Chapter III

CHAPTER III

LINEAR OPERATORS PRESERVING FACTOR RANK OF MATRICES OVER SEMIRINGS

Let $\mathbf{M}_{m,n}(\mathbf{S})$ denote the set of all $(m \times n)$-matrices with entries from the semiring **S**.

Definition : 3.1

The matrix $A \in \mathbf{M}_{m,n}(\mathbf{S})$ is said to be of **factor rank** k (rank(A) = k) if there exist matrices $B \in \mathbf{M}_{m,k}(\mathbf{S})$ and $C \in \mathbf{M}_{k,n}(\mathbf{S})$ such that A = BC and k is the smallest positive integer such that such a factorization exists. By definition, the unique matrix with factor rank equal to 0 is the zero matrix O.

If **S** is a subsemiring of a certain field, then there is the usual rank function ρ(A) for any matrix $A \in \mathbf{M}_{m,n}(\mathbf{S})$. These functions are not equal in general. However, the equality rank (A) ≥ ρ(A) always holds.

The behaviour of the function ρ with respect to matrix multiplication addition is given by the following inequalities :

- the rank-sum inequalities :

$$|\rho(A) - \rho(B)| \leq \rho(A+B) \leq \rho(A) + \rho(B);$$

- Sylvester's laws :

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\},$$

- and the Frobenius inequality :

$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B),$$

where A, B, and C are conformal matrices with entries from a field.

The arithmetic properties of factor rank depend on the structure of the semiring of entries. It is restricted by the following list of inequalities :

If the semiring is arbitrarily antinegative, then

(1) $rank(A+B) \le rank(A) + rank(B)$;

(2) $rank(AB) \le \min\{rank(A), rank(B)\}$.

If the semiring uses Boolean arithmetics, then

$$(3)\ rank(A+B) \ge \begin{cases} rank(A) & \text{if } B = O, \\ rank(B) & \text{if } A = O, \\ 1 & \text{if } A \ne O \text{ and } B \ne O; \end{cases}$$

$$(4)\ rank(AB) \ge \begin{cases} 0 & \text{if } rank(A) + rank(B) \le n, \\ 1 & \text{if } rank(A) + rank(B) > n. \end{cases}$$

If the semiring is a subsemiring of the set R_+ of positive real numbers, we have

(5) $rank(A+B) \ge |\rho(A) - \rho(B)|$;

$$(6)\ rank(AB) \ge \begin{cases} 0 & \text{if } \rho(A) + \rho(B) \le n, \\ \rho(A) + \rho(B) - n & \text{if } \rho(A) + \rho(B) > n; \end{cases}$$

(7) $\rho(AB) + \rho(BC) \le rank(ABC) + rank(B)$,

where $\rho(A)$ is the usual rank function for any matrix A .

Notation : 3.2

In order to denote the sets of matrices that arise as extremal cases in the inequalities listed above, the following notations are used :

$\mathbf{F_1(S)}=\{(X,Y) \in \mathbf{M}_{m,n}(S)^2 \mid rank(X+Y)=rank(X)+rank(Y)\}$;

$\mathbf{F_{2B}(S)}=\{(X,Y) \in \mathbf{M}_{m,n}(S)^2 \mid rank(X+Y)=1\}$;

$\mathbf{F_{2R}(S)}=\{(X,Y) \in \mathbf{M}_{m,n}(S)^2 \mid rank(X+Y)= \mid \rho(X)-\rho(Y) \mid \}$;

$\mathbf{F_3(S)}=\{(X,Y) \in \mathbf{M}_n(S)^2 \mid rank(XY)=\min\{rank(X)+rank(Y)\}\}$;

$\mathbf{F_{4N}(S)}=\{(X,Y) \in \mathbf{M}_n(S)^2 \mid rank(XY)=0\}$;

$\mathbf{F_{4B}(S)}=\{(X,Y) \in \mathbf{M}_n(S)^2 \mid rank(XY)=1\}$;

$\mathbf{F}_{4R}(\mathbf{S})=\{(X,Y)\in \mathbf{M}_n(\mathbf{S})^2 \mid \text{rank}(XY)=\rho(X)+\rho(Y)-n\}$;

$\mathbf{F}_5(\mathbf{S})=\{(X,Y,Z)\in \mathbf{M}_n(\mathbf{S})^3 \mid \text{rank}(XYZ)+\text{rank}(Y)=\rho(XY)+\rho(YZ)\}$.

Definition : 3.3

We say that an operator T **preserves** a set **P** if X∈**P** or if **P** is any set of ordered pairs (triples) such that (X,Y)∈**P** (respectively, (X,Y,Z)∈**P**) implies (T(X),T(Y))∈**P** (respectively, (T(X),T(Y),T(Z))∈**P**).

Definition : 3.4

An operator T **strongly preserves** the set **P** if X∈**P** if and only if T(X)∈**P** or if **P** is the set of ordered pairs (triples) such that (X,Y)∈**P** (respectively, (X,Y,Z)∈**P**) if and only if (T(X),T(Y))∈**P** (respectively, (T(X),T(Y),T(Z))∈**P**).

Definition : 3.5

An operator T : $\mathbf{M}_{m,n}(\mathbf{S})\rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ is called a **(U,V)-operator** if there exist invertible matrices U and V of appropriate orders such that T(X) = UXV for all X∈ $\mathbf{M}_{m,n}(\mathbf{S})$ or if for m=n, T(X) = UX^tV for all X∈ $\mathbf{M}_{m,n}(\mathbf{S})$ or if for m=n, T(X) = UX^tV for all X∈ $\mathbf{M}_{m,n}(\mathbf{F})$, where X^t denotes the transpose of X.

Lemma : 3.6

Let **S** be a semiring and B = $(b_{i,j})\in \mathbf{M}_{m,n}(\mathbf{S})$, where m,n≥2. Assume that $b_{i,j}$ is a unit for any $1\le i\le n$ and $1\le j\le m$. Let (k,l) be any fixed pair of integers such that $2\le k\le n$ and $2\le l\le m$. Assume that the factor rank of each $(l\times k)$-submatrix of B is 1. Then the factor rank of each $((l+1)\times k)$-submatrix (if any) is 1 and the factor rank of each $(l\times (k+1))$-submatrix (if any) is 1.

Corollary : 3.7

Let **S** be a semiring and $B = (b_{i,j}) \in \mathbf{M}_{m,n}(\mathbf{S})$, where $m,n \geq 2$. Assume that $b_{i,j}$ is a unit for any $1 \leq i \leq n$ and $1 \leq j \leq m$. Let $rank(B') = 1$ for any (2×2)-submatrix B' of B. Then $rank(B) = 1$.

Corollary : 3.8

Let **S** be a semiring and $B = (b_{i,j}) \in \mathbf{M}_{m,n}(\mathbf{S})$, where $m,n \geq 2$. Assume that $b_{i,j}$ is a unit for any $1 \leq i \leq n$ and $1 \leq j \leq m$. Let $rank(B') > 1$. Then there exists a (2×2)-submatrix of B of factor rank 2.

The following theorem will be crucial for our considerations.

Theorem : 3.9

Let **S** be an antinegative semiring without zero divisors and $T : \mathbf{M}_{m,n}(\mathbf{S}) \rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ be a linear operator. Then the following conditions are equivalent :

(1) T is bijective ;

(2) T is surjective ;

(3) there exists a permutation σ on $\{(i,j) \mid i=1,2,\ldots\ldots,m,\ j=1,2,\ldots\ldots,n\}$ and units $b_{i,j} \in \mathbf{Z}(\mathbf{S})$, $i=1,2,\ldots\ldots,m$, $j=1,2,\ldots\ldots,n$ such that

$$T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$$

for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Proof :

The implications (1) ⇒ (2) and (3) ⇒ (1) are verified by straightforward calculations. The fact that $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ follows immediately from the linearity of T. We prove the implication (2) ⇒ (3).

We assume that T is surjective. Then for any pair (i,j), there exists some X such that $T(X) = E_{i,j}$. Clearly, $X \neq O$ by the linearity of T. Thus there is a pair of subscripts (r,s) such that $X = x_{r,s}E_{r,s} + X'$, where the (r,s)-entry of X' is zero and the following two conditions hold : $x_{r,s} \neq 0$ and $T(E_{r,s}) \neq O$. Indeed, if, on the contrary, for all pairs (r,s) either $x_{r,s} = 0$ or $T(E_{r,s}) = O$, then T(X) = $E_{i,j}$. Since **S** is antinegative without zero divisors, if follows that

$$T(x_{r,s}E_{r,s}) \leq T(x_{r,s}E_{r,s}) + T(X \setminus (x_{r,s}E_{r,s})) = T(X) = E_{i,j}.$$

Hence, $x_{r,s}T(E_{r,s}) = T(x_{r,s}E_{r,s}) \leq E_{i,j}$ and $T(E_{r,s}) \neq O$ by the above. Therefore, $T(E_{r,s}) \leq E_{i,j}$. Indeed, if, on the contrary, $T(E_{r,s})$ is the sum of certain multiples of cells, then $x_{r,s}T(E_{r,s})$ is, since **S** is antinegative and without zero divisors.

Let $P_{i,j} = \{E_{r,s} | T(E_{r,s}) \leq E_{i,j}\}$. By the above, $P_{i,j} \neq \phi$ for all (i,j). By its definition, $P_{i,j} \cap P_{u,v} = \phi$ whenever $(i,j) \neq (u,v)$, i.e., $\{P_{i,j}\}$ is the set of mn nonempty sets which partition the set of cells. By the pigeonhole principle, we have that $|P_{i,j}| = 1$ for all (i,j). Necessarily, for each pair (r,s) there is a unique pair (i,j) such that $T(E_{r,s}) = b_{r,s}E_{i,j}$, i.e., there is some permutation σ on {(i,j)| i=1,2,……,m; j=1,2,……,n} such that for some scalars $b_{i,j}$, we have $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$. Now we only need to show that the $b_{i,j}$ are all units. Since T is surjective and $T(E_{r,s}) \leq E_{\sigma(i,j)}$ for $(r,s) \neq (i,j)$, there is some α such that $T(\alpha E_{i,j}) = E_{\sigma(i,j)}$. But then, since T is linear, $T(\alpha E_{i,j}) = \alpha T(E_{i,j}) = \alpha b_{i,j}E_{\sigma(i,j)} = E_{\sigma(i,j)}$, i.e., $\alpha b_{i,j} = 1$. Similarly, $b_{i,j}\alpha = 1$. Hence $b_{i,j}$ is a unit.

Remark : 3.10

One can easily verify that if m=1 or n=1, then all operators under consideration are (P,Q,B)-operators and if m=n=1, then all operators under consideration are (P,P^t,B)-operators.

Note : 3.11

Henceforth we will always assume that $m,n \geq 2$.

Theorem : 3.12

Let **S** be an antinegative semiring, T : $\mathbf{M}_{m,n}(\mathbf{S}) \rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ be an operator which maps lines to lines and is defined by the rule $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, where σ is a permutation on the set {(i,j)| i=1,2,……,m; j=1,2,……,n}, and $b_{i,j} \in$ **S**, i=1,2,……,m; j=1,2,……,n, are certain nonzero elements. Then T is a (P,Q,B)-operator.

Proof :

Since no combination of a rows and b columns can dominate J, where a+b = m unless b = 0 (or m=n if a=0), we have that either the image of each row is a row and the image of each column is a column, or m=n and the image of each row is a column and the image of each column is a row. Thus, there are permutation matrices P and Q such that

$$T(R_i) \leq PR_iQ \ , \quad T(C_j) \leq PC_jQ$$

or, if m=n,

$$T(R_i) \leq P(R_i)^t Q \ , \quad T(C_j) \leq P(C_j)^t Q .$$

Since each cell lies in the intersection of a row and a column and T maps nonzero cells to nonzero (weighted) cells, it follows that

$$T(E_{i,j}) = Pb_{i,j}E_{i,j}Q = P(E_{i,j} \circ B)^t Q ,$$

where B = $(b_{i,j})$ is defined by the action of T on the cells.

Theorem : 3.13

Let **S** be a commutative semiring. If T(X) = X∘B for all $X \in \mathbf{M}_{m,n}(\mathbf{S})$ and rank (B) = 1 then there exist diagonal matrices D and E such that T(X) = DXE for all $X \in \mathbf{M}_{m,n}(\mathbf{S})$.

Proof :

If rank (B) = 1, then there exist vectors d = $[d_1,d_2,\ldots\ldots,m]$ and e = $[e_1,e_2,\ldots\ldots,e_n]$ such that B = de^t or $b_{i,j} = d_ie_j$. Let D = diag $\{d_1,d_2,\ldots\ldots,d_m\}$ and

$E = \text{diag}\{e_1, e_2, \ldots\ldots, e_n\}$. Now the (i,j)-entry of T(X) is $b_{i,j}x_{i,j}$ and the (i,j)-entry of DXE is $d_i x_{i,j} e_j = b_{i,j}x_{i,j}$. The theorem is proved.

Theorem : 3.14

Let **S** be a semiring. Consider the linear operator $T : \mathbf{M}_{m,n}(\mathbf{S}) \rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ defined by the formula $T(X) = X \circ B$ for all $X \in \mathbf{M}_{m,n}(\mathbf{S})$, where rank (B) = 1 and all entries $b_{i,j} \in \mathbf{Z}(S)$ are units. Then T preserves the factor rank.

Proof :

Consider a matrix $X \in \mathbf{M}_{m,n}(\mathbf{S})$, rank (X) = k. By definition, there exist $Y \in \mathbf{M}_{m,k}(\mathbf{S})$ and $Z \in \mathbf{M}_{k,n}(\mathbf{S})$ such that X=YZ, i.e., $x_{i,j} = \sum_{l=1}^{k} y_{i,l} z_{l,i}$ for all i,j. Since rank (B) = 1, it follows that there exist vectors $d = [d_1, d_2, \ldots\ldots, d_m]$ and $e = [e_1, e_2, \ldots\ldots, e_n]$ such that $B = de^t$ or $b_{i,j} = d_i e_j$. Consider the matrices $D = (d_{i,j}) \in \mathbf{M}_{m,k}(\mathbf{S})$ and $E = (e_{i,j}) \in \mathbf{M}_{k,n}(\mathbf{S})$ such that $d_{i,j} = d_i$ for all i=1,2,……,m and j=1,2,……,k and $e_{i,j} = e_j$ for all i=1,2,……,k and j=1,2,……,n. Since $b_{i,j} \in \mathbf{Z}(\mathbf{S})$, it is easy

$$T(X) = (Y \circ D)(E \circ Z).$$

Thus, rank (T(X)) $\leq$ rank (X). Since $b_{i,j}$ are units, T is bijective. Applying similar considerations with T^{-1}, we obtain the required result.

Linear operators of special types that preserve $\mathbf{F}_1$

Lemma : 3.15

Let **S** be an antinegative semiring, σ be a permutation of the set $\{(i,j) | 1 \leq i \leq m, 1 \leq j \leq n\}$, and $T : \mathbf{M}_{m,n}(\mathbf{S}) \rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ be defined by the formula $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$ for some scalars $b_{i,j}$, i=1,2,……,m; j=1,2,……,n, which are not zero divisors. If T preserves $\mathbf{F}_1$, then T is a (P,Q,B)-operator.

Proof :

We examine the action of T on rows and columns of a matrix. Suppose that the image of two cells are in the same line, but the cells are not, say E and F are cells such that rank (E+F) = 2 and rank (T(E+F)) = 1. Then (E,F)$\in$ $\mathbf{F}_1$ but (T(E),T(F))$\in$ $\mathbf{F}_1$, a contradiction. Thus, T maps lines to lines.

By Lemma 3.12, we obtain the result.

Lemma : 3.16

If **S** is an antinegative semiring and for some B = ($b_{i,j}$), where $b_{i,j}$, i=1,2,……,m; j=1,2,……,n, are invertible, T(X) = X$\circ$B preserves $\mathbf{F}_1$, then rank (B) = 1. If **S** is commutative then T(X) = DXE for diagonal matrices D and E of appropriate sizes.

Proof :

If rank (B)$\geq$2, then by Corollary 3.8, there is a (2×2)-submatrix B[i,j][k,l] such that

$$\text{rank }(B[i,j][k,l]) = 2.$$

Let

$$J' = E_{i,k} + E_{j,k} + E_{i,l} + E_{j,l}\ .$$

Thus

$$T(J') = b_{i,k}E_{i,k} + b_{j,k}E_{j,k} + b_{i,l}E_{i,l} + b_{j,l}E_{j,l} = B'\ .$$

Then for q$\neq$k,l, we have

$$\text{rank }(E_{i,q}+J') = 2 = \text{rank }(E_{i,q}) + \text{rank }(J'),$$

so that ($E_{i,q}$, J')$\in$$\mathbf{F}_1$ while

$$\text{rank }(T(E_{i,q}+J')) = \text{rank }(b_{i,q}E_{i,q}+B') = 2 \neq \text{rank }(b_{i,q}E_{i,q}) + \text{rank }(B') = 1+2 = 3,$$

a contradiction. Thus rank (B) = 1. If **S** is commutative, then by Lemma 3.13, there exist diagonal matrices D and E such that T(X) = DXE.

Theorem : 3.17

Let **S** be an antinegative semiring without zero divisors and $T : \mathbf{M}_{m,n}(\mathbf{S}) \to \mathbf{M}_{m,n}(\mathbf{S})$ be a surjective linear operator. If T preserves $\mathbf{F}_1$, then T is a (P,Q,B)-operator, where $B = (b_{i,j}) \in \mathbf{M}_{m,n}(\mathbf{Z}(\mathbf{S}))$, the entries $b_{i,j}$ are units for all i=1,2,……,m and j=1,2,……,n and rank (B) = 1.

Proof :

If T is surjective and **S** is an antinegative semiring without zero divisors, then by Theorem 3.9, we have that T is defined by a permutation σ on the set $\{(i,j) | 1 \le i \le m,\ 1 \le j \le n\}$, i.e., $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some scalars $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ which are units.

By Lemma 3.15, we have that T is a (P,Q,B)-operator. By Lemma 3.16, we have that rank (B) = 1.

Theorem : 3.18

Let **S** be a commutative antinegative semiring without zero divisors and $T : \mathbf{M}_{m,n}(\mathbf{S}) \to \mathbf{M}_{m,n}(\mathbf{S})$ be a surjective linear operator. The operator T preserves $\mathbf{F}_1$ if and only if T is a (U,V)-operator, where U and V are invertible monomial matrices.

Proof :

It is easy to verify that multiplication with invertible matrices does not change the factor rank. Also, over a commutative semiring, the transposition transformation does not change the factor rank. Thus, they preserves $\mathbf{F}_1$.

By the subsequent application of Theorem 3.17 and Lemma 3.16, one obtains that T is of the form T(X) = PDXEQ for all $X \in \mathbf{M}_{m,n}(\mathbf{S})$ or m=n and $T(X) = PDX^tEQ$ for all $X \in \mathbf{M}_{m,n}(\mathbf{S})$, where D and E are diagonal matrices of appropriate sizes. Since T is surjective, one has that the diagonal entries of D and E are units and the corollary is proved.

Over chain or finite semirings, the assumption of surjectivity from the previous theorem can be replaced with the assumption that T is a strong preserver.

Theorem : 3.19

Let **S** be a finite antinegative or a chain semiring and $T : \mathbf{M}_{m,n}(\mathbf{S}) \to \mathbf{M}_{m,n}(\mathbf{S})$ be a linear operator that strongly preserves $\mathbf{F}_1$. Then T is a (P,Q,B)-operator, where $B \in \mathbf{M}_{m,n}(\mathbf{Z}(\mathbf{S}))$.

Linear operators of special types that preserve $\mathbf{F}_{2B}$

Theorem : 3.20

Let **S** be an antinegative semiring without zero divisors and $T : \mathbf{M}_{m,n}(\mathbf{S}) \to \mathbf{M}_{m,n}(\mathbf{S})$ be a surjective linear operator. If T preserves $\mathbf{F}_{2B}$, then T is a (P,Q,B)-operator, where $B = (b_{i,j}) \in \mathbf{M}_{m,n}(\mathbf{Z}(\mathbf{S}))$, the entries $b_{i,j}$ are units for all $i=1,2,\ldots\ldots,m$ and $j=1,2,\ldots\ldots,n$, and rank (B) = 1.

Proof :

If T is surjective ans **S** is an antinegative semiring without zero divisors, then by Theorem 3.9, we have that $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some scalars $(b_{i,j}) \in \mathbf{Z}(\mathbf{S})$ which are units. It is easy to see that the weighted cells $\alpha E_{i,j}$ and $\beta E_{r,s}$ are in the same line if and only if rank $(\alpha E_{i,j} + \beta E_{r,s}) = 1$ if and only if $(\alpha E_{i,j} , \beta E_{r,s}) \in \mathbf{F}_{2B}$. Thus, lines are mapped to lines, and we have that T is a (P,Q,B)-operator by Theorem 3.12.

Assume that rank (B)>1. Then by Corollary 3.8, there is a (2×2)-submatrix B[i,j][k,l] such that rank (B[i,j][k,l]) = 2. Let

$$A = E_{j,k} + E_{i,l} + E_{j,l}\,, \quad B = E_{i,k}\,.$$

Thus,

$$\text{rank } (A+B) = \text{rank } (E_{j,k} + E_{i,l} + E_{j,l} + E_{i,k}) = 1,$$

i.e., $(A,B) \in \mathbf{F}_{2B}$, while,

$$\text{rank } (T(A)+T(B)) = \text{rank } (b_{i,k}E_{i,k} + b_{j,k}E_{j,k} + b_{i,l}E_{i,l} + b_{j,l}E_{j,l}) = 2,$$

i.e., $(T(A),T(B)) \notin \mathbf{F}_{2B}$. This contradiction concludes the proof that rank (B) = 1.

One can easily see that for a commutative semiring, the transposition transformation preserves $\mathbf{F}_{2B}$. Hence, all (P,Q,B)-operators, where all elements of B are inverible, preserve the set $\mathbf{F}_{2B}$. This allows one to improve Theorem 3.20 as follows.

Theorem : 3.21

Let **S** be a commutative, antinegative semiring without zero divisors and T : $\mathbf{M}_{m,n}(\mathbf{S}) \rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ be a surjective linear operator. Then T preserves the set $\mathbf{F}_{2B}$ if and only if T is a (P,Q,B)-operator, where M = $(b_{i,j}) \in \mathbf{M}_{m,n}(\mathbf{Z}(\mathbf{S}))$, the entries $b_{i,j}$ are units for all i=1,2,……,m and j=1,2,……,n, and rank (B) = 1.

One chain semirings, this theorem can be generalized as follows.

Theorem : 3.22

Let **S** be a chain semiring and T : $\mathbf{M}_{m,n}(\mathbf{S}) \rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ be a linear operator preserving $\mathbf{F}_{2B}$. Then T is surjective if and only if T strongly preserves $\mathbf{F}_{2B}$ if and only if T is a (U,V)-operator.

Linear operators of special types that preserve $\mathbf{F}_{2R}$

Lemma : 3.23

Let E_1, E_2, E_3, and E_4 be distinct (weighted) cells. Assume that rank (E_1+E_2) = 2 and rank $(E_1+E_2+E_3+E_4)$ = 1. Then the nonzero entries of $E_1+E_2+E_3+E_4$ lie in the intersection of two rows and two columns (i.e., the nonzero entries lie in a (2×2)-submatrix).

Theorem : 3.24

Let $\mathbf{S} \subseteq \mathbb{R}_+$ be a semiring and T : $\mathbf{M}_{m,n}(\mathbf{S}) \rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ be a surjective linear operator. Then T preserves $\mathbf{F}_{2R}$ if and only if T(X) = PDXEQ for all $X \in \mathbf{M}_{m,n}(\mathbf{S})$, or m=n and $T(X) = PDX^tEQ$ for all $X \in \mathbf{M}_{m,n}(\mathbf{S})$, where D and E are diagonal matrices P and Q are permutational matrices of appropriate sizes.

Proof :

One can easily see that operators of the type described preserve the set $\mathbf{F}_{2R}$. By Theorem 3.9, we have that $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some permutation σ of set $\{(i,j)|1 \le i \le m,\ 1 \le j \le n\}$, and $b_{i,j}$ is a unit for each (i,j). Let us verify that T transforms lines to lines.

If m=n=2, by multiplying with permutational matrices on the left and on the right, one can assume that $T(E_{1,1}) = b_{1,1}E_{1,1}$. Thus, if T does not transform lines to lines, then, without loss of generality, we may assume that $T(E_{1,2}) = b_{1,2}E_{2,2}$ (the other case with $T(E_{2,1}) = b_{2,1}E_{2,2}$ can be considered similarly). Without loss of generality, one can assume that $T(E_{2,1}) = b_{2,1}E_{2,1}$ and $T(E_{2,2}) = b_{2,2}E_{1,2}$ (the case $T(E_{2,1}) = b_{2,1}E_{1,2}$ and $T(E_{2,2}) = b_{2,2}E_{2,1}$ can be considered similarly). Since $1 \in \mathbf{S}$ by definition and $\mathbf{S} \subseteq \mathbb{R}_+$, we have that the elements $2,3 \in \mathbf{S}$. Consider the pair of matrices $(A,B) \in \mathbf{F}_{2R}$, where

$$A = E_{1,1} + E_{1,2} + 2E_{2,1} + E_{2,2}\ ,\ B = E_{2,2}\ .$$

Since $\rho(T(B)) = 1$, $T(A+B) \neq 0$, and T preserves the set $\mathbf{F}_{2R}$, it follows that rank (T(A+B)) = 1 and $\rho(T(A)) = 2$. Therefore, $\rho(T(A+B)) = 1$. Applying, now, T to the pair $(T(A),T(B)) \in \mathbf{F}_{2R}$, one obtains in a similar way that

$$\rho\left(T^2(A+B)\right) = \rho\left(b_{1,1}^2 E_{1,1} + b_{1,2}b_{2,2}E_{1,2} + 2b_{2,1}^2 E_{2,1} + 2b_{1,2}b_{2,2}E_{2,2}\right) = 1.$$

Thus, $b_{1,1} = b_{2,1}$ and since $\rho(T(A+B)) = 1$, it follows that $b_{1,2} = 4b_{2,2}$. Consider now the pair of matrices $(C,D) \in \mathbf{F}_{2R}$, where

$$C = E_{1,1} + E_{1,2} + 3\,E_{2,1} + E_{2,2}\ ,\ D = 2\,E_{2,2}\ .$$

One can easily verify that $\rho(T(C+D)) = 2$, which contradicts the assumption $(T(C),T(D)) \in \mathbf{F}_{2R}$.

Now we assume that $m + n \ge 5$. Assume that there is some row, say R_i , such that $T(R_i)$ is not dominated by some row or column. Then there are two cells in R_i whose images are not in any line, that is, for some k and l, rank $(T(E_{i,k}+E_{i,l})) = 2$, i.e.,

$$T(E_{i,k}+E_{i,l}) = b_{i,k}E_{r,s} + b_{i,l}E_{p,q}$$

for some $p \neq r$ and $q \neq s$. Now given any $j \neq i$, $(E_{i,k} + E_{i,l} + E_{j,k} + E_{j,l}) \in \mathbf{F}_{2R}$. By Lemma 3.23, we have

$$T(E_{i,k} + E_{i,l} + E_{j,k}) + T(E_{j,l}) = b_{i,k}E_{r,s} + b_{i,l}E_{p,q} + \alpha E_{r,q} + \beta E_{p,s},$$

where $\alpha = b_{j,k}$ or $b_{i,l}$ and $\beta = b_{j,l}$ or $b_{j,k}$, respectively. Since σ is a permutation, we have that $m \leq 2$. Similarly, $n \leq 2$. This contradicts the assumption that $m+n \geq 5$; thus, the image of a row is dominated by a row or a column. By Theorem 3.12, it follows that T is a (P,Q,B)-operator where $b_{i,j}$ is a unit for all (i,j).

If rank $(B) \neq 1$, then by Corollary 3.8, there is a (2×2)-submatrix of B whose rank is two. That is, there are cells $E_{i,k}$, $E_{i,l}$, $E_{j,k}$, and $E_{j,l}$ such that rank $(T(E_{i,k} + E_{i,l} + E_{j,k} + E_{j,l})) = 2$. But $(E_{i,k} + E_{i,l} + E_{j,k} + E_{j,l}) \in \mathbf{F}_{2R}$, while $(T(E_{i,k} + E_{i,l} + E_{j,k}), T(E_{j,l})) \notin \mathbf{F}_{2R}$, a contradiction. Thus, rank (B) = 1. By Theorem 3.13, the theorem is proved.

Linear operators of special types that preserve $\mathbf{F}_3$

Theorem :3.25

Let **S** be an antinegative semiring without zero divisors and $T : \mathbf{M}_n(\mathbf{S}) \to \mathbf{M}_n(\mathbf{S})$ be a surjective linear operator which preserves $\mathbf{F}_3$. Then there exists a permutation matrix P such that $T(X) = P(X \circ B)P^t$ for all $X \in \mathbf{M}_n(\mathbf{S})$, where the entries of $B \in X \in \mathbf{M}_n(\mathbf{Z}(\mathbf{S}))$ are all units and rank (B) = 1.

Proof :

By Theorem 3.9, we have that $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some scalars $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ which are units and a permutation σ on $\{(i,j) \mid 1 \leq i, j \leq n\}$. Consider $(E_{i,j}, E_{j,k}) \in \mathbf{F}_3$ for all k. Thus,

$$\text{rank}\,(T(E_{i,j})T(E_{j,k})) = \min\{\text{rank}\,(T(E_{i,j})), \text{rank}\,(T(E_{j,k}))\} = 1,$$

but, $T(E_{i,j})T(E_{j,k}) = b_{i,j}b_{j,k}E_{\sigma(i,j)}E_{\sigma(j,k)}$.

It follows that $E_{\sigma(j,k)}$ is in the same row as $E_{\sigma(j,1)}$ for any k = 1,2,……,n. That is, T maps rows to rows; similarly T maps columns to columns. By Theorem 3.12, it follows that $T(X) = P(X \circ B)Q$ for some permutation matrices P and Q.

Let us show that $Q = P^t$. Indeed, $T(E_{i,j}) = b_{i,j}E_{\sigma(i)\tau(j)}$, where σ is the permutation corresponding to Q^t. But $(E_{1,i}, E_{i,1}) \in \mathbf{F}_3$ and hence $\sigma \equiv \tau$. Therefore, $Q = P^t$.

It remains to show that rank (B) = 1. Assume that rank (B)≥2. Then by Corollary 3.8, there exist subscripts i, j, k, and l such that rank (B[i,j | k,l]) = 2.

Let

$$A = E_{1,i} + E_{2,q} \quad , \quad J' = E_{i,k} + E_{i,l} + E_{j,k} + E_{j,l} \, ,$$

where $q \neq i,j$. Then rank $(A, J') = 1 = \min\{ \text{rank } (A), \text{rank } (J')\}$ and therefore, $(A, J') \in \mathbf{F}_3$. However, it follows from the relation $J' \circ B = B[\, i,j \mid k,l] = 2$ and the fact that multiplication with a permutation matrix preserves the factor rank that

$$\text{rank } (T(J')) = \text{rank } (B[\, i,j \mid k,l]) = 2.$$

Also, rank (T(A)) = 2 since T transforms lines to lines.

However,

$$\text{rank } (T(A)T(J')) = \text{rank } (P(A \circ B)P^tP(J' \circ B)^t)$$
$$= \text{rank } (P(b_{1,i}\, b_{i,k}\, E_{1,k} + b_{1,i}\, b_{i,l}\, E_{1,l}\,)P^t) = 1.$$

Then $(T(A),T(J')) \notin \mathbf{F}_3$, a contradiction. Thus, rank (B) = 1.

Lemma : 3.26

Let **S** be an antinegative semiring satisfying the condition $1+1 \neq 1$ in **S**. Assume that T : $\mathbf{M}_n(\mathbf{S}) \to \mathbf{M}_n(\mathbf{S})$, n>4, is defined by T(X) = DXE for all $X \in \mathbf{M}_n(\mathbf{S})$, where $D,E \in \mathbf{M}_n(\mathbf{S})$ are invertible diagonal matrices. Then T preserves $\mathbf{F}_3$ if and only if $E = \alpha D^{-1}$ for some invertible $\alpha \in \mathbf{S}$.

Theorem : 3.27

Let **S** be a commutative antinegative semiring without zero divisors satisfying the condition $1+1\neq 1$. Assume that T : $\mathbf{M}_n(\mathbf{S})\rightarrow \mathbf{M}_n(\mathbf{S})$, n>4, is a surjective linear operator. Then T preserves $\mathbf{F}_3$ if and only if $T(X) = \alpha PDXD^{-1}P^t$ for all $X\in\mathbf{M}_n(\mathbf{S})$, where $D\in\mathbf{M}_n(\mathbf{S})$ is an invertible diagonal matrix, α is invertible in **S**, and $P\in\mathbf{M}_n(\mathbf{S})$ is a permutation matrix.

Proof :

It is easy to see that operators of the form $T(X) = \alpha PDXD^{-1}P^t$ preserve $\mathbf{F}_3$. Assume that T preserves $\mathbf{F}_3$. Then by applying Theorem 3.25 and Theorem 3.13, we have that $T(X) = PDXEP^t$ for invertible diagonal matrices D and E. By Lemma 3.26, we obtain the result.

Over chain semirings, the above results have the following improvement.

Theorem : 3.28

Let **S** be a chain semiring and T : $\mathbf{M}_n(\mathbf{S})\rightarrow \mathbf{M}_n(\mathbf{S})$ be a linear operator. The operator T strongly preserves $\mathbf{F}_3$ if and only if there exists a permutation matrix P such that $T(X) = P(X\circ B)P^t$ for all $X\in\mathbf{M}_n(\mathbf{S})$ and $b_{i,j}\in\mathbf{Z}(\mathbf{S})$ is nonzero and not a zero divisor for i and j.

Linear operators of special types that preserve $\mathbf{F}_{4N}$

Theorem : 3.29

Let **S** be an antinegative semiring without zero divisors and T : $\mathbf{M}_n(\mathbf{S})\rightarrow \mathbf{M}_n(\mathbf{S})$ be a nonsingular ($T(X) = O \Rightarrow X = O$) additive operator. Assume that T(J) has a nonzero entry in each row and column. Then T preserves $\mathbf{F}_{4N}$ if and only if $T(X) = P(X\circ B)P^t$ for all $X\in\mathbf{M}_n(\mathbf{S})$ and all entries of B are nonzero and not zero divisors.

Proof :

One can easily see that such transformations preserve the set $\mathbf{F}_{4N}$.

Since T(J) has a nonzero entry in each row and column, there are n different cells whose images have nonzero entries in every column. Assume that these cells can be chosen such that their nonzero entries are in fewer than n columns, say $X = E_1 + E_2 + \ldots\ldots + E_n$ is the sum of n such cells and X has no nonzero entry in column k. Then $(X,R_k)\in\mathbf{F}_{4N}$ and hence $(T(X),T(R_k))\in\mathbf{F}_{4N}$. But $T(X)T(R_k)\neq O$, a contradiction.

Thus, T must map columns to columns, and, further, T induces a permutation on the set of columns. Similarly, T induces a permutation on the set rows, i.e., $T(X) = P(X\circ B)Q$ for all $X\in\mathbf{M}_n(S)$ for some permutation matrices P and Q. Let us show that $Q = P^t$. Indeed, we have that $T(E_{i,j}) = b_{i,j}E_{\pi(i),\tau(j)}$. If $Q\neq P^t$, then $\pi\neq\tau$. Thus, for some i, we have $\pi(i)\neq\tau(j)$ and hence for some $j\neq i$, we have $\pi(i)=\tau(j)$. Here, $(E_{i,i}\ , E_{j,i})\in\mathbf{F}_{4N}$ but

$$\begin{aligned}T(E_{i,i})T(E_{j,i}) &= b_{i,i}\, b_{j,i}\, E_{\pi(i),\tau(i)}\, E_{\pi(j),\tau(i)}\\ &= b_{i,i}\, b_{j,i}\, E_{\pi(i),\tau(i)} \neq O,\end{aligned}$$

a contradiction. Thus, $\pi = \tau$ and $T(X) = P(X\circ B)P^t$ for all $X\in\mathbf{M}_n(S)$. Since T is nonsingular, all entries of B are nonzero and not zero divisors.

Corollary : 3.30

Let **S** be an antinegative semiring without zero divisors and $T : \mathbf{M}_n(\mathbf{S})\rightarrow \mathbf{M}_n(\mathbf{S})$ be a surjective linear operator. Then T preserves $\mathbf{F}_{4N}$ if and only if there exists a permutation matrix P such that $T(X) = P(X\circ B)P^t$ for all $X\in\mathbf{M}_n(\mathbf{S})$ and $b_{i,j}\in\mathbf{Z}(\mathbf{S})$ is a unit for all i and j.

Proof :

The result follows by the subsequent application of Theorems 3.9 and 3.29.

Corollary : 3.31

Let $\mathbf{S}$ be an antinegative semiring and $T : \mathbf{M}_n(\mathbf{S}) \to \mathbf{M}_n(\mathbf{S})$ be a linear operator. Then T strongly preserves $\mathbf{F}_{4N}$ if and only if there exists a permutation matrix P such that $T(X) = P(X \circ B)P^t$ for all $X \in \mathbf{M}_n(\mathbf{S})$ and $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ is nonzero and not a zero divisor for all i and j.

Linear operators of special types that preserve $\mathbf{F}_{4B}$

Lemma : 3.32

Let $\mathbf{S}$ be a chain semiring and $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some permutation σ of $\{(i,j) \mid 1 \leq i, j \leq n\}$ and nonzero scalars $b_{i,j} \in \mathbf{S}$. Then T preserves $\mathbf{F}_{4B}$ if and only if there exists a permutation matrix $P \in \mathbf{M}_n(\mathbf{S})$ such that $T(X) = PXP^t$ for all $X \in \mathbf{M}_n(\mathbf{S})$.

Theorem : 3.33

Let $\mathbf{S}$ be a chain semiring and $T : \mathbf{M}_n(\mathbf{S}) \to \mathbf{M}_n(\mathbf{S})$ be a surjective linear operator. Then T strongly preserves $\mathbf{F}_{4B}$ if and only if there exists a permutation matrix $P \in \mathbf{M}_n(\mathbf{S})$ such that $T(X) = PXP^t$ for all $X \in \mathbf{M}_n(\mathbf{S})$.

Proof :

By Theorem 3.9, we have that for all i and j such that $1 \leq i, j \leq n$, the relation $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ holds for some scalars $b_{i,j}$ which are units. By Lemma 3.32, we obtain the result.

Theorem : 3.34

Let $\mathbf{S}$ be a chain semiring and $T : \mathbf{M}_n(\mathbf{S}) \to \mathbf{M}_n(\mathbf{S})$ be a linear operator. Then T strongly preserves $\mathbf{F}_{4B}$ if and only if there exists a permutation matrix $P \in \mathbf{M}_n(\mathbf{S})$ such that $T(X) = PXP^t$ for all $X \in \mathbf{M}_n(\mathbf{S})$.

Proof :

Since operators of the form $T(X) = PXP^t$ preserve $\mathbf{F}_{4B}$, we assume that T strongly preserves $\mathbf{F}_{4B}$ and show that T is of the form $T(X) = PXP^t$. Let M be a (0,1)-matrix such that

$$|T(M)| = |T(J)| \text{ and if } |T(N)|\ |T(J)|,$$

then $|M| \leq |N|$ ------------------------------(1)

(i.e., M is a minimal matrix with respect to (1)).

Let α be the smallest nonzero entry in T(M), so that $T(\alpha M) = T(\alpha J)$.

Let us assume that there exists an index j such that the j^{th} column of M is zero. Then $\alpha M \alpha E_{j,k} = O$ so $(\alpha M, \alpha E_{j,k}) \notin \mathbf{F}_{4B}$, and hence

$$(T(\alpha M), T(\alpha E_{j,k})) = (T(\alpha J), T(\alpha E_{j,k})) \notin \mathbf{F}_{4B} .$$

That is, since T strongly preserves $\mathbf{F}_{4B}$, $(\alpha J, \alpha E_{j,k}) \notin \mathbf{F}_{4B}$, a contradiction. Thus M has no zero column. Similarly, M has no zero row.

Now $(\alpha J, I) \in \mathbf{F}_{4B}$ and, therefore, $(T(\alpha M), I) \in \mathbf{F}_{4B}$ and hence $(\alpha M, I) \in \mathbf{F}_{4B}$. That is, rank (M) = 1, and hence $\alpha M = \alpha J$. Since M was chosen according to (1), T induces a bijection on the set of cells, that is, $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some permutation σ of $\{(i,j) \mid 1 \leq i, j \leq n\}$. By Lemma 3.32, the theorem is proved.

Linear operators of special types that preserve $\mathbf{F}_{4R}$

Lemma : 3.35

Let $S \subseteq R_+$ be a subsemiring and $T : \mathbf{M}_n(S) \rightarrow \mathbf{M}_n(S)$ be defined by $T(X) = DXE$ for all $X \in \mathbf{M}_n(S)$, where $D,E \in \mathbf{M}_n(S)$ are invertible diagonal matrices, $n > 4$. Then T preserves $\mathbf{F}_{4R}$ if and only if $E = \alpha D^{-1}$ for some invertible $\alpha \in S$.

Theorem : 3.36

Let $S \subseteq R_+$ be a semiring and $T : \mathbf{M}_n(S) \rightarrow \mathbf{M}_n(S)$, $n > 4$, be a surjective linear operator. Then T preserves $\mathbf{F}_{4R}$ if and only if there exists a permutation matrix $P \in \mathbf{M}_n(\mathbf{S})$, invertible $\alpha \in \mathbf{S}$, and an invertible diagonal matrix $D \in \mathbf{M}_n(\mathbf{S})$ such that $T(X) = \alpha PDXD^{-1}P^t$ for all $X \in \mathbf{M}_n(\mathbf{S})$.

Proof :

One can easily see that all operators of the form $T(X) = \alpha PDXD^{-1}P^t$ for all $X \in \mathbf{M}_n(\mathbf{S})$ and some permutation matrix $P \in \mathbf{M}_n(\mathbf{S})$, invertible diagonal matrix $D \in \mathbf{M}_n(\mathbf{S})$ preserve $\mathbf{F}_{4R}$. Now we assume that T preserves $\mathbf{F}_{4R}$ and show that T is of this form.

Since T is surjective, by Theorem 3.9 we have that $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some permutation σ and matrix B of units. If rank (A) = n, then $(E_{i,j}, A) \in \mathbf{F}_{4R}$ and since rank $(T(E_{i,j})) = 1$ and T preserves $\mathbf{F}_{4R}$, it follows that rank (T(A)) = n. Therefore, T preserves rank-n matrices.

If the preimage of a row is not dominated by any line, then there are cells $E_{i,k}$ and $E_{i,l}$ such that $T(E_{i,k} + E_{i,l}) \leq E_{r,s} + E_{p,q}$ to a permutation matrix by adding n-2 cells, we find a matrix which is the image of a permutation matrix but is dominated by n-1 lines; a contradiction since T preserves rank-n matrices. Thus, the preimage of every row is a row or column and, similarly, the preimage of every column is a row or a column. Hence, T maps lines to lines. By Lemma 3.12, we have that T is a (P,Q,B)-operator. Since $(E_{1,1} + E_{2,1} + E_{3,2} + \ldots\ldots + E_{n,n-1}) \in \mathbf{F}_{4R}$ while $(E_{1,1} + E_{1,2} + E_{2,3} + \ldots\ldots + E_{n-1,n}) \notin \mathbf{F}_{4R}$, we have that the transpose operator does not preserve $\mathbf{F}_{4R}$, thus there exist permutation matrices P and Q such that $T(X) = P(X \circ B)Q$ for some matrix B of units.

Without loss of generality, we may assume that P = I. If $Q \neq I$, we assume that Q corresponds to the permutation π and $\pi(1) \neq 1$. Without loss of generality, $T(E_{1,1}) = b_{1,1}E_{1,2}$. Then $(E_{1,1} + E_{2,2} + E_{3,3} + \ldots\ldots + E_{n,n}) \in \mathbf{F}_{4R}$, while $(T(E_{1,1}), T(E_{2,2} + E_{3,3} + \ldots\ldots + E_{n,n})) \notin \mathbf{F}_{4R}$ since

$$(b_{1,1}E_{1,1}),(b_{2,2}E_{2,\pi(2)} + E_{3,\pi(3)} + \ldots\ldots + E_{n,\pi(n)}) = b_{1,1}E_{1,2}\, b_{2,2}E_{2,\pi(2)} \neq O$$

This contradiction gives that $Q = P^t$ and we assume that $T(X) = X \circ B$.

Assume that rank (B) > 1. Then by Corollary 3.8, there exist subscripts i, j, k, and l such that rank (B[i,j][k,l]) = 2. Let

$$X = b_{i,k}^{-1}E_{i,k} + b_{i,l}^{-1}E_{i,l} + b_{j,k}^{-1}E_{j,k} + b_{j,l}^{-1}E_{j,l} + E_3 + E_4 + \dots\dots + E_n,$$

where E_3 , E_4 , ……, E_n are chosen so that rank (X) = n. But rank (T(X)) ≤ n-1; a contradiction since T preserves rank-n matrices. Thus, rank (B) = 1 and the theorem follows by subsequent application of Lemmas 3.16 and 3.35.

Linear operators of special types that preserve $\mathbf{F_5}$

Lemma : 3.37

Let $\mathbf{S} \subseteq \mathbb{R}_+$ be a subsemiring and T : $\mathbf{M_n(S)} \rightarrow \mathbf{M_n(S)}$ be a surjective linear preserver of $\mathbf{F_5}$. Then there exists a permutation matrix $P \in \mathbf{M_n(S)}$ and a matrix $B \in \mathbf{M_n(Z(S))}$ with invertible entries such that $T(X) = P(X \circ B)P^t$ for all $X \in \mathbf{M_n(S)}$.

Lemma : 3.38

Let $\mathbf{S} \subseteq \mathbb{R}_+$ be a subsemiring and T : $\mathbf{M_n(S)} \rightarrow \mathbf{M_n(S)}$ be defined by T(X) = DXE for all $X \in \mathbf{M_n(S)}$, where $D, E \in \mathbf{M_n(S)}$ are invertible diagonal matrices, n > 4. Then T preserves $\mathbf{F_5}$ if and only if $E = \alpha D^{-1}$ for some invertible $\alpha \in \mathbf{S}$.

Proof :

If $E = \alpha D^{-1}$, then it is easy to show that T preserves $\mathbf{F_5}$. Now we assume that $E \neq \alpha D^{-1}$ for any invertible α. Let $L(X) = ET(X)E^{-1} = EDX$ and G = ED; therefore, G is not a scalar matrix. As in Lemma 3.26, without loss of generality we can assume that G = diag {g_1 , g_2 , ……, g_n} and $g_3 \neq g_4$. Let A and B be as in the proof of Lemma 3.26, $X = A \oplus O_{n-4}$, $Y = I_4 \oplus O_{n-4}$, and $Z = B \oplus O_{n-4}$. Then

$$XYZ = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 2 & 2 & 4 & 4 \end{bmatrix} \oplus O_{n-4}$$

and, therefore,

ρ(L(X)L(Y)) = ρ(XY) = 3, ρ(L(Y)L(Z)) = ρ(YZ) = 2,

rank (L(Y)) = rank (Y) = 4, rank (XYZ) = 1.

Thus, $(X,Y,Z) \in \mathbf{F}_5$. But

$$L(X)L(Y)L(Z) = G\left(\begin{bmatrix} g_2^2 & g_2^2 & g_3^2+g_4^2 & g_3^2+g_4^2 \\ g_1^2 & g_1^2 & g_3^2+g_4^2 & g_3^2+g_4^2 \\ g_1^2+g_2^2 & g_1^2+g_2^2 & 4g_4^2 & 4g_4^2 \\ g_1^2+g_2^2 & g_1^2+g_2^2 & 4g_3^2 & 4g_3^2 \end{bmatrix} \oplus O_{n-4}\right)$$

and, therefore, rank (L(X)L(Y)L(Z)) = 2 since $g_3 \neq g_4$. Thus, $(L(X),L(Y),L(Z)) \notin \mathbf{F}_5$. Thus, L does not preserve $\mathbf{F}_5$. The lemma is proved.

Theorem : 3.39

Let $\mathbf{S} \subseteq \mathbb{R}_+$ be a subsemiring and $T : \mathbf{M}_n(\mathbf{S}) \rightarrow \mathbf{M}_n(\mathbf{S})$ be a surjective linear operator, n > 4. Then T preserves $\mathbf{F}_5$ if and only if $T(X) = \alpha PDXD^{-1}P^t$ for all $X \in \mathbf{M}_n(\mathbf{S})$, where $D \in \mathbf{M}_n(\mathbf{S})$ is an invertible diagonal matrix, $\alpha \in \mathbf{S}$ is invertible, and $P \in \mathbf{M}_n(\mathbf{S})$ is a permutation matrix.

Proof :

It is easy to prove that operators of the form $T(X) = \alpha PDXD^{-1}P^t$ preserve $\mathbf{F}_5$.

Assume that T preserves $\mathbf{F}_5$. Then by Lemma 3.37, $T(X) = P(X \circ B)P^t$, where all entries of B are invertible.

Let us verify that rank $(B) \geq 2$. Then by Corollary 3.8, there exist subscripts i, j, k, and l such that rank (B[i,j][k,l]) = 2. Let

$$X = 0,\ Y = E_{i,k} + E_{i,l} + E_{j,k} + E_{j,l}\ ,\ Z = E_{k,l}\ .$$

Then

$$P(XY) + \rho(YZ) = 0+1 = \text{rank}\ (XYZ) + \text{rank}\ (Y)$$

i.e., $(X,Y,Z) \in \mathbf{F}_5$. On the other hand, rank $(Y \circ B) = 2$ since rank (B[i,j][k,l]) = 2 and $\rho(Y \circ B . Z \circ B) = 1$ since Z is a cell. Hence $(T(X),T(Y),T(Z)) \notin \mathbf{F}_5$, a contradiction. The theorem follows by subsequent applications of Lemmas 3.16 and 3.38.