the sets {φ, {c}, {b, c}, {c, a}, X} are gα - closed sets. The sets {a} and {b} are b* - closed sets but not gα - closed sets in X.

Example : 1.2.48

Let X = {a, b, c} with topology τ = {φ, {a, b}, X}. Then the sets {φ, {c}, {b, c}, {c, a}, X} are gα - closed sets whereas the sets {φ, {c}, X} are b* - closed sets. Hence the sets {b, c} and {c, a} are gα - closed sets but not b* - closed sets in X

Remark : 1.2.49

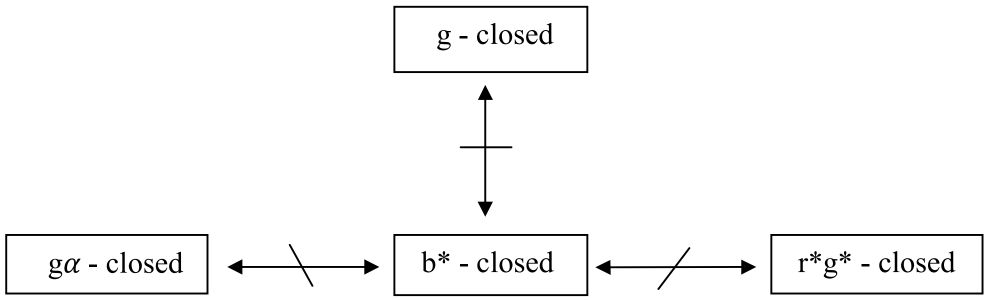
The following example show that b* - closed set is independent of r*g* - closed set.

Example : 1.2.50

Let X = {a, b, c} with topology τ = {φ, {a}, {a, b}, X}. Then the sets {φ, {b}, {c}, {b, c}, X} are b* - closed sets whereas the sets {φ, {c}, {b, c}, {c, a}, X} are r*g* - closed sets. Thus the set {b} is b* - closed set but not r*g* - closed set in X and the set {c, a} is r*g* - closed set but not b* - closed set in X.

Remark : 1.2.51

The following diagram shows that b* - closed sets are independent of some of the existing closed sets.



where A ←|→ B indicates A is independent of B.

Section - 1.3

b* - Open Sets

Definition : 1.3.1 [23]

A subset A of a topological space (X, τ) is called b* - open set if its complement A^c is b* - closed set in X.

Theorem : 1.3.2

If a subset A of a topological space (X, τ) is open then it is b* - open set in X.

Proof :

Let A be an open set in X. Let U be an open set contained in A. Since A is open, we get

$$int(A) \subseteq A \Rightarrow U \subseteq int(A) \subseteq A$$

$$U \subseteq cl(int(A)) \subseteq cl(A) \Rightarrow U \subseteq cl(int(A))$$

Hence A is b*- open set in X.

Remark : 1.3.3

The converse of the above theorem need not be true as seen from the following example.

Example : 1.3.4

Let X = {a, b, c} with topology τ = {φ, {a}, X}. Then the sets {φ, {a}, {a, b}, {c, a}, X} are b* - open sets. The sets {a, b} and {c, a} are b* - open set but not open sets in X.

Theorem : 1.3.5

A subset A of a topological space (X, τ) is b* - open if and only if $F \subseteq cl(int(A))$ whenever F is closed and $F \subseteq A$.

Proof :

Necessity :

Assume that A is b* - open set in X. Then A^c is b* - closed set in X. Let F be a closed set in X contained in A. Then F^c is an open set in X containing A^c. Since A^c is b* - closed, $int(cl(A^c)) \subseteq F^c$. Taking complement on both sides, we get $cl(int(A)) \supseteq F$, that is $F \subseteq cl(int(A))$.

Sufficiency :

Assume that $F \subseteq cl(int(A))$ whenever $F \subseteq A$ and F is closed in X. Let U be an open set containing A^c in X.

Then $U^c \subseteq cl(int(A)) \Longrightarrow U \supseteq int(cl(A^c)) \Longrightarrow int(cl(A^c)) \subseteq U$.

Hence A^c is b* - closed set in X which implies that A is b* - open set in X.

Hence A is b* - open set in X.

Chapter – 2

CHAPTER - 2

b* - CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

Section - 2.1

Preliminaries

Definition : 2.1.1 [23]

A subset A of a topological space (X, τ) is called a **b* - closed** if int(cl(A)) ⊆ U, whenever A ⊆ U and U is b - open in (X, τ).

Definition : 2.1.2 [11]

A function f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is said to be **b - continuous** if for each x ϵ X and for each open set of V of Y containing f (x),there exists U ϵ bO (X, x) such that f (U) ⊆ V.

Section - 2.2

b* - Continuous Functions

Definition : 2.2.1 [24]

A map f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is called b* - continuous map if the inverse image of every closed set in Y is b* - closed in X.

Theorem : 2.2.2

If a map f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is continuous then it is b* - continuous.

Proof :

Let f : X → Y be continuous. Let M be any closed set in Y. Then the inverse image $f^{-1}(M)$ is closed in Y. Since every closed set is b* - closed set in X, $f^{-1}(M)$ is b* - closed set in X. Therefore, f is b* - continuous.

Remark : 2.2.3

The converse of the above theorem need not be true as seen from the following example.

Example : 2.2.4

Let X = {a, b, c} and Y = {a, b, c} with topologies τ = {φ, {c}, {a, c}, X} and σ = {φ, {a, b}, Y}. A function f : X → Y is defined by f ({a}) = {c}, f ({b}) = {b}, f ({c}) = {a} then {a} = f^{-1}({c}), {b} = f^{-1}({b}), {c} = f^{-1}({a}). Then f is b* - continuous. But f is not continuous since the inverse image of the closed set {c} in Y (i.e) f^{-1}({c}) = {a} is not closed set in X.

Theorem : 2.2.5

Let (X, τ) and (Y, σ) be topological spaces. If a map f : X → Y is b* - continuous then it is b - continuous.

Proof :

Assume that a map f : X → Y is b* - continuous. Let V be a closed set in Y. Then $f^{-1}(V)$ is a b* - closed set in X. Since every b* - closed set is b - closed, $f^{-1}(V)$ is a b - closed set in X. Therefore f is b - continuous.

Remark : 2.2.6

The converse of the above theorem need not be true as seen from the following example.

Example : 2.2.7

Let X = {a, b, c} and Y = {a, b, c} with topologies τ = {φ, {b}, {a, c}, X} and σ = {φ, {a}, {a, b}, Y} respectively. Let f : X → Y be defined as f ({a}) = {b}, f ({b}) = {c}, f ({c}) = {a} then {a} = f^{-1}({b}), {b} = f^{-1}({c}), {c} = f^{-1}({a}).Then f is b - continuous but not b* - continuous as the inverse image of the closed set {b, c} in Y (i.e) f^{-1}({b, c}) = {a, b} which is not b* - closed set in X.

Theorem : 2.2.8

Let $f : X \rightarrow Y$ be a single valued function where (X, τ) and (Y, σ) are topological spaces. Then the following are equivalent.

(a) The function f is b* - continuous.

(b) The inverse image of each open set in Y is b* - open in X.

(c) If $f : X \rightarrow Y$ is b* - continuous, then $f(cl^*(A)) \subseteq cl(f(A))$ for every subset A of X.

Proof :

(a) ⟹ (b)

Assume that $f : X \rightarrow Y$ is b* - continuous. Let M be open in Y. Then M^c is closed in Y. Since Y is b* - continuous $f^{-1}(M^c)$ is b* - closed in X. But $f^{-1}(M^c) = X - f^{-1}(M)$. Thus $X - f^{-1}(M)$ is b* - closed in X and so $f^{-1}(M)$ is b* - open in X.

Therefore (a) ⟹ (b).

(b) ⟹ (a)

Assume that the inverse image of each b - open set in Y is b* - open in X. Let B be any closed set in Y. Then B^c is open in Y. By assumption, $f^{-1}(B^c)$ is b* - open in X. But $f^{-1}(B^c) = X - f^{-1}(B)$. Thus $X - f^{-1}(B)$ is b*-open in X and so $f^{-1}(B)$ is b* - closed in X. Therefore f is b*-continuous.

Hence (b) ⟹ (a).

Thus (a) and (b) are equivalent.

(a) ⟹ (c)

Assume that f is b* - continuous. Let A be any subset of X. Then cl(f(A)) is a closed set in Y. Since f is b* - continuous, $f^{-1}(cl(f(A)))$ is b* - closed in X and it contains A. But cl*(A) is the intersection of all b* - closed sets containing A. Therefore $cl^*(A) \subseteq f^{-1}(cl(f(A)))$ and so $f(cl^*(A)) \subseteq cl(f(A))$.

Thus (a) ⟹ (c).

Section - 2.3

b* - Open maps and b* - Closed maps

Definition : 2.3.1 [24]

Let (X, τ) and (Y, σ) be two topological spaces. A map f : X → Y is called **b* - open map** if the image of every open set in X is b* - open in Y.

Definition : 2.3.2 [24]

Let (X, τ) and (Y, σ) be two topological spaces. A map f : X → Y is called **b* - closed map** if the image of every closed set in X is b* - closed in Y.

Theorem : 2.3.3

Let (X, τ) and (Y, σ) be two topological spaces. If a map f : X → Y is an open map then it is b* - open map.

Proof :

Let f : X → Y be an open map and V be an open set in X. Then f (V) is open. Since every open set is b* - open, we get f (V) is b* - open in Y. Thus f is b* - open map.

Remark : 2.3.4

The converse of the above theorem need not be true as seen from the following example.

Example : 2.3.5

Let X = {a, b, c} and Y = {a, b, c} with the topologies τ = {φ, {a}, {a, b}, {a, c}, X} and σ = {φ, {a}, Y} respectively. Let f : X → Y be an identity map. Then f is b* - open but not open as the image of the open set {a, c} in X (i.e) f ({a, c}) = {a, c} is not open in Y.

Theorem : 2.3.6

Let (X, τ) and (Y, σ) be two topological spaces. If a map f : X → Y is a closed map then it is b* - closed map.

Proof :

Let a map $f : X \to Y$ be a closed map and V be a closed set in X. Then f (V) is closed. Since every closed set is b* - closed, we get f (V) is b* - closed in Y. Thus f is b* - closed map.

Remark : 2.3.7

The converse of the above theorem need not be true as seen from the following example.

Example : 2.3.8

Let X = {a, b, c} and Y = {a, b, c} with the topologies $\tau = \{\varphi, \{a, b\}, X\}$ and $\sigma = \{\varphi, \{a\}, Y\}$ respectively. Let $f : X \to Y$ be an identity map. Then f is b* - closed but not closed as the image of the closed set {c} in X (i.e) f ({c}) = {c} is not closed in Y.

Theorem : 2.3.9

A map $f : X \to Y$ is b* - closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$ there is a b* - open set V of Y such that $S \subseteq U$ and $f^{-1}(V) \subseteq U$.

Proof:

Necessity :

Let S be a subset of Y and U be an arbitrary open set in X containing $f^{-1}(S)$. It is enough we produce a b* - open set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Choose V = Y - (f (X - U))

As U is open, X - U is closed and by the definition of b* - closed map, f (X - U) is b* - closed in Y. Hence V is a b* - open set and $f^{-1}(S) \subseteq U$.

Hence $X - U \subseteq X - f^{-1}(S) \subseteq f^{-1}(Y - S) \Longrightarrow f(X - U) \subseteq Y - S$.

Thus $S \subseteq Y - f(X - U) = V$

Now $V = Y - f(X - U) \Longrightarrow f(X - U) \subseteq Y - V$

$$\Rightarrow X - U \subseteq f^{-1}(Y - V) = X - f^{-1}(V)$$

$$\Rightarrow f^{-1}(V) \subseteq U$$

Sufficiency :

Let S be closed in X. Then X - S is open. In the given criteria, put U = X - S and S = Y - f(S). As $f^{-1}(Y - f(S)) \subseteq X - S = U$, there exists a b* - open set V of Y such that $Y - f(S) \subseteq V$ and $f^{-1}(V) \subseteq X - S \Rightarrow S \subseteq X - f^{-1}(V)$.

$$\text{Now } Y - f(S) \subseteq V \Rightarrow Y - V \subseteq f(S) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$$

Therefore f(S) = Y – V. Since Y – V is b* - closed, f(S) is b* - closed and hence f is a b* - closed map.

Theorem : 2.3.10

If a map f : X → Y is continuous and b* - closed and if A is a b* - closed set of X then f (A) is b* - closed in Y.

Proof:

Let f (A) ⊆ U where U is an open set of Y. Since f is continuous $f^{-1}(U)$ is an open set containing A. Hence int(cl(A)) ⊆ f^{-1} (U) as A is b* - closed. Since f is b* - closed, f(int(cl(A))) is a b* - closed set contained in the open set U which implies int(cl(f(int(cl(A))))) ⊆ U. Hence A = int(cl(A)) ⊆ U. So f (A) is b* - closed in Y.

Corollary : 2.3.11

If a map f : X → Y is continuous and closed and if A is a b* - closed set of X then f (A) is b* - closed in Y.

Corollary : 2.3.12

If a map f : X → Y is b* - closed and if A is a closed set of X then $f_A : A \rightarrow Y$ is b* - closed in Y.

Corollary : 2.3.13

If a map f : X → Y is b* - closed and continuous and if A is a b* - closed set of X then $f_A : A \rightarrow Y$ is continuous and b* - closed in Y.

Proof:

Let F be a closed set of A then F is b* - closed set of X. From **theorem 2.3.10**, it follows that $f_A(F)$ = f (F) is a b* - closed set of Y. Hence f_A is b* - closed and also continuous.

Theorem : 2.3.14

If a map f : X → Y is open, continuous, b* - closed and surjection where X is regular then Y is regular.

Proof :

Let V be an open set containing a point x of X, such that f (x) = p. Since X is regular and f is continuous, there is an open set V such that x ϵ V ⊆ f^{-1} (V). Hence p ϵ f (V) ⊆ f (cl(V)))) ⊆ U. Since f is b* - closed, f (cl(V)) is b* - closed set contained in the open set U. It follows that int (cl(f(cl(V)))) ⊆ U and hence p ϵ f (V) ⊆ cl(f (V)) ⊆ U and f (V) is open, since f is open. Hence Y is regular.

Theorem : 2.3.15

If a map f : X → Y is closed map and a map g : Y → Z is b* - closed then g ∘ f : X → Z is b* - closed.

Proof :

Let H be a closed set in X. Then f (H) is closed. Since g is b* - closed, g(f(H)) is closed in Z. But (g ∘ f) (H) = g(f(H)). Hence g ∘ f is b* - closed.

Chapter – 3

CHAPTER - 3

STRONGLY b* - CLOSED SETS IN TOPOLOGICAL SPACES

Section - 3.1

Preliminaries

Definition : 3.1.1 [37]

A subset A of a topological space (X, τ) is called **w - closed** if cl(A) ⊆ U whenever A ⊆ U and U is semi open in (X, τ).

Definition : 3.1.2 [25]

A subset A of a topological space (X, τ) is called **weakly generalized closed** (briefly wg - closed) if cl(int(A)) ⊆ U whenever A ⊆ U and U is open in (X, τ).

Definition : 3.1.3 [4]

A subset A of a topological space (X, τ) is called **semi - pre open** (briefly β - open set) if A ⊆ cl(int(cl(A))) and **semi - pre closed** if int(cl(int(A))) ⊆ A.

Definition : 3.1.4 [43]

A subset A of a topological space (X, τ) is called **strongly generalized closed** (briefly g* - closed) if cl(A) ⊆ U, whenever A ⊆ U and U is g - open in (X, τ).

Definition : 3.1.5 [23]

A subset A of a topological space (X, τ) is called **b** - open** if A ⊆ (int(cl(int(A)))) ∪ (cl(int(cl(A)))).

Section - 3.2

Strongly b* - Closed Sets

Definition : 3.2.1 [30]

A subset A of a topological space (X, τ) is called a **strongly b* - closed** (briefly sb* - closed) if $cl(int(A)) \subseteq U$, whenever $A \subseteq U$ and U is b - open in (X, τ).

Theorem : 3.2.2

If a subset A of a topological space (X, τ) is sb* - closed then it is b - closed set in X.

Proof :

Assume that A is sb* - closed in X and let U be an open set such that $A \subseteq U$. Since every open set is b - open set and A is sb* - closed set, $cl(int(A)) \subseteq (cl(int(A))) \cup (int(cl(A))) \subseteq U$. Therefore A is b - closed set in X.

Remark : 3.2.3

The converse of the above theorem need not be true as seen from the following example.

Example : 3.2.4

Let X = {a, b, c} with the topology τ = {φ, {a}, {c}, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, X} are b - closed sets whereas the sets {φ, {b}, {a, b}, {b, c}, X} are sb* - closed sets. The sets {a} and {c} are b - closed sets but not sb* - closed sets in X.

Theorem : 3.2.5

A set A is sb* - closed if and only if $cl(int(A)) - A$ contains no non empty b - closed sets.

Proof :

Necessity:

Suppose that F is a non empty b - closed subset of cl(int(A)) - A such that $F \subseteq cl(int(A)) - A$. Then $F \subseteq cl(int(A))$.

Therefore $F \subseteq cl(int(A))$ and $F \subseteq A^c$. Since F^c is b - open set and A is sb* - closed, $cl(int(A)) \subseteq F^c$. Thus $F \subseteq (cl(int(A)))^c$.

This implies that $F \subseteq [cl(int(A))] \cap [cl(int(A))]^c = \varphi$. Therefore $F = \varphi$. Hence cl(int(A)) - A contains no non empty b - closed sets.

Sufficiency :

Let A be a b - open set contained in U. Suppose that cl(int(A)) is not contained in U, then $[cl(int(A))] \cap U^c$ is a non empty b - closed set and contained in cl(int(A)) - A, which is a contradiction.

Therefore $cl(int(A)) \subseteq U$ and hence A is sb* - closed set.

Theorem : 3.2.6

Let $B \subseteq Y \subseteq X$, if B is sb* - closed set relative to Y and Y is open and sb* - closed set in (X, τ) then B is sb* - closed set in (X, τ).

Proof :

Let U be a b - open set in (X, τ) such that $B \subseteq U$. Given that $B \subseteq Y \subseteq X$. Therefore $B \subseteq Y$ and $B \subseteq U$. This implies that $B \subseteq Y \cap U$. Since B is sb* - closed set relative to Y, then $cl(int(B)) \subseteq U$. Hence $Y \cap cl(int(B)) \subseteq Y \cap U$ implies that $Y \cap cl(int(B)) \subseteq U$. Thus

$$[\ Y \cap cl(int(B))\ ] \cup [\ cl(int(B))\ ]^c \subseteq U \cup [\ cl(int(B))\ ]^c.$$

This implies that

$$[\ Y \cup (cl(int(B)))^c\ ] \cap [\ (cl(int(B))) \cup (cl(int(B)))^c\ ] \subseteq U \cup (cl(int(B)))^c.$$

Therefore $[\ Y \cup (cl(int(B)))^c\ ] \subseteq U \cup (cl(int(B)))^c$.

Since Y is sb* - closed set in X, $cl(int(Y)) \subseteq U \cup [\ cl(int(B))\ ]^c$. Also $B \subseteq Y$ implies that $cl(int(B)) \subseteq cl(int(Y))$. Thus

$$cl(int(B)) \subseteq cl(int(Y)) \subseteq U \cup [\ cl(int(B))\ ]^c.$$

Since cl(int(B)) is not contained in $[\ cl(int(B))\ ]^c$,we get $cl(int(B)) \subseteq U$. Hence B is sb* - closed set relative to X.

Theorem : 3.2.7

Let $A \subseteq Y \subseteq X$ and suppose that A is sb* - closed set in X then A is sb* - closed set relative to Y.

Proof :

Assume that $A \subseteq Y \subseteq X$ and A is sb* - closed set in X.

Let $A \subseteq Y \cap U$ where U is b - open in X. Since A is sb* - closed set in X, $A \subseteq U$ implies $cl(int(A)) \subseteq U$. That is $Y \cap cl(int(A)) \subseteq Y \cap U$, where $Y \cap cl(int(A))$ is the closure interior of A in Y. Therefore $cl(int(A)) \subseteq U$. Thus A is sb* - closed set relative to Y.

Theorem : 3.2.8

If A is a sb* - closed set and $A \subseteq B \subseteq cl(int(A))$ then B is a sb* - closed set.

Proof :

Let A be a sb* - closed set, such that $A \subseteq B \subseteq cl(int(A))$. Let U be a b - open set of X such that $B \subseteq U$. Since A is sb* - closed set, we have $cl(int(A)) \subseteq U$ whenever $A \subseteq U$. Since $A \subseteq B$ and $B \subseteq cl(int(A))$ then

$$cl(int(B)) \subseteq cl(int(cl(int(A)))) \subseteq cl(int(A)) \subseteq U.$$

Therefore $cl(int(B)) \subseteq U$. Thus B is sb* - closed set in X.

Theorem : 3.2.9

The intersection of a sb* - closed set and a closed set is a sb* - closed set.

Proof :

Let A be a sb* - closed set and F be closed set. Since A is sb* - closed set, $cl(int(A)) \subseteq U$ whenever $A \subseteq U$, where U is b - open set. To show that $A \cap F$ is sb* - closed set, it is enough to show that $cl(int(A \cap F)) \subseteq U$ whenever $A \cap F \subseteq U$, where U is b - open set. Let $G = X - F$ then $A \subseteq U \cup G$. Since G is open set, $U \cup G$ is b - open set and A is sb* - closed, $cl(int(A)) \subseteq U \cup G$. Now

$$cl(int(A \cap F)) \subseteq cl(int(A)) \cap cl(int(F)) \subseteq cl(int(A)) \cap F \subseteq (U \cup G) \cap F$$

Since $(U \cup G) \cap F \subseteq (U \cap F) \cup (G \cap F) \subseteq (U \cap F) \cup \varphi \subseteq U$.

This implies that $A \cap F$ is sb* - closed set.

Theorem : 3.2.10

If A and B are two sb* - closed sets in X, then their intersection $A \cap B$ is sb* - closed set in X.

Proof :

Let A and B are sb* - closed sets and also consider U be a b - open set in X such that $A \cap B \subseteq U$. Now $cl(int(A \cap B)) \subseteq cl(int(A)) \cap cl(int(B)) \subseteq U$. Hence $A \cap B$ is sb* - closed set.

Remark : 3.2.11

The Union of two sb* - closed sets need not be sb* - closed set.

Example : 3.2.12

Let $X = \{a, b, c\}$ with topology $\tau = \{\varphi, \{a, c\}, X\}$. Then the sets $\{\varphi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$ are sb* - closed sets in X. The union of the two sb* - closed sets $\{a\}$ and $\{c\}$ is $\{a, c\}$ is not a sb* - closed set in X.

Theorem : 3.2.13

If a subset A of a topological space (X, τ) is closed then it is sb* - closed set in X.

Proof :

Let A be a closed set of X. Let U be an open set containing A in X. Since A is closed, A ⊆ cl (A) ⊆ U. Since cl(int(A)) ⊆ cl (A), and since every open set is b - open set, we get cl(int(A)) ⊆ U. Thus A is sb* - closed set in X.

Remark : 3.2.14

The converse of the above theorem need not be true as seen from the following example.

Example : 3.2.15

Let X = {a, b, c} with topology τ = {φ, {b}, X}. Then the sets {φ, {a}, {c}, {a, c}, X} are sb* - closed sets whereas the sets {φ, {a, c}, X} are closed sets. The sets {a} and {c} are sb* - closed sets but not closed sets in X.

Theorem : 3.2.16

If a subset A of a topological space (X, τ) is g* - closed then it is sb* - closed set in X.

Proof :

Let A be a g* - closed set of X. Let U be a g - open set containing A in X. Since A is g* - closed, cl(A) ⊆ U. Since cl(int(A)) ⊆ cl(A), and since every g - open set is b - open set, we get cl(int(A)) ⊆ U. Thus A is sb* - closed set in X.

Remark : 3.2.17

The converse of the above theorem need not be true as seen from the following example.

Example : 3.2.18

Let X = {a, b, c} with topology τ = {φ, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, X} are sb* - closed sets whereas the sets {φ, {b}, {a, b}, {b, c}, X} are g* - closed sets. The sets {a} and {c} are sb* - closed sets but not g* - closed sets in X.

Theorem : 3.2.19

If a subset A of a topological space (X, τ) is sb* - closed then it is b** - closed set in X.

Proof :

Assume that A is sb* - closed set in X.

Let U be an open set containing A in X. Since A is sb* - closed, cl(int(A)) ⊆ A ⊆ U. This implies that int(cl(int(A))) ⊆ int(A). Taking complement on both sides, we get cl(int(cl(A))) ⊇ cl(A).

Hence we get int(cl(int(A))) ⊆ int(A) ⊆ cl(A) ⊆ cl(int(cl(A)))

⟹ int(cl(int(A))) ⊆ A ⊆ cl(int(cl(A))). Thus cl(int(cl(A))) ∩ int(cl(int(A))) ⊆ A.

Thus A is b** - closed set in X.

Remark : 3.2.20

The converse of the above theorem need not be true as seen from the following example.

Example : 3.2.21

Let X = {a, b, c} with topology τ = {φ, {a}, {c}, {a, c}, X}. Then the sets {φ, {b}, {a, b}, {b, c}, X} are sb* - closed sets whereas the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, X} are b** - closed sets. The sets {a} and {c} are b** - closed sets but not sb* - closed sets in X.

Theorem : 3.2.22

If a subset A of a topological space (X, τ) is α - closed then it is sb* - closed set in X.

Proof :

Let A be a α - closed set of X. Let U be an open set containing A in X. Since A is α - closed, αcl(A) ⊆ U. Since αcl(A) ⊆ cl(int(A)) ⊆ U. Thus A is sb* - closed set in X.

Remark : 3.2.23

The converse of the above theorem need not be true as seen from the following example.

Example : 3.2.24

Let X = {a, b, c} with topology τ = {φ, {b}, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, {c, a}, X} are sb* - closed sets whereas the sets {φ, {b}, {a, c}, X} are α - closed sets. The sets {a}, {c}, {a, b}, {b, c} are sb* - closed sets but not α - closed sets in X.

Theorem : 3.2.25

If a subset A of a topological space (X, τ) is sb* - closed then it is wg - closed set in X.

Proof :

Assume that A is sb* - closed in X. Let U be an open set containing A in X. Then cl(int(A)) ⊆ U. Thus A is wg - closed set in X.

Remark : 3.2.26

The converse of the above theorem need not be true as seen from the following example.

Example : 3.2.27

Let X = {a, b, c} with topology τ = {φ, {a}, {a, b}, X}. Then the sets {φ, {b}, {c}, {b, c}, {c, a}, X} are wg - closed sets whereas the sets {φ, {b}, {c}, {b, c}, X} are sb* - closed sets. The set {c, a} is wg - closed set but not sb* - closed set in X.

Theorem : 3.2.28

If a subset A of a topological space (X, τ) is w - closed then it is sb* - closed set in X.

Proof :

Let A be a w - closed set of X. Let U be an open set containing A in X. This implies that int(A) ⊆ U, where U is semi - open in X. Since every semi - open set is b - open and cl(int(A)) ⊆ cl(A) ⊆ U. Thus A is sb* - closed set in X.

Remark : 3.2.29

The converse of the above theorem need not be true as seen from the following example.

Example : 3.2.30

Let X = {a, b, c} with topology τ = {φ, {a}, X}. Then the sets {φ, {b}, {c}, {b, c}, X} are sb* - closed sets whereas the sets {φ, {b, c}, X} are w - closed sets. The sets {b} and {c} are sb* - closed sets but not w - closed sets in X.

Theorem : 3.2.31

If a subset A of a topological space (X, τ) is sb* - closed then it is semi - pre closed set in X.

Proof :

Assume that A is sb* - closed in X. Let U be an open set containing A in X. Then cl(int(A)) ⊆ U. Hence cl(int(A)) ⊆ A ⊆ U. This implies that int(cl(int(A))) ⊆ int(A) ⊆ A. Thus A is semi - pre closed set in X.

Remark : 3.2.32

The converse of the above theorem need not be true as seen from the following example.

Example : 3.2.33

Let X = {a, b, c} with topology τ = {φ, {a}, {c}, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, X} are semi - pre closed sets whereas the sets {φ, {b}, {a, b}, {b, c}, X} are sb* - closed sets. The sets {a} and {c} are semi - pre closed sets but not sb* - closed sets in X.

Theorem : 3.2.34

A subset A of a topological space (X, τ) is sb* - closed set if and only if A is pre - closed set in X.

Proof :

Necessity :

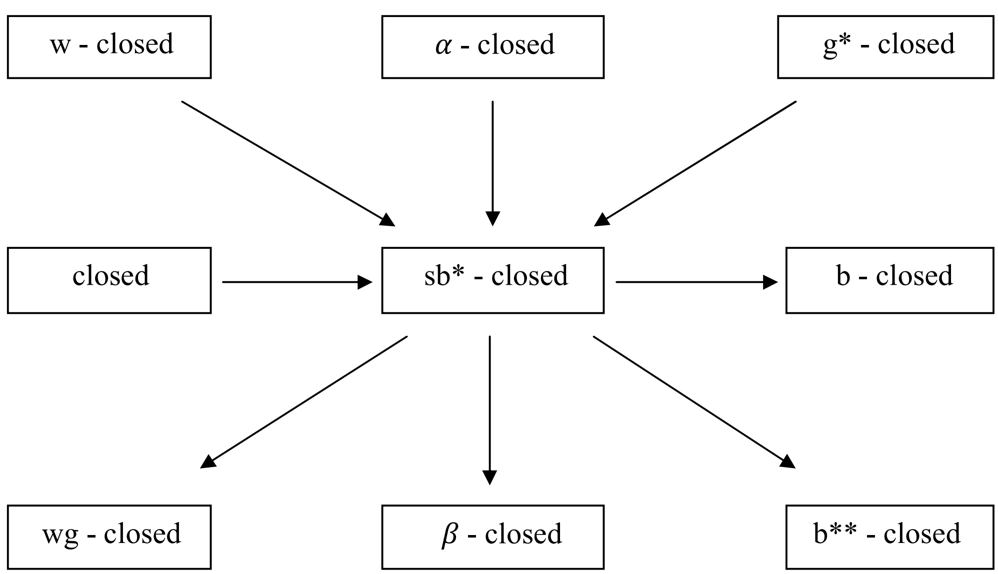
Let A be a sb* - closed set of X. Let U be an open set containing A in X. Then cl(int(A)) ⊆ U which implies that cl(int(A)) ⊆ A ⊆ U. Thus A is pre - closed set in X.

Sufficiency :

Let A be a pre - closed set of X. Let U be an open set containing A in X. Since A is pre - closed, cl(int(A)) ⊆ A ⊆ U. Hence A is sb* - closed set in X.

Remark : 3.2.35

The following diagram summaries the above discussions.



where A ⟶ B represents A implies B but not conversely.

Section - 3.3

Independency of sb* - Closed Sets

Remark : 3.3.1

The following examples show that the concept of g - closed sets and sb* - closed sets are independent.

Example : 3.3.2

Let X = {a, b, c} with topology τ = {φ, {b}, X}. Then the sets {φ, {a}, {c}, {a,b}, {b, c}, {c, a}, X} are g - closed sets whereas the sets {φ, {a}, {c}, {c, a}, X} are sb* - closed sets. The set {a, b} is g - closed set but not sb* - closed set in X.

Example : 3.3.3

Let X = {a, b, c} with topology τ = {φ, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, X} are sb* - closed sets whereas the sets {φ, {b}, {a, b}, {b, c}, X} are g - closed sets. The set {a} is sb* - closed set but not g - closed set in X.

Remark : 3.3.4

The following examples show that the concept of αg - closed sets and sb* - closed sets are independent.

Example : 3.3.5

Let X = {a, b, c} with topology τ = {φ, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, X} are sb* - closed sets whereas the sets {φ, {b}, {a, b}, {b, c}, X} are αg - closed sets. The set {a} is sb* - closed set but not αg - closed set in X.

Example : 3.3.6

Let X = {a, b, c} with topology τ = {φ, {b}, X}. Then the sets {φ, {a}, {c}, {a, b}, {b, c}, {c, a}, X} are αg - closed sets whereas the sets {φ, {a}, {c}, {c, a}, X} are sb* - closed sets. The set {a, b} is αg - closed set but not sb* - closed set in X.

Remark : 3.3.7

The following examples show that the concept of semi - closed sets and sb* - closed sets are independent.

Example : 3.3.8

Let X = {a, b, c} with topology τ = {φ, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, X} are sb* - closed sets whereas the sets {φ, {b}, X} are semi - closed sets. The set {a} is sb* - closed set but not semi - closed set in X.

Example : 3.3.9

Let X = {a, b, c} with topology τ = {φ, {a}, {c}, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, X} are semi - closed sets whereas the sets {φ, {b}, {a, b}, {b, c}, X} are sb* - closed sets. The set {a} is semi - closed set but not sb* - closed set in X.

Remark : 3.3.10

The following examples show that the concept of sg - closed sets and sb* - closed sets are independent.

Example : 3.3.11

Let X = {a, b, c} with topology τ = {φ, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, X} are sb* - closed sets whereas the sets {φ, {b}, {a, b}, {b, c}, X} are sg - closed sets. The set {a} is sb* - closed set but not sg - closed set in X.

Example : 3.3.12

Let X = {a, b, c} with topology τ = {φ, {a}, {c}, {a, c}, X}. Then the sets {φ, {a}, {b}, {c}, {a, b}, {b, c}, X} are sg - closed sets whereas the sets {φ, {b}, {a, b}, {b, c}, X} are sb* - closed sets. The set {a} is sg - closed set but not sb* - closed set in X.

Remark : 3.3.13

The following examples show that the concept of $\psi g^{\#}$ - closed sets and sb* - closed sets are independent.

Example : 3.3.14

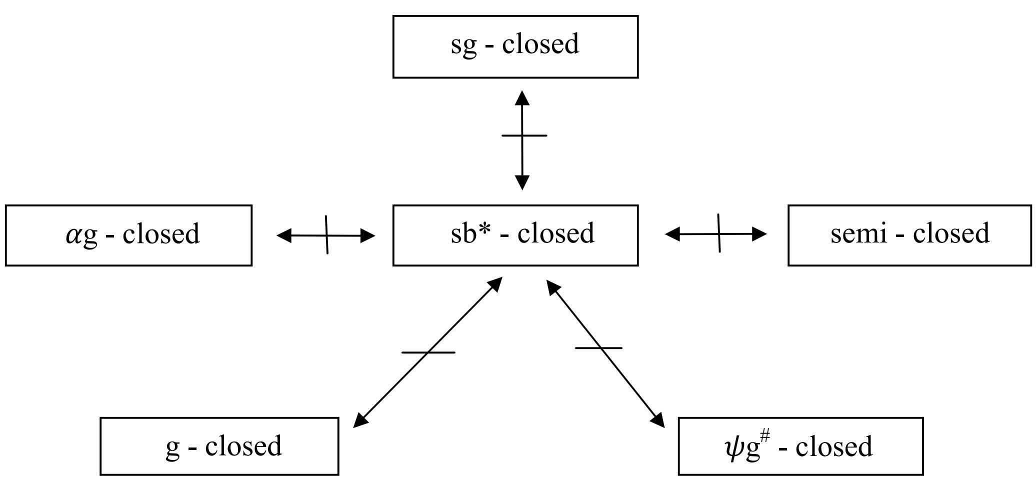
Let X = {a, b, c} with topology τ = {φ, {a, b}, X}. Then the sets {φ, {a}, {b}, {c}, {b, c}, {c, a}, X} are sb* - closed sets whereas the sets {φ, {c}, {b, c}, {c, a}, X} are $\psi g^{\#}$ - closed sets. The set {a} is sb* - closed set but not $\psi g^{\#}$ - closed set in X.

Example : 3.3.15

Let X = {a, b, c} with topology τ = {φ, {b}, X}. Then the sets {φ, {a}, {c}, {a, b}, {b, c}, {c, a}, X} are $\psi g^{\#}$ - closed sets whereas the sets {φ, {a}, {c}, {c, a}, X} are sb* - closed sets. The set {a, b} is $\psi g^{\#}$ - closed set but not sb* - closed set in X.

Remark : 3.3.16

The following diagram illustrates the independency of sb* - closed sets with already existing closed sets.



where A ←|→ B represents A is independent of B.

Chapter – 4

CHAPTER - 4

STRONGLY b* - CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

Section - 4.1

Preliminaries

Definition : 4.1.1 [30]

A subset A of a topological space (X, τ) is called **strongly b* - closed** (briefly sb* - closed) if cl(int(A)) ⊆ U, whenever A ⊆ U and U is b - open in (X, τ).

Definition : 4.1.2 [16]

A map f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is said to be **semi - continuous** if and only if for every closed set B of Y, $f^{-1}(B)$ is semi - closed in X.

Definition : 4.1.3 [6]

A function f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is said to be **generalised continuous** (briefly g - continuous) if $f^{-1}(V)$ is g - open in X for each open set V of Y.

Definition : 4.1.4 [38]

A function f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is said to be **semi - generalised continuous** (briefly sg - continuous) if $f^{-1}(V)$ is semi - generalized closed (briefly sg - closed) in X for every closed set V of Y.

Definition : 4.1.5 [37]

A function f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is said to be **w - continuous** if $f^{-1}(V)$ is w - open in X for each open set V of Y.

Definition : 4.1.6 [20]

A function f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is said to be **α - continuous** if $f^{-1}(V)$ is α - open in X for each open set V of Y.

Definition : 4.1.7 [25]

Let (X, τ) and (Y, σ) be topological spaces. A function f : X → Y is said to be **weakly generalized - continuous** (briefly wg - continuous) if the inverse image of every open set in Y is wg - open in X.

Definition : 4.1.8 [10]

A function f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is said to be **αg - continuous** if $f^{-1}(V)$ is αg - open in X for each open set V of Y.

Definition : 4.1.9 [1]

A function f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is **semi pre - continuous** if and only if for every closed set B of Y, $f^{-1}(B)$ is semi pre - closed in X.

Section - 4.2

Strongly b^* - Continuous Functions

Definition : 4.2.1 [31]

Let (X, τ) and (Y, σ) be topological spaces. A map f : X → Y is called strongly b* - continuous (sb* - continuous) if the inverse image of every open set in Y is sb* - open in X.

Theorem : 4.2.2

If a map f : X → Y is continuous then it is sb* - continuous.

Proof :

Let $f : X \to Y$ be continuous. Let F be any open set in Y. The inverse image of F is open in X. Since every open set is sb^* - open set, the inverse image of every open set in Y is sb^* - open set in X. Therefore f is sb^* - continuous.

Remark : 4.2.3

The converse of the above theorem need not be true as seen from the following example.

Example : 4.2.4

Consider $X = Y = \{a, b, c\}$ with topologies $\tau = \{\varphi, \{a, c\}, X\}$ and $\sigma = \{\varphi, \{b\}, \{a, c\}, Y\}$. Let $f : X \to Y$ be defined by $f(\{a\}) = \{a\}$, $f(\{b\}) = \{c\}$, $f(\{c\}) = \{b\}$. Then f is sb^* - continuous. But f is not continuous since for the open set $U = \{a, c\}$ in Y, $f^{-1}(U) = \{a, b\}$ is not open in X.

Theorem : 4.2.5

Let $f : X \to Y$ be a map from a topological space (X, τ) into a topological space (Y, σ). Then the following statements are equivalent.

(a) f is sb^* - continuous

(b) The inverse image of every closed set in Y is sb^*- closed in X.

Proof:

(a) ⟹ (b)

Assume that $f : X \to Y$ is sb^* - continuous. Let F be closed in Y. Then F^c is open in Y. Since f is sb^* - continuous, $f^{-1}(F^c)$ is sb^* - open in X. But $f^{-1}(F^c) = X - f^{-1}(F)$ and so $f^{-1}(F)$ is sb^* - closed in X.

Therefore (a) ⟹ (b).

(b) ⟹ (a)

Assume that the inverse image of every closed set in Y is sb^* - closed in X. Let V be an open set in Y and V^c is closed in Y. Then $f^{-1}(V^c)$ is sb^* - closed in X. But

$f^{-1}(V^c) = X - f^{-1}(V)$. Thus $f^{-1}(V)$ is sb^*- open in X.

Hence (b) ⟹ (a).

Thus (a) and (b) are equivalent.

Theorem : 4.2.6

Let $f : X \to Y$ be sb* - continuous map from a topological space (X, τ) into a topological space (Y, σ) and let H be a closed subset of X. Then the restriction $f / H : H \to Y$ is sb* - continuous where H is endowed with the relative topology.

Proof :

Let H be any closed set in X and F be any closed subset in Y.

Since f is sb* - continuous, $f^{-1}(F)$ is sb* - closed in X. Since the intersection of two sb* - closed sets is sb* - closed, $f^{-1}(F) \cap H = H_1$, then H_1 is sb* - closed set in X.

Since $(f / H)^{-1}(F) = H_1$, it is sufficient to show that H_1 is sb* - closed set in H. Let G_1 be any open set of H such that $H_1 \subseteq G_1$. Let $G_1 = G \cap H$ where G is open in X. Now $H_1 \subseteq G \cap H$. Since H_1 is sb* - closed in X, $\overline{H}_1 \subseteq G$. Now

$cl_H (H_1) = \overline{H}_1 \cap H \subseteq G \cap H = G_1$, where $cl_H (A)$ is the closure of a subset $A \subset H$ in a subspace H of X.

Therefore f / H is sb* - continuous.

Remark : 4.2.7

In the above theorem, the assumption of closedness of H cannot be removed as seen from the following example.

Example : 4.2.8

Let X = {a, b, c}, Y = {a, b, c} with topologies $\tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\sigma = \{\varphi, \{b\}, \{a, c\}, Y\}$. Let $f : X \to Y$ be defined as f({a}) = {b}, f({b}) = {a}, f({c}) = {c}. Now H = {a, b} is not closed in X. Then f is sb* - continuous but the restriction f / H is not sb* - continuous. Since for the closed set F = {a, c} in Y, $f^{-1}(F) = \{b, c\}$ and $f^{-1}(F) \cap H = \{b, c\} \cap \{a, b\} = \{b\}$ is not sb* - closed in X.

Theorem : 4.2.9

If a function $f : X \rightarrow Y$ from a topological space (X, τ) into a topological space (Y, σ) is sb* - continuous then it is b - continuous.

Proof :

Assume that a map $f : X \rightarrow Y$ is sb* - continuous. Let V be an open set in Y. Since f is sb* - continuous, $f^{-1}(V)$ is sb* - open and hence b - open in X. Therefore f is b - continuous.

Remark : 4.2.10

The converse of the above theorem need not be true as seen from the following example.

Example : 4.2.11

Let $X = Y = \{a, b, c\}$ with topologies $\tau = \{\varphi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\varphi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$. Let $f : X \rightarrow Y$ be defined by $f(\{a\}) = \{b\}$, $f(\{b\}) = \{c\}$, $f(\{c\}) = \{a\}$. Then f is b - continuous. But f is not sb* - continuous since for the open set $U = \{a\}$ in Y, $f^{-1}(U) = \{c\}$ is not sb* - open in X.

Theorem : 4.2.12

If a map $f : X \rightarrow Y$ from a topological space (X, τ) into a topological space (Y, σ) is α - continuous then it is sb* - continuous.

Proof :

Assume that f is α - continuous. Let V be an open set in Y. Since f is α - continuous, $f^{-1}(V)$ is α - open and hence it is sb* - open in X. Then f is sb* - continuous.

Remark : 4.2.13

The converse of the above theorem need not be true as seen from the following example.

Example : 4.2.14

Let X = Y = {a, b, c} with topologies τ = {φ, {b}, {a, c}, X} and σ = {φ, {c}, {a, b}, Y}. Let f : X → Y be an identity map. Then f is sb* - continuous. But f is not α - continuous since for the open set U = {c} in Y, $f^{-1}(U)$ = {c} is not α - open in X.

Theorem : 4.2.15

If a map f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is sb* - continuous then it is wg - continuous.

Proof :

Assume that f : X → Y is sb* - continuous. Let V be an open set in Y. Since f is sb* - continuous, $f^{-1}(V)$ is sb* - open and hence it is wg - open in X. Then f is wg - continuous.

Remark : 4.2.16

The converse of the above theorem need not be true as seen from the following example.

Example : 4.2.17

Let X = Y = {a, b, c} with topologies τ = {φ, {a}, {a, b}, X} and σ = {φ, {b}, Y}. Let f : X → Y be an identity map. Then f is wg - continuous. But f is not sb* - continuous, since for the open set U = {b} in Y, $f^{-1}(U)$ = {b} is not sb* - open in X.

Theorem : 4.2.18

If a map f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is w - continuous then it is sb* - continuous.

Proof :

Let f : X → Y is w - continuous and Let V be an open set in Y. Since f is w - continuous, $f^{-1}(V)$ is w - open and hence it is sb* - open in X. Then f is sb* - continuous.

Remark : 4.2.19

The converse of the above theorem need not be true as seen from the following example.

Example : 4.2.20

Let X = Y = {a, b, c} with topologies τ = {φ, {b, c}, X} and σ = {φ, {b}, {a, c}, Y}. Let f : X → Y be an identity map. Then f is sb* - continuous. But f is not w - continuous, since for the open set U = {a, c} in Y, $f^{-1}(U)$ = {a, c} is not w - open in X.

Theorem : 4.2.21

If a map f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is sb* - continuous then it is semi pre - continuous.

Proof :

Assume that f : X → Y is sb* - continuous. Let V be an open set in Y. Since f is sb* - continuous, $f^{-1}(V)$ is sb* - open and hence it is semi pre - open set in X. Then f is semi - pre continuous.

Remark : 4.2.22

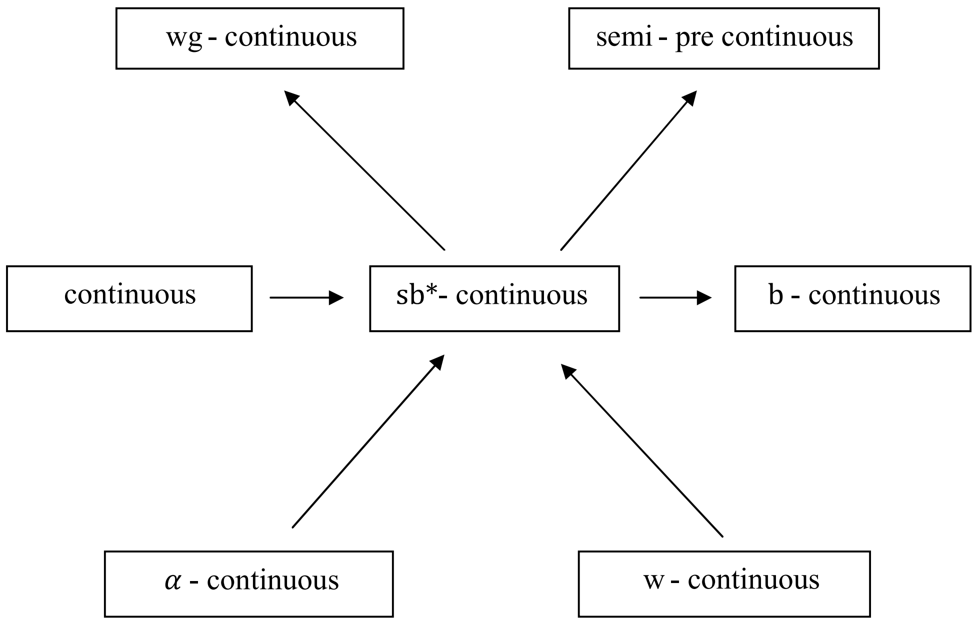
The converse of the above theorem need not be true as seen from the following example.

Example : 4.2.23

Let X = Y = {a, b, c} with topologies τ = {φ, {a}, {c}, {a, c}, X} and σ = {φ, {b, c}, Y}. Let f : X → Y be an identity map. Then f is semi - pre continuous. But f is not sb* - continuous, since for the open set U = {b, c} in Y, $f^{-1}(U)$ = {b, c} is not sb* - open in X.

Remark : 4.2.24

From the above results the diagram follows.



where A ⟶ B indicates A implies B but not conversely.

Remark : 4.2.25

The following examples show that the g - continuous functions and sb* - continuous functions are independent.

Example : 4.2.26

Let X = Y = {a, b, c} with topologies τ = {φ, {b}, X} and σ = {φ, {a}, {b, c}, Y}. Let f : X → Y be defined by f ({a}) = {b}, f ({b}) = {c}, f ({c}) = {a}. Then f is g - continuous, but f is not sb* - continuous, since for the open set U = {a} in Y, $f^{-1}(U)$ = {c} is g - open but not sb* - open in X. Therefore the given function is g - continuous but not sb* - continuous.

Example : 4.2.27

Let X = Y = {a, b, c} with topologies τ = {φ, {a}, {a, b}, X} and σ = {φ, {a, b}, Y}. Let f : X → Y be defined by f ({a}) = {a}, f ({b}) = {c}, f ({c}) = {b}. Then f is sb* - continuous. But f is not g - continuous, since for the open set U = {a, b} in Y, $f^{-1}(U)$ = {a, c} is sb* - open but not g - open in X. Therefore the given function is sb* - continuous but not g - continuous.

Remark : 4.2.28

The following examples show that the αg - continuous functions and sb* - continuous functions are independent.

Example : 4.2.29

Let X = Y = {a, b, c} with topologies τ = {φ, {b}, X} and σ = {φ, {c}, Y}. Let f : X → Y be an identity map. Then f is αg - continuous. But f is not sb* - continuous, since for the open set U = {c} in Y, $f^{-1}(U)$ = {c} is αg - open set but not sb* - open in X. Therefore the defined function is αg - continuous but not sb* - continuous.

Example : 4.2.30

Let X = Y = {a, b, c} with topologies τ = {φ, {a, c}, X} and σ = {φ, {a, b}, Y}. Let f : X → Y be an identity map. Then f is sb* - continuous. But f is not αg - continuous, since for the open set U = {a, b} in Y, $f^{-1}(U)$ = {a, b} is sb* - open but not αg - open in X. Therefore the given function is sb* - continuous but not αg - continuous.

Remark : 4.2.31

The following examples show that the sb* - continuous functions and sg - continuous functions are independent.

Example : 4.2.32

Let X = Y = {a, b, c} with topologies τ = {φ, {a}, {c}, {a, c}, X} and σ = {φ, {a, c}, Y}. Let f : X → Y be defined by f ({a}) = {b}, f ({b}) = {a}, f ({c}) = {c}. Then f is sg - continuous. But f is not sb* - continuous, since for the open set U = {a, c} in Y, $f^{-1}(U)$ = {b, c} is sg - open set but not sb* - open in X. Therefore the defined function is sg - continuous but not sb* - continuous.

Example : 4.2.33

Let X = Y = {a, b, c} with topologies τ = {φ, {a, c}, X} and σ = {φ, {a}, {a, b}, Y}. Let f : X → Y be defined by f ({a}) = {c}, f ({b}) = {b}, f ({c}) = {a}. Then f is sb* - continuous. But f is not sg - continuous, since for the open set U = {a, b} in Y, f^{-1}(U) = {b, c}is sb* - open but not sg - open in X. Therefore the given function is sb* - continuous but not sg - continuous.

Remark : 4.2.34

The following examples show that the sb* - continuous functions and semi - continuous functions are independent.

Example : 4.2.35

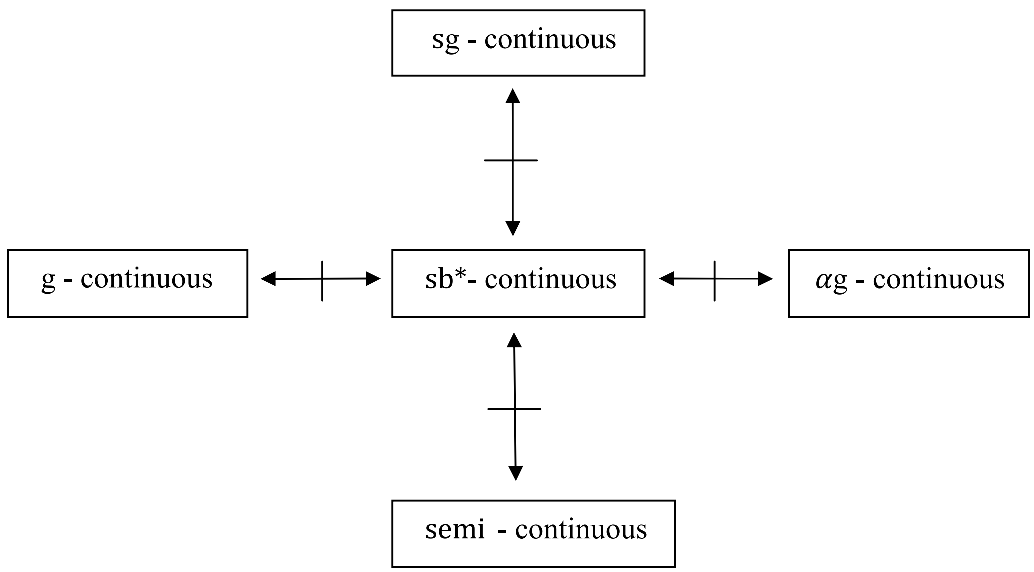
Let X = Y = {a, b, c} with topologies τ = {φ, {a, c}, X} and σ = {φ, {a}, {b}, {a, b}, Y}. Let f : X → Y be defined by f ({a}) = {a}, f ({b}) = {c}, f ({c}) = {b}. Then f is sb* - continuous. But f is not semi - continuous, since for the open set U = {a} in Y, f^{-1}(U) = {a} is sb* - open but not semi - open in X. Therefore the given function is sb* - continuous but not semi - continuous.

Example : 4.2.36

Let X = Y = {a, b, c} with topologies τ = {φ, {a}, {c}, {a, c}, X} and σ = {φ,{b, c}, Y}. Let f : X → Y be defined by f ({a}) = {a}, f ({b}) = {c}, f ({c}) = {b}. Then f is semi - continuous. But f is not sb* - continuous, since for the open set U = {b, c} in Y, f^{-1}(U) = {b, c} is semi - open set but not sb* - open in X. Therefore the defined function is semi - continuous but not sb* - continuous.

Remark : 4.2.37

The following diagram illustrates the above discussions.



where A ←|→ B represents A is independent of B.

Section - 4.3

Strongly b* - Open Maps and Strongly b* - Closed Maps

Definition : 4.3.1 [30]

Let (X, τ) and (Y, σ) be topological spaces. A map f : X → Y is called **strongly b* - closed** (briefly sb* - closed) map if the image of every closed set in X is sb* - closed set in Y.

Definition : 4.3.2 [31]

Let (X, τ) and (Y, σ) be topological spaces. A map f : X → Y is called **strongly b* - open** (briefly sb* - open) map if the image of every open set in X is sb* - open set in Y.

Theorem : 4.3.3

If a map f : X → Y from a topological space (X, τ) into a topological space (Y, σ) is closed then it is strongly b* - closed map.

Proof :

Let f : X → Y be a closed map and V be a closed set in X. Then f (V) is closed and hence sb* - closed in Y. Thus f is sb* - closed map.

Remark : 4.3.4

The converse of the above theorem need not be true as seen from the following example.

Example : 4.3.5

Let X = Y = {a, b, c} with topologies τ = {φ, {a}, X} and σ = {φ, {a}, {a, b}, Y}. Let f : X → Y be defined by f ({a}) = {a}, f ({b}) = f ({c}) = {b}. Then f is sb* - closed but f is not closed, since for the closed set U = {b, c} in X, f (U) = {b} is not closed in Y.

Theorem : 4.3.6

If a map f : X → Y is continuous and sb* - closed, and if A be a sb* - closed set of X then f (A) is sb* - closed in Y.

Proof :

Let f (A) ⊆ U, where U is b - open set of Y. Since f is continuous $f^{-1}(U)$ is b - open set containing A. Hence cl (int(A)) ⊆ $f^{-1}(U)$, as A is sb* - closed. Since f is sb* - closed, f (cl(int(A))) is a sb* - closed set contained in the b - open set U, which implies cl(int(f(A))) ⊆ U. So f (A) is sb* - open in Y.

Theorem : 4.3.7

A map f : X → Y is sb* - closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$ there is a sb* - open set V of Y such that S ⊆ V and $f^{-1}(V)$ ⊆ U.

Proof :

Necessity :

Let S be a subset of Y and U be an arbitrary open set in X containing $f^{-1}(S)$. It is enough we produce a sb* - open set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Choose $V = Y - (f(X - U))$

As U is open, X - U is closed and by the definition of sb* - closed map, $f(X - U)$ is sb* - closed in Y. Hence V is a sb* - open set and $f^{-1}(S) \subseteq U$.

Hence $X - U \subseteq X - f^{-1}(S) \subseteq f^{-1}(Y - S) \Rightarrow f(X - U) \subseteq Y - S$.

Thus $S \subseteq Y - f(X - U) = V$

$$\begin{aligned} \text{Now } V = Y - f(X - U) &\Rightarrow f(X - U) \subseteq Y - V \\ &\Rightarrow X - U \subseteq f^{-1}(Y - V) = X - f^{-1}(V) \\ &\Rightarrow f^{-1}(V) \subseteq U \end{aligned}$$

Sufficiency :

Let S be closed in X. Then X - S is open. In the given criteria, put U = X - S and S = Y - f(S). As $f^{-1}(Y - f(S)) \subseteq X - S = U$, there exists a sb* - open set V of Y such that $Y - f(S) \subseteq V$ and $f^{-1}(V) \subseteq X - S \Rightarrow S \subseteq X - f^{-1}(V)$.

Now $Y - f(S) \subseteq V \Rightarrow Y - V \subseteq f(S) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$

Therefore $f(S) = Y - V$. Since Y - V is sb* - closed, f(S) is sb* - closed and hence f is a sb* - closed map.

Theorem : 4.3.8

If a map $f : X \to Y$ is closed and a map $g : Y \to Z$ is sb* - closed then $g \circ f : X \to Z$ is sb* - closed.

Proof :

Let V be a closed set in X. Since $f : X \to Y$ is closed, f (V) is closed set in Y. Since $g : Y \to Z$ is sb* - closed, h (f (V)) is sb* - closed set in Z. Therefore $g \circ f : X \to Z$ is sb* - closed map.

Summary and Conclusion

SUMMARY AND CONCLUSION

In Chapter 1, b* - closed sets and b* - open sets in topological spaces are studied and their properties are analyzed.

Chapter 2 deals with b* - continuous functions, b* - open maps and b* - closed maps in topological spaces due to Muthuvel and Parimelazhagan [24].

In Chapter 3, Strongly b* - closed sets in topological spaces are studied. Properties and characterizations of strongly b* - closed sets are analyzed.

In Chapter 4, Strongly b* - continuous functions, Strongly b* - closed maps, properties and characterizations due to Poongothai and Parimelazhagan [31] are studied.

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