4. Batch Arrival Retrial G-Queue with Multistage and Multi-Optional Services, Admission Control, Feedback and Vacation

$M^X/G/1$ two phase retrial G-queue with feedback and vacation is considered. Admission of each individual customer to the system depends upon the state of the server. Positive customers arrive in batches according to Poisson process. If the server is idle, one of the admitted customers enters the service immediately and the rest joins the orbit, otherwise all the admitted customers enter the orbit. After the completion of essential or optional services, the customer may join the queue if they are dissatisfied with the service. After providing service to a customer, the server may go for a vacation with certain probability. The arrival of negative customers makes the server to breakdown. The joint distributions of the server state and the number of customers in the orbit are obtained using the supplementary variable technique. Stochastic decomposition property is verified and the special cases are deduced. The impact of parameters in the system performance measures is examined.

4.1 Model Description

The basic assumptions of the model under study are described here.

Arrival Process

Negative customers arrive according to the Poisson process with rate λ^-. The positive customers arrive in batches according to the Poisson process with rate λ^+. The batch size Y is a random variable with distribution function $P(Y=k) = C_k$, $k \geq 1$ and probability generating function C(z) with first two moments m_1 and m_2. Let ω_1, ω_2, ω_3 and ω_4 be the probabilities for admitting a customer when the server is idle, busy, repair and on vacation respectively. Then the corresponding probabilities of admitting n customers from the arriving batch of k customers is given by

$$a_{j,n} = \begin{cases} \sum\limits_{k=1}^{\infty} C_k (1-\omega_j)^k , & n=0, \quad j=1,2,3,4 \\ \sum\limits_{k=1}^{\infty} C_k \binom{k}{n} \omega_j^n (1-\omega_j)^{k-n} , & n \geq 1, \quad j=1,2,3,4 \end{cases}$$

Let $\lambda_j^+ = \lambda^+ \sum_{n=1}^{\infty} a_{j,n}$ for $j = 1,2,3,4$

The probability generating function $a_j(z)$ of the sequence $\{a_{j,n}, n \geq 0\}$ is given by $a_j(z) = C(\omega_j z + (1-\omega_j))$ with first two moments $\omega_j m_1$ and $\omega_j^2 m_2$, j = 1,2,3,4

Retrial Process

If an arriving batch of customers finds the server busy, on vacation or down, the batch joins the orbit. If the server is free, the first stage service commences for one of the arriving customers and others join the orbit. The inter-retrial times follow general distribution with distribution function A(x), Laplace-Stieltjes transform $A^*(s)$ and the hazard rate function $\eta(x)$.

Service Process and Feedback Rule

The server renders service in two phases which include first phase of essential service and the second phase of multistages of service. In each stage, there are multi-optional services. If the server is idle upon the arrival of a batch, one of the customers in the batch commences the first stage of essential service and the rest join the orbit. After completion of essential service, the customer moves to first stage of second phase and opts j_1^{th} (j_1=1,2,....,k_1) option with probability p_{j_1} , departs the system with probability q_0 or joins the orbit as a feedback customer with probability δ_0.

After the completion of first stage, the customer moves to second stage and opts j_2^{th} (j_2=1,2,....,k_2) option with probability p_{j_2}, departs the system with probability q_1 or joins the orbit as a feedback customer with probability δ_1. In general, after the completion of i^{th} stage (i=1,2,….,M) service, the customer may opt j_{i+1}^{th} (j_{i+1}=1,2,....,k_{i+1}) option in $(i+1)^{th}$ stage with probability $p_{j_{i+1}}$, depart from the system with probability q_i or join the orbit with probability δ_i . After final stage (M^{th} stage) service, the customer either leaves the system with probability q_M or joins the orbit with probability δ_M .

The essential service time is generally distributed with distribution function $B_0(x)$, Laplace-Stieltjes transform $B_0^*(s)$, first two moments $\mu_0^{(1)}$ and $\mu_0^{(2)}$ and the conditional completion rate $\mu_0(x)$. The service time of ith stage j$_i^{th}$ optional service is generally distributed with distribution function $B_{i,j_i}(x)$, Laplace-Stieltjes transform $B_{i,j_i}^*(s)$, first two moments $\mu_{i,j_i}^{(1)}$ and $\mu_{i,j_i}^{(2)}$ and the conditional completion rate $\mu_{i,j_i}(x),\ i=1,2,3,\ldots..,M\ ,\ j_i=1,2,3,\ldots...,k_i$

Removal Rule and Repair process

The arrival of a negative customer eliminates the positive customer being in service from the system and makes the server down. The repair of the failed server commences immediately. The repair time of the server failed during essential service is generally distributed with distribution function $R_0(x)$, Laplace-Stieltjes transform $R_0^*(s)$, first two moments $\beta_0^{(1)}$, $\beta_0^{(2)}$ and the conditional completion rate $\beta_0(x)$. The repair time of the server failed during ith stage j$_i^{th}$ optional service is generally distributed with distribution function $R_{i,j_i}(x)$, Laplace-Stieltjes transform $R_{i,j_i}^*(s)$, moments $\beta_{i,j_i}^{(1)}$, $\beta_{i,j_i}^{(2)}$ and the conditional completion rate $\beta_{i,j_i}(x)$, $i=1,2,\ldots..,M$, $j_i=1,2,\ldots..,k_i$.

Vacation process

After completing service to a customer, the server may take a single vacation with probability ν or remain in the system with probability $\bar{\nu}=1-\nu$. The vacation times are arbitrarily distributed with distribution function $V(x)$, LST $V^*(s)$ with first two moments γ_1 and γ_2 respectively and the hazard rate function $\gamma(x)$.

The schematic representation of the proposed model is shown in Fig. 4.1.

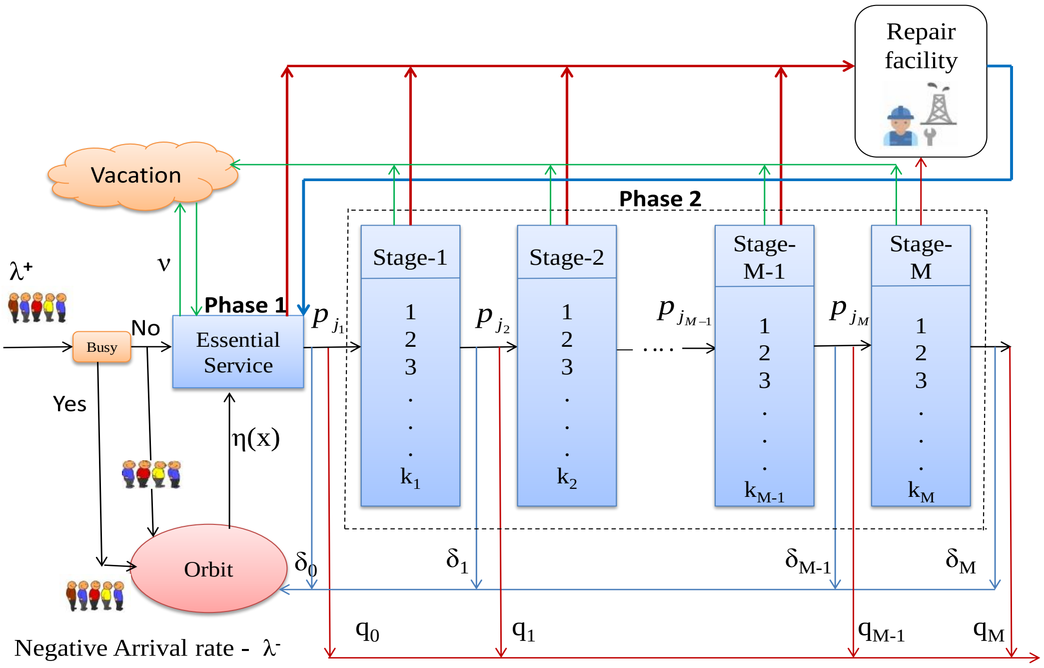

Fig. 4.1 Batch Arrival Retrial G-Queue with Multistage and Multi-Optional Services, Admission Control, Feedback and Vacation

4.2 Steady State Distributions

In this section, the steady state difference differential equations for the retrial system are developed by treating the elapsed retrial time, elapsed service time, elapsed repair time and elapsed vacation time as supplementary variables. Then the probability generating functions for the server state and the number of customers in the orbit and in the system are derived.

The probabilities used in this chapter are defined below.

$I_0(t)$ is the probability that the server is idle at time t and the orbit is empty.

$I_n(x,t)dx$ is the probability that the server is idle at time t with n (≥ 1) customers in the orbit and the elapsed retrial time is between x and x+dx.

$P_{0,n}(x,t)dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server is busy in first phase of essential service and the elapsed service time is between x and x+dx.

$P_{i,j_i,n}(x,t)dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server is busy in ith stage j_i^{th} optional service and the elapsed service time is between x and x+dx, $i=1,2,....,M,\ j_i=1,2,....,k_i$

$R_{0,n}(x,t)dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server in first phase is under repair and the elapsed repair time is between x and x+dx.

$R_{i,j_i,n}(x,t)dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server in ith stage j_i^{th} optional state is under repair and the elapsed repair time is between x and x+dx, $i=1,2,....,M, j_i=1,2,....,k_i$

$V(x,t)dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server is on vacation and the elapsed vacation time is between x and x+dx.

Let I_0, $I_n(x)$, $P_{0,n}(x)$, $P_{i,j_i,n}(x)$, $R_{0,n}(x)$, $R_{i,j_i,n}(x)$ and $V_n(x)$ be respectively the steady state probabilities of $I_0(t)$, $I_n(x,t)$, $P_{0,n}(x,t)$, $P_{i,j_i,n}(x,t)$, $R_{0,n}(x,t)$, $R_{i,j_i,n}(x,t)$ and $V_n(t,x)$ ($i = 1, 2,\ldots, M,\quad j_i=1,2,\ldots,k_i$).

4.2.1 Steady State Equations

The system of steady state equations that governs the model under consideration is

$$\lambda_1^+ I_0 = \bar{\nu}\left[q_0 \int_0^\infty P_{0,0}(x)\mu_0(x)\,dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i} \int_0^\infty P_{i,j_i,0}(x)\mu_{i,j_i}(x)\,dx \right] + \int_0^\infty R_{0,0}(x)\beta_0(x)\,dx$$

$$+ \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} \int_0^\infty R_{i,j_i,0}(x)\beta_{i,j_i}(x)\,dx + \int_0^\infty V_0(x)\gamma(x)\,dx \qquad (4.1)$$

$$\frac{d}{dx}I_n(x) = -(\lambda_1^+ + \eta(x))I_n(x),\ n \geq 1 \qquad (4.2)$$

$$\frac{d}{dx}P_{0,n}(x) = -(\lambda_2^+ + \lambda^- + \mu_0(x))P_{0,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} a_{2,k}P_{0,n-k}(x),\ n \geq 0 \qquad (4.3)$$

$$\frac{d}{dx}P_{i,j_i,n}(x) = -(\lambda_2^+ + \lambda^- + \mu_{i,j_i}(x))\,P_{i,j_i,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} a_{2,k}P_{i,j_i,n-k}(x), \qquad (4.4)$$

$$n \geq 0,\ i=1,2,....,M,\ j_i=1,2,....,k_i$$

$$\frac{d}{dx}R_{0,n}(x) = -(\lambda_3^+ + \beta_0(x))\, R_{0,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} a_{3,k} R_{0,n-k}(x), \quad n \geq 0 \tag{4.5}$$

$$\frac{d}{dx}R_{i,j_i,n}(x) = -(\lambda_3^+ + \beta_{i,j_i}(x))\, R_{i,j_i,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} a_{3,k} R_{i,j_i,n-k}(x), n \geq 0, \quad i = 1,2,....,M, \;\; j_i = 1,2,....,k_i \tag{4.6}$$

$$\frac{d}{dx}V_n(x) = -(\lambda_4^+ + \gamma(x))\, V_n(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n} a_{4,k} V_{n-k}(x), \quad n \geq 0 \tag{4.7}$$

with boundary conditions

$$\begin{aligned} I_n(0) = {} & \bar{\nu}\,[\delta_0 \int_0^\infty P_{0,n-1}(x)\mu_0(x)dx + \sum_{i=1}^{M}\delta_i \sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n-1}(x)\mu_{i,j_i}(x)dx + q_0\int_0^\infty P_{0,n}(x)\mu_0(x)dx \\ & + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n}(x)\mu_{i,j_i}(x)dx\,] + \int_0^\infty R_{0,n}(x)\beta_0(x)dx \\ & + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,n}(x)\beta_{i,j_i}(x)dx + \int_0^\infty V_n(x)\gamma(x)dx, \; n \geq 1 \end{aligned} \tag{4.8}$$

$$P_{0,0}(0) = \lambda^+ a_{1,1} I_0 + \int_0^\infty I_1(x)\eta(x)dx \tag{4.9}$$

$$P_{0,n}(0) = \lambda^+ a_{1,n+1} I_0 + \lambda^+ \sum_{k=1}^{n} a_{1,k}\int_0^\infty I_{n-k+1}(x)dx + \int_0^\infty I_{n+1}(x)\eta(x)dx, \; n \geq 1 \tag{4.10}$$

$$P_{1,j_1,n}(0) = p_{j_1}\int_0^\infty P_{0,n}(x)\mu_0(x)dx, \;\; n \geq 0, \;\; j_1 = 1,2,....,k_1 \tag{4.11}$$

$$P_{i,j_i,n}(0) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^\infty P_{i-1,j_{i-1},n}(x)\mu_{i-1,j_{i-1}}(x)dx, \quad n \geq 0, i = 2,3,......,M, \;\; j_i = 1,2,....,k_i \tag{4.12}$$

$$R_{0,n}(0) = \lambda^- \int_0^\infty P_{0,n}(x)dx, \; n \geq 0 \tag{4.13}$$

$$R_{i,j_i,n}(0) = \lambda^- \int_0^\infty P_{i,j_i,n}(x)dx, \quad n \geq 0, \; i = 1,2,3,......,M, \;\; j_i = 1,2,....,k_i \tag{4.14}$$

$$\begin{aligned} V_n(0) = {} & \nu\,[q_0\int_0^\infty P_{0,n}(x)\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n}(x)\mu_{i,j_i}(x)\,dx \\ & + \delta_0\int_0^\infty P_{0,n-1}(x)\mu_0(x)dx + \sum_{i=1}^{M}\delta_i \sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n-1}(x)\mu_{i,j_i}(x)\,dx\,], \quad n \geq 0 \end{aligned} \tag{4.15}$$

4.2.2 Steady State Solutions

The following probability generating functions are defined to solve the equations that govern the model.

$$\left.\begin{aligned} I(x,z) &= \sum_{n=1}^{\infty} I_n(x)z^n ; & P_0(x,z) &= \sum_{n=0}^{\infty} P_{0,n}(x)z^n \\ P_{i,j_i}(x,z) &= \sum_{n=0}^{\infty} P_{i,j_i,n}(x)z^n ; & R_0(x,z) &= \sum_{n=0}^{\infty} R_{0,n}(x)z^n \\ R_{i,j_i}(x,z) &= \sum_{n=0}^{\infty} R_{i,j_i}(x)z^n ; & V(x,z) &= \sum_{n=0}^{\infty} V_n(x)z^n \end{aligned}\right\} \quad (4.16)$$

By using the definition of probability generating functions, the equations (4.1) to (4.15) give

$$\left(\frac{\partial}{\partial x} + \lambda_1^+ + \eta(x)\right) I(x,z) = 0 \quad (4.17)$$

$$\left(\frac{\partial}{\partial x} + \lambda_2^+(1-a_2(z)) + \lambda^- + \mu_0(x)\right) P_0(x,z) = 0 \quad (4.18)$$

$$\left(\frac{\partial}{\partial x} + \lambda_2^+(1-a_2(z)) + \lambda^- + \mu_{i,j_i}(x)\right) P_{i,j_i}(x,z) = 0, \quad i = 1,2,....,M, \ j_i = 1,2,....,k_i \quad (4.19)$$

$$\left(\frac{\partial}{\partial x} + \lambda_3^+(1-a_3(z)) + \beta_0(x)\right) R_0(x,z) = 0 \quad (4.20)$$

$$\left(\frac{\partial}{\partial x} + \lambda_3^+(1-a_3(z)) + \beta_{i,j_i}(x)\right) R_{i,j_i}(x,z) = 0, \quad i = 1,2,....,M, \ j_i = 1,2,....,k_i \quad (4.21)$$

$$\left(\frac{\partial}{\partial x} + \lambda_4^+(1-a_4(z)) + \gamma(x)\right) V(x,z) = 0 \quad (4.22)$$

$$\begin{aligned} I(0,z) = \ & \bar{\nu}\,[\left(\delta_0 z + q_0\right)\int_0^{\infty} P_0(x,z)\mu_0(x)dx + \sum_{i=1}^{M}\left(\delta_i z + q_i\right)\sum_{j_i=1}^{k_i}\int_0^{\infty} P_{i,j_i,n}(x,z)\mu_{i,j_i}(x)dx \\ & + \int_0^{\infty} R_0(x,z)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^{\infty} R_{i,j_i}(x,z)\beta_{i,j_i}(x)dx + \int_0^{\infty} V(x,z)\gamma(x)dx - \lambda_1^+ I_0 \end{aligned} \quad (4.23)$$

$$P_0(0,z) = \frac{1}{z}[\ \lambda^+\left(a_1(z) - a_{1,0}\right)I_0 + \int_0^{\infty} I(x,z)\eta(x)dx + \lambda^+\left(a_1(z) - a_{1,0}\right)\int_0^{\infty} I(x,z)dx] \quad (4.24)$$

$$P_{1,j_1}(0,z) = p_{j_1}\int_0^{\infty} P_0(x,z)\mu_0(x)dx, \quad j_1 = 1,2,......,k_1 \quad (4.25)$$

$$P_{i,j_i}(0,z) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^{\infty} P_{i-1,j_{i-1}}(x,z)\mu_{i-1,j_{i-1}}(x)dx, \quad i = 2,3,......,M, \ j_i = 1,2,......,k_i \quad (4.26)$$

$$R_0(0,z) \;=\; \lambda^- \int_0^\infty P_0(x,z)dx \tag{4.27}$$

$$R_{i,j_i}(0) = \lambda^- \int_0^\infty P_{i,j_i}(x,z)dx, \quad i = 1,2,3,\ldots\ldots,M, \; j_i = 1,2,\ldots\ldots,k_i \tag{4.28}$$

$$V(0,z) = \nu\,[(\delta_0\, z + q_0)\int_0^\infty P_0(x,z)\mu_0(x)dx + \sum_{i=1}^{M}(\delta_i\, z + q_i)\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n}(x,z)\mu_{i,j_i}(x)dx\,] \tag{4.29}$$

Solving the partial differential equations (4.17) to (4.22), we get

$$I(x,z) \;=\; I(0,z)\, e^{-\lambda_1^+ x}(1 - A(x)) \tag{4.30}$$

$$P_0(x,z) \;=\; P_0(0,z)\, e^{-(\lambda^+ + \lambda^- - \lambda^+ a_2(z))x}(1 - B_0(x)) \tag{4.31}$$

$$P_{i,j_i}(x,z) = P_{i,j_i}(0,z)\, e^{-(\lambda^+ + \lambda^- - \lambda^+ a_2(z))x}(1 - B_{i,j_i}(x))) \tag{4.32}$$

$$R_0(x,z) \;=\; R_0(0,z)\, e^{-(\lambda^+ - \lambda^+ a_3(z))x}(1 - R_0(x)) \tag{4.33}$$

$$R_{i,j_i}(x,z) = R_{i,j_i}(0,z)\, e^{-(\lambda^+ - \lambda^+ a_3(z))x}(1 - R_{i,j_i}(x))) \tag{4.34}$$

$$V(x,z) \;=\; V(0,z)\, e^{-(\lambda^+ - \lambda^+ a_4(z))x}(1 - V(x)) \tag{4.35}$$

Using equations (4.30) to (4.35), the equation (4.23) becomes

$$\begin{aligned} I(0,z) &= \bar{\nu}\,[(q_0 + \delta_0\, z)\,P_0(0,z)\,B_0^*(g(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}(q_i + \delta_i\, z)\,P_{i,j_i}(0,z)\,B_{i,j_i}^*(g(z))] \\ &\quad + R_0(0,z)\,R_0^*(h(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(0,z)\,R_{i,j_i}^*(h(z)) + V(0,z)\,V^*(h(z)) - \lambda^+ I_0 \end{aligned} \tag{4.36}$$

where

$g(z) \;=\; \lambda^+(1 - a_2(z)) + \lambda^-$

$h_1(z) \;=\; \lambda^+ (1 - a_3(z))$

$h_2(z) \;=\; \lambda^+ (1 - a_4(z))$

Substituting the expression of I(x, z), the equation (4.24) yields

$$z\ P_0(0,z) = \lambda^+(a_1(z)-a_{1,0})I_0 + \left(A^*(\lambda_1^+) + \frac{\lambda^+}{\lambda_1^+}\left(a_1(z)-a_{1,0}\right)(1-A^*(\lambda_1^+)) \right) I(0,z) \tag{4.37}$$

Inserting the equation (4.31) in equation (4.25), we obtain

$$P_{1,j_1}(0,z) = p_{j_1}\ P_0(0,z)\ B_0^*(g(z)), \quad j_1 = 1,2,......,k_1 \tag{4.38}$$

Using the expression of $P_{i,j_i}(x,z)$, the equation (4.26) gives

$$\begin{aligned}
P_{i,j_i}(0,z) &= p_{j_i} \sum_{j_{i-1}=1}^{k_{i-1}} P_{i-1,j_{i-1}}(0,z) B^*_{i-1,j_{i-1}}(g(z)) \\
&= p_{j_i} \sum_{j_{i-1}=1}^{k_{i-1}} P_{j_{i-1}} \sum_{j_{i-2}=1}^{k_{i-2}} P_{j_{i-2}} P_{i\text{-}2,j_{i-2}}(0,z)\ B^*_{i-2,j_{i-2}}(g(z)) B^*_{i-1,j_{i-1}}(g(z)) \\
&= p_{j_i} \sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}} \sum_{j_{i-2}=1}^{k_{i-2}} p_{j_{i-2}} \cdots \sum_{j_2=1}^{k_2} p_{j_2} \sum_{j_1=1}^{k_1} p_{j_1} P_0(0,z)\ B_0^*(g(z))\ B^*_{1,j_1}(g(z)) \\
&\qquad B^*_{2,j_2}(g(z)).....B^*_{i-2,j_{i-2}}(g(z)) B^*_{i-1,j_{i-1}}(g(z))
\end{aligned}$$

$$= p_{j_i}\left[\sum_{j_1=1}^{k_1} p_{j_1}\ B^*_{1,j_1}(g(z)) \sum_{j_2=1}^{k_2} p_{j_2}\ B^*_{2,j_2}(g(z)) .... \sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}}\ B^*_{i-1,j_{i-1}}(g(z)) \right] B_0^*(g(z))\ P_0(0,z)$$

$$= p_{j_i}\left[\prod_{l=1}^{i-1} \sum_{j_l=1}^{k_l} p_{j_l} B^*_{l,j_l}(g(z)) \right] B_0^*(g(z))\ P_0(0,z)$$

$$= p_{j_i} \Lambda^*_{i-1}(g(z)) B_0^*(g(z))\ P_0(0,z)\ ,\ i = 2,3,......,M,\ j_i = 1,2,......,k_i \tag{4.39}$$

where

$$\Lambda_0^*(g(z)) = 1\ , \qquad \Lambda_i^*(g(z)) = \prod_{l=1}^{i} \sum_{j_l=1}^{k_l} p_{j_l} B^*_{l,j_l}(g(z))$$

Using the equations (4.31) and (4.32), the equations (4.27) to (4.29) give

$$R_0(0,z) = \lambda^-\ P_0(0,z)(1-B_0^*(g(z)))/g(z) \tag{4.40}$$

$$R_{i,j_i}(0,z) = \lambda^-\ P_{i,j_i}(0,z)\,(1-B^*_{i,j_i}(g(z)))/g(z),\ i = 1,2,3,......,M,\ j_i = 1,2,......,k_i \tag{4.41}$$

$$V(0,z) = \nu\,[(q_0 + \delta_0\, z)\, P_0(0,z) B_0^*(g(z)) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} (q_i + \delta_i\, z)\, P_{i,j_i}(0,z) B_{i,j_i}^*(g(z))] \quad (4.42)$$

Substituting the expression of $P_{i,j_i}(0,z)$ in the equations (4.41) and (4.42), we get

$$R_{i,j_i}(0,z) = \lambda^-\, p_{j_i} \Lambda_{i-1}^*(g(z)) B_0^*(g(z))\; P_0(0,z)\; (1 - B_{i,j_i}^*(g(z)))\big/g(z), \qquad (4.43)$$
$$i = 1,2,3,\ldots\ldots,M, \quad j_i = 1,2,\ldots\ldots,k_i$$

$$V(0,z) = \nu\,[(q_0 + \delta_0\, z)\, B_0^*(g(z)) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} (q_i + \delta_i\, z) \Lambda_{i-1}^*(g(z)) B_0^*(g(z))]\; P_0(0,z)$$

$$(4.44)$$

Substituting the equations (4.38), (4.39), (4.43) and (4.44) in equation (4.36), then

$$I(0,z) = T_1(z) P_0(0,z) - \lambda_1^+ I_0 \qquad (4.45)$$

where

$$\begin{aligned} T_1(z) = {} & \bar{\nu} \sum_{i=0}^{M} (q_i + \delta_i\, z)\, \Lambda_i^*(g(z)) B_0^*(g(z)) + \lambda^-((1 - B_0^*(g(z)))/g(z)) R_0^*(h_1(z)) \\ & + \lambda^- \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i\text{-}1}^*(g(z))((1 - B_{i,j_i}^*(g(z)))/g(z)) R_{i,j_i}^*(h_1(z)) B_0^*(g(z)) \\ & + \nu \sum_{i=0}^{M} (q_i + \delta_i\, z)\, \Lambda_i^*(g(z)) B_0^*(g(z)) V^*(h_2(z)) \end{aligned}$$

Inserting the expression of I(0, z) from the equation (4.45) in the equation (4.37) and on simplification, we get

$$P_0(0,z) = \frac{I_0\, \lambda^+ (a_1(z) - 1)\, A^*(\lambda_1^+)}{D(z)} \qquad (4.46)$$

where

$$D(z) = z - \left(A^*(\lambda_1^+) + \frac{\lambda^+}{\lambda_1^+}\big(a_1(z) - a_{1,0}\big)(1 - A^*(\lambda_1^+)) \right) T_1(z)$$

Substituting $P_0(0,z)$ in (4.45), (4.38), (4.39), (4.40), (4.43) and (4.44), we have respectively

$$I(0,z) = \frac{\lambda^+\, I_0\, [(a_1(z) - a_{1,0}) T_1(z) - z\,(1 - a_{1,0})]}{D(z)} \qquad (4.47)$$

$$P_{i,j_i}(0,z) = \frac{I_0 \lambda^+ (a_1(z)-1)\, A^*(\lambda_1^+)\, p_{j_i}\, \Lambda^*_{i-1}(g(z))\, B_0^*(g(z))}{D(z)} \tag{4.48}$$

$$R_0(0,z) = \frac{\lambda^- \lambda^+ I_0\, A^*(\lambda_1^+)\, (a_1(z)-1)\, (1-B_0^*(g(z)))\,/\,g(z)}{D(z)} \tag{4.49}$$

$$R_{i,j_i}(0,z) = \frac{\lambda^- \lambda^+ I_0\, A^*(\lambda_1^+)\, (a_1(z)-1)\, p_{j_i}\, \Lambda^*_{i-1}(g(z))\, B_0^*(g(z))\, (1-B^*_{i,j_i}(g(z)))\,/\,g(z)}{D(z)} \tag{4.50}$$

$$V(0,z) = \frac{\lambda^+ I_0\, A^*(\lambda_1^+)\, (a_1(z)-1)\, \nu \sum_{i=0}^{M} (\delta_i z + q_i)\, \Lambda_i^*(g(z))\, B_0^*(g(z))}{D(z)} \tag{4.51}$$

The probability generating function of the orbit size when the server is idle in the non-empty system is given by

$$\begin{aligned} I(z) &= \int_0^\infty I(x,z)\,dx \\ &= I(0,z) \int_0^\infty e^{-\lambda_1^+ x}(1-A(x))\,dx \\ &= \frac{\lambda^+ I_0\, (1-A^*(\lambda_1^+))\,[(a_1(z)-a_{1,0})\,T_1(z) - z\,(1-a_{1,0})]}{\lambda_1^+ D(z)} \end{aligned} \tag{4.52}$$

The probability generating function of the orbit size when the server is busy in first phase is given by

$$\begin{aligned} P_0(z) &= \int_0^\infty P_0(x,z)\,dx \\ &= P_0(0,z) \int_0^\infty e^{-g(z)\,x}(1-B_0(x))\,dx \\ &= \frac{\lambda^+ I_0\, A^*(\lambda_1^+)\, (a_1(z)-1)\, (1-B_0^*(g(z)))}{g(z)\; D(z)} \end{aligned} \tag{4.53}$$

The probability generating function of the orbit size when the server is busy in second phase is given by

$$\begin{aligned} P(z) &= \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} P_{i,j_i}(z) \\ &= \int_0^\infty \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} P_{i,j_i}(x,z)\,dx \end{aligned}$$

$$= \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(0,z)\int_0^{\infty} e^{-g(z)x}(1-B_{i,j_i}(x))\,dx$$

$$= \frac{\lambda^{+} I_0 (a_1(z)-1)\, A^{*}(\lambda_1^{+})\, B_0^{*}(g(z)) \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i\text{-}1}^{*}(g(z))\,(1-B_{i,j_i}^{*}(g(z)))}{g(z)\, D(z)} \tag{4.54}$$

The probability generating function of the orbit size when the server in first phase is under repair is given by

$$R_0(z) = \int_0^{\infty} R_0(x,z)\,dx$$

$$= R_0(0,z)\int_0^{\infty} e^{-h_1(z)x}(1-R_0(x))\,dx$$

$$= \frac{\lambda^{-}\lambda^{+} I_0\, A^{*}(\lambda_1^{+})\,(a_1(z)-1)\,(1-B_0^{*}(g(z)))\,(1-R_0^{*}(h_1(z))))}{g(z)\, h_1(z)\, D(z)} \tag{4.55}$$

The probability generating function of the orbit size when the server in second phase is under repair is given by

$$R(z) = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^{\infty} R_{i,j_i}(x,z)\,dx$$

$$= \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(0,z)\int_0^{\infty} e^{-h_1(z)x}(1-R_{i,j_i}(x))\,dx$$

$$= \frac{\lambda^{-}\lambda^{+} I_0\, A^{*}(\lambda_1^{+})\,(a_1(z)-1) B_0^{*}(g(z)) \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^{*}(g(z))\,(1-B_{i,j_i}^{*}(g(z)))\,(1-R_{i,j_i}^{*}(h_1(z)))}{g(z)\, h_1(z)\, D(z)} \tag{4.56}$$

The probability generating function of the orbit size when the server is on vacation is given by

$$V(z) = \int_0^{\infty} V(x,z)\,dx$$

$$= V(0,z)\int_0^{\infty} e^{-h_2(z)x}(1-V(x))\,dx$$

$$= \frac{\lambda^{+} I_0 A^{*}(\lambda_1^{+})(a_1(z)-1)\nu \sum_{i=0}^{M}(\delta_i z+q_i)\Lambda_i^{*}(g(z))B_0^{*}(g(z))(1-V^{*}(h_2(z)))}{h_2(z)D(z)} \tag{4.57}$$

Using the normalizing condition, the unknown constant I_0 can be obtained as

$$I_0 = (1 - \frac{\lambda^{+}\omega_1 m_1}{\lambda_1^{+}}(1-A^{*}(\lambda_1^{+})) - T_2) / A^{*}(\lambda_1^{+})\{1+\sum_{j_1=1}^{k_1} p_{j_1} f_0^{(1)} - T_4 - T_3$$

$$-\bar{\nu}\lambda^{+}\omega_1 m_1 \sum_{i=0}^{M}(\delta_i+q_i)\Lambda_i^{*}(\lambda^{-}) + \lambda^{+}(\omega_1-\omega_2)m_1T_5 - \lambda^{+}(\omega_1-\omega_3)m_1T_6\}$$

where

$$T_2 = \lambda^{+}\omega_4 m_1\gamma_1\nu B_0^{*}(\lambda^{-})\sum_{i=0}^{M}(\delta_i+q_i)\Lambda_i^{*}(\lambda^{-}) - \sum_{j_1=1}^{k_1} p_{j_1} f_0^{(1)} + T_4 + \lambda^{+}m_1(\omega_2T_5+\omega_3T_6) + T_3$$

$$T_3 = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}(M_{i-1}^{(1)}B_0^{*}(\lambda^{-}) + \Lambda_{i-1}^{*}(\lambda^{-})f_0^{(1)}(1-B_{i,j_i}^{*}(\lambda^{-})) - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^{*}(\lambda^{-})B_0^{*}(\lambda^{-})f_{i,j_i}^{(1)}$$

$$T_4 = B_0^{*}(\lambda^{-})\sum_{i=0}^{M}\delta_i\Lambda_i^{*}(\lambda^{-}) + \sum_{i=1}^{M}(\delta_i+q_i)(M_i^{(1)}B_0^{*}(\lambda^{-}) + \Lambda_i^{*}(\lambda^{-})f_0^{(1)})$$

$$T_5 = (1/\lambda^{-})[1-B_0^{*}(\lambda^{-}) + B_0^{*}(\lambda^{-})\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^{*}(\lambda^{-})(1-B_{i,j_i}^{*}(\lambda^{-}))]$$

$$T_6 = (1-B_0^{*}(\lambda^{-}))\beta_0^{(1)} + B_0^{*}(\lambda^{-})\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^{*}(\lambda^{-})(1-B_{i,j_i}^{*}(\lambda^{-}))\beta_{i,j_i}^{(1)}$$

$$f_0^{(1)} = \lambda^{+}\omega_2 m_1\int_0^{\infty} x e^{-\lambda^{-}x} b_0(x)dx, \qquad f_{i,j_i}^{(1)} = \lambda^{+}\omega_2 m_1\int_0^{\infty} x e^{-\lambda^{-}x} b_{i,j_i}(x)dx$$

$$M_i^{(1)} = \lim_{z\to 1}\Lambda_i^{*\prime}(g(z)), \qquad M_i^{(2)} = \lim_{z\to 1}\Lambda_i^{*\prime\prime}(g(z))$$

The probability generating function of the orbit size is

$$P_q(z) = I_0 + I(z) + P_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) + V(z)$$

$$= I_0 A^{*}(\lambda_1^{+})\{(a_3(z)-1)(a_4(z)-1)[z-T_1(z)+\lambda^{+}(a_1(z)-1)T_7(z)]$$

$$-\lambda^{-}(a_1(z)-1)(a_4(z)-1)T_8(z) + \nu(a_1(z)-1)(a_3(z)-1)\sum_{i=0}^{M}(\delta_i z+q_i)$$

$$\Lambda_i^{*}(g(z))B_0^{*}(g(z))(1-V^{*}(h_2(z)))\} / D(z)(a_3(z)-1)(a_4(z)-1) \tag{4.58}$$

The probability generating function of the system size is

$$
\begin{aligned}
P_s(z) &= I_0 + I(z) + z\,P_0(z) + z\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) + V(z) \\
&= I_0\,A^*(\lambda_1^+)\{(a_3(z)-1)(a_4(z)-1)[z - T_1(z) + z\lambda^+\,(a_1(z)-1)T_7(z)] \\
&\quad -\lambda^-\,(a_1(z)-1)(a_4(z)-1)T_8(z) + \nu(a_1(z)-1)(a_3(z)-1)\sum_{i=0}^{M}(\delta_i\, z + q_i) \\
&\quad \Lambda_i^*(g(z))B_0^*(g(z))(1-V^*(h_2(z)))\}\,/\;D(z)\,(a_3(z)-1)(a_4(z)-1) \qquad (4.59)
\end{aligned}
$$

where

$$T_7(z) \;=\; 1 - B_0^*(g(z)) + B_0^*(g(z))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\,\Lambda_{i-1}^*(g(z))\,(1 - B_{i,j_i}^*(g(z)))$$

$$
\begin{aligned}
T_8(z) \;=\; &(1 - B_0^*(g(z)))(1 - R_0^*(h_1(z))) + B_0^*(g(z))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\,\Lambda_{i-1}^*(g(z)) \\
&(1 - B_{i,j_i}^*(g(z)))(1 - R_{i,j_i}^*(h_1(z)))
\end{aligned}
$$

4.3 Stability Condition

The necessary and sufficient condition for the system to be stable is

$$\frac{\lambda^+\,\omega_1\,m_1}{\lambda_1^+}\,(1 - A^*(\lambda_1^+)) + T_2 \;<\; 1$$

4.4 Performance Measures

In this section, probabilities and expected queue length corresponding to different states of the server are derived.

4.4.1 System state probabilities

- The steady state probability that the server is idle in the non-empty system is

$$
\begin{aligned}
I &= \lim_{z\to 1} I(z) \\
&= \frac{I_0\,(1 - A^*(\lambda_1^+))[\,(\lambda^+\,\omega_1\,m_1/\,\lambda_1^+) + T_2 - 1]}{D'} \qquad (4.60)
\end{aligned}
$$

where

$$D' \;=\; 1 - \frac{\lambda^+\,\omega_1\,m_1}{\lambda_1^+}\,(1 - A^*(\lambda_1^+)) - T_2$$

- The steady state probability that the server is busy in providing essential service

$$P_0 = \lim_{z \to 1} P_0(z)$$

$$= \frac{\lambda^+ \omega_1 m_1 I_0 A^*(\lambda_1^+)(1 - B_0^*(\lambda^-))}{\lambda^- D'} \tag{4.61}$$

- The steady state probability that the server is busy in providing second phase services is

$$P = \lim_{z \to 1} P(z)$$

$$= \frac{\lambda^+ \omega_1 m_1 I_0 A^*(\lambda_1^+) B_0^*(\lambda^-) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))}{\lambda^- D'} \tag{4.62}$$

- The steady state probability that the server in first phase is under repair is given by

$$R_0 = \lim_{z \to 1} R_0(z)$$

$$= \frac{\lambda^+ \omega_1 m_1 I_0 A^*(\lambda_1^+)(1 - B_0^*(\lambda^-)) \beta_0^{(1)}}{D'} \tag{4.63}$$

- The steady state probability that the server in second phase is under repair is given by

$$R = \lim_{z \to 1} R(z)$$

$$= \frac{\lambda^+ \omega_1 m_1 I_0 A^*(\lambda_1^+) B_0^*(\lambda^-) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \beta_{i,j_i}^{(1)}}{D'} \tag{4.64}$$

- The steady state probability that the server is on vacation is

$$V = \lim_{z \to 1} V(z)$$

$$= \frac{\nu \lambda^+ \omega_1 m_1 I_0 A^*(\lambda_1^+) \sum_{i=0}^{M} (\delta_i + q_i) \Lambda_i^*(\lambda^-) B_0^*(\lambda^-) \gamma_1}{D'} \tag{4.65}$$

4.4.2 Mean Queue Length

Let $N_I(z), N_{P_0}(z), N_P(z), N_{R_0}(z), N_R(z)$ and $N_V(z)$ denotes the numerators of $I(z)$, $P_0(z), P(z), R_0(z), R(z)$ and $V(z)$ respectively.

- Mean number of customers in the orbit when the server is idle is given by

$$L_I = \lim_{z\to 1}\frac{d}{dz}I(z)$$

$$= \frac{D'N_I'' - N_I'D''}{2D'^2} \tag{4.66}$$

where

$$N_I' = I_0(1-A^*(\lambda_1^+))\left[\frac{\lambda^+\omega_1 m_1}{\lambda_1^+} + T_2 - 1\right]$$

$$N_I'' = \frac{\lambda^+}{\lambda_1^+} I_0(1-A^*(\lambda_1^+))\left[\omega_1^2 m_2 + 2\omega_1 m_1 T_2 + (1-a_{1,0})T_9\right]$$

$$D'' = -\frac{\lambda^+\omega_1}{\lambda_1^+}\left(\omega_1 m_2 + 2m_1T_2\right)\left(1-A^*(\lambda_1^+)\right) - T_9$$

$$\begin{aligned}
T_9 = {} & 2\sum_{i=0}^{M}[\delta_i\Lambda_i^*(\lambda^-)f_0^{(1)} + M_i^{(1)}B_0^*(\lambda^-)] - \sum_{j_1=1}^{k_1} p_{j_1} f_0^{(2)} + \sum_{i=1}^{M}(\delta_i + q_i) \\
& [M_i^{(2)}B_0^*(\lambda^-) + M_i^{(1)}f_0^{(1)} + \Lambda_i^*(\lambda^-)f_0^{(2)}] - 2(\lambda^+\omega_2 m_1/\lambda^-)(f_0^{(1)} - T_3) \\
& - 2\lambda^+\omega_3 m_1(f_0^{(1)}\beta_0^{(1)} - T_{11}) + [2((\lambda^+\omega_2 m_1)^2/\lambda^-) - \lambda^+\omega_2^2 m_2]T_5 \\
& + [2((\lambda^+ m_1)^2\omega_2\omega_3/\lambda^-) + \lambda^+\omega_3^2 m_2]T_6 + (\lambda^+\omega_3 m_1)^2 T_{10} \\
& + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}[M_{i-1}^{(2)}B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-)f_0^{(2)} + 2M_{i-1}^{(1)}f_0^{(1)}](1-B_{i,j_i}^*(\lambda^-)) \\
& - 2B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)} f_{i,j_i}^{(1)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(2)} B_0^*(\lambda^-) \\
& - 2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}(M_{i-1}^{(1)}B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-)f_0^{(1)})f_{i,j_i}^{(1)} + 2\nu\lambda^+\omega_4 m_1\gamma_1 \\
& ((\delta_0 + q_0)f_0^{(1)} + T_4) + \nu[(\lambda^+\omega_4 m_1)^2\gamma_2 + \lambda^+\omega_4^2 m_2\gamma_1]\sum_{i=0}^{M}\delta_i\Lambda_i^*(\lambda^-)B_0^*(\lambda^-)
\end{aligned}$$

$$T_{10} = (1-B_0^*(\lambda^-)))\beta_0^{(2)} + B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(2)}$$

$$
\begin{aligned}
T_{11} \;=\;& B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)}(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)} - B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) f_{i,j_i}^{(1)} \beta_{i,j_i}^{(1)} \\
&+ \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)} f_0^{(1)}
\end{aligned}
$$

- Mean number of customers in the orbit when the server is busy in providing essential service is given by

$$
\begin{aligned}
L_{P_0} &= \lim_{z\to 1}\frac{d}{dz}P_0(z) \\
&= \frac{D'N''_{P_0} - N'_{P_0}D''}{2D'^2} \qquad (4.67)
\end{aligned}
$$

where

$$
N'_{P_0} = \frac{\lambda^+ \omega_1 m_1}{\lambda^-} I_0 \, A^*(\lambda_1^+)\,(1-B_0^*(\lambda^-))
$$

$$
N''_{P_0} = \lambda^+ I_0 \, A^*(\lambda_1^+)\left(\frac{\omega_1^2 m_2}{\lambda^-}(1-B_0^*(\lambda^-)) - 2\frac{\omega_1 m_1}{\lambda^-} f_0^{(1)} + 2\frac{\lambda^+ \omega_1 \omega_2 m_1^2}{(\lambda^-)^2}(1-B_0^*(\lambda^-))\right)
$$

- Mean number of customers in the orbit when the server is busy in providing optional services is given by

$$
\begin{aligned}
L_P &= \lim_{z\to 1}\frac{d}{dz}P(z) \\
&= \frac{D'N''_P - N'_P D''}{2D'^2} \qquad (4.68)
\end{aligned}
$$

where

$$
N'_P = \frac{\lambda^+ \omega_1 m_1}{\lambda^-} I_0 \, A^*(\lambda_1^+) B_0^*(\lambda^-) \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)\,(1-B_{i,j_i}^*(\lambda^-))
$$

$$
\begin{aligned}
N''_P =\;& \lambda^+ I_0 \, A^*(\lambda_1^+)\,\{\,(\frac{\omega_1^2 m_2}{\lambda^-} + 2\frac{\lambda^+\omega_1\omega_2 m_1^2}{(\lambda^-)^2}) B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)\,(1-B_{i,j_i}^*(\lambda^-)) \\
&+ 2\,\frac{\omega_1 m_1}{\lambda^-} T_3 \,\}
\end{aligned}
$$

- Mean number of customers in the orbit when the server is under repair during essential service is

$$
L_{R_0} = \lim_{z\to 1}\frac{d}{dz}R_0(z)
$$

$$= \frac{D_1'' N_{R_0}''' - N_{R_0}'' D_1'''}{3D_1''^2} \tag{4.69}$$

where

$$N_{R_0}'' = -2(\lambda^+ m_1)^2 \omega_1 \omega_3 I_0 A^*(\lambda_1^+)(1-B_0^*(\lambda^-))\beta_0^{(1)}$$

$$\begin{aligned} N_{R_0}''' = & -\lambda^+\omega_3 m_1 I_0 A^*(\lambda_1^+)[3\lambda^+\omega_1\omega_3 m_2(1-B_0^*(\lambda^-))\beta_0^{(1)} - 4\lambda^+\omega_3 m_1 f_0^{(1)}\beta_0^{(1)} \\ & + 4\frac{(\lambda^+ m_1)^2\omega_1\omega_3}{\lambda^-}(1-B_0^*(\lambda^-))\beta_0^{(1)} + 3(\lambda^+\omega_3 m_1)^2(1-B_0^*(\lambda^-))\beta_0^{(2)} \\ & + 3\lambda^+\omega_3^2 m_2(1-B_0^*(\lambda^-))\beta_0^{(1)}] \end{aligned}$$

$$D_1'' = -2\lambda^+\omega_3 m_1 D'$$

$$D_1''' = -3\lambda^+\omega_3[\omega_3 m_2 D' + m_1 T_{12}]$$

$$T_{12} = -\frac{\lambda^+}{\lambda_1^+}\left(\omega_1^2 m_2 + 2\omega_1 m_1 T_2\right)\left(1 - A^*(\lambda_1^+)\right) - T_9$$

- Mean number of customers in the orbit when the server is under repair during optional services is

$$\begin{aligned} L_R &= \lim_{z\to 1}\frac{d}{dz}R(z) \\ &= \frac{D_1'' N_R''' - N_R'' D_1'''}{3D_1''^2} \end{aligned} \tag{4.70}$$

where

$$N_R'' = -2(\lambda^+ m_1)^2 \omega_1 \omega_3 I_0 A^*(\lambda_1^+) B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i\text{-}1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)}$$

$$\begin{aligned} N_R''' = & -\lambda^+\omega_3 m_1 I_0 A^*(\lambda_1^+)\{[3\lambda^+\omega_1\omega_3 m_2 + 4((\lambda^+ m_1)^2\omega_1\omega_3/\lambda^-) + 3\lambda^+\omega_3^2 m_2] \\ & B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i\text{-}1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)} - 4\lambda^+\omega_3 m_1 T_{11} \\ & + 3(\lambda^+\omega_3 m_1)^2 B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i\text{-}1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(2)}\} \end{aligned}$$

- Mean number of customers in the orbit when the server is on vacation is

$$
\begin{aligned}
L_V &= \lim_{z\to 1}\frac{d}{dz}V(z) \\
&= \frac{D_2'' N_V''' - N_V'' D_2'''}{3D_2''^{2}} \qquad (4.71)
\end{aligned}
$$

where

$$N_V'' = -2\nu(\lambda^+ m_1)^2\,\omega_1\,\omega_4\,I_0\,A^*(\lambda_1^+)\sum_{i=0}^{M}(\delta_i+q_i)\Lambda_i^*(\lambda^-)B_0^*(\lambda^-)\,\gamma_1$$

$$
\begin{aligned}
N_V''' = &-\nu\lambda^+\omega_1 m_1 I_0\,A^*(\lambda_1^+)\{3\lambda^+\omega_1\omega_4 m_2\sum_{i=0}^{M}(\delta_i+q_i)\Lambda_i^*(\lambda^-)B_0^*(\lambda^-)\,\gamma_1 \\
&+4\lambda^+\omega_4 m_1\gamma_1[(\delta_0+q_0)f_0^{(1)}+T_4]+3((\lambda^+\omega_4 m_1)^2\gamma_2 \\
&+\lambda^+\omega_4^2 m_2\gamma_1)\sum_{i=0}^{M}(\delta_i+q_i)\Lambda_i^*(\lambda^-)B_0^*(\lambda^-)
\end{aligned}
$$

$$D_2'' = -2\lambda^+\omega_4 m_1 D'$$

$$D_2''' = -3\lambda^+\omega_4[\omega_4 m_2 D' + m_1 T_{12}]$$

Let $N_q(z)$ be the numerator of $P_q(z)$.

- Expected orbit size is given by

$$
\begin{aligned}
L_q &= \lim_{z\to 1}\frac{d}{dz}P_q(z) \\
&= \frac{D_3''' \,N_q''''' - N_V''' D_q'''''}{4D_3'''^{\,2}} \qquad (4.72)
\end{aligned}
$$

where

$$N_q''' = I_0\,A^*(\lambda^+)6\omega_3\omega_4 m_1^2[1-T_2+\lambda^+\omega_1 m_1(T_5+T_6)+\nu B_0^*(\lambda^-)\sum_{i=0}^{M}(\delta_i+q_i)\Lambda_i^*(\lambda^-)\gamma_1]$$

$$
\begin{aligned}
N_q'''' = \; & I_0\,A^*(\lambda^+)12\{(\omega_3+\omega_4)\omega_3\omega_4 m_1 m_2[1-T_2+\lambda^+\omega_1 m_1 T_5]+\omega_3\omega_4 m_1^2[-T_9 \\
&+\lambda^+\omega_1(\omega_1 m_2+\frac{2\lambda^+\omega_2 m_1^2}{\lambda^-})T_5-\frac{2\lambda^+\omega_1 m_1}{\lambda^-}(f_0^{(1)}-T_3)]+\lambda^+\omega_1\omega_3\omega_4 m_1^2 \\
&[(\omega_1+\omega_4)m_2+4\lambda^+\omega_2 m_1^2]T_6-\lambda^+\omega_1\omega_3\omega_4 m_1^2[2m_1(f_0^{(1)}\beta_0^{(1)}-T_{11}) \\
&-\lambda^+\omega_3 m_1 T_{10}-\omega_3 m_2 T_6]+(\omega_1+\omega_3)\nu\lambda^+\omega_1\omega_3\omega_4 m_1^2 m_2 B_0^*(\lambda^-)\gamma_1 \\
&\sum_{i=0}^{M}(\delta_i+q_i)\Lambda_i^*(\lambda^-)-\nu\,\omega_1\omega_3 m_1^2((\delta_0+q_0)f_0^{(1)}+T_4) \\
&+\nu\omega_1\omega_3 m_1^2((\lambda^+\omega_4 m_1)^2\gamma_2+\lambda^+\omega_4^2 m_2\gamma_1)B_0^*(\lambda^-)\sum_{i=0}^{M}(\delta_i+q_i)\Lambda_i^*(\lambda^-)
\end{aligned}
$$

$$D_3''' = 6\,\omega_3\omega_4 m_1^2\, D'$$

$$D_3'''' = 3\,\omega_3\omega_4 m_1[4m_1T_{12} + 3(\omega_3+\omega_4)m_2D']$$

$$f_0^{(2)} = (\lambda^+ m_1)^2 \int_0^\infty x^2 e^{-\lambda^- x} b_0(x)dx + \lambda^+ m_2 \int_0^\infty x e^{-\lambda^- x} b_0(x)dx$$

$$f_{i,j_i}^{(2)} = (\lambda^+ m_1)^2 \int_0^\infty x^2 e^{-\lambda^- x} b_{i,j_i}(x)dx + \lambda^+ m_2 \int_0^\infty x e^{-\lambda^- x} b_{i,j_i}(x)dx$$

- Expected system size is given by

$$L_s = L_q + P_0 + P \tag{4.73}$$

4.5 Stochastic Decomposition

Theorem 4.1

The number of customers in the system L_s can be expressed as the sum of two independent random variables one of which is the mean number of customers (L_C) in the classical batch arrival G-queue with multistage and multi-optional services, feedback and admission control and the other is the mean number of customers in the orbit (L_D) given that the server is idle or on vacation.

Proof

The probability generating function $\phi(z)$ of the number of customers in the classical batch arrival G-queue with multistage and multi-optional services, feedback and admission control is

$$\phi(z) = \frac{[1-T_2]\,T_{13}}{[1+\sum\limits_{j_1=1}^{k_1} p_{j_1} f_0^{(1)} - T_4 - T_3]\;[z-T_1(z)]} \tag{4.74}$$

where

$$\begin{aligned} T_{13} = \; & z - T_1(z) + z\lambda^+ (a_1(z)-1)T_7(z)] - \lambda^- [(a_1(z)-1)/(g(z)(a_3(z)-1))]\,T_8(z) \\ & + \nu[(a_1(z)-1)/(a_4(z)-1)]\sum_{i=1}^{M} (\delta_i\, z + q_i)\,\Lambda_i^*(g(z))B_0^*(g(z))\,(1-V^*(h_2(z))) \end{aligned}$$

The probability generating function $\psi(z)$ of the number of customers in the orbit given that the server is idle or on vacation is

$$\psi(z) = \frac{I_0 + I(z) + V(z)}{I_0 + I + V}$$

$$= \frac{A^*(\lambda_1^+) D' \{z - T_1(z) + ((a_1(z) - 1) / (a_4(z) - 1)) \; \nu \sum_{i=0}^{M} (\delta_i \, z + q_i) \, \Lambda_i^*(g(z)) \, B_0^*(g(z)) \, (1 - V^*(h_2(z)))\}}{A^*(\lambda_1^+) [1 - T_2 + \nu \lambda^+ \omega_4 \, m_1 \sum_{i=0}^{M} (\delta_i + q_i) \, \Lambda_i^* (\lambda^+) \, B_0^*(\lambda^-) \, \gamma_1] \, D(z)} \qquad (4.75)$$

From equations (4.59), (4.74) and (4.75), we see that

$$P_s(z) = \phi(z) \, \psi(z) \qquad (4.76)$$

Differentiating equation (4.77) with respect to z and taking limit as $z \to 1$, we get

$$L_s = L_C + L_D$$

4.6 Reliability Measures

Theorem 4.2

The steady state availability of the server is given by

$$\mathcal{A} = \frac{I_0 \, A^*(\lambda_1^+) [1 - T_2 + \lambda^+ \omega_1 \, m_1 \, T_5]}{D'} \qquad (4.77)$$

Proof

$$\mathcal{A} = I_0 + \lim_{z \to 1} [I(z) + P_0(z) + P(z)]$$

Substituting the expressions of I(z), $P_0(z)$ and P(z) in the above equation, we obtain the result in equation (4.77).

Theorem 4.3

The steady state failure frequency of the server is

$$\mathcal{F} = \frac{I_0 \, A^*(\lambda_1^+) \; \lambda^- \lambda^+ \omega_1 \, m_1 \, T_5}{D'} \qquad (4.78)$$

Proof

$$\mathcal{F} \;=\; \lambda^{-} \lim_{z\to 1}\,[P_0(z)+P(z)]$$

Using the expressions P$_0$(z) and P(z) from the equations (4.53) and (4.54) and by direct calculation, we get the equation (4.78).

4.7 Special Cases

Case (i) : If $\lambda^{-}=0$, M = 0 and ν=0, then the system reduces to batch arrival retrial queue with feedback and admission control. In this case, the probability generating functions of the queue size distribution when the server is idle and busy are obtained as

$$I(z) \;=\; I_0\,(1-A^{*}(\lambda_1^{+}))\left(\frac{\dfrac{\lambda^{+}}{\lambda_1^{+}}(a_1(z)-a_{1,0})\,(\delta_0 z+q_0)\,B_0^{*}(\lambda^{+}(1-a_2(z)))-z}{D(z)}\right)$$

$$P_0(z) \;=\; \frac{I_0\,A^{*}(\lambda_1^{+})\,(a_1(z)-1)\,(1-B_0^{*}\,(g(z)))}{(1-a_2(z))\;D(z)}$$

where

$$D(z) \;=\; z-\left(A^{*}(\lambda_1^{+})+\frac{\lambda^{+}}{\lambda_1^{+}}\big(a_1(z)-a_{1,0}\big)(1-A^{*}(\lambda_1^{+}))\right)(\delta_0 z+q_0)\,B_0^{*}(\lambda^{+}(1-a_2(z)))$$

The above results agree with Wang and Zhou (2010) without starting failures.

Case (ii) : If $A^{*}(\lambda^{+})\to 1$, $C(z)\to z$, $\lambda^{-}=0$, ν=0, j_i = 1 (i=1,2,…,M) and $\omega_1=\omega_2=\omega_3=\omega_4=1$, then the model under study reduces to M/G/1 queueing model with M-phase optional services and Bernoulli feedback. In this case, the probability generating functions of the queue size distribution when the server is busy in essential and optional services are obtained respectively as

$$P_0(z) \;=\; \frac{I_0\;(1-B_0^{*}(\lambda^{+}(1-z)))}{\sum\limits_{i=0}^{M}(\delta_i z+q_i)\,\Lambda_i^{*}(\lambda^{+}(1-z))\;(1-B_0^{*}(\lambda^{+}(1-z)))-z}$$

$$P_i(z) \;=\; \frac{I_0\;p_i\,\Lambda_i^{*}(\lambda^{+}(1-z))\;B_0^{*}(\lambda^{+}(1-z))\;(1-B_i^{*}(\lambda^{+}(1-z)))}{\sum\limits_{i=0}^{M}(\delta_i z+q_i)\,\Lambda_i^{*}(\lambda^{+}(1-z))\;(1-B_0^{*}(\lambda^{+}(1-z)))-z}$$

where $\Lambda_i^*(\lambda^+(1-z)) = \prod_{l=1}^{i} p_l \, B_l^*(\lambda^+(1-z))$

The above results coincide with the results of Abdollahi and Salehi Rad (2012).

4.8 Practical Justification of the Model

This model has potential applications in the performance of Wireless Local Area Networks (WLAN) operating under transmission protocols like the CSMA/CD. In such a context, messages of a variable length arrive at the stations and then are divided into a number of packets (Positive Customers) in order to be transmitted to the destination station through wireless channels. Before transmission, the availability of the wireless channels will be detected according to specific transmission protocols such as IEEE 802.11. If the channel is found available, one packet is selected to be transmitted automatically and the rest are stored in a buffer (Orbit). On the other hand, if the server is busy, all the packets must be stored in the buffer and will retry their transmissions later (Retrial). In this framework, the unsuccessful transmitted messages are sent again (Feedback). But not all the packets can be transmitted successfully. To avoid this congestion, each individual incoming packet is admitted to join the system with some specified probability (Admission Control).

Each sensor node is capable of sensing the intended physical phenomenon and routing the monitoring information toward the destination node through multi-optional path with no infrastructure. All possible paths for the selected source and destination pair are divided into two phases, first phase of essential path and the second phase of multi-optional paths. In second phase, the multi-optional paths are subdivided into stages consisting of all possible node to node transmission. Here the selection of nodes in each stage is collectively known as the multistage and multi-optional services. After the completion of the first essential path, the transmitted data may reach the destination directly. Otherwise, it is allowed to pass through the stage by stage multi-optional paths.

Some sort of transmission disturbances related to radio propagation in wireless sensor networks is identified (Negative Customers). By computing the amplitude histograms of the received signal strength and characteristics of the histograms, the interference causes the signal degradation in radio link can be deduced (Repair).

When there are no incoming requests, the server may go for self-upgradation maintenance activity (Vacation).

4.9 Numerical Results

In this section, for the purpose of understanding the effect of parameters on the system performance measures, it is assumed that the retrial time, first phase service time, second phase service time, repair time during first phase, repair time during second phase and vacation time are exponentially distributed with parameters η, μ_0, μ_{i,j_i}, β_0, β_{i,j_i} and ν respectively and the batch size follows geometric distribution with mean $1/\sigma$.

The input parameters are $\lambda^+ = 1.7$, $\lambda^- = 0.4$, $\eta = 50$, $M = 3$, $k_1 = 2$, $k_2 = 3$, $k_3 = 2$, $p = 0.3$, $p_1 = [0.3, 0.2]$, $p_2 = [0.3, 0.2, 0.2]$, $p_3 = [0.4, 0.2]$, $\delta_0 = 0.3$, $\delta = [0.1, 0.3, 0.6]$, $q_0 = 0.2$, $q = [0.2, 0.1, 0.4]$, $\nu = 0.4$, $\mu_0 = 20$, $\beta_0 = 4$, $\mu_1 = [40, 30]$, $\mu_2 = [35, 42, 52]$, $\mu_3 = [25, 30]$, $\beta_1 = [1, 3]$, $\beta_2 = [5, 7, 10]$, $\beta_3 = [12, 14]$, $\gamma = 7$, $\theta = 0.5$, $\sigma = 0.5$, $\omega_1 = 0.6$, $\omega_2 = 0.7$, $\omega_3 = 0.6$, $\omega_4 = 0.4$.

Table 4.1 shows the way in which the orbit size changes for different values of retrial rate (η), essential service rate (μ_0) and admission probability (ω_1) at idle state of the server. From the table it is observed that mean orbit size moderately decreases with increase in retrial rate and essential service rate and mean orbit size sharply increases with increase in ω_1.

Table 4.2 reveals the changes in mean values L_I, L_{P_0}, L_P, L_{P_0}, L_R, L_V and L_s for different values of repair completion rate at first phase (β_0) and vacation probability (ν). As expected, the increase in β_0 decreases the mean values and increase in ν increases the mean values.

Performance measures are calculated by varying the state dependent admission probabilities ω_1, ω_2, ω_3 and ω_4 and presented in the Table 4.3 to 4.6. From the results it is observed that, increase in admission probabilities gives a decreasing trend in I_0 and an increasing trend on the other performance measures.

Table 4.1 Mean Orbit Size for varying values of η , μ_0 and ω_1

η	μ_0	**Mean Orbit Size (L_q)**			
		$\omega_1 = 0.6$	$\omega_1 = 0.7$	$\omega_1 = 0.8$	$\omega_1 = 0.9$
20	**20**	3.0926	4.5496	7.6046	18.1964
	25	2.7803	3.7405	5.2872	8.2195
	30	2.6445	3.4383	4.6034	6.4965
25	**20**	2.4950	3.2507	4.3667	6.1927
	25	2.3664	2.9598	3.7488	4.8566
	30	2.3027	2.8301	3.5005	4.3872
30	**20**	2.2247	2.759	3.4545	4.4021
	25	2.1606	2.6130	3.1662	3.8623
	30	2.1254	2.5416	3.0361	3.6368
35	**20**	2.0706	2.5004	3.0244	3.6804
	25	2.0375	2.4170	2.8598	3.3859
	30	2.0170	2.3733	2.7802	3.2520
40	**20**	1.9710	2.3409	2.7741	3.2904
	25	1.9556	2.2911	2.6708	3.1064
	30	1.9437	2.2629	2.6181	3.0180

Table 4.2 Mean Values by varying β_0 and ν

β_0	ν	L_I	L_{P_0}	L_P	L_{R_0}	L_R	L_V	L_s
4	**0.4**	0.1035	0.1199	0.1849	0.0211	0.0101	0.1985	2.2993
	0.6	0.1450	0.2223	0.3202	0.0310	0.0159	0.4679	3.2358
	0.8	0.2578	0.5038	0.6937	0.0589	0.0321	1.2555	5.1821
6	**0.4**	0.1000	0.1118	0.1742	0.0113	0.0096	0.1894	2.2678
	0.6	0.1386	0.2067	0.2994	0.0175	0.0150	0.4414	3.1618
	0.8	0.2380	0.4547	0.6284	0.0339	0.0293	1.1445	4.9185
8	**0.4**	0.0986	0.1086	0.1699	0.0075	0.0094	0.1858	2.2560
	0.6	0.1359	0.2003	0.2909	0.0120	0.0146	0.4306	3.1319
	0.8	0.2298	0.4344	0.6014	0.0237	0.0281	1.0987	4.8101
10	**0.4**	0.0979	0.1069	0.1677	0.0056	0.0094	0.1839	2.2501
	0.6	0.1345	0.1968	0.2863	0.0091	0.0144	0.4247	3.1161
	0.8	0.2254	0.4234	0.5867	0.0181	0.0274	1.0738	4.7515
12	**0.4**	0.0974	0.1059	0.1664	0.0044	0.0093	0.1828	2.2467
	0.6	0.1336	0.1947	0.2835	0.0073	0.0143	0.4211	3.1064
	0.8	0.2226	0.4165	0.5776	0.0147	0.0270	1.0583	4.7150

Table 4.3 Performance Measures by varying ω_1

ω_1	I_0	I	P_0	P	R_0	R	V
0.2	0.5306	0.0346	0.1202	0.1598	0.0120	0.0070	0.1357
0.4	0.3357	0.0583	0.1676	0.2228	0.0168	0.0097	0.1892
0.6	0.2314	0.0710	0.1929	0.2565	0.0193	0.0112	0.2178
0.8	0.1664	0.0789	0.2087	0.2775	0.0209	0.0121	0.2356
1	0.1221	0.0843	0.2194	0.2918	0.0219	0.0127	0.2478

Table 4.4 Performance Measures by varying ω_2

ω_2	I_0	I	P_0	P	R_0	R	V
0.2	0.3959	0.0510	0.1529	0.2034	0.0153	0.0088	0.1727
0.4	0.3390	0.0579	0.1668	0.2217	0.0167	0.0096	0.1883
0.6	0.2708	0.0662	0.1833	0.2438	0.0183	0.0106	0.2070
0.8	0.1876	0.0763	0.2035	0.2706	0.0204	0.0118	0.2298
1	0.0837	0.0889	0.2288	0.3042	0.0229	0.0132	0.2583

Table 4.5 Performance Measures by varying ω_3

ω_3	I_0	I	P_0	P	R_0	R	V
0.2	0.2472	0.0691	0.1891	0.2514	0.0189	0.0109	0.2135
0.4	0.2393	0.0700	0.1910	0.2539	0.0191	0.0110	0.2156
0.6	0.2314	0.0710	0.1929	0.2565	0.0193	0.0112	0.2178
0.8	0.2232	0.0720	0.1949	0.2591	0.0195	0.0113	0.2200
1	0.2149	0.0730	0.1969	0.2618	0.0197	0.0114	0.2223

Table 4.6 Performance Measures by varying ω_4

ω_4	I_0	I	P_0	P	R_0	R	V
0.2	0.2851	0.0645	0.1798	0.2391	0.018	0.0104	0.2031
0.4	0.2314	0.071	0.1929	0.2565	0.0193	0.0112	0.2178
0.6	0.1692	0.0785	0.208	0.2766	0.0208	0.012	0.2349
0.8	0.0965	0.0874	0.2257	0.3001	0.0226	0.0131	0.2548
1	0.0102	0.0979	0.2466	0.3279	0.0247	0.0143	0.2784