A STUDY OF DERIVATIONS AND INTUITIONISTIC L-FUZZY G-SUBALGEBRAS OF G-ALGEBRAS

REVATHI R

(17PMA017)

Thesis Submitted to
Avinashilingam Institute for Home Science and Higher Education for Women
Coimbatore-641 043

In Partial Fulfilment of the Requirement for the Degree of

Master of Science in Mathematics

April, 2019

A STUDY OF DERIVATIONS AND INTUITIONISTIC L-FUZZY G-SUBALGEBRAS OF G-ALGEBRAS

REVATHI R

(17PMA017)

Thesis Submitted to
Avinashilingam Institute for Home Science and Higher Education for Women
Coimbatore-641 043

In Partial Fulfilment of the Requirement for the Degree of

Master of Science in Mathematics

April, 2019

P. Jeyalaksl

Signature of the

Head of the Department

P. Jeyalaksl

Signature of the

Supervisor

ACKNOWLEDGEMENT

First and foremost, I am extremely thankful to the **LORD ALMIGHTY** for his graces and blessings showered on me.

I take immense pleasure in thanking **Dr.P.R.KRISHNAKUMAR,** Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for providing the conducive infrastructure for the conduct of the research study.

I would like to thank **Dr.(Tmt.)PREMAVATHY VIJAYAN,** Vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for providing the opportunity to develop and establish my skills.

I extend my heartfelt thanks to **Dr. (Tmt.) S. KOWSALYA,** Registrar, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the encouragement given by her during the investigation.

My words never fail to express my deep sense of gratitude to **Hony.COL. Dr. (Tmt.) SAROJA PRABHAKARAN,** Former Vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, and The Director , Halls of Residence, Avinashilingam Education Trust Institution Hostel for Women, Coimbatore, for all the necessary support and guidance towards the completion of the study.

I express my heartfelt thanks to **Dr. (Tmt.) K.UDAYA CHANDRIKA,** Professor of Mathematics, Dean, School of Physical Sciences and Computational Sciences, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her excellent support, unflinching encouragement and guidance during the course of the investigation.

I express my heartfelt thanks to **Dr. (Tmt.) P.JEYALAKSHMI,** Professor and Head, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her inspiring guidance, innovative ideas, critical suggestions and constant encouragement throughout the completion of this work.

I would like to express my sincere thanks to all the **Staff Members of the Department of Mathematics** who were responsible for the good finish of this dissertation.

I owe my special thanks to my **Beloved Parents, Friends and also the Graceful relatives,** who helped me by providing full strength, support and encouragement to complete my project successfully.

CONTENT PAGE NO

INTRODUCTION

In 1965, Zadeh [41] introduced the concept of fuzzy set which is a generalization of an ordinary set. In 1999 the notion of intuitionistic fuzzy set (IFS) was defined by Atanassov [12] as a generalization of Zadeh's fuzzy set. In 1984 this concept was generalized to "Intuitionistic L -fuzzy sets" by Atanassov and Stoeva [10]. In intuitionistic fuzzy set theory, membership and non-membership values are drawn from the unit interval and intersection, union are modeled by minimum and maximum respectively. Atanassov and Stoeva replaced the structure of the totally ordered unit interval by an arbitrary bounded lattice L to allow incomparabilities among elements.

In 1966, Y. Imai and K. Iseki [17] introduced two classes of abstract algebras: BCK - algebras and BCI - algebras. These algebras have been extensively studied since their introduction. In 1999 Ahn et al. [3] introduced a new notion, called QS - algebras and discussed some properties of the G -part of QS - algebras. In 2012 Bandaru and Rafi [13] introduced a new notion, called G - algebras, which is a generalization of QS – algebras and discussed relationship between these algebras with other related algebras such as Q - algebras, BCI - algebras, BCH - algebras, BF - algebras and B - algebras. They introduced the concept of 0 - commutative, G - part and medial of G - algebras and studied their related properties.

In 2014, Senapati et al.[39] applied the L -fuzzy notions to G - subalgebras and introduced the notions of L - fuzzy G - subalgebras of G - algebras.

This thesis is devoted to "Study of derivations and intuitionistic L - fuzzy G - subalgebras of G - algebras".

The following articles are chosen for our discussion:

1. "ON G-ALGEBRAS" by Bandaru, R.K., [13].
2. "NOTES ON G-ALGEBRAS AND ITS DERIVATIONS" by AL-Kadi, D., and Hosny, R., [7].
3. "f_q – DERIVATIONS OF G-ALGEBRAS" by AL-Kadi, D., [6].
4. "ON (f,g)- DERIVATIONS OF G-ALGEBRAS" by AL-Kadi, D., [8].
5. "ON INTUITIONISTIC FUZZY G-SUBALGEBRAS OF G-ALGEBRAS" by Jana, C., Pal, M., Senapati, T., Bhowmik, M.,[22].
6. "L- FUZZY G-SUBALGEBRAS OF G-ALGEBRAS" by Senapati, T., Jana, C., Bhowmik, M., Pal, M., [36].

7. “ATANASSOV’S L- INTUITIONISTIC FUZZY G-SUBALGEBRAS OF G-ALGEBRAS” by Jana, C., Pal, M., Senapati, T., Bhowmik, M., [20].

This thesis divided into four chapters.

In the first section of first chapter, preliminary definitions and properties of G-algebras are collected. In the second section of this chapter, the concept of 0 - commutative, G - part and medial of a G - algebras are studied and their properties are discussed.

The important results studied in this chapter are given as follows:

1. Let $(X,*,0)$ be a G-algebra. Then the following are equivalent.

   (i) $(x * y) * z = (x * z) * y$ for all $x ,y ,z \in X$

   (ii) $(x * y) * (x * z) = z * y$ for all $x, y, z \in X$.

2. If S is a subalgebra of a G - algebra $(X, *, 0)$, then $G(X) \cap S = G(S)$.

3. A G-algebra X is medial if and only if it satisfies the following conditions:

   (i) $y * x = 0 * (x * y)$ for all $x, y \in X$

   (ii) $x * (y * z) = z * (y * x)$ for all $x ,y, z \in X$.

4. Every G-algebra $(X, *,0)$ satisfying the associative law is a group under the operation “$*$ ”.

Chapter 2 deals with the study of derivations of G-algebras. In the first section of this chapter, properties of derivations of a G - algebras are studied .In the second section, the concept of f_q – derivations of G - algebras are studied and their properties are obtained. In the third section, the concepts on (f,g) –derivations of G-algebras are studied and their properties are discussed. Also the concept of regular (f,g) – derivations of G-algebras are provided and some results are obtained. Moreover, a condition for the composition of two (f,g) – derivations of a G- algebras to be a (f,g) – derivations is given.

The interesting results studied in this chapter are given as follows:

1. Let X be a G-algebra and d_1,d_2 be two derivations of X. Then $d_1 \circ d_2$ is a

derivation of X.

2. Let X be an associative medial G-algebra, then d_q^f is a f_q - derivation of X, for all $q \in X$.
3. Let d be a (r,l)-(f,g)-derivation of a G-algebra X. Then d(x) = f(x) ∧ d(x), for all x ∈ X if and only if d is regular.
4. Let X be a G-algebra. If d_1 and d_2 are (f,g)-derivations of X such that $f^2 = f$. Then $d_1 \circ d_2$ is an (f, g)-derivation of X.

Chapter 3 deals with the study of intuitionistic fuzzy G-subalgebras of G-algebras.

In the first section of this chapter, preliminary definitions of fuzzy sets and intuitionistic fuzzy sets are presented. In the second section, the intuitionistic fuzzification of G-subalgebras and their related properties are investigated. Also some characterization of intuitionistic fuzzy G-algebras are given. The G-subalgebras are classified by their family of level subalgebras of G-algebras.

The interesting results discussed in this chapter are given as follows:

1. Let A_1 and A_2 be two intuitionistic fuzzy G -subalgebras of X .Then $A_1 \cap A_2$ is an intuitionistic fuzzy G -subalgebra of X .
2. Let A = (α_A , β_A) be an IFS in X, such that the sets $\cup(\alpha_A ; s)$ and $L(\beta_A ; t)$ are G -subalgebras of X for every s, t ∈ [0,1].Then A=(α_A , β_A) is an intuitionistic fuzzy G -subalgebra of X .
3. Let $T_1 \supseteq T_2 \supseteq T_3 \ldots\ldots$ be a descending chain of G -subalgebras of X which terminates at finite step. For an intuitionistic fuzzy G - subalgebra A=$(\alpha_A , \beta_A)$of X , if a sequence of elements of Im(α_A) is strictly increasing and Im(β_A) strictly decreasing, then A=$(\alpha_A , \beta_A)$is finite valued.

Chapter 4 deals with the study of intuitionistic L-fuzzy G-subalgebras of G-algebras. In the first section of this chapter, the L- fuzzification of G-subalgebras are considered and some related properties are investigated. A characterization of L-fuzzy G-algebras are given .The G-subalgebras are classified by their family of level subalgebras of G-algebras. In the second section of this chapter, the intuitionistic L - fuzzification of G-subalgebras and some related properties are investigated. A characterization of intuitionistic L-fuzzy G-algebras are given. The G-subalgebras are classified by their family of level subalgebras of G-algebras.

The interesting results studied in this chapter are given as follows:

1. Let B be a non-empty subset of X and A be a L-fuzzy set in X defined by

$$\alpha_A(\mathrm{x}) = \begin{cases} \lambda, \text{ if x} \in B \\ \tau \text{ , otherwise} \end{cases}$$

   For all $\lambda, \tau \in L$ with $\lambda \geq \tau$. Then A is a L-fuzzy G -subalgebra of X if and only if B is a G -subalgebra of X. Moreover, $I_{\alpha A}$ = B.

2. Let X be a G -algebra. Then given any chain of G – subalgebras $P_0 \subset P_1 \subset P_2 \subset \ldots\ldots\ldots\ldots \subset P_r = X$, there exists a L-fuzzy G -subalgebra A of X whose level subalgebras are exactly the G -subalgebras of this chain.
3. Let A be a L-fuzzy G -subalgebra of a G -algebra X. Then

   (i) Two L-upper s-level subalgebras $\cup(\alpha_A\,; s_1)$, $\cup(\alpha_A\,; s_2)$ (with $s_1 < s_2$)

   of A are equal if and only if there is no x$\in X$ such that $s_1 \leq \alpha_A(\mathrm{x}) < s_2$.

   (ii) Two L-lower t -level subalgebras $L(\beta_A\,; t_1)$, $L(\beta_A\,; t_2)$ (with $t_1 > t_2$)

   of A are equal if and only if there is no x$\in X$

   such that $t_1 \geq \alpha_A(\mathrm{x}) > t_2$.

4. Let A=(α_A , β_A) and B=(α_B , β_B) be two intuitionistic fuzzy G -subalgebras of a finite G -algebra X with identical family of level subalgebras.
   If $\mathrm{Im}(\alpha_A) = \{l_0, l_1, \ldots\ldots.., l_r\}$, $\mathrm{Im}(\alpha_B) = \{s_0, s_1, \ldots\ldots.., s_k\}$,

Im$(\beta_A) = \{m_0, m_1, \dots \dots \dots, m_r\}$ and Im$(\beta_A) = \{t_0, t_1, \dots \dots, t_k\}$

where $l_0 > l_1 > \dots \dots > l_r$, $s_0 > s_1 > \dots \dots \dots s_k$,

$m_0 < m_1 < \dots \dots < m_r$ and $t_0 < t_1 < \dots \dots < t_k$ then we have

(i) r=k

(ii) U$(\alpha_A; l_i)$= U$(\alpha_B; s_i)$ and L$(\beta_A\,; m_i)$= L$(\beta_A\,; t_i)$, $0 \leq$ i $\leq k$,

(iii) If x$\in X$ such that α_A(x) $= l_i$ and β_A(x) $= m_i$, then α_A(x) $= s_i$ and β_A(x) $= t_i$, $0 \leq$ i $\leq k$.

REVIEW OF LITERATURE

In 1965 the concept of fuzzy set was initiated by Zadeh [41]. After the introduction of fuzzy sets by Zadeh [41], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [11] is one among them. Later this concept was generalized to “Intuitionistic L -fuzzy sets” by Atanassov and Stoeva [10]. In intuitionistic fuzzy set theory, membership and non-membership values are drawn from the unit interval and intersection, union are modeled by minimum and maximum respectively. Atanassov and Stoeva replaced the structure of the totally ordered unit interval by an arbitrary bounded lattice L to allow incomparabilities among elements.

In 1966, Y. Imai and K. Iseki [17] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. These algebras have been extensively studied since their introduction. In 1983, Hu and Li [16] introduced the notion of a BCH-algebra which is a generalization of the notion of BCK and BCI-algebras and studied a few properties of these algebras. In 1999 Ahn et al. [3] introduced a new notion, called QS - algebras and discussed some properties of the G -part of QS -algebras.

In 2001, J. Neggers, S. S. Ahn and H.S. Kim [32] introduced a new notion, called a Q-algebra and generalized some theorems discussed in BCI/BCK-algebras. In 2002, J. Neggers and H. S. Kim [33] introduced a new notion, called a B-algebra and obtained several results. In 2007, A. Walendziak [39] introduced a new notion, called a BF-algebra which is a generalization of B-algebra.

In 2004 the notion of derivations in rings and near-rings theory was applied to BCI-algebra by Jun and Xin [25]. Many researches have been done on derivations of BCI- algebra in different aspects. In 2012,(α,β)-derivations of BCI-algebra was introduced by G. Muhiuddin and M. AI –Roqi [29] and some related properties were investigated. Also, the notion of t-derivations of BCI-algebra was given and the study was extended to t-derivations of a p-semisimple BCI-algebra [28].

In 2012 Bandaru and Rafi [13] introduced a new notion, called G -algebras, which is a generalization of QS –algebras and discussed relationship between these algebras with other related algebras such as Q - algebras, BCI - algebras, BCH -algebras, BF - algebras and B -algebras. They introduced the concept of

0-commutative, G -part and medial of G -algebras and studied their related properties. In 2014, Senapati et al. [38] applied the L -fuzzy notions to G-subalgebras and introduced the notions of L -fuzzy G - subalgebras of G -algebras.

Here we present abstracts of some important articles on various algebras related to this topic.

1. An introduction to the theory of BCK –algebras

K. Iseki and S. Tanaka (1978) [20]

In this article the definition of BCK –algebras and its fundamental properties are studied. Various ideals of BCK –algebras are discussed in detailed manner. Also the homomorphism properties on BCK –algebras are discussed.

2. On BCI - algebras

I . Iseki (1980) [19]

In the article the authors proved that, for a BCI-algebra X with condition(S), if the p-semi simple part of X is a subalgebra of X then X is isomorphic to direct product of a BCK algebra with conditions and a p-semisimple BCI algebra, and obtained other results on such algebras. Moreover they showed that the concepts of regular ideals, closed p-ideals and stong ideals coincide.

3. On QS - algebras

S.S . Ahn and H.S. kim (1999) [3]

In this article, the authors introduced a new notion, called an QS-algebra, which is related to the areas of BCI /BCK-algebras and discussed the G- part of QS-algebra.

4. Intuitionistic fuzzy ideals of BCK-algebras

Y . B. Jun and K . H. Kim (2000) [24]

The authors considered the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigated some of their properties. They introduced the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigated some related properties.

5. On derivations of BCI-algebras:

Y.B Jun and X. L.Xin (2004) [25]

The notion of left–right (resp. right–left) derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular derivation the authors gave characterizations of a *p*-semisimple BCI-algebra. They also give a condition for a derivation to be regular.

6. On *f* -derivations of BCI-algebras

J. Zhan and Y.L. Liu (2005) [42]

The notion of left-right (resp., right-left) *f* -derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular *f*-derivation, the authors gave characterizations of a *p*-semisimple BCI-algebra.

7. Some results on derivations of BCI-algebras

H.A. S. Abujabal and N .O. Al-Shehri (2006) [2]

In this note, the authors investigated some fundamental properties and proved some results on derivations of BCI-algebras.

8. Intuitionistic Fuzzy Lie Algebras

M. Akram and K. P. Shum (2007) [4]

The authors introduced the concept of intuitionistic fuzzy Lie subalgebras of a Lie algebra and investigate some of their properties. The Cartesian product of intuitionistic fuzzy Lie subalgebras will be discussed. In particular, the homomorphisms between the Lie subalgebras of a Lie algebra and their relationship between the domains and the co-domains of the fuzzy subalgebras under these homomorphisms will be investigated.

9. On left derivation of BCI algebra

H.A.S. Abujabal and N.O. Al-Shehri (2007) [1]

In this article, the authors introduced the notion of left derivation of a *BCI* algebra and investigated some related properties. A condition for left derivation to be regular is given. Finally, they gave a characterization of a *p*-semisimple *BCI*-algebra which admits left derivation.

10. On Derivations of BCC-algebras

C. Prabpayak and U. Leerawat (2009) [34]

In this article, the notions of left-right (resp. right-left) derivations of BCC-algebras are studied and some properties on derivations of BCC-algebras are investigated. Regular derivations and the d-invariant on ideals of BCC-algebras are also considered.

11. Derivations of B- algebras

N.O. AL-Shehrie (2010) [9]

The notion of left-right (resp. right-left) derivation of B-algebra is introduced and some related properties are investigated. Also the notion of derivation of 0-commutative B-algebra is studied and some of its properties are investigated.

12. On (α,β) derivation in BCI - algebra

G. Muhiuddin and A. M. Al-Roqi (2012) [29]

The notion of (regular) (α, β)-derivations of a BCI-algebra X is introduced, some useful examples are discussed, and related properties are investigated. The condition for a (α, β)-derivation to be regular is provided. The concepts of a $d_{(\alpha,\beta)}$-invariant (α, β)-derivation and α-ideal are introduced, and their relations are discussed. Finally, some results on regular (α, β)-derivations are obtained.

13. On *t*-Derivations of BCI-Algebras

G. Muhiuddin and A. M. Al-Roqi (2012) [28]

In this article, the authors introduced the notion of t-derivation of a BCI-algebra and investigated related properties. Moreover, they studied t-derivations in a p-semisimple BCI-algebra and established some results on t-derivations in a p-semisimple BCI-algebra.

14. Interval-valued Intuitionistic Fuzzy BG –subalgebras

T. Senapati, M. Bhowmik and M. Pal (2012) [35]

In this article, the authors introduced the notion of interval-valued intuitionistic fuzzy subalgebras of BG -algebra. Also the concept of homomorphism of interval-valued intuitionistic fuzzy BG -subalgebras and some of their properties are discussed.

15. A new kind of derivations in BCI-algebras

K .J. Lee (2013) [27]

A new kind of derivation in BCI-algebras is introduced, and related properties are investigated. For a self map d_{f_q} of a BCI-algebra X, conditions for the kernel of d_{f_q} to be both a subalgebra and an ideal of X are provided.

16. Generalized Derivations of BCC-algebras

S. M. Bawazeer, N.O. Al - shehri, and Rawia Saleh Babusail (2013) [15]

The notion of generalized derivations of BCC-algebras is introduced, and some related properties are investigated. Also, the authors considered regular generalized derivations and the D-invariant on ideals of BCC-algebras.

17. Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy H - ideals in BCK/BCI-algebras.

T. Senapati, M. Bhowmik and M. Pal (2013) [37]

In this article, the concepts of intuitionistic fuzzy translation to intuitionistic fuzzy H-ideals in BCK/BCI-algebras are introduced. The notion of intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy H-ideals with several related properties are investigated. Also the relationships between intuitionistic fuzzy translations, intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy H-ideals are investigated.

18. On Generalized (α, β) Derivations in BCI-algebras

A. M. AL – Roqi (2014) [5]

The notion of generalized (regular) (α, β)- derivations of a BCI-algebra is introduced some useful examples are discussed and related properties are investigated. The conditions for a generalized (α, β) - derivation to be regular is provide. The concepts of a generalized F invariant (α, β) - derivation and α- ideals are introduced, and their relations are discussed. Moreover, some results on regular generalized (α, β) - derivations are proved.

19. On Generalized Left Derivations in BCI-Algebras

G. Muhiuddin and A. M. Al-Roqi (2014) [30]

In this article, the authors introduced the notion of generalized left derivation of a BCI-algebra X, construct several examples, and investigated related properties. They established some results on regular generalized left derivation. Furthermore, for a generalized left derivation H, the concept of a H-invariant generalized left derivation is introduced, and examples are discussed. Using this concept a condition for a generalized left derivation to be regular is provided. Finally, some results on p-semisimple BCI-algebra are obtained.

20. On Generalized Derivations of BCI-algebras and Their Properties

L. Kamali Ardekani and B. Davvaz (2014) [26]

The authors introduced the concept of (f ,g)-derivations of BCI-algebras and they investigated some fundamental properties and established some results on (f,g) - derivations. Also, they treated to generalization of right derivation and left derivation of BCI-algebras and considered some related properties.

21. Intuitionistic Q-fuzzy Ideals of BG-Algebras

S. R. Barbhuiya (2014) [14]

The concept of intuitionistic fuzzy subset was introduced by K .T. Atanassov and the notion of intuitionistic fuzzy ideals of BG-algebra was developed by A. Zarandi and A. Borum and Saeid in 2005.In this article, for a set Q, the notion of intuitionistic Q-fuzzy ideals of BG-algebra is introduce and investigated some of their basic properties.

22. Cubic *G*-subalgebras of *G*-algebras

C. Jana and T. Senapati (2015) [23]

In this article, the notion of cubic *G*-subalgebras of *G*-algebras are introduced. Some characterization of cubic *G*-subalgebras of *G*-algebras are given. The homomorphic image and inverse image of cubic *G*-subalgebras are studied and investigated some related properties.

CHAPTER- 1

PRELIMINARIES OF G-ALGEBRAS AND ASSOCIATIVE G-ALGEBRAS

Section 1.1

Preliminary definitions and properties of G-algebras

Definition: 1.1.1

A BCI - algebra $(X,*,0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

BCI_1: $((x * y) * (x * z)) * (z * y) = 0$

BCI_2: $(x * (x * y)) * y = 0$

BCI_3: $x * x = 0$

BCI_4: $x * y = 0$ and $y * x = 0$ implies $x = y$

BCI_5: $x * 0 = 0$ implies $x = 0$

Definition: 1.1.2

A BCK - algebra $(X,*,0)$ is a non - empty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

BCK_1 : $((x * y) * (x * z)) * (z * y) = 0$

BCK_2: $(x * (x * y)) * y = 0$

BCK_3 : $x * x = 0$

BCK_4 : $x * y = 0$ and $y * x = 0$imply $x = y$

BCK_5 : $0 * x = 0$

Note:

Every BCK - algebra is a BCI - algebra but not conversely.

Definition: 1.1.3

A BCH - algebra $(X,*,0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

BCH_1: $x * x = 0$

BCH_2: $x * y = 0$ and $y * x = 0$ imply $x = y$

BCH_3: $(x * y) * z = (x * z) * y$

Note:

Every BCI-algebra is a BCH-algebra but not conversely.

Definition: 1.1.4

A Q - algebra $(X,*,0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

Q_1: $x * x = 0$

Q_2: $(x * y) * z = (x * z) * y$

Q_3: $x * 0 = x$

Note:

Every BCH - algebra is a Q - algebra but not conversely.

Definition: 1.1.5

A Q - algebra X is said to be a QS - algebra if it satisfies the additional relation

QS_1: $(x * y) * (x * z) = z * y$ for any $x, y, z \in X$.

Definition: 1.1.6

A B – algebra $(X,*,0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

B_1 : $x * x = 0$

B_2 : $x * 0 = x$

B_3 : $(x * y) * z = x *(z * (0*y))$

A B - algebra X is said to be 0-commutative if $a * (0 * b) = b * (0 * a)$

for any $a, b \in X$.

Note:

Q - algebras and B - algebras are different notions.

Definition: 1.1.7

A BF - algebra $(X,*,0)$) is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

BF_1: $x * x = 0$

BF_2: $x * 0 = x$

BF_3 : $0 * (x * y) = (y * x)$

Note:

Every B-algebra is BF-algebra but not conversely.

Definition: 1.1.8

An algebra $(X,*,0)$ is called a BP-algebra if it satisfies the following conditions:

BP_1: $x* x = 0$

BP_2: $x*(x* y) = y$

BP_3: $(x *z)*(y *z)= x* y$, for all $x, y, z \in X$.

Note :

In BCI / BCK / BCH / Q / QS / B /BF – algebra, $x \leq y$ is defined by $x * y = 0$

Definition: 1.1.9

A G-algebra $(X,*,0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying axioms:

G_1: $x * x = 0$

G_2: $x * (x * y) = y$ for all $x, y, z \in X$.

Example: 1.1.10

Let $X = R - \{-n\}, 0 \neq n \in Z+$ where R is the set of all real numbers and Z^+ is the set of all positive integers. Define a binary operation $*$ on X by

$$x * y = \frac{n(x-y)}{n+y}$$

Then $(X,*,0)$ is a G - algebra.

Note:

(i) Every commutative B-algebra is a G-algebra but converse need not be true

(ii)Every QS-algebra is a G-algebra but converse need not be true.

Example: 1.1.11

Let $X = \{0,1,2\}$ in which $*$ is defined by

*	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Then $(X,*,0)$ is a G-algebra but not a QS-algebra because

$(0 * 1) * 2 = 1 * 2 = 2 \neq 1 = 2 * 1 = (0 * 2) * 1$

Example: 1.1.12

Let X = {0,1,2,3,4,5,6,7} in which $*$ is defined by

$*$	0	1	2	3	4	5	6	7
0	0	2	1	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	2	1	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Then $(X,*,0)$ is a G-algebra which is not a BCK/BCI/BCH/Q/QS/B - algebras.

Example: 1.1.13

Let X = {0,1,2,3} in which $*$ is defined by

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X,*,0)$ is a Q-algebra, but not a G-algebra.

Note:

The two axioms G_1 and G_2 are independent. For consider the set X = {0,1,2} with the following table:

$*$	0	1	2
0	0	1	2
1	1	1	2
2	2	1	2

Table(i)

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

Table(ii)

The set X with table (i) the axiom G_2 holds but not G_1, since $2 * 2 \neq 0$.

Similarly, the set X with table(ii) the axiom G_1 but not G_2,

Since $1 * (1 * 2) = 1 * 1 = 0 \neq 2$.

Note:

In any G-algebra, define a partial order $\leq$ by putting $x \leq y$ if and only if $y * x=0$. This means that a G-algebra can be considered as a partially ordered set with $x \leq x$ and $x \leq y* (y*x)$.

Proposition: 1.1.14

If $(X,*,0)$ is a G-algebra, then for any x, y $\in X$ the following conditions hold:

P_1: $x * 0 = x$

P_2: $0 * (0 * x) = x$

P_3: $(x * (x * y)) * y = 0$

P_4: $x * y = 0$ implies $x = y$

P_5: $0 * x = 0 * y$ implies $x = Y$

Proof

Let $(X,*,0)$ be a G− algebra and $x ,y \in X$.

Then $x * 0 = x * (x * x) = x$ (by G_2). Put $x = 0$ and $y = x$ in $x * (x * y) = y$,

then $0 * (0 * x) = x$. Hence P_2 is obtained.

To Prove P_3: $(x * (x * y)) * y = y * y$ (by G_2) $= 0$ (by G_1).

To Prove P_4: Let $x * y = 0$. Then, by (P_1), we have $y = x * (x * y) = x * 0 =x$.

To Prove P_5: Let $0 * x = 0 * y$. Then $0 * (0 * x) = 0 * (0 * y)$ and hence $x = y$ by(P_2).

Theorem: 1.1.15

If $(X,*,0)$ be a G-algebra satisfying $(x * y) * (0 * y) = x$ for any $x,y \in X$ then $x * z = y * z$ implies $x = y$.

Proof

Let $(X,*,0)$ be a G-algebra and for any $x ,y \in X$. $(x * y) * (0 * y) = x$. Then $x * z = y * z \Rightarrow (x * z) * (0 * z) = (y * z) * (0 * z) \Rightarrow x = y$.

Note:

The following theorems shows that some relations between G−algebras and BCI/BCH/Q/BF-algebras.

Theorem: 1.1.16

(i) Every G- algebra satisfying $(x * y) * (x * z) = z * y$ is a

BCI / BCH / Q-algebra.

(ii) Every G- algebra satisfying $(x * y) * z = (x * z) * y$ is a BF-algebra.

Proof

Obvious.

Theorem: 1.1.17

Let $(X,*,0)$ be a G-algebra. Then the following are equivalent.

(i) $(x * y) * z = (x * z) * y$ for all $x ,y ,z \in X$

(ii) $(x * y) * (x * z) = z * y$ for all $x, y, z \in X$.

Proof

To Prove (i) ⇒ (ii) : Assume $(x * y) * z = (x * z) * y$ for all $x ,y ,z \in X$

Then $(x * y) * (x * z) = (x * (x * z)) * y$ by (BCH_3) $= z * y$ by (G_2) .

To Prove (ii) ⇒ (i) : Let $x ,y ,z \in X$.

Assume $(x * y) * (x * z) = z * y$ for all $x, y, z \in X$.

Then $(x * y) * z = (x * y) * (x * (x * z))$ by (G_2) $= (x * z) * y$.

Hence (i) and (ii) are equivalent.

In the following, we characterize G - algebra interms of Q − algebra.

Proposition: 1.1.18

Let $(X,*,0)$ be a G - algebra. Then the following are equivalent:

(i) X is a Q - algebra

(ii) X is a QS - algebra

(iii) X is a BCH - algebra.

Proof

Obvious.

Lemma: 1.1.19

Let $(X,*,0)$ be a G-algebra. Then $a * x = a * y$ implies $x = y$

for any $a, x, y \in X$.

Proof

Let $a, x, y \in X$. Then $a * x = a * y \Rightarrow a * (a * x) = a * (a * y) \Rightarrow x = y$.

Theorem: 1.1.20

Let $(X,*,0)$ be a G - algebra. Then the following are equivalent.

(i) $(x * y) * (x * z) = z * y$ for all $x, y, z \in X$

(ii) $(x * z) * (y * z) = x * y$ for all $x, y, z \in X$.

Proof

To Prove (i) $\Rightarrow$ (ii) : Let $x, y, z \in x$ and assume $(x * y) * (x * z) = z * y$.

Then, $(x * y) * ((x * y) * (x * z)) = (x * y) * (z * y) \Rightarrow x * z = (x * y) * (z * y)$

Hence (ii) is obtained by interchanging y and z.

To Prove (ii) $\Rightarrow$ (i) : Let $x, y, z \in X$ and assume $(x * z) * (y * z) = x * y$.

Then, by(G_2) $\Rightarrow (x * z) * (y * z) = (x * z) * ((x * z) * (x * y))$

$\Rightarrow$ (By the above lemma) $y * z = (x * z) * (x * y)$.

Definition: 1.1.21

A non-empty subset N of X is said to be normal of X if:

$(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$.

Note:

Denote $y * (y * x)$ by $x \wedge y$. Therefore by G_2 we have $x \wedge y = x$.

Proposition: 1.1.22

Let $(X,*,0)$ be a G-algebra. Then for any $x, y, z \in X$, we have:

(i) For $x \neq y$, $x \wedge y \neq y \wedge x$,

(ii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,

(iii) $x \wedge 0 = x$ and $0 \wedge x = 0$,

(iv) For $x \neq 0$, $x \wedge (y * z) \neq (x \wedge y) * (x \wedge z)$.

Proof

Obvious.

Theorem: 1.1.23

Let X be a set with $0 \in X$. Define a binary operation $*$ on X for all $x, y \in X$ by:

$$x * y = \begin{cases} x & \text{if } y = 0, \\ 0 & \text{if } y \neq 0,\ x = y, \\ y & \text{if } y \neq 0, x \neq y. \end{cases}$$

Then $(X,*,0)$ is a G-algebra.

Proof

Now we prove $x * (x * y) = y$, in the three cases as follows:

Case (i) If $x \neq y$ then $x * (x * y) = x * y = y$.

Case (ii) If $x = y$ then $x * (x * y) = x * (x * x) = x * 0 = x = y$.

Case (iii) If $y = 0$, we have $x * (x * y) = x * (x * 0) = x * x = 0 = y$.

Theorem: 1.1.24

Let $(X,*,0)$ be a G-algebra. Then every normal subset N of X is a subalgebra of X but the converse is not true.

Proof

Let $x, y \in N$, then by using the condition $x \neq y$, $x \wedge y \neq y \wedge x$, $x * 0$, $y*0 \in N$. Since N is normal, we have $(x* y) * (0*0) \in N$. That is, $x* y \in N$.

Therefore, N is a subalgebra of X. The converse is proved using the following example.

Example: 1.1.25

Consider the G-algebra given by the Cayley table

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Then $N = \{0,2\}$ is a subalgebra of X but is not normal of X as $0*2$, $1*2 \in N$ but $(0*1)*(2*2) = 1*0 = 1 \notin N$.

Definition: 1.1.26

Let $(X,* ,0)$ and $(Y, *' ,0')$ be G-algebras. A mapping $\varphi: X \rightarrow Y$ is called a homomorphism if $\varphi (x * y) = \varphi (x) *' \varphi (y)$, $\forall\, x,y \in X$. The G- homomorphism φ is said to be a G-monomorphism (resp., an G-epimorphism) if it is injective (resp., surjective).

If the map φ is both injective and surjective then X and Y are said to be isomorphic, written $X \cong Y$. For any G-homomorphism $\varphi : X \rightarrow Y$, the set $\{x \in X \mid \varphi (x) = 0'\}$ is called the kernel of φ and denoted by Ker φ.

Lemma: 1.1.27

Let $\varphi: (X,*,0) \rightarrow (Y, *' ,0')$ be a homomorphism of G-algebras, then

(i) $\varphi (0) = 0'$

(ii) Ker φ is a normal G-subalgebra of X

(iii) Im $\varphi = \{y \in Y \mid y = \varphi (x)$, for some $x \in X\}$ is a G-subalgebra.

Proof

Let φ: (X,∗,0) →(Y, ∗ ′ ,0’) be a G-homomorphism of G-algebras.

To Prove (i) : Using the condition x ∗ x = 0 , φ (0) = φ (0 ∗ 0) = φ (0) ∗’ φ (0) = 0’.

To Prove (ii) : Obviously, Ker φ ≠ 0, as 0 ∈ Ker φ . Let x ∗ y, a∗ b ∈ Ker φ .

Then φ (x) ∗’ φ (y) = φ (a) ∗’ φ(b) =0’. This implies φ (x) = φ (y) and φ (a) = φ (b). It follows that, φ ((x∗ a) ∗ (y ∗ b)) = φ (x ∗a) ∗’ φ (y ∗ b)

= (φ (x) ∗’ φ (a)) ∗’ (φ (x) ∗’ φ (a)) = 0’∗’ 0’ = 0’.

Consequently, (x ∗ a) ∗ (y ∗ b) ∈ Ker φ and so Ker φ is a normal G-subalgebra of X.

(iii) obvious.

Section: 1.2

G-part of G-algebras And Associative G-algebras

Definition: 1.2.1

A G-algebra (X, ∗ ,0) is said to be 0−commutative If x ∗ (0 ∗ y) = y ∗ (0 ∗ x)

for any x, y ∈ X.

Definition: 1.2.2

A non-empty subset S of a G-algebras X is called a subalgebra of X

if x ∗ y ∈ S for any x, y ∈ S.

Example: 1.2.3

Let X = {0,1,2} be a set with the following table:

∗	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then (X,∗,0) is a 0 - commutative G - algebra.

Theorem: 1.2.4

Let $(X, *, 0)$ be a 0-commutative G-algebra. Then $(0 * x) * (0 * y) = y * x$ for any $x, y \in X$.

Proof

Let $x, y \in X$. Then by the definition of 0-commutative

$(0 * x) * (0 * y) = y * (0 * (0 * x)) = y * x$.

Theorem: 1.2.5

Let $(X, *, 0)$ be a 0-commutative G-algebra satisfying $0 * (x * y) = y * x$. Then $(x * y) * (0 * y) = x$ for any $x, y \in X$.

Proof

Let $x, y \in X$ and $0 * (x * y) = y * x$.
Then $(x * y) * (0 * y) = y * (0 * (x * y)) = y * (y * x) = x$.

Definition 1.2.6

Let X be a G-algebra.

(i) For any subset S of X , define $G(S) = \{x \in S \mid 0 * x = x\}$

In particular, if $S = X$ then $G(X)$ is called the G-part of a G-algebra.

(ii)For any G-algebra X, the set $B(X) = \{x \in A \mid 0 * x = 0\}$ is called a p - radical of X.

(iii)A G-algebra is said to be p - semisimple if $B(X) = \{0\}$.

Note:

The property $G(X) \cap B(X) = \{0\}$ is obvious.

Proposition: 1.2.7

Let $(X,*,0)$ be a G - algebra. Then $x \in G(X)$ if and only if $0 * x \in G(X)$.

Proof

If $x \in G(X)$, then $0 * x = x$ and hence $0 * x \in G(X)$.

Conversely, if $0 * x \in G(X)$,

then $0 * (0 * x) = 0 * x$. and hence $x = 0 * x$. Therefore $x \in G(X)$.

Theorem: 1.2.8

If S is a subalgebra of a G - algebra $(X,*,0)$, then $G(X) \cap S = G(S)$.

Proof

Clearly $G(X) \cap S \subseteq G(S)$. If $x \in G(S)$, then $0 * x = x$ and $x \in S \subseteq A$. Hence $x \in G(X)$.

Therefore $x \in G(X) \cap S$. Thus $G(X) \cap S = G(S)$.

Hence the Proof.

Theorem: 1.2.9

Let $(X,*,0)$ be a G-algebra. If $G(X) = X$ then A is p - semisimple.

Proof

Obvious.

Definition: 1.2.10

An element $x \in X$ is minimal if $x * y = 0$ implies $y = x$.

Theorem: 1.2.11

Let $(X,*,0)$ be a G-algebra. Then

(i) X is p-semisimple.

(ii) Every element in X is a minimal element.

Proof

To Prove (i) : Let $(X,*,0)$ be a G-algebra. Then $B(X) = \{x \in X \mid 0*x = 0\}$. It follows from the condition $x * y = 0 \Rightarrow y = x$. that $B(X)=\{x \in X \mid x=0\}=\{0\}$. Hence every G - algebra is a p - semisimple.

To Prove (ii) : Let x be an element in X such that $y \leq x$ for some $y \in X$. Hence, $x * y = 0 \Rightarrow x = y$. Thus x is a minimal element in X.

Definition: 1.2.12

A G-algebra $(X,*,0)$ satisfying $(x * y) * (z * u) = (x * z) * (y * u)$ for any x ,y,z and $u \in X$, is called a medial G-algebra.

Example: 1.2.13

Example 1.1.9 is a medial G-algebra.

Lemma: 1.2.14

If X is a medial G-algebra, then, for any $x, y, z \in X$, the following holds:

(i) $(x * y) * x = 0 * y$

(ii) $x * (y * z) = (x * y) * (0 * z)$

(iii) $(x * y) * z = (x * z) * y$

Proof

Let X be a medial G-algebra and $x, y, z \in X$. Then

(i) $(x * y) * x = (x * y) * (x * 0) = (x * x) * (y * 0) = 0 * y$

(ii) $(x * y) * (0 * z) = (x * 0) * (y * z) = x * (y * z)$

(iii) $(x * y) * z = (x * y) * (z * 0) = (x * z) * (y * 0) = (x * z) * y$.

Theorem: 1.2.15

Every medial G-algebra is a QS−algebra.

Proof

Obvious.

Theorem: 1.2.16

Let X be a medial G-algebra. Then the right cancellation law holds in G(x).

Proof

Let a, b,x ∈ G(X) with $a * x = b * x$

Then, for any y ∈ G(X), $x * y = (0 * x) * y = (0 * y) * x = y * x$.

Therefore $a = x * (x * a) = x * (a * x) = x * (b * x) = x * (x * b) = b$.

The following theorems gives a necessary and sufficient condition for a G-algebra to become medial G-algebra.

Theorem: 1.2.17

A G-algebra X is medial if and only if it satisfies the following conditions:

(i) $y * x = 0 * (x * y)$ for all x, y ∈ X

(ii) $x * (y * z) = z * (y * x)$ for all x ,y, z ∈ X.

Proof

Suppose (X,∗,0) is medial.

(i) $0 * (x * y) = (y * y) * (x * y) = (y * x) * (y * y) = (y * x) * 0 = y * x$

(ii) $x*(y *z) = 0*((y *z)*x) = 0*((y *z)*(x*0)) = 0*((y* x)*z) = z*(y*x)$.

Conversely assume that the conditions hold.

Then $(x * y) * (z * u) = u * (z * (x * y)) = u * (y * (x * z)) = (x * z) * (y * u)$.

The following example shows that a G-algebra may not satisfy the associative law.

Example: 1.2.18.

Let X = {0,1,2} be a set with the following table:

∗	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Then (X,∗,0) is a G-algebra, but associativity does not hold,

since $(1 * 2) * 1 = 2 * 1 = 1$.But $1 * (2 * 1) = 1 * 1=0$.

Theorem: 1.2.19

Let X be an associative G-algebra. Then, for any $x \in B(X)$, $x = 0$.

Proof

Let $x \in B(X)$. Then $0 = 0*x = (x* x)*x = x*(x *x) = x*0 = x$.

Theorem: 1.2.20

If X is an associative G-algebra then $G(X) = X$.

Proof

Let X be an associative G–algebra. Clearly $G(X) \subseteq X$. Let $x \in X$.

Then $0 * x = x * (x * (0 * x)) = (x * x) * (0 * x) = 0 * (0 * x) = x$.

Hence $x \in G(X)$. Therefore $G(X) = X$.

Theorem: 1.2.21

Every G-algebra $(X,*,0)$ satisfying the associative law is a group under the operation " $*$ ".

Proof

Putting $x = y = z$ in the associative law. $x * (y * z) = (x * y) * z$ and using (G_2) and (P_3), we obtain $0 * x = x * 0 = x$. This means that 0 is the zero element of X. By (G_1), every element x of X has as its inverse the element x itself. Therefore $(X,*,0)$ is a group.

Theorem: 1.2.22

Every associative 0 - commutative G-algebra is commutative i.e., $x * y = y * x$

Proof

Let $(X,*,0)$ be an associative 0-commutative G-algebra and $x, y \in X$. Then $x * y = (x*0) * y = x*(0*y) = y*(0*x) = (y*0)*x = y *x$. Therefore X is commutative.

Corollary: 1.2.23

Every associative 0 - commutative G-algebra is an abelian group.

Proof

Obvious.

CHAPTER-2

f_q-DERIVATIONS AND (f, g) DERIVATIONS ON G-ALGEBRAS

Section 2.1

Derivations of G-algebras

Definition: 2.1.1

Let X be a G-algebra and d a self-map of X. Then d is called

(i) (l, r)-derivation of X if $d(x * y)=(d(x) * y) \wedge (x * d(y))$.

(ii) (r, l)-derivation of X if $d(x * y)=(x * d(y)) \wedge (d(x) * y)$.

If d is both (l, r)-derivation and (r, l)-derivation of X then d is a derivation of X.

Note:

In G-algebra, $x \wedge y = x$. Thus, to check that d is (l, r)-derivation of X, it is enough to check that $d(x * y) = d(x) * y$. Similarly, if $d(x * y) = x * d(y)$ then d is (r, l)-derivation of X.

Definition: 2.1.2

Let X be a G-algebra and d a self-map of X. Then d is called a derivation of X. if d is (l, r)-derivation of X and (r,l)-derivation of X. That is, for all $x, y \in X$:

$d(x * y) = d(x) * y$ and $d(x * y) = x * d(y)$, respectively.

Example: 2.1.3

Consider the G-algebra given by Cayley table

∗	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Define a map $d : X \to X$ by: $d(x) = \begin{cases} 1 & \text{if } x \in \{0,2\}, \\ 0 & \text{if otherwise.} \end{cases}$

Then $d(2*2) = d(0) = 1$ and $d(2) * 2 = 1 * 2 = 2$. Therefore $d(2 * 2) \neq d(2) * 2$. $\Rightarrow$ d is not (l, r)-derivation of X. Similarly, d is not (r, l)-derivation of X as $d(1 * 2) \neq 1 * d(2)$. Hence, d is not a derivation of X.

Example: 2.1.4

Let $X = \{0,1,2,3\}$ in which $*$ is defined by the Table .

∗	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Define a map $d: X \rightarrow X$ by:

$$d(x) = \begin{cases} 2 \text{ if } x = 0, \\ 3 \text{ if } x = 1, \\ 0 \text{ if } x = 2, \\ 1 \text{ if } x = 3. \end{cases}$$

Then d is a derivation of X.

Theorem: 2.1.5

In G-algebra X, the identity map is a derivation on X.

Proof

Consider the two cases to prove that d is a derivation of X. We will show that d is a (l, r)-derivation of X and it can be shown similarly that d is a (r, l)-derivation of X.

Case (i) Let $x,y \in X$. If $x = y$ then $d(x * x) = d(0) = 0.d(x) * x = x * x = 0$.

Case (ii) If $x \neq y$ then either $d(x * y) = d(x)$ or $d(x * y) = d(y)$. If $d(x * y) = d(x)$ then $d(x * y) = x$.

On the other hand, $d(x * y) = d(x) * y$. $\Rightarrow x = x * y \Rightarrow y = 0$. Therefore, $d(x) * y = x$. d is a (l, r)-derivation of X. if $d(x * y) = d(y)$. Then $d(x * y) = y$.

We have, $d(x * y) = d(x) * y = x * y$.

Therefore, $y = x * y$. Thus, $d(x * y) = d(x * (x * y)) = d(y) = y$.

Hence, d is a (l, r)-derivation of X.

Definition: 2.1.6

A derivation d of a G-algebra is said to be regular if $d(0) = 0$.

Lemma: 2.1.7

If d is a regular derivation of a G-algebra X, then d is the identity map on X.

Proof

If d is a regular derivation of a G-algebra X then $d(0) = 0$, $d(x * x) = d(0)$. Hence, $d(x) * x = 0$. $\Rightarrow$ $d(x) = x$. That is, d is the identity map on X.

Definition: 2.1.8

Let X be a G-algebra and d_1, d_2 be two self-maps of X.

We define $d_1 \circ d_2 \colon X \to X$, as $(d_1 \circ d_2)(x) = d_1(d_2(x))$, for all $x \in X$.

Theorem: 2.1.9

Let X be a G-algebra and d_1, d_2 be two derivations of X. Then $d_1 \circ d_2$ is a derivation of X.

Proof

Assume that d_1, d_2 are (l ,r) - derivations. Let $x, y \in X$.

Then $(d_1 \circ d_2)(x * y) = d_1(d_2(x * y)) = d_1(d_2(x) * y) = d_1(d_2(x)) * y = ((d_1 \circ d_2)(x) * y)$.

Hence $d_1 \circ d_2$ is a (l, r)-derivation. Similarly, $d_1 \circ d_2$ is a (r,l)-derivation of X. and thus, $d_1 \circ d_2$ is a derivation of X.

Lemma: 2.1.10

Let X be a G-algebra and d_1, d_2 are either two (l, r)-derivations or (r, l)-derivations of X. Then the composition of the derivations is not necessary commutative.

Proof

Suppose that d_1, d_2 are (l, r)-derivations, then
$(d_1 \circ d_2)(x * y) = d_1(d_2(x * y))= d_1(d_2(x) * y) = d_1\ d_2(x) * y$.
On the other hand,$(d_2 \circ d_1)(x * y) = d_2(d_1(x * y)) = d_2(d_1(x) * y) = d_2\ d_1(x) * y$. Hence, $(d_1 \circ d_2)(x * y) \neq (d_2 \circ d_1)(x * y)$.

Similarly, we can show that if d_1,d_2 are (r,l)-derivations, then
$(d_1 \circ d_2)\ (x * y) = x * d_1d_2(y)$ and $(d_2 \circ d_1)\ (x * y) = x * d_2d_1(y)$.
Therefore, $(d_1 \circ d_2)(x * y) \neq (d_2 \circ d_1)(x * y)$.
That is, the composition of the derivations is not necessary commutative.

Theorem: 2.1.11

Let X be a G-algebra and suppose that d_2 is a (l,r)-derivation and d_1 is a (r,l)-derivation of X. Then the composition of the derivations is commutative.

Proof

To Prove $d_1 \circ d_2 = d_2 \circ d_1$ for all x,y ∈ X. suppose that d_2 is a (l,r)-derivation, Then($d_1 \circ d_2)\ (x * y) = d_1(d_2(x * y)) = d_1(d_2(x) * y) = (d_2(x) * d_1(y))$ as d_1 is a (r,l)-derivation. Thus,

$$(d_1 \circ d_2)\ (x * y) = d_2\ (x) * d_1\ (y) \tag{1}$$

Similarly, suppose that d_1 is a (r,l)-derivation,
then $(d_2 \circ d_1)(x * y) = d_2\ (d_1\ (x * y)) = d_2\ (x * d_1(y)) =(d_2(x) * d_1(y))$ as d_2 is a (l,r)-derivation.

Hence, $$(d_2 \circ d_1)\ (x * y) = d_2(x) * d_1(y) \tag{2}$$

From (1) and (2), we have $(d_1 \circ d_2)(x * y)=(d_2 \circ d_1)(x * y)$.
Let y = 0, then $(d_1 \circ d_2)\ (x) = (d_2 \circ d_1)(x)$ as $x * 0 = x$. Therefore, $d_1 \circ d_2 = d_2 \circ d_1$. Therefore the composition of the derivations is commutative.

Section 2.2

f_q - Derivations of G-algebras

Definition: 2.2.1

Let X be a G-algebra and let f be an endomorphism self-map of X. Then

d_q^f is a self-map of G-algebra X defined by $d_q^f(x) = f(x) * q$ for all $x \in X$, $q \in X$.

Definition: 2.2.2

(i) A map d_q^f is called an outside f_q- derivation of X if

$d_q^f\ (x * y) = f\ (x) * d_q^f\ (y)$, $\forall x, y \in X$.

(ii) A map d_q^f is called an inside f_q-derivation of X if

$d_q^f\ (x * y) = d_q^f\ (x) * f\ (y)$, $\forall x, y \in X$.

(iii) If d_q^f is both an outside and inside f_q- derivation of X, then d_q^f is a f_{q} - derivation of X.

Note:

If d_q^f is f_q- derivation of X , then $d_q^f\ (x * y) = f\ (x) * d_q^f\ (y) = d_q^f\ (x) * f\ (y)$.

Example: 2.2.3

Consider the G-algebra given by the cayley table 1

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Define an endomorphism : $f: X \to X$,such that $x \to \begin{cases} 0 & \text{if } x=0 \\ 2 & \text{if } x=1 \\ 1 & \text{if } x=2 \end{cases}$

Consider the following table 2

x	0	0	0	1	1	1	2	2	2
y	0	1	2	0	1	2	0	1	2
x $*$ y	0	1	2	1	0	2	2	1	0
F(x $*$ y)	0	2	1	2	0	1	1	2	0
d_0^f(x $*$ y)	0	2	1	2	0	1	1	2	0
f(x)	0	0	0	2	2	2	1	1	1
d_0^f(x)	0	0	0	2	2	2	1	1	1
f(y)	0	2	1	0	2	1	0	2	1
d_0^f(y)	0	2	1	0	2	1	0	2	1
f(x)$*$ d_0^f(y)	0	2	1	2	0	1	1	2	0
d_0^f(x) $*$ F(y)	0	2	1	2	0	1	1	2	0

If q=0, then above table shows that d_0^f is an outside and an inside f_q- derivation of X. Hence, d_0^f is a f_q -derivation of X. If we take q=2, then d_2^f is not an outside fq-derivation of X or an inside f_q - derivation of X since $d_2^f(1 * 2) = 2$ while $f(1) * d_2^f(2) = 0$ and $d_2^f(1) * f(2) = 1$.

Example: 2.2.4

Let $X = \{0, a, b, c\}$.Consider the G-algebra given by Cayley table

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define an endomorphism:

$$f : X \rightarrow X, \text{ such that } x \rightarrow \begin{cases} 0 \text{ if } x = 0, \\ b \text{ if } x = a, \\ a \text{ if } x = b, \\ c \text{ if } x = c. \end{cases}$$

Then d_q^f is f_q- derivation of X for all $q \in X$.

Proposition: 2.2.5

For any G-algebra X, there exists at least one f_q-derivation of X, that is, the map d_0^f.

Proof

Given X be a G-algebra. Then To prove d_q^f is a f_q-derivation of X. Let $q = 0$, then $d_0^f(x * y) = (f(x * y)) * 0 = (f(x) * f(y)) * 0 = f(x) * f(y)$ and $f(x) * d_0^f(y) = f(x) * (f(y) * 0) = f(x) * f(y)$.

Also we have $d_0^f(x) * f(y) = (f(x) * 0) * f(y) = f(x) * f(y)$.

Hence, d_0^f is f_q-derivation of X.

Proposition: 2.2.6

If X is an associative G-algebra, then d_q^f is an outside f_q - derivation of X, for all $q \in X$.

Proof

Let X be an associative G-algebra.

Then $d_q^f(x * y) = ((x * y)) * q = (f(x) * f(y)) * q$ and

$(x) * d_q^f(y) = (x) * ((y) * q) = ((x) * (y)) * q$, as X is associative.

Hence, d_q^f is an outside f_q- derivation of X.

Proposition: 2.2.7

If X is a medial G-algebra, then is an inside f_q-derivation of X, for all $q \in X$.

Proof

Let X be a medial. Since $d_q^f(x * y)$ =f $(x * y) * q = (f(x) * f(y)) * q$ and $d_q^f(x) *$ f$(y) = ((x) * q) * (y)) = ((x) * (y)) * q$, as$X$ is medial, therefore, d_q^f is an inside f_q - derivation of X.

Theorem: 2.2.8

Let X be an associative medial G-algebra. then d_q^f is a f_q - derivation of X, for all $q \in X$.

Proof

Since X is associative G-algebra . d_q^f is an outside f_q - derivation of X. Since X is associative G-algebra . d_q^f is an inside f_q - derivation of X. Then d_q^f is a

f_q - derivation of X .

Alternative proof

Let $x, y, q \in X$. Then

(i) $d_q^f\ (x * y) = f\ (x * y) * q = ((f\ (x) * f\ (y)) * q) * 0 = ((f\ (x) * q) * f\ (y)) * 0$

$= ((f\ (x) * q) * f\ (y)) * (((f\ (x) * q) * f\ (y)) * ((f\ (x) * q) * f\ (y)))$

$= (d_q^f\ (x) * f\ (y)) * ((d_q^f\ (x) * f\ (y)) * ((f\ (x) * f\ (y)) * q))$

$= (d_q^f\ (x) * f\ (y)) * ((d_q^f\ (x) * f\ (y)) * (f\ (x) * (f\ (y) * q)))$

$= (d_q^f\ (x) * f\ (y)) * ((d_q^f\ (x) * f\ (y)) * (f\ (x) * d_q^f\ (y)))$

$=$f$(x) *\ d_q^f\ (y)$, as $y * (y * x) = x$.

Therefore d_q^f is an outside f_q - derivation of X.

(ii) $d_q^f\ (x * y) = f\ (x * y) * q = (f\ (x * y) * q) * 0$

$= (f\ (x * y) * q) * ((f\ (x * y) * q) * (f\ (x * y) * q))$

$= ((f\ (x) * f\ (y)) * q) * (((f\ (x) * f\ (y)) * q) * ((f\ (x) * f\ (y)) * q))$

$= (f(x) * (f\ (y) * q)) * ((f\ (x) * (f\ (y) * q)) * ((f\ (x) * q) * f\ (y)))$

$= (f(x) * d_q^f\ (y)) * ((f(x) * d_q^f\ (y)) * (d_q^f\ (x) * f\ (y)))$

$= d_q^f\ (x) * f\ (y)$.

Therefore, d_q^f is an inside f_q -derivation of X. Therefore, d_q^f is a f_q - derivation of X.

Proposition: 2.2.9

If d_q^f is an outside (resp., inside) f_q - derivation of X, then

d_q^f (0) = $(x) *$ d_q^f (x), $\forall x \in X$ (resp., d_q^f (0) = d_q^f $(x) * (x)$, $\forall x \in X$).

Proof

obvious.

Theorem: 2.2.10

Let X be a medial G-algebra. If $d_q^f$is an outside f_q -derivation of X, then d_q^f is a f_q- derivation of X.

Proof

Let X be a medial G-algebra. Then d_q^f is an inside fq- derivation of X.Given $d_q^f$is an outside f_q- derivation of X. Therefore, d_q^f is a f_q - derivation of X.

Definition: 2.2.11

A map d_q^f is said to be regular if d_q^f (0) = 0.

Proposition: 2.2.12

Let d_q^f be a f_q- derivation of X. If either $(x) * d_q^f$ (y) = 0 or d_q^f $(x) * (y)$ = 0, then d_q^f is a regular derivation.

Proof

Since d_q^f is a f_q- derivation,

we have d_q^f $(x * y)$ = $f(x) * d_q^f$ (y) = d_q^f $(x) * f(y)$.

Consider that $(x) * d_q^f$ (y) = 0; then d_q^f (0) = d_q^f $(x * x)$ = $(x) * d_q^f$ (x) = 0.

Similarly, if d_q^f $(x) * (y) = 0$, we have d_q^f $(0) = d_q^f$ $(x * x) = d_q^f$ $(x) * (x) = 0$. This proves that $d_q^f$is a regular derivation.

Proposition: 2.2.13

Let d_q^f be a regular f_q - derivation of X. Then $d_q^f(x) = f(x)$, $\forall x \in X$.

Proof

Since d_q^f is a regular f_q - derivation of X, then d_q^f $(0) = 0$.

d_q^f $(0) = d_q^f$ $(x * x) = d_q^f$ $(x) * f(x) = 0$. Therefore, d_q^f $(x) = (x)$.

Definition: 2.2.14

Let X be a G-algebra and let d_q^f , d'^f_q be two self-maps of X.

Define $d_q^f \circ d'^f_q : X \rightarrow X$ by ($d_q^f \circ d'^f_q$) $(x) = d_q^f($ $d'^f_q(x))$ $\forall x \in X$.

Proposition: 2.2.15

Let X be a G-algebra and let f be the identity endomorphism of X.If d_q^f , d'^f_q are outside f_q- derivations of X, then $d_q^f \circ d'^f_q$ is also an outside f_q- derivation.

Proof

Consider the element $x*y$. Then $(d_q^f \circ d'^f_q)$ $(x * y) = d_q^f(d'^f_q$ $(x * y))$. Since d_q^f and d'^f_q are outside f_q- derivations,

we have d_q^f (d'^f_q $(x * y)) =$ d_q^f (f$(x) * d'^f_q$ $(y)) = f(x) *$ ($d_q^f \circ d'^f_q$)(y).

Thus, d_q^f $\circ d'^f_q$ is an outside f_q - derivation.

Proposition: 2.2.16

For a G-algebra X, let f be the identity endomorphism of X. If d_q^f , d'^f_q are inside f_q- derivations of X, then $d_q^f \circ d'^f_q$ is also an inside f_q- derivation.

Proof

Obvious.

Theorem: 2.2.17

Let X be a G-algebra and let f be the identity endomorphism of X. If d_q^f , d'^f_q are both outside (resp., inside) f_q- derivations of X, then the composition is an outside (resp., inside) f_q- derivation of X.

Proof

By using proposition 2.2.15 and 2.2.16 proof follows.

Proposition: 2.2.17

Let X be a G-algebra and let d_q^f , d'^f_q be f_q - derivations of X such that

$d_q^f \circ f = f \circ d_q^f$ and $d'^f_q \circ f = f \circ d'^f_q$; then $d_q^f \circ d'^f_q = d'^f_q \circ d_q^f$.

Proof

Given X be a G-algebra and let d_q^f , d'^f_q be f_q - derivations of X such that $d_q^f \circ f = f \circ d_q^f$ and $d'^f_q \circ f = f \circ d'^f_q$. Consider d'^f_q as an outside f_q- derivation of X and d_q^f as an inside f_q- derivation of X; then for all $x, y \in X$ we have

$$\begin{aligned}(d_q^f \circ d'^f_q)(x * y) &= d_q^f\ (d'^f_q\ (x * y)) \\ &= d_q^f\ (f(x) * d'^f_q\ (y)) \\ &= d_q^f\ (f(x)) * f(d'^f_q\ (y)) \\ &= (d_q^f \circ f)(x) * (f \circ d'^f_q)(y) \qquad (1)\end{aligned}$$

On the other hand,

$$\begin{aligned}(d'^f_q \circ d_q^f)(x * y) &= d'^f_q\ (d_q^f\ (x * y)) \\ &= d'^f_q\ (d_q^f\ (x) * f(y)) \\ &= f(d_q^f\ (x)) * d'^f_q\ (f(y)) \\ &= (f \circ d_q^f)(x) * (d'^f_q \circ f)(y)\end{aligned}$$

$$= (d_q^f \circ f)(x) * (f \circ d'^f_q)(y) \tag{2}$$

From (1) and (2), we get $(d_q^f \circ d'^f_q)(x * y) = (d'^f_q \circ d_q^f)(x * y)$.

By putting y=0, we get

$(d_q^f \circ d'^f_q)(x) = (d'^f_q \circ d_q^f)(x)$.

Hence, $d_q^f \circ d'^f_q = d'^f_q \circ d_q^f$.

Section 2.3

(f, g)-Derivations of G-algebras

Definition: 2.3.1

Let X be a G-algebra. A left-right (f,g)-derivation (briefly (l,r)-(f,g)derivation) of X is a self-map d of X satisfying the following identity, for all x,y ∈ X:

d(x ∗ y) = (d(x) ∗ f(y)) ∧ (g(x) ∗ d(y)).If d satisfies the identity:

d(x ∗ y) = (f(x) ∗ d(y)) ∧ (d(x) ∗ g(y)),

then d is a right-left (f,g)-derivation (briefly (r,l)-(f,g)-derivation) of X. If d is both (l, r)-(f,g)-derivation and (r,l)-(f,g)-derivation of X then d is an (f,g)-derivation of X.

Example: 2.3.2

Consider the G-algebra (X,∗,0) given by Cayley Table

∗	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Let d be the zero map and f and g endomorphisms defined as follow: f(x) = 0 and

$$g(x) = \begin{cases} 0, & \text{if } x = 0; \\ 2, & \text{if } x = 1; \\ 1, & \text{if } x = 2. \end{cases}$$

Then d is an (f, g) - derivation of X.

Corollary: 2.3.3

Let X be a G-algebra and f be the zero map. If d is the zero map then d is an (f,g)-derivation of X.

Proof

obvious .

Example: 2.3.4

Consider example 2.3.2 with f an identity endomorphism. As d is the zero map, we have $d(x * y) = 0$ but $d(x) * f(y) = 0 * y \neq 0$ for some $y \in X$. Therefore, d is not a (l, r) - (f, g)-derivation of X neither a (r,l)-(f, g)- derivation of X as
$f(x) * d(y) = x * 0 = x \neq 0$, for all $x \in X \backslash \{0\}$.

Proposition: 2.3.5

Let d be a (l, r) - (f, g) -derivation of a medial G-algebra X.

Then $d(x) = d(x) \wedge g(x)$, for all $x \in X$.

Proof

Let d be a (l,r)-(f,g)-derivation of a medial G-algebra X.

Then $d(x) = d(x * 0) = (d(x) * f(0)) \wedge (g(x) * d(0)) = (d(x) * 0) \wedge (g(x) * d(0))$

$= d(x) \wedge (g(x) * d(0)) = (g(x) * d(0)) * ((g(x) * d(0)) * d(x))$

$= (g(x) * d(0)) * ((g(x) * d(x)) * d(0)) = (g(x) * d(0)) * ((g(x) * d(x)) * d(0))$

$= g(x) * (g(x) * d(x)) = d(x) \wedge g(x)$.

Definition: 2.3.6

An (f,g)-derivation d of a G-algebra is said to be regular if $d(0) = 0$.

Proposition: 2.3.7

Let d be a (r,l)-(f,g)-derivation of a G-algebra X. Then $d(x) = f(x) \wedge d(x)$, for all $x \in X$ if and only if d is regular.

Proof

Suppose that $d(x) = f(x) \wedge d(x)$. Then $d(0) = f(0) \wedge d(0) = 0 \wedge d(0) = 0$ and hence d is regular.

Conversely, Suppose that d is regular.

Then $d(x) = d(x * 0) = (f(x) * d(0)) \wedge (d(x) * g(0))$

$$= (f(x) * 0) \wedge (d(x) * 0) = f(x) \wedge d(x).$$

Theorem: 2.3.8

Let X be a G-algebra. If d is a (l ,r)-(f, g)-derivation of X then $d(a + b) = d(a) + f(b)$, for all $a, b \in X$. Similarly, if d is a (r, l)-(f, g)- derivation of X then $d(a + b) = f(a) + d(b)$.

Proof

Let d is a (l ,r)-(f, g)-derivation of a G-algebra X. Then

$d(a + b) = d(a * (0 * b)) = d(a) * f(0 * b) = d(a) * (f(0) * f(b))$

$$= d(a) * (0 * f(b)) = d(a) + f(b).$$

Similarly, if d is a (r,l)-(f, g)- derivation of X then $d(a + b) = f(a) + d(b)$.

Corollary: 2.3.9

If G-algebra is 0-commutative then $d(a + b) = d(b + a)$.

Proof

obvious.

Proposition: 2.3.10

Let X be a G-algebra and d be an (f,g)-derivation of X. Then

(i) $d(0) \in Lp(X)$,

ii) For all $a \in Lp(X)$, we have $f(a)$ and $g(a) \in Lp(X)$.

Proof

(i) If $x * d(0) = 0$ then $x = d(0) \Rightarrow d(0) \in Lp(X)$.

(ii) Let $a \in Lp(X)$. Then $a = 0 * (0 * a)$.

Therefore, $f(a) = f(0 * (0 * a))$

$$= f(0) * f(0 * a) = 0 * (f(0) * f(a)) = 0 * (0 * f(a)).$$

Hence $f(a) \in Lp(X)$.

Similarly $g(a) \in Lp(X)$.

Proposition: 2.3.11

Let X be a G-algebra and let d be a (l,r)-(f,g)- derivation of X. Then

(i) For all $a \in Lp(X)$, $d(a) = d(0) + f(a)$,

(ii) For all $a \in Lp(X)$, $d(a) \in Lp(X)$,

(iii) For all $a, b \in Lp(X)$, $d(a + b) = d(a) + d(b) - d(0)$.

Proof

To Prove (i) : Let X be a G-algebra. Let $a \in Lp(X)$. Then $a = 0 * (0 * a)$.

Therefore, $d(a) = d(0 * (0 * a)) = (d(0) * f(0 * a)) \wedge (g(0) * d(0*a))$

$$= d(0)*(f(0)*f(a)) = d(0) * (0 * f(a)) \quad = d(0) + f(a).$$

To Prove (ii) : Let $a \in Lp(X)$. Using the above condition (i),

we have $d(a) = d(0) + f(a)$,

hence $d(a) \in Lp(X)$ as $d(0)$ and $f(a) \in Lp(X)$.

To Prove (iii) Let $a,b \in Lp(X)$. Then, using (i), $d(a + b) = d(0) + f(a + b)$

$$= d(0) +f(a) + f(b) = f(a) + d(b) + d(0) - d(0) = d(a) + d(b) - d(0).$$

Proposition: 2.3.12

Let X be a G-algebra and d be a (r,l)-(f,g)-derivation of X. Then

(i) For all $a \in G(X)$, $d(a) \in G(X)$,

(ii) For all $a \in Lp(X)$, $d(a) \in Lp(X)$,

(iii) For all $a \in X$, $d(a) = f(a) + d(0)$,

(iv) For all $a,b \in X$, $d(a + b) = d(a) + d(b) - d(0)$.

Proof

To Prove (i) Let $a \in G(X)$. Then $a = 0 * a$ and

$$d(a) = d(0 * a) = f(0) * d(a) = 0 * d(a).$$

Therefore $d(a) \in G(X)$.

To Prove (ii) : Let a ∈ Lp(X). Then a = 0 ∗ (0 ∗ a).

Hence d(a) = f(0) ∗ d(0 ∗ a) = 0 ∗ (0 ∗ d(a)). Therefore, d(a) ∈ Lp(X).

To Prove (ii) : Let a ∈ X. Then d(a) = d(a ∗ 0) = f(a) ∗ d(0). As d(0) ∈ G(X), then d(a) = f(a) ∗ (0 ∗ d(0)) = f(a) + d(0).

To Prove (iv) proof obvious .

Proposition: 2.3.13

Let X be a G-algebra, d a (r,l)-(f,g)-derivation of X and f the identity map on X. Then d is the identity map on X if and only if d is regular.

Proof

Let d be the identity map on X. d(a) = d(0) + f(a), and so a = d(0) + a which gives d(0) = 0. On the other hand, let d(0) = 0.

Then d(a) = d(a ∗ 0) = f(a) ∗ d(0) = a ∗ 0 = a.

Theorem: 2.3.14

Let X be a G-algebra and d an (f,g)-derivation of X. If there exists a ∈ X such that, for all x ∈ X, either d(x) ∗ f(a) = 0 or f(x) ∗ d(a) = 0 then d is regular.

Proof

Let a ∈ X such that d(x) ∗ f(a) = 0, for all x ∈ X.

Then 0 = d(x) ∗ f(a) = f(x) ∗ d(a) = d(x ∗ a).

Hence d(0) = 0. This implies d is regular.

Definition: 2.3.15

Let X be a G-algebra and d_1,d_2 be an (f,g)-derivation of X.

We define $d_1 \circ d_2 : X \to X$ by $(d_1 \circ d_2)(x) = d_1(d_2(x))$, for all x ∈ X.

Theorem: 2.3.16

Let d be a regular (f, g)-derivation of a G-algebra X. If d^2(x) = 0, for all x ∈ Lp(X) then (f ∘d)(x) = 1/2 ((f ∘d)(0)) for all x ∈ Lp(X).

Proof

Let x ∈ Lp(X) such that d^2 (x) = 0.

Therefore $x + x \in Lp(X)$ and so $d^2(x + x) = 0$.

So we have $0 = d^2(x + x) = d(d(x + x)) = d(0) + f(d(x + x))$

$$= 0 + f(d(x) + d(x) - d(0)) = 2f(d(x)) - f(d(0)) \text{ and so } (f \circ d)(x) = 1/2\,((f \circ d)(0)).$$

Corollary: 2.3.17

Let d_1 and d_2 be two regular (f,g)-derivations of a G- algebra X. If $(d_1 \circ d_2)(x) = 0$, for all $x \in Lp(X)$ then $(f \circ d_2)(x) = 1/2\,((f \circ d_2)(0))$ for all $x \in Lp(X)$.

Definition: 2.3.18

A G-algebra X is said to be torsion free if it satisfies $x + x = 0$ implies $x = 0$, for all $x \in X$. If there exists a nonzero element $x \in X$ such that $x + x = 0$ then X is not a torsion free.

Example: 2.3.19

Consider the G-algebra X given by Cayley table

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Define $d : X \rightarrow X$ by $d(x) = \begin{cases} 1, & \text{if } x \text{ in } 0, \\ 0, & \text{otherwise.} \end{cases}$

is not a torsion free as there exist a non zero element $1 \in X$ such that $1 + 1 = 1 * (0 * 1) = 0$.

Theorem: 2.3.20

Let X be a torsion free G-algebra and d be a (l,r)-(f,g)- derivation of X such that $(f \circ d)(x) = d(x)$. Then for all $x \in Lp(X)$, if $d^2(x) = 0$ we have $d(x) = 0$.

Proof

Let $x \in Lp(X)$ such that $d^2(x) = 0$. As $x \in Lp(X)$, we have $x + x \in Lp(X)$. Thus $0 = d^2(x + x) = d(d(x + x)) = d(0) + f(d(x + x))$

$= d(0) + d(x + x) = d(0) + d(x) + d(x) - d(0) = d(x) + d(x).$

As X is torsion free, then $d(x) + d(x) = 0$ implies $d(x) = 0$.

Therefore d is the zero map.

Theorem: 2.3.21

Let X be a torsion free G-algebra, d_2 an (f,g)-derivation and d_1 a (l,r)-(f,g)derivation of X such that for all $x \in X$, $(f \circ d_2)(x) = d_2(x)$. If $(d_1 \circ d_2)(x) = 0$, for all $x \in X$, then $d_2(x) = 0$, for all $x \in X$.

Proof

Let X be a torsion free G-algebra. Suppose that $(d_1 \circ d_2)(x) = 0$.

Then $0 = (d_1 \circ d_2)(x + x) = d_1(d_2(x + x)) = d_1(0)+f(d_2(x + x)) = d_1(0)+ d_2(x +x)$

$= d_1(0) + (d_2(x) + d_2(x) - d_2(0))$

$= d_1(0) - d_2(0) + (d_2(x) + d_2(x)) = (d_1(0) * d_2(0)) + (d_2(x)+ d_2(x))$

$= (d_1(0) * (0 * d_2(0)) + (d_2(x) + d_2(x)) = (d_1(0) + d_2(0))+(d_2(x) + d_2(x))$

$= (d_1(0) + f(d_2(0))) + (d_2(x) + d_2(x)) = d_1(d_2(0)) + (d_2(x) + d_2(x))$

$= (d_1 \circ d_2)(0) + (d_2(x) + d_2(x)).$

Having $(d_1 \circ d_2)(0) = 0$ implies $d_2(x) + d_2(x) = 0$ and so $d_2(x) = 0$ as X is torsion free.

Theorem: 2.3.22

Let X be a G-algebra. If d_1 and d_2 are (f,g)-derivations of X such that $f^2 = f$.

Then $d_1 \circ d_2$ is an (f, g)-derivation of X.

Proof

Let $x,y \in X$. Then $(d_1 \circ d_2)(x * y) = d_1(d_2(x * y))$

$= d_1((d_2(x) * f(y)) \wedge (g(x) * d_2(y)))$

$= d_1(d_2(x) * f(y)) = (d_1(d_2(x)) * f(f(y))) = d_1(d_2(x)) * f(y) = (d_1 \circ d_2(x) * f(y)).$

Therefore, $d_1 \circ d_2$is a (r,l)-(f,g)-derivation and similarly we can prove that $d_1 \circ d_2$ is a (l, r)-(f, g)-derivation of X.

CHAPTER-3

INTUITIONISTIC FUZZY G-SUBALGEBRAS OF G-ALGEBRAS

Section 3.1

Preliminaries on Fuzzy sets and Intuitionistic Fuzzy sets

Definition: 3.1.1

Let X be a non-empty set. Then a fuzzy set A in X is defined as A = {< x, α_A(x)>| x ∈ X}, where α_A(x) is called the membership value of x in A and $0 \leq \alpha_A(x) \leq 1$.

Note:

In the above definition, when A is a set in the ordinary sense of the term. Its membership function can take on only two values 0 and 1, with α_A(x) = 1 or 0 according as x does or does not belong to A. Thus, in this case α_A(x) reduces to the familiar characteristic function of a set A.

Definition: 3.1.2

A fuzzy set A : X → [0,1] is said to be empty if α_A(x) = 0 for all x in X and it is denoted by φ_x .

Definition: 3.1.3

A fuzzy set A : X → [0,1] is said to be whole fuzzy set if α_A(x) = 1 for all x in X and it is denoted by 1_x .

Definition: 3.1.4

For c ∈ [0,1] , a fuzzy set k_c : X→ [0,1] is said to be constant fuzzy set if α_{k_c} (x) = c for all x in X .

Definition: 3.1.5

The complement of a fuzzy set A , denoted by A^c is defined as α_A^c (x) = 1- α_A(x).

Definition: 3.1.6

Let A : X → [0,1] be a fuzzy set. Then the support of A ,denoted by supp A is defined as supp A={ x / α_A(x) > 0 for all x in X.

Definition: 3.1.7

Two fuzzy sets A and B are equal ,written as A=B if and only if

α_A(x) = α_B(x).

Definition: 3.1.8

Let A and B are fuzzy sets in X .A is contained in B (or equivalently , A is a subset of B, or A is smaller than or equal to B) if and only if α_A(x) ≤ α_B(x) for all x in X .

Note:

In symbols $A \leq B \Leftrightarrow \alpha_A \leq \alpha_B$.

Definition: 3.1.9

The union of two fuzzy sets A and B with respective membership functions α_A(x) and α_B(x) is a fuzzy set C ,written as C = A∨B , whose member functions is related to those of A and B by , $\alpha_{A\vee B}$ (x) = α_C (x) = max { α_A(x), α_A(x) }, ∀ x ∈ X or in abbreviated form $\alpha_C = \alpha_A \vee \alpha_B$.

Definition: 3.1.10

The intersection of two fuzzy sets A and B with respective membership functions α_A(x) and α_B(x) is a fuzzy set C ,written as C = A ∧ B , whose member functions is related to those of A and B by , $\alpha_{A\wedge B}$ (x) = α_C (x) = min { α_A(x), α_A(x)}, ∀ x ∈ X or in abbreviated form $\alpha_C = \alpha_A \wedge \alpha_B$.

Definition: 3.1.11

For a family of fuzzy sets A={ A_α / $\alpha \in \Lambda$, the index set}, the union

$C = \bigvee_{\alpha\in\Lambda} A_\alpha$ and the intersection $D = \bigwedge_{\alpha\in\Lambda} A_\alpha$ are respectively defined by

α_C (x) = sup { αA_α / α ∈ Λ, the index set}, x ∈ X and α_D (x) = inf{ αA_α / α ∈ Λ, the index set}, x ∈ X

Definition: 3.1.12

A fuzzy set A in X with the membership function α_A is said to have

(i). The sup property if for any subset T of x, there exist $t_0 \in$ T such that $\alpha_A(t_0) = \underset{t \in T}{sup} \alpha_A(t)$.

(ii). The inf property if for any subset T of x, there exist $t_0 \in$ T such that $\alpha_A(t_0) = \underset{t \in T}{inf} \alpha_A(t)$.

Definition: 3.1.13

Let A be a fuzzy set in a set X. For t ∈ [0,1]

(i). The set U $(\alpha_A; t) = \{x \in X \ / \ \alpha_A(x) \geq t\}$ is called upper level cut (upper t - level set, upper level set) of A.

(ii). The set L $(\alpha_A; t) = \{x \in X \ / \ \alpha_A(x) \leq t\}$ is called lower level cut (lower t - level set, lower level set) of A.

Definition: 3.1.14

A fuzzy set A is called a fuzzy relation on any set X, if A is a fuzzy set A:X x X → [0,1].

Definition: 3.1.15

If A is a fuzzy relation on set of X and B is a fuzzy set of X, then A is a fuzzy relation on B if α_B (x,y) ≤ min { α_B(x), α_B(y) } for all x, y ∈ X.

Lemma: 3.1.16

Let A and B be fuzzy set of a set X, then

(i) A x B is a fuzzy relation on X.

(ii) U $(\alpha_{A\,X\,B}; t) =$ U $(\alpha_A; t)\ X$ U $(\alpha_B; t)$ for all t ∈ [0, 1].

Proof

Obvious.

Definition: 3.1.17 If B is a fuzzy set of a set X, the strongest fuzzy relation on X, that is fuzzy relation on B is denoted by α_B given by

Sα_B (x,y) = min { α_B(x), α_B(y) } for all x, y ∈ X.

Lemma: 3.1.18

For a given fuzzy subset X, let S_{α_β} be the strongest fuzy relation on X , then for t ∈ [0,1] ,we have ∪(S_{α_β}; t) = ∪(α_B ; t) × ∪(α_B ; t) .

Proof

obvious .

Definition: 3.1.19

Let A and B be two fuzzy sets in the set X .The cartesian product

A × B **:** X × X → [0,1] is defined by (A × B) (x,y) =min { α_A (x), α_B (x) }.

Definition: 3.1.20

Let f be mapping from the set X to the set Y.

(i) Let B be a fuzzy set in Y with membership function α_B .the inverse image of B,denoted $f^{-1}(B)$, is the fuzzy set in X ,with membership function $\alpha_{f^{-1}(B)}$ defined as, $\alpha_{f^{-1}(B)}$ (x) = α_B (f(x)) for all x ∈ X.

(ii) Let A be a fuzzy set in X with membership function α_A .Then the image of A, denoted by $f(A)$, is the fuzzy set in Y defined as

$$\alpha_{f(A)}(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \alpha_A(z), & f^{-1}(y) \neq \varphi \\ 0 & \text{, otherwise} \end{cases}$$

(iii) Let A be a fuzzy set in X with membership function α_A .Then the (inf) image of A denoted by $f(A)$, is the fuzzy set in Y defined as,

$$\alpha_{f(A)}\text{(y)} = \begin{cases} \underset{z \in f^{-1}(y)}{inf} \alpha_A\text{(z)} , f^{-1}(y) \neq \varphi \\ 0 \qquad\qquad\qquad , \text{ otherwise} \end{cases}$$

Definition: 3.1.21

An intuitionistic fuzzy set A over X is an object having the form A={$< x,\ \alpha_A$ (x)$, \beta_A$ (x) $> / x \in X$ }.where α_A (x) / X→ [0,1] and β_A (x) / X→ [0,1] , with the condition 0 ≤ β_A (x) + β_A (x) ≤ 1 for all x ≤ X. The numbers α_A (x) and β_A (x) denote respectively ,the degree of membership and the degree of non membership of the element x in the set A .

Note:

The symbol A = (α_A , β_A) for the intuitionistic fuzzy set

A ={$< x,\ \alpha_A$ (x)$, \beta_A$ (x) $> / x \in X$ }.

Definition: 3.1.22

Let A= (α_A , β_A) and B= (α_B , β_B) be two intuitionistic fuzzy sets on X. Then the intersection of A and B is denoted by A∩B and is given by A∩B = {min (α_A , α_A), (β_B , β_B) }. Also the complement of A is denoted by $\bar{A}$ is defined by $\bar{A}$ = (β_A , α_A) .

Definition: 3.1.23

Let A = (α_A , β_A) be an intuitionistic fuzzy set defined on X. The operators ⊕A and ⊗A are defined as ⊕A= ($\alpha_{A ,}\overline{\alpha_A}$) , and ⊗A = ($\beta_{A ,}\overline{\beta_A}$) respectively.

Definition: 3.1.24

Let A={$< x,\ \alpha_A$ (x)$, \beta_A$ (x) $> / x \in X$ } be a fuzzy set in X , where X is a G –subalgebra. Then the set A is a fuzzy G – subalgebra over the binary operator $*$ if it satisfies the condition $\alpha_A\ (x * y) \geq$ min{α_A (x) , α_A (y)}for all x ,y ∈ X.

Example: 3.1.25

Let X ={ 0,1,2,3,4,5,6,7}be a G-algebra with the following Cayley table :

$*$	0	1	2	3	4	5	6	7
0	0	2	1	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	2	1	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Define a fuzzy set A in X by α_A (0)=α_A (5)=0.8 and α_A (3)=α_A (6)=0.7 and α_A (x) =0.6 for all x $\in$\{ 1,2,4,7\}. Then A is a fuzzy G – subalgebra of X.

Definition: 3.1.26

Let A = (α_A , β_A) be an intuitionistic fuzzy G – subalgebra of X . For

s,t $\in$ [0,1] , the set $\cup(\alpha_A\,; s)$ = \{ x / x $\in X$ *and* α_A (x) $\geq s$\}is called upper s-level of A and $L(\beta_A\,; t)$ = \{ x / x $\in X$ *and* β_A (x) $\leq t$\}is called t- level of A .

Section 3.2

Intuitionistic Fuzzy G – subalgebras of G-algebras

Definition: 3.2.1

An IFS A=(α_A , β_A) in X is called an intuitionistic fuzzy G – subalgebra of X if for all x,y $\in X$ it satisfies

IFGS$_1$: $\alpha_A(x * y) \geq \min \{ \alpha_A(x), \alpha_A(y) \}$

IFGS$_2$: $\beta_A(x * y) \leq \max \{ \beta_A(x), \beta_A(y) \}$.

Example: 3.2.1

Let X= \{0,1,2,3,4,5\} be a set with the following Cayley table

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	4	0	2	1
4	4	3	5	1	0	2
5	5	1	2	4	3	0

Let A=(α_A, β_A) be an IFS in X defined by $\alpha_A(x) = \begin{cases} s_0, & \text{if } x \in \{0,3\} \\ s_1, & \text{otherwise} \end{cases}$ and

$$\beta_A(x) = \begin{cases} t_0, & \text{if } x \in \{0,3\} \\ t_1, & \text{otherwise} \end{cases}$$

where $s_0, s_1, t_0, t_1 \in [0,1]$ with $s_0 > s_1, t_0 < t_1, s_0 + t_0 \leq 1, s_1 + t_1 \leq 1$. Then A= (α_A, β_A) is an intuitionistic fuzzy G – subalgebra of X .

Proposition: 3.2.3

If A = (α_A, β_A) is an intuitionistic fuzzy G -subalgebra in X , then for all x ∈ X , $\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$. Thus, $\alpha_A(0)$ and $\beta_A(0)$ are the upper bounds and lower bounds of $\alpha_A(x)$ and $\beta_A(x)$ respectively.

Proof

Let x ∈ X. Then $\alpha_A(0) = \alpha_A(x * x) \geq \min\{\alpha_A(x), \alpha_A(x)\} = \alpha_A(x)$ and $\beta_A(0) = \beta_A(x * x) \leq \max\{\beta_A(x), \beta_A(x)\} = \beta_A(x)$.

Theorem: 3.2.4

Let A=(α_A, β_A) be an intuitionistic fuzzy G -subalgebra of X . If there exists a sequence$\{x_n\}$ in X such that $\lim_{n \to \infty} \alpha_A(x_n) = 1$ and,

$\lim_{n \to \infty} \beta_A(x_n) = 0$ then $\alpha_A(0) = 1$ and $\beta_A(0) = 0$.

Proof

By the above Proposition, $\alpha_A(0) \geq \alpha_A(x)$ for all x ∈ X ,therefore $\alpha_A(0) \geq \alpha_A(x_n)$ for every positive integer n.

Consider, $1 \leq \beta_A(0) \leq \lim_{n \to \infty} \alpha_A(x_n) = 1$. Hence $\alpha_A(0) = 1$. Again, by the above Proposition, $\beta_A(0) \leq \beta_A(x)$ for all x ∈ X , thus $\beta_A(0) \leq \beta_A(x_n)$ for every positive integer n. Now, $0 \geq \beta_A(0) \geq \lim_{n \to \infty} \beta_A(x_n) = 0$. Hence, $\beta_A(0) = 0$.

Proposition: 3.2.5

If an IFS A=(α_A , β_A) in X is an intuitionistic fuzzy G - subalgebra, then for all x ∈ X, $\alpha_A (0 * x) \geq \alpha_A (x)$ and $\beta_A (0 * x) \leq \beta_A (x)$.

Proof

For all x ∈ X, $\alpha_A (0 * x) \geq \min \{ \alpha_A(0), \alpha_A(x)\}$

$\alpha_A (0 * x) \geq \min \{ \alpha_A(0), \alpha_A(x)\} = \min\{\min \{ \alpha_A(0), \alpha_A(x), \alpha_A (x)\}$

$= \alpha_A (x)$

and $\beta_A (0 * x) \leq \max \{ \beta_A(x), \beta_A(x) \} = \max\{ \max \{ \beta_A(x), \beta_A(x), \beta_A(x)\}$

$= \beta_A(x)$.

Definition: 3.2.6

Let A=(α_A , β_A) and B=(α_B , β_B) be two IFSs on X . Then the intersection of A and B is defined by $A \cap B$ and is given by $A \cap B$ = {min (α_A , α_B), max(β_A , β_B).Also the complement of A is denoted by $\bar{A}$ and is defined by $\bar{A}$ = (β_A , α_A) .

Note :

The intersection of two intuitionistic fuzzy G -subalgebras is also an intuitionistic fuzzy G –subalgebra.

Theorem: 3.2.7

Let A_1 and A_2 be two intuitionistic fuzzy G -subalgebras of X .Then $A_1 \cap A_2$ is an intuitionistic fuzzy G -subalgebra of X .

Proof.

Let x , y ∈ $A_1 \cap A_2$. Then x , y ∈ A_1 and A_2 .

Now $\alpha_{A1\cap A2} (x * y) = \min\{ \alpha_{A_1} (x * y), \alpha_{A_2}(x * y)\}$

$\geq \min\{\min\{\alpha_{A_1} (x), \alpha_{A_1} (y)\}, \min\{\alpha_{A_2} (x), \alpha_{A_2} (y)\}\}$

$= \min\{\min\{\alpha_{A_1} (x), \alpha_{A_2} (x)\}, \min\{\alpha_1 (y), \alpha_{A_2} (y)\}\}$

$= \min\{ \alpha_{A1\cap A2} (x), \alpha_{A1\cap A2} (y)\}$

and $\beta_{A1\cap A2}(x*y) = \max\{\beta_{A_1}(x*y), \beta_{A_2}(x*y)\}$

$$\leq \max\{\max\{\beta_{A_1}(x), \beta_{A_1}(y)\}, \max\{\beta_{A_2}(x), \beta_{A_2}(y)\}\}$$

$$= \max\{\max\{\beta_{A_1}(x), \beta_{A_2}(x)\}, \max\{\beta_1(y), \beta_{A_2}(y)\}\}$$

$$= \max\{\beta_{A1\cap A2}(x), \beta_{A1\cap A2}(y)\}.$$

Hence, $A_1 \cap A_2$ is an intuitionistic fuzzy G -subalgebra of X .

Theorem: 3.2.8

Let $\{A_i \ / \ i = 1,2,3\ldots\ldots\}$be a family of intuitionistic fuzzy G - subalgebras of X. Then $\cap A_i$ is also an intuitionistic fuzzy G -subalgebra of X,
where $\cap A_i = (\min(\alpha_{A_i}(x), \beta_{A_i}(x))$.

Proof

obvious

Theorem: 3.2.9

An IFS A=(α_A, β_A) is an intuitionistic fuzzy G -subalgebra of X if and only if the fuzzy sets $A_1 = \{\alpha_A(x) \ / \ x \in A\}$ and $A_2 = \{\bar{\beta}_A(x) \ / \ x \in A\}$ are fuzzy G -subalgebras of X .

Proof

Let A=(α_A, β_A) be an intuitionistic fuzzy G -subalgebra of X .Clearly, A_1 is a fuzzy G -subalgebra of X . For every ,x y $\in X$.

We have $\bar{\beta}_A(x*y) = 1 - \beta_A(x*y)$

$$\geq 1\text{-max}\{\beta_A(x), \beta_A(y)\} = \min\{1\text{-}\beta_A(x), 1 - \beta_A(y)\}$$

$$= \min\{\bar{\beta}_A(x), \bar{\beta}_A(y)\}.$$

Hence A_2 is a fuzzy G -subalgebra of X .

Conversely assume that A_1 and A_2 be two intuitionistic fuzzy G -subalgebras of X. For every ,x y $\in X$. $\alpha_A(x*y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$ and

$1 - \beta_A(x*y) = \bar{\beta}_A(x*y) \geq \min\{\bar{\beta}_A(x), \bar{\beta}_A(y)\}$

$$= \min\{1-\beta_A(x), 1-\beta_A(y)\}$$

$$=1-\max\{\beta_A(x), \beta_A(y)\}$$

That is, $\beta_A(x * y) \leq \max\{\beta_A(x), \beta_A(y)\}$. Hence A= (α_A, β_A) be an intuitionistic fuzzy G -subalgebra of X .

Note:

for any element x and y of X, let us write

$\prod^n$ x∗y for $x * (\ldots\ldots * (x * (x * y)))$ where occurs n times.

Theorem: 3.2.10.

Let A= (α_A, β_A) be an intuitionistic fuzzy G -subalgebra of X and let n $\in N$ (the set of natural numbers). Then for all x $\in X$.

(i) $\alpha_A(\prod^n x * x) \geq \alpha_A(x)$, for any odd number n, and $\beta_A(\prod^n x * x) \leq \beta_A(x)$, for any odd number n ,

(ii) $\alpha_A(\prod^n x * x) = \alpha_A(x)$, for any even number n, and $\beta_A(\prod^n x * x) = \beta_A(x)$, for any even number n .

Proof

To prove (i) Let x $\in X$ and assume that n is odd. Then n=2p-1 for some positive integer p . Now we prove this theorem by induction

$\alpha_A(x * x) = \alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(x * x) = \beta_A(0) \leq \beta_A(x)$. Suppose that

$\alpha_A(\prod^{2p-1} x * x) \geq \alpha_A(x)$ and $\beta_A(\prod^{2p-1} x * x) \leq \beta_A(x)$. Then by assumption,

$$\alpha_A(\textstyle\prod^{2(p+1)-1} x * x) = \alpha_A(\textstyle\prod^{2p+1} x * x)$$

$$= \alpha_A(\textstyle\prod^{2p-1} x * (x * (x * x)))$$

$$= \alpha_A(\textstyle\prod^{2p-1} x * x) \geq \alpha_A(x)$$

$$\beta_A(\textstyle\prod^{2(p+1)-1} x * x) = \beta_A(\textstyle\prod^{2p+1} x * x)$$

$$= \beta_A(\textstyle\prod^{2p-1} x * (x * (x * x)))$$

$$= \alpha_A (\prod^{2p-1} x * x) \le \beta_A(x)$$

Which proves (i) similarly (ii) can be proved .

Theorem: 3.2.11

If A=(α_A , β_A) is an intuitionistic fuzzy G -subalgebra of X , then (i) $\oplus A$,and (ii) $\otimes$A , both are intuitionistic fuzzy G -subalgebras.

Proof

To Prove (i) It is sufficient to show that $\bar{\alpha}_A$ satisfies the condition IFGS$_2$. Let ,x,y∈ X. Then $\bar{\alpha}_A(x * y) = 1 - \alpha_A(x * y) \le 1 - \min\{\alpha_A(x), \alpha_A(y)\}$

$$= \max\{1 - \alpha_A(x) , 1 - \alpha_A(y) = \max \{\bar{\alpha}_A(x), \bar{\alpha}_A(y)\}.$$

Hence, $\oplus A$ is an intuitionistic fuzzy G -subalgebra of X .

To Prove (ii) It is sufficient to show that $\bar{\beta}_A$ satisfies the condition IFGS$_2$. Let ,x, y ∈ X. Then$\bar{\beta}_A(x * y) = 1 - \beta_A(x * y) \ge 1 - \max\{\beta_A(x), \beta_A(y)\}$

$$= \min\{1 - \beta_A(x) ,\ 1 - \beta_A(y)\} = \min \{\bar{\beta}_A(x), \bar{\beta}_A(y)\}.$$

Hence, $\otimes A$ is also an intuitionistic fuzzy G -subalgebra of X .

Note:

The sets {x / x∈ X and$\alpha_A(x) = \alpha_A(0)$ } and{x / x∈ X and $\beta_A(x) = \beta_A(0)$ } are denoted by I_{α_A} and I_{β_A} respectively. These two sets are also G -subalgebra of X.

Theorem: 3.2.12

Let A= (α_A , β_A) be an intuitionistic fuzzy G -subalgebra of X , then the sets I_{α_A} and I_{β_A} are G -subalgebras of X.

Proof.

Let x, y ∈ I_{α_A}.

Then$\alpha_A(x) = \alpha_A(0)$ =$\alpha_A(y)$ and $\alpha_A(x * y) \ge \min\{ \alpha_A(x), \alpha_A(y)\} = \alpha_A(0)$. By using proposition 3.2.3,we know that $\alpha_A (x * y)$= $\alpha_A(0)$ .That is $x * y \in I_{\alpha_A}$.

Let x, y ∈ I_{β_A}.

Then $\beta_A(x) = \beta_A(0)$= $\beta_A(y)$and $\beta_A (x * y) \le \max\{ \beta_A(x), \beta_A(y)\}$ =$\beta_A(0)$.

By using proposition 3.2.3,we know that $\beta_A(x * y) = \beta_A(0)$ .That is $x * y \in I_{\beta_A}$. Hence ,the sets I_{α_A} and I_{β_A} are G -subalgebras of X .

Theorem: 3.2.13

Let B be a nonempty subset of X and A=(α_A , β_A)be an IFS in X defined by

$$\alpha_A(x) = \begin{cases} \lambda, \text{ if } x \in B \\ \tau \text{ , otherwise} \end{cases} \quad \text{and} \quad \beta_A(x) = \begin{cases} \gamma \text{ , if } x \in B \\ \delta \text{ , otherwise} \end{cases}$$

For all λ , τ ,γ and $\delta \in [0,1]$ with $\lambda \geq \tau$ and $\gamma \leq \delta$ and $\lambda + \gamma \leq 1$; $\tau + \delta \leq 1$. Then A is an intuitionistic fuzzy G -subalgebra of X if and only if B is a G -subalgebra of X . Moreover, $I_{\alpha_A} = B = I_{\beta_A}$.

Proof

Let A be an intuitionistic fuzzy G -subalgebra of X .Let, x, y ∈ X be such that x, y ∈ B. Then $\alpha_A(x * y) \geq \min\{\alpha_A(x), \alpha_A(y)\} = \min\{\lambda, \lambda\} = \lambda$

and $\beta_A(x * y) \leq \max\{\beta_A(x), \beta_A(y)\} = \max\{\gamma, \gamma\} = \gamma$. This implies $x * y \in B$. Hence B is a G -subalgebra of X.

Conversely, suppose that B is a G -subalgebra of X . Let ,x y $\in X$.

Consider two cases:

Case (i) If x ,y $\in B$, then $x * y \in B$, thus $\alpha_A(x * y) = \lambda = \min\{\alpha_A(x), \alpha_A(y)\}$ and

$\beta_A(x * y) = \gamma = \max\{\beta_A(x), \beta_A(y)\}$.

Case (ii) If x∉ B or, y∉ B, then $\alpha_A(x * y) \geq \tau = \min\{\alpha_A(x), \alpha_A(y)\}$ and $\beta_A(x * y) \leq \delta = \max\{\beta_A(x), \beta_A(y)\}$.

Hence A is intuitionistic fuzzy G -subalgebra of X.

Also $I_{\alpha_A} = \{x \in X$ / and $\alpha_A(x) = \alpha_A(0)\} = \{x \in X$ / and $\alpha_A(x) = \lambda\}$= B. and

$I_{\beta_A} = \{x \in X$ / and $\beta_A(x) = \beta_A(0)\} = \{x \in X$ / and $\beta_A(x) = \delta\}$=B.

Definition: 3.2.14

Let A=(α_A , β_A) is an intuitionistic fuzzy G-subalgebra of X. For s, t∈ [0,1] , the set $\cup(\alpha_A ; s) = \{x \in X$ / and $\alpha_A(x) \geq s\}$ is called upper s-level of A and $L(\beta_A ; t) = \{x \in X$ / and $\beta_A(x) \leq t\}$ is called lower t -level of A.

Theorem: 3.2.15

If A=(α_A , β_A) is an intuitionistic fuzzy G -subalgebra of X , then the upper s-level and lower t -level of A are G -subalgebras of X .

Proof

Let x, y $\in \cup(\alpha_A\,; s)$.Then α_A(x) $\geq s$ and α_A(y) $\geq s$.
It follows that $\alpha_A\,(x * y) \geq$ min { α_A(x), α_A(y)} $\geq s$. so that $x * y \in \cup(\alpha_A\,; s)$ is a G -subalgebra of X.
Let x,y ∈ L($\beta_A\,; t$) Then β_A(x) $\leq t$ and β_A(y) $\geq t$.
It follows that $\beta_A\,(x * y) \leq$ max { β_A(x), β_A(y)} $\leq t$. so that $x * y \in$ L($\beta_A\,; t$) is a G -subalgebra of X.

Theorem: 3.2.16

Let A=(α_A , β_A) be an IFS in X such that the sets $\cup(\alpha_A\,; s)$ and L($\beta_A\,; t$) are G -subalgebras of X for every s, t ∈ [0,1].Then A=(α_A , β_A) is an intuitionistic fuzzy G -subalgebra of X .

Proof

Let for every s,t∈[0,1] ,$\cup(\alpha_A\,; s)$ and L($\beta_A\,; t$) be two subalgebras of X . In contrary, Let $x_0, y_0 \in$ X be such that $\alpha_A\,(x_0 * y_0)$ <min{$\alpha_A\,(x_0)$, $\alpha_A\,(y_0)$}. Let $\alpha_A\,(x_0) = \theta_1$ and $\alpha_A\,(y_0) = \theta_2$ and $\alpha_A\,(x_0 * y_0) =$ s .Then s < min {θ_1, θ_2}. Let us consider $s_1 = \frac{1}{2}$ ($\alpha_A\,(x_0 * y_0)$ + min{$\alpha_A\,(x_0)$, $\alpha_A\,(y_0)$})
we get $s_1 = \frac{1}{2}$ (s+ min{θ_1, θ_2}) .Therefore $\theta_1 > s_1 = \frac{1}{2}$ (s+ min {θ_1, θ_2}) $>$ s and $\theta_2 > s_1 = \frac{1}{2}$ (s+ min{θ_1, θ_2}) $>$ s. Hence , min {θ_1, θ_2} $> s_1 >$ s =$\alpha_A\,(x_0 * y_0)$, so that $x_0 * y_0 \notin \cup(\alpha_A\,; s)$ which is a contradiction.
Since $\alpha_A\,(x_0) = \theta_1 \geq$ min {θ_1, θ_2} $> s_1$ and $\alpha_A\,(y_0) = \theta_2 \geq$ min {θ_1, θ_2} $> s_1$. This implies $x_0, y_0 \in \cup(\alpha_A\,; s)$.Thus $\alpha_A\,(x * y) \geq$ min { α_A(x), α_A(y)} for all x,y ∈ X .
Again Let x_0, y_0 ∈ X be such that $\beta_A\,(x_0 * y_0)$ >max {$\beta_A\,(x_0)$, $\beta_A\,(y_0)$}.
Let $\beta_A\,(x_0) = \eta_1$ and $\beta_A\,(y_0) = \eta_2$. and $\beta_A\,(x_0 * y_0) =$ t. Then t>max{ η_1 , η_2 }. consider $t_1 = \frac{1}{2}$ [$\beta_A\,(x_0 * y_0)$ + max{$\beta_A\,(x_0)$, $\beta_A\,(y_0)$}].

we get that $t_1 = \frac{1}{2}$ (t+ max $\{\eta_1, \eta_2\}$). Therefore $\eta_1 < t_1 = \frac{1}{2}$ (t+ max $\{\eta_1, \eta_2\}$) < t and $\eta_2 < t_1 = \frac{1}{2}$ (t+ max $\{\eta_1, \eta_2\}$) < t. Hence max $\{\eta_1, \eta_2\} < t_1 <$ t = $\beta_A(x_0 * y_0)$.

So that $x_0 * y_0 \notin$ L($\beta_A; t$) which is a contradiction.

Since $\beta_A(x_0) = \eta_1 \leq$ max $\{\eta_1, \eta_2\} < t_1$ and $\beta_A(y_0) = \eta_2 \leq$ max $\{\eta_1, \eta_2\} < t_1$This implies $x_0, y_0 \in$ L($\beta_A; t$). Thus $\beta_A(x * y) \leq$ max { β_A(x), β_A(y)} for all x,y ∈ X. Hence A = (α_A, β_A) is an intuitionistic fuzzy G -subalgebra of X .

Theorem: 3.2.17

Any subalgebra of X can be realized as both the upper level and lower level of some intuitionistic fuzzy G -subalgebra of X .

Proof

Let P be an intuitionistic fuzzy G -subalgebra of X , and A be an IFS on X defined by

$$\alpha_A(x) = \begin{cases} \lambda, & \text{if } x \in P \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_A(x) = \begin{cases} \tau, & \text{if } x \in P \\ 1, & \text{otherwise} \end{cases}$$

For all $\lambda, \tau \in [0,1]$ and $\lambda + \tau \leq 1$. We consider the following cases:

Case(i) If x ,y ∈ P ,then α_A(x) = λ , β_A(x) = τ and α_A(y) = λ , β_A(y) = τ.Thus α_A(x ∗ y) = λ= min $\{\lambda, \lambda\}$=min{ α_A(x), α_A(y)} and

β_A(x ∗ y)= τ =max{ τ, τ}=max{ β_A(x), β_A(y)}.

Case(ii) If x∈ P and y∉ P, then α_A(x) = λ , β_A(x) = τ and α_A(y) = 0 , β_A(y)= 1. Thus α_A(x ∗ y) ≥ 0 = min{λ, 0 } = min{ α_A(x), α_A(y)} and

β_A(x ∗ y) ≤ 1=max{τ, 1}= max{β_A (x), β_A(y)}.

Case(iii) If x ∉ P and y ∈ P, then α_A(x) = 0 , β_A(x) = 1 and α_A(y) = λ , β_A(y)= τ. Thus α_A(x ∗ y) ≥ 0 = min{0, λ} = min{ α_A(x), α_A(y)} and

β_A(x ∗ y) ≤ 1=max{1, τ}= max{β_A (x), β_A(y)}.

Csae(iv) If x ∉ P and y ∉ P, then α_A(x) = 0 , β_A(x) = 1 and α_A(y) = 0 , β_A(y)= 1. Thus α_A(x ∗ y) ≥ 0 = min{0, 0} = min{ α_A(x), α_A(y)} and

β_A(x ∗ y) ≤ 1=max{1, 1}= max{β_A (x), β_A(y)}.

Therefore, A is an intuitionistic fuzzy G -subalgebra of X.

Theorem: 3.2.18

Let P be a subset of X and A be an IFS on X which is given by

$$\alpha_A(x) = \begin{cases} \lambda \text{ , if } x \in P \\ 0 \text{ ,otherwise} \end{cases} \quad \text{and} \quad \beta_A(x) = \begin{cases} \tau\text{, if } x \in P \\ 1 \text{ ,otherwise} \end{cases}$$

For all $\lambda , \tau \in [0,1]$ and $\lambda + \tau \leq 1$.If A be realized as lower level subalgebra and upper level subalgebra of some intuitionistic fuzzy G -subalgebra of X , then P is a intuitionistic fuzzy G-subalgebra of X .

Proof

Let A be an intuitionistic fuzzy G-subalgebra of X, and x,y ∈ P.

Then $\alpha_A (x) = \lambda = \alpha_A(y)$ and $\beta_A (x) = \tau = \beta_A(y)$.
Thus$\alpha_A(x * y) \geq \min\{ \alpha_A(x), \alpha_A(y)\} = \min\{\tau, \lambda\} = \lambda$. And
$\beta_A(x * y) \leq \max \{ \beta_A(x), \beta_A(y)\} = \min\{\tau, \tau\} = \tau$.which imply that x∗ y ∈ P .
Hence the proof is completed.
As a generalization of Theorem 3.2.17, we prove the following theorem:

Theorem: 3.2.19

Let X be a G -algebra. Then any given chain of G –subalgebras

$P_0 \subset P_1 \subset P_2 \subset$ ……….$\subset P_r = X$, there exists an intuitionistic fuzzy G -subalgebra A of X whose level subalgebras are exactly the G -subalgebras of this chain.

Proof

Consider set of numbers $s_0 > s_1 >$ ……….> s_r and $t_0 < t_1 <$………$< t_r$, where each $s_i , t_i \in [0,1]$ and $s_i + t_i \leq 1$ .Let A = (α_A , β_A) an IFS defined by

$$\alpha_A(x) = \begin{cases} s_0 \text{ , if } x \in P_0 \\ s_1 \text{ ,if } x \in P_i - P_{i-1} \end{cases} \qquad (1)$$

And

$$\beta_A(x) = \begin{cases} t_0 \text{ , if } x \in P_0 \\ t_1 \text{ , if } x \in P_i - P_{i-1} \text{ , } 0 < i \leq r \text{ .} \end{cases} \qquad (2)$$

We consider the following two cases:

Case (i). Let x, y $\in P_i - P_{i-1}$ .Therefore, by (1) and (2), α_A(x) $=$ α_A(y)=s_i and β_A(x) = β_A(y) $= t_i$. Since P_i is a G –subalgebra, we have x∗y $\in P_i$, and so either x∗y $\in P_i - P_{i-1}$ or x∗y $\in P_{i-1}$. In any case we conclude that $\alpha_A(x * y) \geq S_i = \min\{\alpha_A(x), \alpha_A(y)\}$ and $\beta_A(x * y) \leq t_i = \max\{\beta_A(x), \beta_A(y)\}$.

Case (ii) Let x $\in P_i - P_{i-1}$ and y $\in P_j - P_{j-1}$ for i > j.Therefore, by (1)and (2), α_A(x)=s_i, α_A(y) $= s_j$, β_A(x)= t_i, β_A(y) $= t_j$. Then x∗y $\in P_i$ Since P_i is a G –subalgebra of X and $P_j \subset P_i$. Hence $\alpha_A(x * y) \geq S_j = \min\{\alpha_A(x), \alpha_A(y)\}$ and $\beta_A(x * y) \leq t_j = \max\{\beta_A(x), \beta_A(y)\}$.

Thus A is an intuitionistic fuzzy G -subalgebra of X . From (1) and(2), it follows that Im(α_A) = $\{s_0, s_1, \ldots, s_r\}$ and Im(β_A)= $\{t_0, t_1, \ldots, t_r\}$.

Hence, the level subalgebras of A are given by the chain of G -subalgebras

$$\cup(\alpha_A\,; s_0) \subset \cup(\alpha_A\,; s_1) \subset \ldots\ldots\ldots\ldots \subset \cup(\alpha_A\,; s_r) = X \qquad (3)$$

$$L(\beta_A\,; t_0) \subset L(\beta_A\,; t_1) \subset \ldots\ldots\ldots\ldots \subset L(\beta_A\,; t_r) = X \qquad (4)$$

Now , $\cup(\alpha_A\,; s_0)$ ={ x$\in X$ / and α_A(x) $\geq s_0$} = P_0

= { x$\in X$ / and β_A(x) $\leq t_0$} = $L(\beta_A\,; t_0)$.

Finally we prove that $\cup(\alpha_A\,; s_i)$ = P_i = $L(\beta_A\,; t_i)$ for 0 < i ≤ r .clearly, $P_i \subset \cup(\alpha_A\,; s_i)$ and $L(\beta_A\,; t_i)$. If x$\in \cup(\alpha_A\,; s_i)$ and $L(\beta_A\,; t_i)$, then α_A(x) $\geq s_i$ and β_A(x) $\leq t_i$. Which implies that x $\notin P_j$ for j>i .

Hence α_A(x) $\in \{s_0, s_1, \ldots, s_i\}$ and β_A(x) $\in\{t_0, t_1, \ldots, t_i\}$and x$\in P_k$ for some k $\leq i$. As $P_k \subseteq P_i$, it follows that x$\in P_i$. Therefore $\cup(\alpha_A\,; s_i)$ = P_i = $L(\beta_A\,; t_i)$ for 0 ≤ i ≤ r. This completes the proof.

Note:

If X is a finite G -algebra, then the number of G -subalgebras of X is finite but on the other hand the number of level subalgebras of an intuitionistic fuzzy G - subalgebra A appears to be infinite. But since every level subalgebra is indeed a G - subalgebra of X , not all these G -subalgebras are distinct. The next theorem characterizes this aspect.

Theorem: 3.2.20

Let A be a fuzzy G -subalgebra of a G -algebra X. Then

(i) Two upper s-level subalgebras $\mathrm{U}(\alpha_A\,;s_1)$, $\mathrm{U}(\alpha_A\,;s_2)$ (with $s_1<s_2$) of A are equal if and only if there is no x$\in X$ such that $s_1 \leq \alpha_A$(x) $< s_2$.

(ii) Two lower t -level subalgebras $L(\beta_A\,;t_1)$,$L(\beta_A\,;t_2)$ (with $t_1 > t_2$) of A are equal if and only if there is no x$\in X$ such that $t_1 \geq \alpha_A$(x) $> t_2$.

Proof

To prove (i) Assume that $\mathrm{U}(\alpha_A\,;s_1)$ = $\mathrm{U}(\alpha_A\,;s_2)$ for $s_1 < s_2$ and assume that there exists x$\in X$ such that $s_1 \leq \alpha_A$(x) $< s_2$ .Then $\mathrm{U}(\alpha_A\,;s_2)$ is a proper subset of $\mathrm{U}(\alpha_A\,;s_1)$ which is a contradiction.

Conversely, suppose that there is no x$\in X$ such that $s_1 \leq \alpha_A$(x) $< s_2$.Since $s_1 < s_2$, we have $\mathrm{U}(\alpha_A\,;s_2) \subseteq \mathrm{U}(\alpha_A\,;s_1)$. If x $\in \mathrm{U}(\alpha_A\,;s_1)$, then α_A(x) $\geq s_1$ and so α_A(x) $\geq s_2$, because α_A(x) does not lie between s_1 and s_2 .
Hence x $\in \mathrm{U}(\alpha_A\,;s_2)$, which implies that $\mathrm{U}(\alpha_A\,;s_1) \subseteq \mathrm{U}(\alpha_A\,;s_2)$.
This completes the proof of (i). Similarly (ii) can be proved .

Remark: 3.2.21

As a consequence of Theorem 3.2.20, the level subalgebras of an intuitionistic fuzzy G -subalgebra A of a finite G -algebra X form a chain. But α_A(x) $\leq \alpha_A(0)$ andβ_A(x) $\geq \beta_A(0)$ for all x$\in X$. Therefore$\mathrm{U}(\alpha_A\,;s_0)$, where $s_0 = \alpha_A(0)$,is the smallest upper s-level subalgebra and $L(\beta_A\,;t_0)$,where $t_0 = \beta_A(0)$, is the smallest lower t -level subalgebra but not always $\mathrm{U}(\alpha_A\,;s_0)$ and $L(\beta_A\,;t_0)$, as shown in the following example.

Example: 3.2.22

Let P be a non-trivial G -subalgebra of X and A be an intuitionistic fuzzy G –subalgebra in Theorem 3.2.17. Then Im(α_A) ={0,λ} and Im(β_A)={τ,1}. Furthermore, the two upper s -level subalgebras of A are U(α_A; 0) = X and U(α_A; λ) = P ≠{0}, and the two lower t -level subalgebras of A are L(β_A; 1) = X and L(β_A; τ) = P ≠{0}.

Theorem: 3.2.23

Let X be a finite G -algebra and A be an intuitionistic fuzzy G-subalgebra of X .

(i) If Im(α_A) = $\{s_1, s_2,\ldots\ldots, s_n\}$, then the family of G-subalgebras U(α_A ; s_i) , 1≤i≤ n constitutes all the upper s-level subalgebras of A.

(ii) If Im(β_A) = $\{t_1, t_2,\ldots\ldots, t_n\}$, then the family of G-subalgebras L(β_A ; t_i), 1≤i≤ n constitutes all the lower t -level subalgebras of A.

Proof

To prove (i) Let s∈ [0,1]and s ∉ Im(α_A). Suppose $s_1 < s_2 <\ldots\ldots< s_n$ without loss of generality. If s ≤ s_1, then U(α_A; s_1) = X =U(α_A; s) , if s >s_n then obviously U(α_A; s) =φ . If $s_{i-1} < s < s_i$, then by Theorem 3.2.20,
we get U(α_A; s) = X =U(α_A; s_i).
Thus for any s ∈ [0,1], the level subalgebra is one of { U(α_A; s_i) / i= 1,2,…..,n}. similarly (ii) can be proved .
The following examples show that two intuitionistic fuzzy G -subalgebras of a G -algebra may have an identical family of level subalgebras but the intuitionistic fuzzy G -subalgebras may not be equal.

Example: 3.2.24

In Example 3.2.1, the upper s-level subalgebras of A are
U(α_A; s_0) $= \{0,3\}$ and U(α_A; s_1) $= X$, and the lower t -level subalgebras of A are L(β_A; t_0) $= \{0,3\}$ and L(β_A; t_1) $= X$. Now, let $p_0, p_1, k_0, k_1 \in [0,1]$ be such that $p_0 > p_1$ and $k_0 < k_1$ with $p_i \neq s_j$ and $k_i \neq t_j$ for i, j =0,3.We define an IFS B =(α_B, β_B) with membership values $\alpha_B(0) = \alpha_B(3)=$ p_0, α_B(x) $= p_1$, and non-membership values$\beta_B(0) = \beta_B(3)= k_0, \beta_B$(x) $= k_1$ for all x ∈ {1,2,4,5}. Then B is an intuitionistic fuzzy G -subalgebra of X . Then family of upper s-level subalgebras of B are U(α_B; p_0) $= \{0,3\}$,U(α_B; p_1) $= X$ and the family of lower t -level subalgebras of B are L(β_B; k_0) $= \{0,3\}$,L(β_B; k_1) $= X$. Thus the two intuitionistic fuzzy G -subalgebras A and B have the same family of level subalgebras. However A is not equal to B .

Theorem: 3.2.25

Let A be an intuitionistic fuzzy G - subalgebra of X . Then

(i) If Im (α_A) is finite, say $\{s_1, s_2,\ldots\ldots, s_n\}$, then for any s_i, s_j $\in$ Im (α_A), $U(\alpha_A; s_i) = U(\alpha_A; s_j)$ implies $s_i = s_j$.

(ii) If Im (β_A) is finite, say $\{t_1, t_2,\ldots\ldots, t_n\}$, then for any t_i, t_j $\in$ Im (β_A), $L(\beta_A\,; t_i) = L(\beta_A\,; t_j)$ implies $t_i = t_j$.

Proof

To prove (i) Assume that $s_i \neq s_j$, says $s_i < s_j$. If x $\in U(\alpha_A; s_j)$, then α_A(x) $\geq s_j > s_i$, which implies that x$\in U(\alpha_A; s_i)$. Let x$\in X$ be such that $s_i < \alpha_A$(x) $< s_j$.Then x$\in U(\alpha_A; s_i)$, but x$\notin U(\alpha_A; s_j)$. Hence $U(\alpha_A; s_j) \subset U(\alpha_A; s_i)$ and $U(\alpha_A; s_j) \neq U(\alpha_A; s_i)$, a contradiction. similarly (ii)can be proved.

Theorem: 3.2.26

Let A=(α_A, β_A) and B=(α_B, β_B) be two intuitionistic fuzzy G -subalgebras of a finite G -algebra X with identical family of level subalgebras. If

Im $(\alpha_A) = \{l_0, l_1, \ldots\ldots.., l_r\}$,Im$(\alpha_B) = \{s_0, s_1, \ldots\ldots.., s_k\}$,

Im$(\beta_A) = \{m_0, m_1, \ldots\ldots.., m_r\}$ and Im (β_A) = $\{t_0, t_1,\ldots\ldots, t_k\}$

where $l_0 > l_1 > \ldots\ldots > l_r$, $s_0 > s_1 > \ldots\ldots\ldots s_k$, $m_0 < m_1 < \ldots\ldots < m_r$ and

$t_0 < t_1 < \ldots\ldots\ldots < t_k$ then we have

(i) r=k

(ii) $U(\alpha_A; l_i)$= $U(\alpha_B; s_i)$ and $L(\beta_A\,; m_i)$= $L(\beta_A\,; t_i)$,0 $\leq$ i $\leq k$,

(iii) If x$\in X$ such that α_A(x) $= l_i$and β_A(x) $= m_i$,then α_A(x) $= s_i$ and β_A(x) $= t_i$, 0 $\leq$ i $\leq k$.

Proof

To prove (i) By Theorem 3.2.23, the only subalgebras of A and B are the families $U(\alpha_A; l_i)$, $U(\alpha_B; s_i)$, $L(\beta_A\,; m_i)$and $L(\beta_A\,; t_i)$.Since A and B have the same family of level subalgebras, it follows that r=k.

(ii) Using Remark 3.2.21 and (i), we have chains of level subalgebras $\mathrm{U}(\alpha_A; l_0) \subset \ldots \subset \mathrm{U}(\alpha_A; l_k)$= X and $\mathrm{U}(\alpha_B; l_0) \subset \ldots \subset \mathrm{U}(\alpha_A; l_k)$= X It follows clearly that if $l_i, l_j \in$ Im (α_A) such that $l_i > l_j$ and s_i, $s_j \in$ Im (α_B) such that $s_i > s_j$, then $\mathrm{U}(\alpha_A; l_i) \subset \mathrm{U}(\alpha_A; l_j)$ and $\mathrm{U}(\alpha_B; s_i) \subset \mathrm{U}(\alpha_B; s_j)$ (1). Since the two families of level subalgebras are identical, $\mathrm{U}(\alpha_A; l_0) = \mathrm{U}(\alpha_B; s_0)$. By hypothesis $\mathrm{U}(\alpha_A; l_1) = \mathrm{U}(\alpha_B; s_j)$ for some j > 0. Assume that $\mathrm{U}(\alpha_A; l_1) \neq \mathrm{U}(\alpha_B; s_1)$. Then $\mathrm{U}(\alpha_A; l_1) = \mathrm{U}(\alpha_B; s_j)$ for some j >1 , and $\mathrm{U}(\alpha_B; s_1) = \mathrm{U}(\alpha_A; l_i)$) for some $l_i < l_1$. Thus by (3) we obtain that $\mathrm{U}(\alpha_B; s_j) = \mathrm{U}(\alpha_A; l_1) \subset \mathrm{U}(\alpha_A; l_i)$ and $\mathrm{U}(\alpha_A; l_i) = \mathrm{U}(\alpha_B; s_1) \subset \mathrm{U}(\alpha_B; s_j)$. This is a contradiction.

Hence $\mathrm{U}(\alpha_A; l_1) = \mathrm{U}(\alpha_B; s_1)$. By induction on i, $0 \leq i \leq k$,we finally obtain that $\mathrm{U}(\alpha_A; l_i) = \mathrm{U}(\alpha_B; s_i)$, $0 \leq i \leq k$ .Similarly we can obtain $\mathrm{L}(\beta_A; m_i) = \mathrm{L}(\beta_B; t_i)$, $0 \leq i \leq k$.

(iii) Let x$\in X$ be such that $\alpha_A(\mathrm{x}) = l_i$ and $\alpha_B(\mathrm{x}) = s_j$, where $0 \leq i \leq k$ and $0 \leq j \leq k$. It is sufficient to show that $s_j = s_i$.
Now x $\in$ $\mathrm{U}(\alpha_A; l_i) = \mathrm{U}(\alpha_B; s_i)$ implies that $\alpha_B(\mathrm{x}) = s_j \geq s_i$. This gives from (3) that $\mathrm{U}(\alpha_B; s_j) \subseteq \mathrm{U}(\alpha_B; s_i)$. Since x $\in \mathrm{U}(\alpha_B; s_i)$,
it follows from (ii) that x$\in$ $\mathrm{U}(\alpha_A; l_j)$ and so $\alpha_A(\mathrm{x}) = l_i \geq l_j$,which implies that $\mathrm{U}(\alpha_A; l_i) \subseteq \mathrm{U}(\alpha_A; l_j)$by (3). Using (ii),
we have $\mathrm{U}(\alpha_B; s_i) = \mathrm{U}(\alpha_A; l_i) \subseteq \mathrm{U}(\alpha_A; l_j) = \mathrm{U}(\alpha_B; s_j)$. Thus $\mathrm{U}(\alpha_B; s_i) = \mathrm{U}(\alpha_B; s_j)$ and by Theorem 3.2.25, $s_j = s_i$. This completes the proof of first part. Proof is similar for the second part.

Theorem: 3.2.27

Let $T_1 \supseteq T_2 \supseteq T_3$ …… be a descending chain of G -subalgebras of X which terminates at finite step. For an intuitionistic fuzzy G -subalgebra A=(α_A , β_A)of X , if a sequence of elements of Im(α_A) a is strictly increasing and Im(β_A) strictly decreasing, then A=(α_A , β_A)is finite valued.

Proof

Assume that A=(α_A , β_A) is infinite valued. Let $\{\phi_n\}$ be a strictly increasing sequence of elements of Im(α_A). Then $0 \leq \phi_1 < \phi_2 <$…….<1.

Note that U$(\alpha_A;\phi_t)$is a G -subalgebra of X for t =1,2,3,…. . Let x ∈ U$(\alpha_A;\phi_t)$ for t =1,2,3,…. .Then $\alpha_A\,(x) \geq \phi_t > \phi_{t-1}$ .which implies that x ∈ U$(\alpha_A;\phi_{t-1})$.

Hence U$(\alpha_A;\phi_t)$ ⊆ U$(\alpha_A;\phi_{t-1})$ for t = 2,3, …. . Since ϕ_{t-1} ∈ Im$(\alpha_A\,)$ there exists x_{t-1} such that $\alpha_A\,(x_{t-1}\,)= \phi_{t-1}$. It follows that x_{t-1} ∈ U$(\alpha_A;\phi_{t-1})$,but x ∉ U$(\alpha_A;\phi_t)$.

Thus U$(\alpha_A;\phi_t)$ ⊊ U$(\alpha_A;\phi_{t-1})$, and so we obtain a strictly descending chain U$(\alpha_A;\phi_1)$ ⊋ U$(\alpha_A;\phi_2)$ ⊋…. of G -subalgebras of X which is not terminating. This is impossible.

Again let $\{\phi_n\}$ be a strictly decreasing sequence of elements of Im$(\beta_A\,)$.Then $1 \geq \phi_1 > \phi_2 >$…….>0. Also L$(\beta_A;\varphi_s)$is a G -subalgebra of X for s =1,2,3,…. . Let x ∈ L$(\beta_A;\varphi_s)$ for s =1,2,3,…. .Then $\beta_A\,(x) \leq \varphi_s < \varphi_{s-1}$.

which implies that x∈ L$(\beta_A;\varphi_{s-1})$ .Hence L$(\beta_A;\varphi_s)$ ⊆ L$(\beta_A;\varphi_{s-1})$ s =2,3,…..Since φ_{s-1} ∈ Im$(\beta_A\,)$ there exists x_{s-1} such that $\beta_A\,(x_{s-1}\,)= \varphi_{s-1}$. It follows that x_{s-1} ∈ L$(\beta_A;\varphi_{s-1})$,but x ∉ L$(\beta_A;\varphi_s)$.

Thus L$(\beta_A;\varphi_s)$ ⊊ L$(\beta_A;\varphi_{s-1})$, and so we obtain a strictly descending chain L$(\beta_A;\varphi_1)$ ⊋ L$(\beta_A;\varphi_2)$ ⊋…. of G -subalgebras of X which is not terminating. This is impossible.

Therefore, A=$(\alpha_A\,,\beta_A\,)$ is finite valued.

Now we consider the converse of Theorem 3.2.27.

Theorem: 3.2.28

If every intuitionistic fuzzy G -subalgebra A of X has the finite image, then every descending chain of G -subalgebras of X terminates at finite step.

Proof

Suppose there exists a strictly descending chain $T_0 \supsetneq T_1 \supsetneq T_2$……. of closed ideals of X which does not terminate at finite step. Define an IFS A in X by

$$\alpha_A\,(x) = \begin{cases} \frac{n}{n+1}\,, \text{ if } x \in T_n \setminus T_{n+1}, \\ 1\ \ , \text{ if } x \in \bigcap_{n=0}^{\infty} T_n \end{cases}$$

and

$$\beta_A\,(x) = \begin{cases} \frac{1}{n+1}\,, \text{ if } x \in T_n \setminus T_{n+1}, \\ 0\ \ , \text{ if } x \in \bigcap_{n=0}^{\infty} T_n \end{cases}$$

where n =0,1,2,…… and T_0 stands for X . Now, we consider the following cases: If x and y $\in T_n$, then x$*$ y $\in T_n$, because T_n is a G -subalgebra of X . Hence,

$$\alpha_A (x * \mathrm{y}) \geq \frac{n}{n+1} = \min\{\alpha_A (x) , \alpha_A (\mathrm{y})\}.$$

$$\beta_A (x * \mathrm{y}) \leq \frac{1}{n+1} = \max\{\beta_A (x) , \beta_A (\mathrm{y})\}.$$

if x $\in T_n \setminus T_{n+1}$ and if y $\in T_m \setminus T_{m+1}$, where n > m , then x$*$ y $\in T_m$. Hence,

$$\alpha_A (x * \mathrm{y}) \geq \frac{m}{m+1} = \min\{\alpha_A (x) , \alpha_A (\mathrm{y})\}.$$

$$\beta_A (x * \mathrm{y}) \leq \frac{1}{m+1} = \max\{\beta_A (x) , \beta_A (\mathrm{y})\}.$$

if x $\in T_n \setminus T_{n+1}$ and if y $\in T_m \setminus T_{m+1}$, where n < m , then x$*$ y $\in T_m$. Hence,

$$\alpha_A (x * \mathrm{y}) \geq \frac{n}{m+1} = \min\{\alpha_A (x) , \alpha_A (\mathrm{y})\}.$$

$$\beta_A (x * \mathrm{y}) \leq \frac{1}{n+1} = \max\{\beta_A (x) , \beta_A (\mathrm{y})\}.$$

This proves that A is an intuitionistc fuzzy G -subalgebra with an infinite number of different values, which is a contradiction. This completes the proof.

Theorem: 3.2.29

Every ascending chain of G -subalgebras of X terminates at finite step if and only if the set of values of any intuitionistic fuzzy G -subalgebra is a well ordered subset of [0,1] .

Proof

Let A be an intuitionistic fuzzy G -subalgebra of X . Suppose that the set of values of A is not a well-ordered subset of [0,1]. Then there exist a strictly decreasing sequence $\{\gamma_n\}$ such that $\alpha_A (x_n) = \gamma_n$.

It follows that $\mathrm{U}(\alpha_A;\gamma_1) \subsetneq \mathrm{U}(\alpha_A;\gamma_2) \subsetneq \mathrm{U}(\alpha_A;\gamma_3) \subsetneq$…… is a strictly ascending chain of G -subalgebras of X which is not terminating. This is impossible. If there exist a strictly increasing sequence $\{\delta_n\}$such that$\beta_A (x_n) = \delta_n$. It follows that $\mathrm{L}(\beta_A;\delta_1) \subsetneq \mathrm{L}(\beta_A;\delta_2) \subsetneq \mathrm{L}(\beta_A;\delta_3) \subsetneq$ ……is a strictly ascending chain of G - subalgebras of X which is not terminating. This is impossible.

To prove the converse suppose that there exist a strictly ascending chain $T_1 \subsetneq T_2 \subsetneq T_3 \subsetneq$...... (1) of G -subalgebras of X which does not terminate at finite step. Note that T = $\bigcup_{n \in \mathbb{N}} T_n$is a closed ideal of X . Define an IFS A =(α_A , β_A) in X by

$$\alpha_A(x) = \begin{cases} \frac{1}{k} & \text{, where k = min\{ } n \in \mathbb{N} \text{ / x} \in T_n\}, \\ 0 & \text{, if x} \notin T_n \end{cases}$$

and $\beta_A(x)$ = 1 - $\alpha_A(x)$.

By using similar method as Theorem 3.2.28, we can prove that A is an intuitionistic fuzzy G -sublagebra of X . Since the chain (1) is not terminating, A has a strictly descending sequence of values. This contradictis that the value set of any intuitionistic fuzzy G -subalgebra is well-ordered. This completes the proof.

CHAPTER- 4

INTUITIONISTIC L-FUZZY G-SUBALGEBRS OF G-ALGEBRAS

Section 4.1

L - Fuzzy G –Subalgebras of G-algebras

Definition: 4.1.1

Let X be a non-empty set. A L-fuzzy set A = {< x, α_A(x)>| x ∈ X} of X is a function α_A : X → L.

Notation:

L = (L,≤,∨,∧) or simply denotes a complete distributive lattice with maximal element 1 and mini mal element 0 respectively.

Definition: 4.1.2

The intersection of two L-fuzzy sets A= {< x, α_A(x)>| x ∈ X} and B= {< x, α_B(x)>| x ∈ X} in X is defined as A ∩ B = α_A(x) ∧ α_A(y) for all x ∈ X.

Definition: 4.1.3

Let A= {< x, α_A(x)>| x ∈ X} be a L-fuzzy set in X, where X is a G-subalgebra, then the set A is L-fuzzy G- subalgebra over the binary operator $*$ if it satisfies the condition $\alpha_A(x * y) \geq \alpha_A(x) \wedge \alpha_A(y)$ for all x,y ∈ X.

Note:

(i) If A is a L-fuzzy G-subalgebra in X, then $\alpha_A(0)$ is the upper bound of α_A(x), for all x ∈ X, i.e. $\alpha_A(0) \geq \alpha_A(x)$. Also, $\alpha_A(0 * x) \geq \alpha_A(x)$for all x ∈ X.

(ii) Let $\{x_n\}$ be a sequence of X.

Then $\alpha_A(0) \geq \alpha_A(x_n)$ or $1 \geq \alpha_A(0) \geq \alpha_A(x_n)$. If $\lim_{n\to\infty} \alpha_A(x_n)$=1

then$\alpha_A(0) = 1$.

(iii) The intersection of two L-fuzzy G-subalgebras of X is also a L-fuzzy G-subalgebra of X. More generally, intersection of infinite number of L-fuzzy G-subalgebras of X is also a L-fuzzy G-subalgebra of X.

(iv) If A is a L-fuzzy G-subalgebra of X, then the set

I_{α_A} = { x ∈ X : α_A(x) $= \alpha_A(0)\}$ is a G-subalgebra of X.

Example: 4.1.4

Let X ={ 0,1,2,3,4,5,6,7}be a G-algebra with the following Cayley table :

$*$	0	1	2	3	4	5	6	7
0	0	2	1	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	2	1	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Define a fuzzy set A in X by α_A (0)=α_A (5)=0.8 and α_A (3)=α_A (6)=0.7 and α_A (x) =0.6 for all x ∈{ 1,2,4,7}. Then A is a fuzzy G – subalgebra of X.

Theorem: 4.1.5

Let B be a non-empty subset of X and A be a L-fuzzy set in X defined by

$$\alpha_A(x) = \begin{cases} \lambda, \text{ if } x \in B \\ \tau \ \text{ otherwise} \end{cases}$$

For all $\lambda\,,\tau \in L$ with $\lambda\ \geq \tau$.Then A is a L-fuzzy G -subalgebra of X if and only if B is a G -subalgebra of X . Moreover, $I_{\alpha A}$ = B.

Proof

Let A be a L-fuzzy G -subalgebra of X .Let ,x y∈ X be such that ,x y∈ B. Then $\alpha_A\ (x * y) \geq \alpha_A$(x) ∧ α_A(y) ={ λ, λ}= λ.so $x * y \in B$. Hence B is a G -subalgebra of X. Conversely, suppose that B is a G -subalgebra of X .
Let x, y ∈ X . Consider two cases:

Case (i).If x,y $\in B$, then $x * y \in B$, thus $\alpha_A(x * y) = \lambda = \alpha_A(x) \wedge \alpha_A(y)$.

Case(ii). If x$\notin$ B or, y$\notin$ B, then $\alpha_A(x * y) \geq \tau = \alpha_A(x) \wedge \alpha_A(y)$.

Hence A is a L-fuzzy G -subalgebra of X.

Also I_{α_A} = {x$\in X$ / and $\alpha_A(x) = \alpha_A(0)$}={x$\in X$ / and $\alpha_A(x) = \lambda$ }= B.

Definition: 4.1.6

Let A is a L-fuzzy G-subalgebra of X. For s $\in L$, the set

$\cup(\alpha_A : s) = \{x \in X: \alpha_A(x) \geq s\}$ is called a L- upper level subset of A.

Note:

Obviously, the L -upper level subset $\cup(\alpha_A : s)$ is a G-subalgebra of X.

Theorem: 4.1.7

Let A be a L-fuzzy set in X, such that the set $\cup(\alpha_A : s)$ is G-subalgebra of X for every s $\in$ L. Then A is a L- fuzzy G-subalgebra of X.

Proof

Let for every s $\in$ L,$\cup(\alpha_A ; s)$ is a subalgebra of X .In contrary, Let $x_0, y_0 \in$ X be such that $\alpha_A(x_0 * y_0) < \alpha_A(x_0) \wedge \alpha_A(y_0)$ .Let $\alpha_A(x_0) = \theta_1$and$\alpha_A(y_0) = \theta_2$ and $\alpha_A(x_0 * y_0)$=s. Then s $< \theta_1 \wedge \theta_2$.

Let us consider $s_1 = \frac{1}{2}$ [$\alpha_A(x_0 * y_0) + \alpha_A(x_0) \wedge \alpha_A(y_0)$].we get $s_1 = \frac{1}{2}$ (s+$\theta_1 \wedge \theta_2$).
Therefore $\theta_1 > s_1 = \frac{1}{2}$ (s+$\theta_1 \wedge \theta_2$) $>$ s and $\theta_2 > s_1 = \frac{1}{2}$ (s+$\theta_1 \wedge \theta_2$) $>$ s.
Hence, $\theta_1 \wedge \theta_2 > s_1 >$ s $= \alpha_A(x_0 * y_0)$, so that $x_0 * y_0 \notin \cup(\alpha_A : s)$ which is a contradiction .But $\alpha_A(x_0) = \theta_1 \geq \theta_1 \wedge \theta_2 > s_1$ and $(y_0) = \theta_2 \geq \theta_1 \wedge \theta_2 > s_1$.This implies $x_0, y_0 \in \cup(\alpha_A : s)$. Thus $\alpha_A(x * y) \geq \alpha_A(x) \wedge \alpha_A(y)$ for all x,y $\in X$.
Hence A is a L-fuzzy G -subalgebra of X .

Theorem: 4.1.8

Any subalgebra of X can be realized as a level subalgebra of some L-fuzzy G-subalgebra of X.

Proof

Let P be a L-fuzzy G-subalgebra of X, and A be a L- fuzzy set on X defined by

$$\alpha_A(x) = \begin{cases} \lambda, \text{ if } x \in P \\ 0 \text{ otherwise} \end{cases}$$

For all $\lambda, \tau \in [0,1]$ and $\lambda + \tau \leq 1$. We consider the following cases:

Case(i) If x ,y $\in$ P ,then $\alpha_A(x) = \lambda$, $\alpha_A(y) = \lambda$ .Thus

$\alpha_A(x * y) = \lambda = \lambda \wedge \lambda = \alpha_A(x) \wedge \alpha_A(y)$.

Case(ii) If x$\in$ P and y$\notin$ P, then $\alpha_A(x) = \lambda$, $\alpha_A(y) = 0$. Thus

$\alpha_A(x * y) \geq 0 = \lambda \wedge 0 = \alpha_A(x) \wedge \alpha_A(y)$.

Case(iii) If x $\notin$ P and y$\in$ P, then $\alpha_A(x) = 0$, $\alpha_A(y) = \lambda$.

Thus $\alpha_A(x * y) \geq 0 = 0 \wedge \lambda = \alpha_A(x) \wedge \alpha_A(y)$.

Csae(iv) If x $\notin$ P and y$\notin$ P, then $\alpha_A(x) = 0$, $\alpha_A(y) = 0$.

Thus $\alpha_A(x * y) \geq 0 = 0 \wedge 0 = \alpha_A(x) \wedge \alpha_A(y)$.

Therefore , A is an intuitionistic fuzzy G -subalgebra of X.

As a generalization of Theorem 4.1.7, we prove the following theorem:

Theorem: 4.1.9

Let X be a G -algebra. Then given any chain of G –subalgebras

$P_0 \subset P_1 \subset P_2 \subset \ldots\ldots\ldots\ldots \subset P_r = X$, there exists a L-fuzzy G -subalgebra A of X whose level subalgebras are exactly the G -subalgebras of this chain.

Proof

Consider set of numbers $s_0 > s_1 > \ldots\ldots\ldots > s_r$, where each $s_i \in [0,1]$ .Let A ={< x, $\alpha_A(x)$>| x $\in$ X} defined by

$$\alpha_A(x) = \begin{cases} s_0, & \text{if } x \in P_0 \\ s_1, & \text{if } x \in P_i - P_{i-1}, 0 < i \le r. \end{cases}$$

(1) We consider the following two cases:

Case (i). Let x,y $\in P_i - P_{i-1}$ .Therefore, by (1), $\alpha_A(x) = \alpha_A(y) = s_i$. Since P_i is a G –subalgebra, we have x∗y $\in P_i$, and so either x∗y $\in P_i - P_{i-1}$ or x∗y $\in P_{i-1}$. In any case we conclude that $\alpha_A(x * y) \ge S_i = \alpha_A(x) \wedge \alpha_A(y)$.

Case (ii) Let x $\in P_i - P_{i-1}$ and y $\in P_j - P_{j-1}$ for i > j.Therefore, by (1), $\alpha_A(x) = s_i, \alpha_A(y) = s_j$. Then x∗y $\in P_i$ Since P_i is a G –subalgebra of X and $P_j \subset P_i$. Hence $\alpha_A(x * y) \ge S_j = \alpha_A(x) \wedge \alpha_A(y)$. Thus A is a L- fuzzy G -subalgebra of X . From (1), it follows that Im(α_A) = $\{s_0, s_1, \ldots, s_r\}$. Hence, the L-upper level subalgebras of A are given by the chain of G -subalgebras

$$\cup(\alpha_A : s_0) \subset \cup(\alpha_A : s_1) \subset \ldots \subset \cup(\alpha_A : s_r) = X$$

Now $\cup(\alpha_A : s_0)$ ={ x$\in X$ / and $\alpha_A(x) \ge s_0\} = P_0$.

Finally we prove that $\cup(\alpha_A : s_i) = P_i$ for 0 < i ≤ r .clearly, $P_i \subseteq \cup(\alpha_A : s_i)$. If x$\in \cup(\alpha_A : s_i)$, then $\alpha_A(x) \ge s_i$. Which implies that x $\notin P_j$ for j>i .

Hence $\alpha_A(x) \in \{s_0, s_1, \ldots, s_i\}$ and so x$\in P_k$ for some k $\le i$. As $P_k \subseteq P_i$, it follows that x$\in P_i$. Therefore $\cup(\alpha_A : s_i) = P_i$ for 0 ≤ i ≤ r.

This completes the proof .

Note:

If X is a finite G-algebra, then the number of G- subalgebras of X is finite where as the number of level subalgebras of a L-fuzzy G-subalgebra A appears to be infinite. But since every level subalgebra is indeed a G-subalgebra of X, not all these G-subalgebras are distinct.

Theorem: 4.1.10

Let A be a L-fuzzy G -subalgebra of a G -algebra X .Then Two L- upper level subalgebras $\cup(\alpha_A : s_1)$, $\cup(\alpha_A : s_2)$ (with $s_1 < s_2$) of A are equal if and only if there is no x $\in X$ such that $s_1 \le \alpha_A(x) < s_2$.

Proof

Assume that $\cup(\alpha_A : s_1) = \cup(\alpha_A : s_2)$ for $s_1 < s_2$ and assume that there exists x$\in X$ such that $s_1 \leq \alpha_A$(x)$<$ s_2 .Then $\cup(\alpha_A : s_2)$ is a proper subset of $\cup(\alpha_A : s_1)$ which is a contradiction.

Conversely, suppose that there is no x$\in X$ such that $s_1 \leq \alpha_A$(x) $< s_2$.Since $s_1 < s_2$, we have $\cup(\alpha_A : s_2) \subseteq \cup(\alpha_A : s_1)$. If x $\in$ $\cup(\alpha_A : s_1)$, then α_A(x) $\geq$ s_1 and so α_A(x) $\geq$ s_2, because α_A(x) does not lie between s_1 and s_2 .

Hence x $\in$ $\cup(\alpha_A : s_2)$, which implies that $\cup(\alpha_A : s_1) \subseteq \cup(\alpha_A : s_2)$.

This completes the proof.

Note:

As a consequence of Theorem 4.1.9, the L –upper level subalgebras of a L – fuzzy G -subalgebra A of a finite G -algebra X form a chain.

But α_A(x) $\leq \alpha_A(0)$ for all x$\in X$. Therefore$\cup(\alpha_A : s_0)$, where $s_0 = \alpha_A(0)$, is the smallest L - upper s-level subalgebra but not always $\cup(\alpha_A : s_0)$ as shown in the following example. and so we have the chain

$\cup(\alpha_A : s_0) \subset \cup(\alpha_A : s_1) \subset \ldots \subset \cup(\alpha_A : s_r)$ = X, where $s_0 > s_1 >$ ………$>$ s_r.

Theorem: 4.1.11

Let X be a finite G -algebra and A be a L-fuzzy G-subalgebra of X. If Im(α_A) = $\{s_1, s_2, \ldots\ldots, s_n\}$, then the family of G-subalgebras U$(\alpha_A : s_i)$, 1≤i≤ n constitutes all the level subalgebras of A.

Proof

Let s $\in$ [0,1]and s $\notin$ Im(α_A). Suppose $s_1 < s_2 < \ldots\ldots < s_n$ without loss of generality. If s $\leq s_1$, then U(α_A: s_1) = X =U(α_A; s) , then obviously U(α_A: s) =φ . If $s_i <$ $s_{i-1} < s_i$, then by Theorem 4.1.9, we get U(α_A: s) = X =U(α_A: s_i).

Thus for any s $\in$ L, the L-upper level subalgebra is one of { U(α_A: s_i) / i= 1,2,…..,n}.

Note:

Two L-fuzzy G-subalgebras of a G-algebra may have an identical family of L-upper level subalgebras but the L-fuzzy G-subalgebras may not be equal.

Theorem: 4.1.12

Let X be a G-algebra and A be a L-fuzzy G-subalgebra of X. If Im (α_A) is finite, say$\{s_1, s_2, \ldots\ldots, s_n\}$, then for any $s_i, s_j \in$ Im (α_A), $\mathrm{U}(\alpha_A : s_i) = \mathrm{U}(\alpha_A : s_j)$ implies $s_i = s_j$.

Proof

Obvious.

Theorem: 4.1.13

Let A= {< x, α_A(x)>| x ∈ X} and B= {< x, α_B(x)>| x ∈ X}be two L-fuzzy G -subalgebras of a finite G -algebra X with identical family of L-upper level subalgebras. If Im$(\alpha_A) = \{t_0, t_1, \ldots\ldots.., t_r\}$,Im$(\alpha_B) = \{s_0, s_1, \ldots\ldots.., s_k\}$where $t_0 > t_1 > \ldots\ldots.>t_r$, $s_0 > s_1 > \ldots\ldots\ldots.s_k$ then we have

(i) r=k

(ii) $\mathrm{U}(\alpha_A; t_i)$= $\mathrm{U}(\alpha_B; s_i)$ $0 \leq i \leq k$,

(iii) If x∈ X such that $\alpha_A(x) = t_i$, then $\alpha_A(x) = s_i$, $0 \leq i \leq k$.

Proof

To prove (i) By Theorem 4.1.10, the only subalgebras of A and B are the families $\mathrm{U}(\alpha_A : t_i)$ and $\mathrm{U}(\alpha_B : s_i)$. Since A and B have the same family of L-upper level subalgebras, it follows that r=k.

(ii) Using note under theorem 4.1.9 and (i), we have chains of L –upper level subalgebras $\mathrm{U}(\alpha_A : t_0) \subset \mathrm{U}(\alpha_A : t_1) \subset \ldots \subset \mathrm{U}(\alpha_A : t_k)$= X and

$\mathrm{U}(\alpha_B : t_0) \subset \mathrm{U}(\alpha_B : t_1) \subset \ldots \subset \mathrm{U}(\alpha_A : t_k)$= X It follows clearly that

if $t_i, t_j \in$ Im (α_A) such that $t_i > t_j$and $s_i, s_j \in$ Im (α_B)such that $s_i > s_j$,then

$$\mathrm{U}(\alpha_A : t_i) \subset \mathrm{U}(\alpha_A : t_j) \text{ and } \mathrm{U}(\alpha_B : s_i) \subset \mathrm{U}\left(\alpha_B : s_j\right). \qquad (2)$$

Since the two families of L-upper level subalgebras are identical, and it is clear that $\mathrm{U}(\alpha_A : t_0)$ = $\mathrm{U}(: s_0)$. By hypothesis $\mathrm{U}(\alpha_A : t_1)$ = $\mathrm{U}(\alpha_B : s_1)$for some j > 0.

Assume that $\mathrm{U}(\alpha_A\colon l_1) \neq \mathrm{U}(\alpha_B\colon s_j)$. Then $\mathrm{U}(\alpha_A\colon t_1) = \mathrm{U}(\alpha_B\colon s_j)$for some j >1 , and $\mathrm{U}(\alpha_B\colon s_1) = \mathrm{U}(\alpha_A\colon t_i)$ for some $t_i < t_1$. Thus by (2) we obtain that

$\mathrm{U}(\alpha_B\colon s_j) = \mathrm{U}(\alpha_A\colon t_1) \subset \mathrm{U}(\alpha_A\colon t_i)$ and $\mathrm{U}(\alpha_A\colon t_i) = \mathrm{U}(\alpha_B\colon s_1) \subset \mathrm{U}(\alpha_B\colon s_j)$.

This is a contradiction. Hence $\mathrm{U}(\alpha_A\colon t_1) = \mathrm{U}(\alpha_B\colon s_1)$. By induction on i ,

$0 \leq i \leq k$,we finally obtain that $\mathrm{U}(\alpha_A\colon t_i) = \mathrm{U}(\alpha_B\colon s_i)$, $0 \leq i \leq k$.

(iii) Let x∈ X be such that $\alpha_A(\mathrm{x}) = t_i$ and $\alpha_B(\mathrm{x}) = s_j$, where $0 \leq i \leq k$ and $0 \leq j \leq k$. It is sufficient to show that $s_j = s_i$.

Now x $\in \mathrm{U}(\alpha_A\colon t_i) = \mathrm{U}(\alpha_B\colon s_i)$ implies that $\alpha_B(\mathrm{x}) = s_j \geq s_i$. This gives from (2) that $\mathrm{U}(\alpha_B\colon s_j) \subseteq \mathrm{U}(\alpha_B\colon s_i)$. Since x $\in \mathrm{U}(\alpha_B\colon s_j)$, it follows from (ii) that x $\in \mathrm{U}(\alpha_A\colon t_j)$ and so $\alpha_A(\mathrm{x}) = t_i \geq t_j$,which implies that $\mathrm{U}(\alpha_A\colon t_i) \subseteq \mathrm{U}(\alpha_A\colon t_j)$by (2). Using (ii), we have $\mathrm{U}(\alpha_B\colon s_i) = \mathrm{U}(\alpha_A\colon t_i) \subseteq \mathrm{U}(\alpha_A\colon t_j) = \mathrm{U}(\alpha_B\colon s_j)$.

Thus $\mathrm{U}(\alpha_B\colon s_i) = \mathrm{U}(\alpha_B\colon s_j)$ and by Theorem 4.1.11, $s_j = s_i$.

This completes the proof.

Theorem: 4.1.14

Let A and B be two L-fuzzy G-subalgebras of a finite G-algebra X such that the families of level subalgebras of A and B are identical. Then A = B if and only if $\mathrm{Im}(\alpha_A) = \mathrm{Im}(\beta_A)$.

Proof

If A = B, then clearly $\mathrm{Im}(\alpha_A) = \mathrm{Im}(\beta_A)$. Conversely, assume that $\mathrm{Im}(\alpha_A) = \mathrm{Im}(\beta_A)$. For convenience,let us denote $\mathrm{Im}(\alpha_A) = \{t_0, t_1, \ldots\ldots.., t_r\}$, $\mathrm{Im}(\alpha_B) = \{s_0, s_1, \ldots\ldots.., s_1\}$where $t_0 > t_1 >$ ……..$>t_r$, $s_0 > s_1 >$ ………..s_r.Then $s_0 \in \mathrm{Im}(\alpha_B) = \mathrm{Im}(\alpha_A)$. Thus $s_0 = t_{n_0}$ for some n_0. Assume that $t_{n_0} \neq t_0$. So $t_{n_0} < t_0$. Now $s_1 \in \mathrm{Im}(\alpha_A)$, and hence $s_1 = t_{n_1}$ for some n_1. Since $s_0 > s_1$,

we have $t_{n_0} > t_1$. Continuing in this way, we have $t_{n_0} > t_{n_1} >$ ……$> t_{n_r}$. Since $s_0 = t_{n_0} < t_0$, this contradicts to the fact that $\mathrm{Im}(\alpha_A) = \mathrm{Im}(\beta_A)$.

Hence we must have $s_0 = t_0$. Proceeding this manner, we get that $s_i = t_i$; $0 \leq i \leq r$. Now let be $x_0 < x_1 <$ ……$< x_r$ be distinct elements of X such that $\alpha_A(x_i) = t_i, 0 \leq i \leq r$. By Theorem 4.1.12, $\alpha_B(x_i) = s_i, 0 \leq i \leq r$. Since $s_i = t_i$, it follows that $\alpha_A(\mathrm{x}) = \alpha_A(\mathrm{y})$ for each x ∈ X. Therefore A = B.

Theorem: 4.1.15

Let $T_1 \supseteq T_2 \supseteq T_3$ …… be a descending chain of G -subalgebras of X which terminates at finite step. For a L-fuzzy G -subalgebra A of X , if a sequence of elements of Im(α_A) a is strictly increasing then A is finite valued.

Proof

Assume that A is infinite valued. Let $\{\phi_n\}$ be a strictly increasing sequence of elements of Im(α_A). Then $0 \leq \phi_1 < \phi_2 <$…….<1. Note that U$(\alpha_A; \phi_t)$is a G -subalgebra of X for t =1,2,3,…. . Let x ∈ U$(\alpha_A; \phi_t)$ for t =2,3,…. .
Then $\alpha_A(x) \geq \phi_t > \phi_{t-1}$ .which implies that x ∈ U$(\alpha_A: \phi_{t-1})$.
Hence U$(\alpha_A: \phi_t) \subseteq$ U$(\alpha_A: \phi_{t-1})$. Since ϕ_{t-1} ∈ Im(α_A) there exists x_{t-1} such that $\alpha_A(x_{t-1}) = \phi_{t-1}$. It follows that x_{t-1} ∈ U$(\alpha_A: \phi_{t-1})$,but x ∉ U$(\alpha_A: \phi_t)$. Thus U$(\alpha_A: \phi_t) \subsetneq$ U$(\alpha_A; \phi_{t-1})$, and so we obtain a strictly descending chain U$(\alpha_A: \phi_1) \supsetneq$ U$(\alpha_A: \phi_2) \supsetneq$…. of G -subalgebras of X which is not terminating. This is impossible. Therefore, A=(α_A , β_A) is finite valued.
Now we consider the converse of above Theorem .

Theorem: 4.1.16

If every L-fuzzy G-subalgebra A of X has the finite image, then every descending chain of G-subalgebras of X terminates at finite step.

Proof

Suppose there exists a strictly descending chain $T_0 \supsetneq T_1 \supsetneq T_2$…… of G- subalgebras of X which does not terminate at finite step. Define a L-fuzzy set A in X by

$$\alpha_A(x) = \begin{cases} \frac{n}{n+1}, & \text{if } x \in T_n \setminus T_{n+1}, \\ 1, & \text{if } x \in \bigcap_{n=0}^{\infty} T_n \end{cases}$$

where n =0,1,2,…… and T_0 stands for X . Let x,y $\in X$. Now, we consider the following cases:

If x and y $\in T_n$, then x∗ y ∈ T_n , because T_n is a G -subalgebra of X . Hence,

$$\alpha_A(x * y) \geq \frac{n}{n+1} = \alpha_A(x) \wedge \alpha_A(y).$$

If x ∈ $T_n \setminus T_{n+1}$ and if y ∈ $T_m \setminus T_{m+1}$, where n > m , then x∗ y ∈ T_m . Hence,

$$\alpha_A(x * y) \geq \frac{m}{m+1} = \alpha_A(x) \wedge \alpha_A(y).$$

if x ∈ $T_n \setminus T_{n+1}$ and if y ∈ $T_m \setminus T_{m+1}$, where n < m , then x∗ y ∈ T_m . Hence,

$$\alpha_A(x * y) \geq \frac{n}{m+1} = \alpha_A(x) \wedge \alpha_A(y).$$

This proves that A is a L-fuzzy G -subalgebra with an infinite number of different values, which is a contradiction. This completes the proof.

Theorem: 4.1.17

Every ascending chain of G-subalgebras of X terminates at finite step if and only if the set of values of any L-fuzzy G-subalgebra is a well ordered subset of L.

Proof

Let A be a L- fuzzy G -subalgebra of X . Suppose that the set of values of A is not a well-ordered subset of L. Then there exist a strictly decreasing sequence $\{\gamma_n\}$ such that $\alpha_A(x_n) = \gamma_n$.

It follows that $U(\alpha_A{:}\gamma_1) \subsetneq U(\alpha_A{:}\gamma_2) \subsetneq U(\alpha_A{:}\gamma_3) \subsetneq \ldots\ldots$ is a strictly ascending chain of G -subalgebras of X which is not terminating. This is impossible.

To prove the converse suppose that there exist a strictly ascending chain

$$T_1 \subsetneq T_2 \subsetneq T_3 \subsetneq \ldots\ldots \quad (1)$$

of closed ideals of X which does not terminate at finite step. Note that T = $\bigcup_{n\in\mathbb{N}} T_n$ is a closed ideal of X . Define an IFS A =(α_A , β_A) in X by

$$\alpha_A(x) = \begin{cases} \frac{1}{k}, & \text{where } k = \min\{ n \in \mathbb{N} \,/\, x \in T_n\}, \\ 0, & \text{if } x \notin T_n \end{cases}$$

By using similar method as Theorem 4.1.15, we can prove that A is an intuitionistic fuzzy G -sublagebra of X . Since the chain (1) is not terminating, A has a strictly descending sequence of values. This contradictis that the value set of a L - fuzzy G -subalgebra is well-ordered.

Section: 4.2

Intuitionistic L- Fuzzy G – subalgebras of G-algebras

Definition: 4.2.1

Let (L , $\leq$) , be the lattice with an involutive order reversing operation N: L → L . Let X be a non-empty set. An intuitionistic L -fuzzy set (ILFS) A in X is defined as an object of the form A= {< x, α_A(x), β_A(x)> : x $\in$ X} where α_A(x): X → L and β_A(x): X → L define the degree of membership and the degree of non membership for every x $\in$ X satisfying α_A(x)$\leq N(\beta_A$(x)). For the sake of simplicity, we shall use the symbol A=(α_A , β_A) for the ILFS A= {< x, α_A(x), β_A(x)> : x $\in$ X} .

Definition: 4.2.2

For any two ILFSs A = { < x , α_A(x) , β_A(x) > : x $\in$ X } and

B= {< x, α_B(x), β_B(x) > : x $\in$ X} on X , the intersection of A and B is denoted by A∩B and is given by A∩B = {< x, α_A(x) $\wedge$ α_B(x) , β_A(x) $\vee$ β_B(x)> : x $\in$ X}. Also the complement of A is denoted by $\bar{A}$ and is defined by $\bar{A}$ ={< x, β_A(x), α_A(x) >: x $\in$ X}.

Our main objective is to investigate the idea of G -subalgebras on ILFS . The ILFS is an extension of L -fuzzy set. Combined the definitions of G -subalgebra over crisp set and the idea of L -fuzzy set Senapati et al. [13] defined L -fuzzy G -subalgebra, which is defined below.

Definition: 4.2.1

Let A=(α_A , β_A) be an ILFS in X , where X is a G – subalgebra. Then the set A is an intuitionistic L- fuzzy G – subalgebra over the binary operator $*$ if it satisfies the following conditions:

$$\text{ILF}_1 : \alpha_A(x * y) \geq \alpha_A(x) \wedge \alpha_A(y)$$

$$\text{ILF}_2 : \beta_A(x * y) \leq \beta_A(x) \vee \beta_A(y)$$

Example: 4.2.4

Let X= {0,1,2,3,4,5} be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	4	0	2	1
4	4	3	5	1	0	2
5	5	1	2	4	3	0

Let A=(α_A , β_A) be an ILFS in X defined by $\alpha_A(x) = \begin{cases} s_0 \text{ if } x \in \{0,3\} \\ s_1 \quad \text{otherwise} \end{cases}$ and

$$\beta_A(x) = \begin{cases} t_0 \text{ ,if } x \in \{0,3\} \\ t_1 \text{ , otherwise} \end{cases}$$

where s_0 , s_1 , t_0 , $t_1 \in$ L with $s_0 > s_1$, $t_0 < t_1$,$s_0 + t_0 \leq 1$, $s_1 + t_1 \leq 1$.

All the conditions of Definition 4.2.3 have been satisfied by the set A with above membership and non-membership values. Thus A=(α_A , β_A) is an intuitionistic L - fuzzy G -subalgebra of X . The bounds of membership and non-membership values of intuitionistic L -fuzzy G - subalgebras are given below

Proposition: 4.2.5

If A = (α_A , β_A) is an intuitionistic L -fuzzy G -subalgebra in X , then for all x $\in$ X , $\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$. Thus, $\alpha_A(0)$ and $\beta_A(0)$ are the upper bounds and lower bounds of $\alpha_A(x)$ and $\beta_A(x)$ respectively.

Proof

Let x $\in$ X. Then $\alpha_A(0) = \alpha_A(x * x) \geq \alpha_A(\mathrm{x}) \wedge \alpha_A(\mathrm{y}) = \alpha_A(\mathrm{x})$ and

$\beta_A(0) = \beta_A(x * x) \leq \beta_A(\mathrm{x}) \vee \beta_A(\mathrm{y}) = \beta_A(\mathrm{x})$.

Theorem: 4.2.6

Let A be an intuitionistic L - fuzzy G -subalgebra of X . If there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} \alpha_A(x_n) = 1$ and $\lim_{n \to \infty} \beta_A(x_n) = 0$,

then $\alpha_A(0)$ =1and $\beta_A(0) = 0$.

Proof

By Proposition 4.2.5 , $\alpha_A(0) \geq \alpha_A(x)$ for all x $\in$ X , therefore $\alpha_A(0) \geq \alpha_A(x_n)$ for every positive integer n .

Consider, 1$\geq$ $\alpha_A(0) \geq \lim_{n \to \infty} \alpha_A(x_n) = 1$. Hence $\alpha_A(0)$ =1. Again, by Proposition 4.2.5 , $\beta_A(0) \leq \beta_A(x)$ for all x $\in$ X , thus $\beta_A(0) \leq \beta_A(x_n)$ for every positive integer n . Now, 0$\geq$ $\beta_A(0) \geq \lim_{n \to \infty} \beta_A(x_n) = 0$. Hence, $\beta_A(0) = 0$.

Proposition: 4.2.7

If an IFS A=(α_A , β_A) in X is an intuitionistic L -fuzzy G - subalgebra, then for all x $\in$ X, $\alpha_A(0 * x) \geq \alpha_A(x)$ and $\beta_A(0 * x) \leq \beta_A(x)$.

Proof

For all x$\in$ X, $\alpha_A(0 * x) \geq \alpha_A(0) \wedge \alpha_A(x) = \alpha_A(x * x) \wedge \alpha_A(x)$

$= \{\alpha_A(x) \wedge \alpha_A(x)\} \wedge \alpha_A(x) = \alpha_A(x)$ and $\beta_A(0 * x) \leq \beta_A(0) \vee \beta_A(x)$

$= \beta_A(x * x) \vee \beta_A(x) = \{\beta_A(x) \vee \beta_A(x)\} \vee \beta_A(x) = \beta_A(x)$.

The intersection of two intuitionistic L -fuzzy G -subalgebras is also an intuitionistic L -fuzzy G -subalgebra, which is proved in the following theorem.

Theorem: 4.2.8

Let A_1 and A_2 be two intuitionistic L-fuzzy G -subalgebras of X .Then $A_1 \cap A_2$ is an intuitionistic L -fuzzy G -subalgebra of X .

Proof

Let x , y ∈ $A_1 \cap A_2$. Then x , y ∈ A_1 and A_2 .

Now $\alpha_{A1\cap A2}(x*y) = \alpha_{A_1}(x*y) \wedge \alpha_{A_2}(x*y)$

$$\geq \{\alpha_{A_1}(x) \wedge \alpha_{A_1}(y)\} \wedge \{\alpha_{A_2}(x) \wedge \alpha_{A_2}(y)\}$$

$$=\{\alpha_{A_1}(x) \wedge \alpha_{A_2}(x)\} \wedge \{\alpha_{A_1}(y) \wedge \alpha_{A_2}(y)\}$$

$$= \alpha_{A1\cap A2}(x) \wedge \alpha_{A1\cap A2}(y)$$

and $\beta_{A1\cap A2}(x*y) = \beta_{A_1}(x*y) \vee \beta_{A_2}(x*y)$

$$\leq \{\beta_{A_1}(x) \vee \beta_{A_1}(y)\} \vee \{\beta_{A_2}(x) \vee \beta_{A_2}(y)\}$$

$$= \{\beta_{A_1}(x) \vee \beta_{A_2}(x)\} \vee \{\beta_{A_1}(y) \vee \beta_{A_2}(y)\}$$

$$= \beta_{A1\cap A2}(x), \ \beta_{A1\cap A2}(y)$$

Hence, $A_1 \cap A_2$ is an intuitionistic L -fuzzy G -subalgebra of X .

Theorem: 4.2.9

Let $\{A_i \ / \ i$ =1,2,3…….}be a family of intuitionistic L -fuzzy G - subalgebra of X . Then $\cap A_i$ is also an intuitionistic L-fuzzy G -subalgebra of X , where $\cap A_i = (\wedge \alpha_{A_i}(x), \vee \beta_{A_i}(x))$.

Proof

obvious

In the same way and by the definition of A we can prove the following result.

Theorem: 4.2.10

An IFS A=(α_A, β_A) is an intuitionistic L -fuzzy G -subalgebra of X if and only if the fuzzy sets $A_1 = \{\alpha_A(x) : x \in A\}$ and $A_2 = \{\bar{\beta}_A(x) : x \in A\}$ are L -fuzzy G -subalgebras of X .

Proof

Let A=(α_A, β_A) be an intuitionistic L -fuzzy G -subalgebra of X .Clearly, A_1 is a L-fuzzy G -subalgebra of X . For every ,x y ∈ X.

We have $\bar{\beta}_A(x * y) = 1 - \beta_A(x * y)$

$\geq 1 - \beta_A(x) \vee \beta_A(y) = \{1 - \beta_A(x)\} \wedge \{1 - \beta_A(y)\}$

$= \bar{\beta}_A(x) \wedge \bar{\beta}_A(y)$.

Hence A_2 is a L -fuzzy G -subalgebra of X .

Conversely assume that A_1 and A_2 be two intuitionistic L -fuzzy G -subalgebras of X. For every ,x y $\in X$. $\alpha_A(x * y) \geq \alpha_A(x) \wedge \alpha_A(y)$ and

$1 - \beta_A(x * y) = \bar{\beta}_A(x * y) \geq \bar{\beta}_A(x) \wedge \bar{\beta}_A(y)$

$= \{1 - \beta_A(x)\} \wedge \{1 - \beta_A(y)\}$

$= 1 - \beta_A(x) \vee \beta_A(y)$.

That is , $\beta_A(x * y) \leq \beta_A(x) \vee \beta_A(y)$. Hence A=$(\alpha_A, \beta_A)$ be an intuitionistic L-fuzzy G -subalgebra of X .

Theorem: 4.2.11

Let A=(α_A, β_A) be an intuitionistic L-fuzzy G -subalgebra of X and let n $\in N$ (the set of natural numbers). Then

(i) $\alpha_A(\prod^n x * x) \geq \alpha_A(x)$, for any odd number n ,and $\beta_A(\prod^n x * x) \leq \beta_A(x)$, for any odd number n ,

(ii) $\alpha_A(\prod^n x * x) = \alpha_A(x)$,for any even number n ,and $\beta_A(\prod^n x * x) = \beta_A(x)$, for any even number n .

Proof

To prove (i) Let x $\in X$ and assume that n is odd. Then n=2p-1 for some positive integer p . Now we prove this theorem by induction
$\alpha_A(x * x) = \alpha_A(0) \geq \alpha_A$ and $\beta_A(x * x) = \beta_A(0) \leq \beta_A(x)$. Suppose that $\alpha_A(\prod^{2p-1} x * x) \geq \alpha_A(x)$ and $\beta_A(\prod^{2p-1} x * x) \leq \beta_A(x)$. Then by assumption,

$\alpha_A(\prod^{2(p+1)-1} x * x) = \alpha_A(\prod^{2p+1} x * x)$

$= \alpha_A(\prod^{2p-1} x * (x * (x * x)))$

$= \alpha_A(\prod^{2p-1} x * x) \geq \alpha_A(x)$

$$\text{and} \quad \beta_A\left(\prod^{2(p+1)-1} \mathrm{x} * \mathrm{x}\right) = \beta_A\left(\prod^{2p+1} \mathrm{x} * \mathrm{x}\right)$$

$$= \beta_A\left(\prod^{2p-1} \mathrm{x} * (\mathrm{x} * (\mathrm{x} * \mathrm{x}))\right)$$

$$= \alpha_A\left(\prod^{2p-1} \mathrm{x} * \mathrm{x}\right) \leq \beta_A(\mathrm{x})$$

Which proves (i) .similarly (ii) can be proved .

Definition: 4.2.12

Let A = (α_A, β_A) be an ILFS defined on X. The operators ⊕A and ⊗A are defined as $\oplus A_L = (\alpha_A, \overline{\alpha_A})$, and $\otimes A_L = (\beta_A, \overline{\beta_A})$ respectively .

Theorem: 4.2.13

If A=(α_A, β_A) is an intuitionistic L-fuzzy G -subalgebra of X , then

(i) $\oplus A_L$,and (ii) $\otimes A_L$, both are intuitionistic L-fuzzy G -subalgebras.

Proof

To Prove (i) It is sufficient to show that $\bar{\alpha}_A$ satisfies the condition ILF$_2$. Let x,y∈ X. Then $\bar{\alpha}_A(x * y) = 1 - \alpha_A(\mathrm{x} * \mathrm{y}) \leq 1 - \alpha_A(\mathrm{x}) \wedge \alpha_A(\mathrm{y})$
$= \{1 - \alpha_A(\mathrm{x})\} \vee \{1 - \alpha_A(\mathrm{y})\} = \bar{\alpha}_A(x) \vee \bar{\alpha}_A(y)$.

Hence, $\oplus A_L$ is an intuitionistic L-fuzzy G -subalgebra of X .

To Prove (ii) It is sufficient to show that $\bar{\beta}_A$ satisfies the condition ILF$_2$. Let x, y ∈ X . Then$\bar{\beta}_A(x * y) = 1 - \beta_A(\mathrm{x} * \mathrm{y}) \geq 1 - \beta_A(\mathrm{x}) \vee \beta_A(\mathrm{y})$
$= \{1 - \beta_A(\mathrm{x})\} \wedge \{1 - \beta_A(\mathrm{y})\} = \bar{\beta}_A(x) \wedge \bar{\beta}_A(y)$.

Hence,$\otimes A_L$ is also an intuitionistic fuzzy G -subalgebra of X .

Notation:

The sets {x / x∈ X and$\alpha_A(\mathrm{x}) = \alpha_A(0)$ } and{x / x∈ X and $\beta_A(\mathrm{x}) = \beta_A(0)$ } are denoted by $I_{\alpha A}$ and $I_{\beta A}$ respectively. These two sets are also G -subalgebra of X.

Theorem: 4.2.14

Let A=(α_A, β_A) be an intuitionistic L-fuzzy G -subalgebra of X , then the sets I_{α_A} and I_{β_A} are G -subalgebras of X .

Proof

Let x, y $\in$ I_{α_A} .Then α_A(x) = $\alpha_A(0)$ = α_A(y) and

$\alpha_A\,(x*y) \geq \alpha_A$(x) $\wedge$ α_A(y)= $\alpha_A(0)$. By using proposition 4.2.5,we know that $\alpha_A\,(x*y)$= $\alpha_A(0)$ .That is $x*y\ \in\ I_{\alpha_A}$. Let x, y$\in$ I_{β_A}.

Then β_A(x) = $\beta_A(0)$ = β_A(y) and $\beta_A\,(x*y) \leq \beta_A$(x) $\vee$ β_A(y)= $\beta_A(0)$. By using proposition 4.2.5,we know that $\beta_A\,(x*y)$= $\beta_A(0)$ .That is $x*y\ \in\ I_{\beta_A}$.

Hence ,the sets I_{α_A} and I_{β_A} are G -subalgebras of X.

Theorem: 4.2.15

Let B be a nonempty subset of X and A=(α_A , β_A)be an IFS in X defined by

$$\alpha_A(x) = \begin{cases} \lambda, & \text{if } x \in B \\ \tau, & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_A(x) = \begin{cases} \gamma, & \text{if } x \in B \\ \delta, & \text{otherwise} \end{cases}$$

For all λ , τ , γ and $\delta \in L$ with $\lambda\ \geq \tau$ and $\gamma \leq \delta$ and $\lambda + \gamma \leq 1$; $\tau +\ \delta\ \leq 1$ .Then A is an intuitionistic L-fuzzy G -subalgebra of X if and only if B is a G -subalgebra of X . Moreover, I_{α_A} = B = I_{β_A}.

Proof

Let A be an intuitionistic L-fuzzy G -subalgebra of X .Let x, y$\in$ X be such that x, y$\in$ B.Then $\alpha_A\,(x*y) \geq \alpha_A$(x) $\wedge$ α_A(y)= $\lambda \wedge \lambda$= λ and

β_A(x $*$ y) $\leq \beta_A$(x) $\vee$ β_A(y)= $\gamma \vee \gamma$= γ . This implies $x*y \in B$. Hence B is a G -subalgebra of X . Conversely, suppose that B is a G -subalgebra of X. Let x, y $\in X$. Consider two cases:

Case (i).If x, y $\in B$, then $x*y \in B$, thus $\alpha_A\,(x*y) = \lambda = \alpha_A$(x) $\wedge$ α_A(y) and

β_A(x$*$ y) = γ =β_A(x) $\vee$ β_A(y).

Case(ii). If x$\notin$ B or, y$\notin$ B, then $\alpha_A\,(x*y) \geq \tau = \alpha_A$(x) $\wedge$ α_A(y) and

β_A(x$*$ y) $\leq \delta$ =β_A(x) $\vee$ β_A(y).

Hence A is intuitionistic L-fuzzy G -subalgebra of X.

Also I_{α_A} = {x∈ X / and α_A(x) =$\alpha_A(0)$}={x∈ X / and α_A(x) = λ }= B. and

I_{β_A} = {x∈ X / and β_A(x) =$\beta_A(0)$}={x∈ X / and β_A(x) = δ }=B.

Definition: 4.2.16

Let A=(α_A , β_A) is an intuitionistic L-fuzzy G-subalgebra of X. For s, t∈ L, the set $\cup(\alpha_A : s) =$ {x∈ X : and α_A(x) $\geq s$} is called L-upper s-level of A and $L(\beta_A : t) =$ { x∈ X : and β_A(x) $\leq t$} is called L-lower t -level of A.

Theorem: 4.2.17

If A=(α_A , β_A) is an intuitionistic L-fuzzy G -subalgebra of X , then the L-upper s-level and L-lower t -level of A are G -subalgebras of X .

Proof

Let x ,y ∈ $\cup(\alpha_A ; s)$. Then α_A(x) $\geq s$ and α_A(y) $\geq s$. It follows that $\alpha_A (x * y) \geq \alpha_A$(x) ∧ α_A(y) $\geq s$. so that $x * y \in \cup(\alpha_A : s)$ is a G -subalgebra of X.

Let x,y ∈ L($\beta_A : t$).Then β_A(x) $\leq t$ and β_A(y) $\leq t$. It follows that $\beta_A (x * y) \leq \beta_A$(x) ∨ β_A(y) $\leq t$. so that $x * y$ ∈ L($\beta_A : t$) is a G -subalgebra of X.

Theorem: 4.2.18

Let A=(α_A , β_A) be an ILFS in X such that the sets $\cup(\alpha_A : s)$ and L($\beta_A : t$) are G -subalgebras of X for every s, t ∈ L. Then A=(α_A , β_A) is an intuitionistic L-fuzzy G -subalgebra of X .

Proof

Let for every s ,t ∈ L ,$\cup(\alpha_A : s)$ and L($\beta_A : t$) be two subalgebras of X .In contrary, Let x_0, y_0 ∈ X be such that $\alpha_A (x_0 * y_0) < \alpha_A(x_0) \wedge \alpha_A(y_0)$.

Let $\alpha_A (x_0) = \theta_1and\alpha_A (y_0) = \theta_2$ and $\alpha_A (x_0 * y_0)$ = s .Then s $< \theta_1 \wedge \theta_2$. Let us consider $s_1 = \frac{1}{2}$ ($\alpha_A (x_0 * y_0)$ +$\alpha_A(x_0) \wedge \alpha_A(y_0)$). we get s_1=$\frac{1}{2}$ (s+$\theta_1 \wedge \theta_2$). Therefore $\theta_1 > s_1$=$\frac{1}{2}$(s + $\theta_1 \wedge \theta_2$) $>$ s and $\theta_2 > s_1$ =$\frac{1}{2}$(s + $\theta_1 \wedge \theta_2$) $>$ s.

Hence, $\theta_1 \wedge \theta_2 > s_1 > s = \alpha_A(x_0 * y_0)$, so that $x_0 * y_0 \notin \cup(\alpha_A : s)$ which is a contradiction. Since $\alpha_A(x_0) = \theta_1 \geq \theta_1 \wedge \theta_2 > s_1$ and $\alpha_A(y_0) = \theta_2 \geq \theta_1 \wedge \theta_2 > s_1$. This implies $x_0, y_0 \in \cup(\alpha_A : s)$.

Thus $\alpha_A(x * y) \geq \alpha_A(x) \wedge \alpha_A(y)$ for all x,y, $\in X$.

Again let $x_0, y_0 \in X$ be such that $\beta_A(x_0 * y_0) > \beta_A(x_0) \vee \beta_A(y_0)$. Let $\beta_A(x_0) = \eta_1$ and $\beta_A(y_0) = \eta_2$. and $\beta_A(x_0 * y_0) = t$. Then $t > \eta_1 \vee \eta_2$. consider $t_1 = \frac{1}{2}[\beta_A(x_0 * y_0) + \beta_A(x_0) \vee \beta_A(y_0)]$. we get that $t_1 = \frac{1}{2}(t + \eta_1 \vee \eta_2)$.

Therefore $\eta_1 < t_1 = \frac{1}{2}(t + \eta_1 \vee \eta_2) < t$ and $\eta_2 < t_1 = \frac{1}{2}(t + \eta_1 \vee \eta_2) < t$.

Hence $\eta_1 \vee \eta_2 < t_1 < t = \beta_A(x_0 * y_0)$. So that $x_0 * y_0 \notin L(\beta_A : t)$ which is a contradiction.

But $\beta_A(x_0) = \eta_1 \leq \eta_1 \vee \eta_2 < t_1$ and $\beta_A(y_0) = \eta_2 \leq \eta_1 \vee \eta_2 < t_1$. This implies $x_0, y_0 \in L(\beta_A : t)$. Thus $\beta_A(x * y) \leq \beta_A(x) \vee \beta_A(y)$ for all x,y $\in X$.

Hence $A = (\alpha_A, \beta_A)$ is an intuitionistic L-fuzzy G -subalgebra of X .

Theorem: 4.2.19

Any subalgebra of X can be realized as both the L- upper level and L-lower level of some intuitionistic L-fuzzy G -subalgebra of X .

Proof

Let P be an intuitionistic L-fuzzy G -subalgebra of X , and A be an ILFS on X defined by

$$\alpha_A(x) = \begin{cases} \lambda, \text{ if } x \in P \\ 0, \text{otherwise} \end{cases} \quad \text{and} \quad \beta_A(x) = \begin{cases} \tau, \text{ if } x \in P \\ 1, \text{otherwise} \end{cases}$$

For all $\lambda, \tau \in [0,1]$ and $\lambda + \tau \leq 1$. We consider the following cases:

Case(i) If x ,y $\in$ P ,then $\alpha_A(x) = \lambda$, $\beta_A(x) = \tau$ and $\alpha_A(y) = \lambda$, $\beta_A(y) = \tau$.Thus $\alpha_A(x * y) = \lambda = \lambda \wedge \lambda = \alpha_A(x) \wedge \alpha_A(y)$ and $\beta_A(x * y) = \tau = \tau \vee \tau = \beta_A(x) \vee \beta_A(y)$.

Case(ii) If x$\in$ P and y$\notin$ P, then $\alpha_A(x) = \lambda$, $\beta_A(x) = \tau$ and $\alpha_A(y) = 0$, $\beta_A(y) = 1$. Thus $\alpha_A(x * y) \geq 0 = \lambda \wedge 0 = \alpha_A(x) \wedge \alpha_A(y)$ and $\beta_A(x * y) \leq 1 = \tau \vee \tau$
$= \beta_A(x) \vee \beta_A(y)$.

Case(iii) If x $\notin$ P and y $\in$ P, then $\alpha_A(x) = 0$, $\beta_A(x) = 1$ and $\alpha_A(y) = \lambda$, $\beta_A(y) = \tau$. Thus $\alpha_A(x * y) \geq 0 = 0 \wedge \lambda = \alpha_A(x) \wedge \alpha_A(y)$ and $\beta_A(x * y) \leq 1 = 1 \vee \tau$
$= \beta_A(x) \vee \beta_A(y)$.

Csae(iv) If x $\notin$ P and y$\notin$ P, then α_A(x) = 0 , β_A(x) = 1 and α_A(y) = 0 , β_A(y)= 1. Thus α_A(x $*$ y) $\geq$0= 0 $\wedge$ 0= α_A(x) $\wedge$ α_A(y) and β_A(x $*$ y) $\leq$ 1=1 $\vee$ 1 =β_A(x) $\vee$ β_A(y) .

Therefore , A is an intuitionistic L-fuzzy G -subalgebra of X.

Theorem: 4.2.20

Let P be a subset of X and A be an IFS on X which is given by

$$\alpha_A(x) = \begin{cases} \lambda , \text{ if } x \in P \\ 0 \text{ otherwise} \end{cases} \quad \text{and} \quad \beta_A(x) = \begin{cases} \tau, \text{ if } x \in P \\ 1 \text{ otherwise} \end{cases}$$

For all $\lambda , \tau \in [0,1]$ and $\lambda + \tau \leq 1$.If A be realized as L- lower level subalgebra and L-upper level subalgebra of some intuitionistic L-fuzzy G -subalgebra of X , then P is a intuitionistic L-fuzzy G-subalgebra of X .

Proof

Let A be an intuitionistic L-fuzzy G-subalgebra of X, and x, y $\in$ P.

Then α_A (x) = λ =α_A(y) and β_A (x) = $\tau = \beta_A$(y) .

Thus α_A(x $*$ y) $\geq \alpha_A$(x) $\wedge$ α_A(y) $= \tau \wedge \lambda = \lambda$. and β_A(x $*$ y) $\leq \beta_A$(x) $\vee$ β_A(y) $= \tau \vee \tau = \tau$. which imply that x $*$ y $\in$ P .

Hence the proof is completed.

Theorem: 4.2.21

Let X be a G -algebra. Then any given chain of G –subalgebras

$P_0 \subset P_1 \subset P_2 \subset$ ……….$\subset P_r = X$, there exists an intuitionistic L-fuzzy G -subalgebra A of X whose level subalgebras are exactly the G -subalgebras of this chain.

Proof

Consider set of numbers $s_0 > s_1 >$ ………$> s_r$ and $t_0 < t_1 <$………$< t_r$, where each $s_i ,t_i \in [0,1]$ and $s_i + t_i \leq 1$ .Let A = (α_A , β_A) an ILFS defined by

$$\alpha_A(x) = \begin{cases} s_0 , \text{ if } x \in P_0 \\ s_1 , \text{ if } x \in P_i - P_{i-1} , 0< i \leq r \end{cases} \tag{1}$$

$$\text{and} \qquad \beta_A(\mathrm{x}) = \begin{cases} t_0 \text{ , if } \mathrm{x} \in P_0 \\ t_1 \text{ , if } \mathrm{x} \in P_i - P_{i-1} \text{ , } 0 < \mathrm{i} \le r \text{ .} \end{cases} \qquad (2)$$

We consider the following two cases:

Case (i). Let x,y $\in P_i - P_{i-1}$ .Therefore, by (1) and (2), $\alpha_A(\mathrm{x}) = \alpha_A(\mathrm{y}) = s_i$and $\beta_A(\mathrm{x}) = \beta_A(\mathrm{y}) = t_i$. Since P_i is a G –subalgebra, we have x$*$y $\in P_i$, and so either x$*$y $\in P_i - P_{i-1}$ or x$*$y $\in P_{i-1}$. In any case we conclude that $\alpha_A(\mathrm{x} * \mathrm{y}) \ge S_i = \alpha_A(\mathrm{x}) \wedge \alpha_A(\mathrm{y})$and $\beta_A(\mathrm{x} * \mathrm{y}) \le t_i = \beta_A(\mathrm{x}) \vee \beta_A(\mathrm{y})$.

Case (ii) Let x $\in P_i - P_{i-1}$ and y $\in P_j - P_{j-1}$ for i > j. Therefore, by (1)and (2), $\alpha_A(\mathrm{x}) = s_i, \alpha_A(\mathrm{y}) = s_j$, $\beta_A(\mathrm{x}) = t_i, \beta_A(\mathrm{y}) = t_j$. Then x$*$y $\in P_i$ Since P_i is a G –subalgebra of X and $P_j \subset P_i$. Hence $\alpha_A(\mathrm{x} * \mathrm{y}) \ge S_j = \alpha_A(\mathrm{x}) \wedge \alpha_A(\mathrm{y})$ and $\beta_A(\mathrm{x} * \mathrm{y}) \le t_j = \beta_A(\mathrm{x}) \vee \beta_A(\mathrm{y})$. Thus A is an intuitionistic L-fuzzy G -subalgebra of X. From (1) and(2), it follows that Im(α_A) = $\{s_0, s_1,\ldots\ldots,s_r\}$ and Im(β_A)= $\{t_0, t_1,\ldots\ldots,t_r\}$.

Hence, the L-level subalgebras of A are given by the chain of G -subalgebras

$$\cup(\alpha_A\,; s_0) \subset \cup(\alpha_A\,; s_1) \subset \ldots\ldots\ldots\ldots \subset \cup(\alpha_A\,; s_r) = \mathrm{X} \qquad (3)$$

$$L(\beta_A\,; t_0) \subset L(\beta_A\,; t_1) \subset \ldots\ldots\ldots\ldots \subset L(\beta_A\,; t_r) = \mathrm{X} \qquad (4)$$

Now , $\cup(\alpha_A : s_0)$ ={ x$\in X$: and $\alpha_A(\mathrm{x}) \ge s_0$} = P_0

= { x$\in X$: and $\beta_A(\mathrm{x}) \le t_0$}= $L(\beta_A\,; t_0)$.

Finally we prove that $\cup(\alpha_A : s_i) = P_i = L(\beta_A : t_i)$ for 0 < i ≤ r .

clearly, $P_i \subset \cup(\alpha_A : s_i)$ and $L(\beta_A : t_i)$. If x$\in \cup(\alpha_A : s_i)$ and $L(\beta_A : t_i)$, then $\alpha_A(\mathrm{x}) \ge s_i$ and $\beta_A(\mathrm{x}) \le t_i$. Which implies that x $\notin P_j$ for j> i.

Hence $\alpha_A(\mathrm{x}) \in \{s_0, s_1,\ldots\ldots,s_i\}$ and $\beta_A(\mathrm{x}) \in \{t_0, t_1,\ldots\ldots,t_i\}$and so x$\in P_k$ for some k $\le i$. As $P_k \subseteq P_i$, it follows that x$\in P_i$.

Therefore $\cup(\alpha_A : s_i) = P_i = L(\beta_A : t_i)$ for 0 ≤ i ≤ r. This completes the proof .

Note:

If X is a finite G -algebra, then the number of G -subalgebras of X is finite where as the number of L-level subalgebras of an intuitionistic L-fuzzy G - subalgebra A appears to be infinite. But since every L-level subalgebra is indeed a G - subalgebra of X , not all these G -subalgebras are distinct.

Theorem: 4.2.22

Let A be a L-fuzzy G -subalgebra of a G -algebra X.Then

(i) Two L-upper s-level subalgebras $\cup(\alpha_A\,;s_1)$, $\cup(\alpha_A\,;s_2)$ (with $s_1<s_2$) of A are equal if and only if there is no x$\in X$ such that $s_1 \le \alpha_A(x) < s_2$.

(ii) Two L-lower t -level subalgebras $L(\beta_A\,;t_1)$,$L(\beta_A\,;t_2)$ (with $t_1 > t_2$) of A are equal if and only if there is no x$\in X$ such that $t_1 \ge \alpha_A(x) > t_2$.

Proof

To prove (i) Assume that $\cup(\alpha_A : s_1) = \cup(\alpha_A : s_2)$ for $s_1 < s_2$ and assume that there exists x$\in X$ such that $s_1 \le \alpha_A(x) < s_2$ .Then $\cup(\alpha_A : s_2)$ is a proper subset of $\cup(\alpha_A : s_1)$ which is a contradiction.

Conversely, suppose that there is no x$\in X$ such that $s_1 \le \alpha_A(x) < s_2$.Since $s_1 < s_2$, we have $\cup(\alpha_A : s_2) \subseteq \cup(\alpha_A : s_1)$. If x $\in \cup(\alpha_A : s_1)$, then $\alpha_A(x) \ge s_1$ and so $\alpha_A(x) \ge s_2$, because $\alpha_A(x)$ does not lie between s_1 and s_2. Hence x $\in \cup(\alpha_A : s_2)$, which implies that $\cup(\alpha_A : s_1) \subseteq \cup(\alpha_A : s_2)$.

This completes the proof of (i). similarly (ii) can be proved .

Remark: 4.2.23.

As a consequence of Theorem 4.2.22, the L-level subalgebras of an intuitionistic L-fuzzy G -subalgebra A of a finite G -algebra X form a chain. But $\alpha_A(x) \le \alpha_A(0)$ and $\beta_A(x) \ge \beta_A(0)$ for all x$\in X$. Therefore $\cup(\alpha_A : s_0)$, where $s_0 = \alpha_A(0)$,is the smallest L-upper s-level subalgebra and $L(\beta_A\,;t_0)$,where $t_0 = \beta_A(0)$, is the smallest L- lower t -level subalgebra but not always $\cup(\alpha_A : s_0)$ and $L(\beta_A : t_0)$, as shown in the following example.

Example: 4.2.24

Let P be a non-trivial G -subalgebra of a G-algebra X and A be an intuitionistic L-fuzzy G –subalgebra in Theorem 4.2.19. Then Im(α_A) ={0,λ} and Im(β_A)={τ,1}. Furthermore, the two L-upper s -level subalgebras of A are U(α_A; 0) = X and U(α_A; λ) = P $\neq${0}, and the two L-lower t -level subalgebras of A are L(β_A; 1) = X and L(β_A; τ) = P $\neq${0}.

Theorem: 4.2.25

Let X be a finite G -algebra and A be an intuitionistic L-fuzzy G-subalgebra of X .

(i)If Im(α_A) = $\{s_1, s_2,……..,s_n\}$, then the family of G-subalgebras U($\alpha_A\,; s_i$) , 1≤i≤ n constitutes all the L-upper s-level subalgebras of A.

(ii) If Im(β_A) = $\{t_1, t_2,……..,t_n\}$, then the family of G-subalgebras L($\beta_A\,; t_i$), 1≤i≤ n constitutes all the L-lower t -level subalgebras of A.

Proof

To prove (i) Let s∈ L and s $\notin$ Im(α_A). Suppose $s_1 < s_2 <……..< s_n$ without loss of generality. If s $\leq s_1$, then U(α_A: s_1) = X =U(α_A: s) ,if s >s_n then obviously U(α_A: s) =φ . If $s_{i-1} < s < s_i$, then by Theorem 4.2.22,
we get U(α_A: s) = X =U(α_A: s_i). Thus for any s ∈ L, the L-level subalgebra is one of { U(α_A; s_i) / i= 1,2,…..,n}. similarly (ii) can be proved .
The following examples show that two intuitionistic L-fuzzy G -subalgebras of a G -algebra may have an identical family of level subalgebras but the intuitionistic L-fuzzy G -subalgebras may not be equal.

Example: 4.2.26

In Example 4.2.4, the L-upper s-level subalgebras of A are U(α_A: s_0) $= \{0,3\}$ and U(α_A: s_1) $= X$, and the L-lower t -level subalgebras of A are L(β_A: t_0) $= \{0,3\}$ and L(β_A: t_1) $= X$. Now, let $p_0, p_1, k_0, k_1 \in L$ be such that $p_0 > p_1$ and $k_0 < k_1$ with$p_i \neq s_j$ and$k_i \neq t_j$ for i, j =0,3.
We define an ILFS B =(α_B, β_B) with membership values
$\alpha_B(0) = \alpha_B(3)$= p_0, α_B(x) $= p_1$, and non-membership values
$\beta_B(0) = \beta_B(3)$= k_0, β_B(x) $= k_1$ for all x ∈ {1,2,4,5}. Then B is an intuitionistic L-fuzzy G -subalgebra of X . Then family of L-upper s-level subalgebras of B are U(α_B; p_0) $= \{0,3\}$,U(α_B; p_1) $= X$ and the family of L-lower t -level subalgebras of B are L(β_B; k_0) $= \{0,3\}$,L(β_B; k_1) $= X$. Thus the two intuitionistic L-fuzzy G -subalgebras A and B have the same family of level subalgebras. However A is not equal to B .

Theorem: 4.2.27

Let x be a G-algebra and Let A be an intuitionistic L-fuzzy G - subalgebra of X . Then

(i) If Im (α_A) is finite, say$\{s_1, s_2,\ldots\ldots, s_n\}$,then
for any $s_i, s_j \in$ Im (α_A), U$(\alpha_A\colon s_i) =$ U$(\alpha_A\colon s_j)$ implies $s_i = s_j$.

(ii) If Im (β_A) is finite, say$\{t_1, t_2,\ldots\ldots, t_n\}$, then
for any $t_i, t_j \in$ Im (β_A), L$(\beta_A : t_i) =$ L$(\beta_A : t_j)$ implies $t_i = t_j$.

Proof

To prove (i) Assume that $s_i \neq s_j$, say $s_i < s_j$. If x $\in$ U $(\alpha_A\colon s_j)$, then α_A(x) $\geq s_j > s_i$, which implies that x$\in$ U$(\alpha_A\colon s_i)$. Let x$\in X$ be such that $s_i < \alpha_A$(x) $< s_j$.Then x$\in$ U$(\alpha_A\colon s_i)$, but x$\notin$ U$(\alpha_A\colon s_j)$. Hence U$(\alpha_A\colon s_j) \subset$ U$(\alpha_A\colon s_i)$ and U$(\alpha_A\colon s_j) \neq$ U$(\alpha_A; s_i)$, a contradiction. similarly (ii)can be proved.

Theorem: 4.2.28.

Let A=(α_A , β_A) and B=(α_B , β_B) be two intuitionistic fuzzy G -subalgebras of a finite G -algebra X with identical family of level subalgebras.

If Im $(\alpha_A) = \{l_0, l_1, \ldots\ldots.., l_r\}$,Im$(\alpha_B) = \{s_0, s_1, \ldots\ldots.., s_k\}$,

Im$(\beta_A) = \{m_0, m_1, \ldots\ldots.., m_r\}$and Im (β_A) =$\{t_0, t_1,\ldots\ldots, t_k\}$

where $l_0 > l_1 >$ ……..$> l_r, s_0$ $> s_1$ $>$ ………..s_k, $m_0 < m_1 <$ ……..$< m_r$ and $t_0 < t_1 <$ ……….$< t_k$ then we have

(i) r = k

(ii)U$(\alpha_A; l_i)$ = U$(\alpha_B; s_i)$ and L$(\beta_A ; m_i)$ = L$(\beta_A ; t_i)$,0 ≤ i ≤ k,

(iii)If x$\in X$ such that α_A(x) $= l_i$ and β_A(x) $= m_i$, then α_A(x) $= s_i$ and β_A(x) $= t_i$, 0 ≤ i ≤ k.

Proof

To prove (i) : By Theorem 4.2.25, the only subalgebras of A and B are the families U$(\alpha_A; l_i)$, U$(\alpha_B; s_i)$, L$(\beta_A ; m_i)$and L$(\beta_A ; t_i)$.Since A and B have the same family of level subalgebras, it follows that r=k.

(ii) Using Remark 4.2.23 and (i), we have chains of level subalgebras U$(\alpha_A; l_0)$ ⊂…⊂ U$(\alpha_A; l_k)$= X and U$(\alpha_B; l_0)$ ⊂… ⊂ U$(\alpha_A; l_k)$= X It follows clearly that if $l_i, l_j \in$ Im (α_A) such that $l_i > l_j$and s_i, s_j ∈ Im (α_B)such that $s_i > s_j$,then U$(\alpha_A; l_i)$ ⊂ U$(\alpha_A; l_j)$ and U$(\alpha_B; s_i)$ ⊂ U$(\alpha_B; s_j)$ (1)

Since the two families of level subalgebras are identical, U$(\alpha_A; l_0)$ = U$(\alpha_B; s_0)$. By hypothesis U$(\alpha_A; l_1)$ = U$(\alpha_B; s_j)$for some j > 0. Assume that U$(\alpha_A; l_1)$ ≠ U$(\alpha_B; s_1)$. Then U$(\alpha_A; l_1)$ = U$(\alpha_B; s_j)$for some j >1 , and U$(\alpha_B; s_1)$ = U$(\alpha_A; l_i)$) for some $l_i < l_1$. Thus by (3) we obtain that U$(\alpha_B; s_j)$ = U$(\alpha_A; l_1)$ ⊂ U$(\alpha_A; l_i)$ and U$(\alpha_A; l_i) =$ U$(\alpha_B; s_1)$ ⊂ U$(\alpha_B; s_j)$This is a contradiction.

Hence U$(\alpha_A; l_1)$ = U$(\alpha_B; s_1)$. By induction on i , 0$\leq i \leq k$,we finally obtain that U$(\alpha_A; l_i)$ = U$(\alpha_B; s_i)$, 0$\leq i \leq k$ .Similarly we can obtain L$(\beta_A; m_i)$ = L$(\beta_B; s_t)$, 0$\leq i \leq k$.

(iii) Let x∈ X be such that α_A(x) $= l_i$ and α_B(x) $= s_j$, where 0$\leq i \leq k$ and 0$\leq j \leq k$. It is sufficient to show that $s_j = s_i$.

Now x∈ U$(\alpha_A; l_i)$ = U$(\alpha_B; s_i)$ implies that α_B(x) = $s_j \geq s_i$. This gives from (3) that U$(\alpha_B; s_j)$ ⊆ U$(\alpha_B; s_i)$. Since x ∈ U$(\alpha_B; s_j)$, it follows from (ii) that x∈ U$(\alpha_A; l_j)$ and so α_A(x) $= l_i \geq l_j$,which implies that U$(\alpha_A; l_i)$ ⊆ U$(\alpha_A; l_j)$by (1). Using (ii), we have U$(\alpha_B; s_i)$ = U$(\alpha_A; l_i)$ ⊆ U$(\alpha_A; l_j)$= U$(\alpha_B; s_j)$. Thus U$(\alpha_B; s_i)$ = U$(\alpha_B; s_j)$ and by Theorem 4.2.27, $s_j = s_i$. This completes the proof of first part.

Proof is similar for the second part.

Theorem: 4.2.29

Let $T_1 \supseteq T_2 \supseteq T_3$ …… be a descending chain of G -subalgebras of X which terminates at finite step. For an intuitionistic L-fuzzy G -subalgebra A=$(\alpha_A , \beta_A)$of X , if a sequence of elements of Im(α_A) a is strictly increasing and Im(β_A) strictly decreasing, then A=$(\alpha_A , \beta_A)$is finite valued.

Proof

Assume that A=(α_A , β_A) is infinite valued. Let $\{\phi_n\}$ be a strictly increasing sequence of elements of Im(α_A).

Then $0 \leq \phi_1 < \phi_2 <$……<1. Note that U($\alpha_A;\phi_t$)is a G -subalgebra of X for t =1,2,3,…. . Let x ∈ U($\alpha_A;\phi_t$) for t =1,2,3,…. .Then $\alpha_A\,(x) \geq \phi_t > \phi_{t-1}$ .which implies that x ∈ U($\alpha_A;\phi_{t-1}$) . Hence U($\alpha_A;\phi_t$) ⊆ U($\alpha_A;\phi_{t-1}$) for $t\ = 1,2,3, \ldots.$. Since ϕ_{t-1} ∈ Im(α_A) there exists x_{t-1} such that $\alpha_A\,(x_{t-1}\,)= \phi_{t-1}$. It follows that x_{t-1} ∈ U($\alpha_A;\phi_{t-1}$) ,but x ∉ U($\alpha_A;\phi_t$). Thus U($\alpha_A;\phi_t$) ⊊ U($\alpha_A;\phi_{t-1}$), and so we obtain a strictly descending chain U($\alpha_A;\phi_1$) ⊋ U($\alpha_A;\phi_2$) ⊋…. of G -subalgebras of X which is not terminating. This is impossible.

Again let $\{\phi_n\}$ be a strictly decreasing sequence of elements of Im(β_A) .

Then $1 \geq \phi_1 > \phi_2 >$……>0. Also L($\beta_A;\varphi_s$)is a G -subalgebra of X for s =1,2,3,…. . Let x ∈ L($\beta_A;\varphi_s$) for s =2,3,…. .Then $\beta_A\,(x) \leq \varphi_s < \varphi_{s-1}$ .which implies that x∈ L($\beta_A;\varphi_{s-1}$) .

Hence L($\beta_A;\varphi_s$) ⊆ L($\beta_A;\varphi_{s-1}$) s = 2 , 3 ,………… . Sinceφ_{s-1} ∈ Im(β_A) there exists x_{s-1} such that $\beta_A\,(x_{s-1}\,)= \varphi_{s-1}$. It follows that x_{s-1} ∈ L($\beta_A;\varphi_{s-1}$) ,but x ∉ L($\beta_A;\varphi_s$) . Thus L($\beta_A;\varphi_s$) ⊊ L($\beta_A;\varphi_{s-1}$), and so we obtain a strictly descending chain L($\beta_A;\varphi_1$) ⊋ L($\beta_A;\varphi_2$) ⊋…. of G -subalgebras of X which is not terminating. This is impossible. Therefore, A=(α_A , β_A) is finite valued.

Now we consider the converse of Theorem 4.2.29.

Theorem: 4.2.30

If every intuitionistic L-fuzzy G -subalgebra A of X has the finite image, then every descending chain of G -subalgebras of X terminates at finite step.

Proof

Suppose there exists a strictly descending chain $T_0 \supsetneq T_1 \supsetneq T_2$……. of closed ideals of X which does not terminate at finite step. Define an IFS A in X by

$$\alpha_A\,(x) = \begin{cases} \frac{n}{n+1}\ , \text{ if } x \in T_n \setminus T_{n+1}, \\ 1\ , \text{ if } x \in \bigcap_{n=0}^{\infty} T_n \end{cases}$$

$$\text{and} \qquad \beta_A(x) = \begin{cases} \frac{1}{n+1}, & \text{if } x \in T_n \setminus T_{n+1}, \\ 0, & \text{if } x \in \bigcap_{n=0}^{\infty} T_n \end{cases}$$

where n =0,1,2,…… and T_0 stands for X . Now, we consider the following cases: If x and y $\in T_n$, then x∗ y ∈ T_n , because T_n is a G -subalgebra of X . Hence,

$$\alpha_A(x * y) \geq \frac{n}{n+1} = \alpha_A(x) \wedge \alpha_A(y).$$

$$\beta_A(x * y) \leq \frac{1}{n+1} = \beta_A(x) \vee \beta_A(y).$$

if x ∈ $T_n \setminus T_{n+1}$ and if y ∈ $T_m \setminus T_{m+1}$, where n > m , then x∗ y ∈ T_m . Hence,

$$\alpha_A(x * y) \geq \frac{m}{m+1} = \alpha_A(x) \wedge \alpha_A(y).$$

$$\beta_A(x * y) \leq \frac{1}{m+1} = \beta_A(x) \vee \beta_A(y).$$

if x ∈ $T_n \setminus T_{n+1}$ and if y ∈ $T_m \setminus T_{m+1}$, where n < m , then x∗ y ∈ T_m . Hence,

$$\alpha_A(x * y) \geq \frac{n}{m+1} = \alpha_A(x) \wedge \alpha_A(y).$$

$$\beta_A(x * y) \leq \frac{1}{n+1} = \beta_A(x) \vee \beta_A(y).$$

This proves that A is an intuitionistic L-fuzzy G -subalgebra with an infinite number of different values, which is a contradiction. This completes the proof.

Theorem: 4.2.31

Every ascending chain of G -subalgebras of X terminates at finite step if and only if the set of values of any intuitionistic L-fuzzy G -subalgebra is a well ordered subset of L .

Proof

Let A be an intuitionistic L-fuzzy G -subalgebra of X . Suppose that the set of values of A is not a well-ordered subset of L. Then there exist a strictly decreasing sequence $\{\gamma_n\}$ such that $\alpha_A(x_n) = \gamma_n$.

It follows that $U(\alpha_A;\gamma_1) \subsetneq U(\alpha_A;\gamma_2) \subsetneq U(\alpha_A;\gamma_3) \subsetneq$…… is a strictly ascending chain of G -subalgebras of X which is not terminating. This is impossible. If there exist a strictly increasing sequence $\{\delta_n\}$such that$\beta_A(x_n) = \delta_n$. It follows that L$(\beta_A; \delta_1) \subsetneq$ L$(\beta_A; \delta_2) \subsetneq$ L$(\beta_A; \delta_3) \subsetneq \ldots\ldots$is a strictly ascending chain of G -subalgebras of X which is not terminating. This is impossible. To prove the converse suppose that there exist a strictly ascending chain $T_1 \subsetneq T_2 \subsetneq T_3 \subsetneq \ldots\ldots$ (1) of G -subalgebras of X which does not terminate at finite step. Note that T = $\bigcup_{n\in\mathbb{N}} T_n$is a closed ideal of X . Define an IFS A =(α_A, β_A) in X by

$$\alpha_A(x) = \begin{cases} \frac{1}{k}, & \text{where } k = \wedge\{ n \in \mathbb{N} \,/\, x \in T_n\}, \\ 0, & \text{if } x \notin T_n \end{cases}$$

and $\beta_A(x) = 1 - \alpha_A(x)$.

By using similar method as Theorem 4.2.30, we can prove that A is an intuitionistic L-fuzzy G -sublagebra of X . Since the chain (1) is not terminating, A has a strictly descending sequence of values. This contradictis that the value set of any intuitionistic l-fuzzy G -subalgebra is well-ordered. This completes the proof.

SUMMARY AND CONCLUSION

In 1995 Zadeh [41] introduced the notion of fuzzy sets. In 1999 the notion of intuitionistic fuzzy set (IFS) was defined by Atanassov [12] as a generalization of Zadeh's fuzzy set. After the concept of intuitionistic fuzzy set was introduced, several articles have been published by mathematicians to extend the classical mathematical concepts and fuzzy mathematical concepts to the case of intuitionistic fuzzy mathematics.

The study of BCK / BCI – algebras was initiated by Imai and Iseki [17 ,18] in 1966 as a generalization of the concept of set – theoretic difference and propositional calculus. In 2012 Bandaru and Rafi [13] introduced a new notion, called G-algebras, which is a generalization of QS-algebras. Senapati et al. [38] applied the fuzzy notions to G-subalgebras and introduced the notions of fuzzy G-subalgebras of G-algebras. In 2014 Jana et al. [22] applied the notion of intuitionistic fuzzy set to G-subalgebras of G-algebras.

In this thesis we have made an attempt to study derivations and intuitionistic L-fuzzy G-subalgebras of G-algebras.

In Chapter 1, properties of G-algebras are discussed. Also the concepts of 0-commutative, G-part and medial of G-algebras are studied due to Bandaru and Rafi[13].

In Chapter 2 , properties of derivations, f_q – derivations and (f,g) – derivations of G-algebras are discussed due to Al-Kadi [7,6,8].

In Chapter 3 , the concepts of intuitionistic fuzzy G-subalgebras of G-algebras are studied and their properties are investigated due to Jana et al.[22].

In Chapter 4, the L-fuzzification and intuitionistic L-fuzzification G-subalgebras are considered and some related properties are investigated. A characterization of intuitionistic L-fuzzy G-algebras are obtained. The G-subalgebras are classified by their family of level subalgebras of G-algebras due to Jana and et al. [36,21].

A deep study of intuitionistic L-fuzzy G-subalgebras of G-algebras can be extended to various algebras. So it provides a lot of scope for further research.

BIBLIOGRAPHY

1. Abujabal, H. A. S., and Al-Shehri, N. O., **On left derivations of BCI-algebras,** Soochow Journal of Mathematics, vol. 33(2007), No. 3, 435–444.
2. Abujabal, H.A.S., and Al-Shehri, N.O., **Some results on derivations of BCI-algebras**, Journal of Natural Sciences and Mathematics, Vol. 46(2006), No.1-2, 13 - 19.
3. Ahn, S.S., and Kim, H.S., **On QS -algebras,** J. Chungcheong Math. Soc., Vol. 12 (1999), 33-41.
4. Akram, M., Shum, K.P., **Intuitionistic Fuzzy Lie Algebras,** Southeast Asian Bulletin of Mathematics,Vol.31(2007), 843-855.
5. Al –Roqi, A.M., **On Generalized (α, β) Derivations in BCIalgebras,** Journal of Applied Mathematics and Informatics ,Vol.32(2014), 27-38.
6. Al-Kadi, D., **f_q-Derivations of G-Algebra,** International Journal of Mathematics and Mathematical Sciences, (2016), 1-5.
7. Al-Kadi, D., and Hosny, R., **Notes on G-algebra and its derivations,** Mathematical Sciences Letters, Vol.6 (2017), No. 3, 287-292.
8. Al-Kadi, D., **On (f,g) - Derivations of G-Algebras,** Pure Mathematical Sciences, Vol .6 (2017), No.1, 95-104.
9. Al-Shehrie, N.O., **Derivations of B-algebras,** JKAU: Sci. Vol.22, (2010), No. 1, 71 - 83.
10. Atanassov, K.T., and Stoeva, S., **Intuitionistic L -fuzzy sets,** Cybernetics and Systems Research, Vol. 2 (1984), 539-540.
11. Atanassov, K.T., **Intuitionistic fuzzy sets,** Central technical library, Bulgarian academy science, (1983).
12. Atanassov, K.T., **Intuitionistic fuzzy sets, Theory and Applications ,** Studies in Fuzziness and Soft Computing , Vol.35 (1999).
13. Bandaru, R.K., and Rafi , N., **On G-algebras,** Scientia Magna, Vol. 8 (2012), No. 3, 1 - 7.
14. Barbhuiya, S.R., **Intuitionistic Q-fuzzy Ideals of BG-Algebras,** International Journal of Computer Applications , Vol . 94 (2014), No.16,1-14.
15. Bawazeer, S.M., Al-shehri, N.O., and Rawia Saleh Babusail, **Generalized Derivations of BCC-algebras,** International Journal of Mathematics and Mathematical science, (2013), 1-4.

16. Hu, Q.P., and Li, X., **On BCH-algebras**, Mathematics Seminar Notes, Vol.11(1983), 313-320.
17. Imai, Y. and Iseki, K., **On axiom systems of propositional calculi. XIV**, Proc. Japan Acad. Vol. 42(1966), 19-22.
18. Iseki, K., **An algebra related with a propositional calculus,** Proceedings of the Japan Academy, Vol. 42(1966), 26-29.
19. Iseki, K., **On BCI-algebras,** Mathematics Seminar Notes, Vol. 8(1980), 125-130.
20. Iseki, K., and Tanaka, S., **An introduction to the theory of BCK-algebras,** Math.Japon. Vol. 23(1978), No .1, 1-26.
21. Jana, C., and Pal, M., Senapati, T., Bhowmik, M., **Atanassov's Intuitionistic L-fuzzy G-subalgebras of G-algebras,** The Journal of Fuzzy Mathematics, Vol.23, (2012), No.2 , 325-340.
22. Jana, C., and Pal, M., Senapati, T., Bhowmik, M., **On Intuitionistic fuzzy G -subalgebras of G-algebras,** Fuzzy Inf . Eng. Vol. 7(2015), 195-209.
23. Jana, C., and Senapati, T., **Cubic *G*-subalgebras of *G*-algebras,** Annals of Pure and Applied Mathematics , Vol.10(2015), No.1, 105-115,.
24. Jun, Y.B and Kim, K.H., **Intuitionistic fuzzy ideals of BCK-algebras,** International Journal of Mathematics and Mathematical Sciences ,Vol.24(2000),No.12, 839-849.
25. Jun, Y.B., and Xin, X.L., **On derivations of BCI-algebras,** Information Sciences, Vol. 159 (2004), No. 3-4, 167–176.
26. Kamali Ardekani, L., and Davvaz, B., **On Generalized Derivations of BCI-algebras and Their Properties,** Journal of Mathematics , (2014), 1-10.
27. Lee, K.J., **A new kind of derivations in BCI-algebras,** Applied Mathematical Sciences, Vol. 7(2013), No. 84, 4185–4194.
28. Muhiuddin, G., and Al-Roqi, A.M., **On t-derivations of BCI- algebras,** Abstract and Applied Analysis, (2012) ,1-12.
29. Muhiuddin, G., and Al-Roqi, A.M., **On (α , β)-derivations in BCI-algebras,** Discrete Dynamics in Nature and Society, (2012), 1-11.
30. Muhiuddin, G., and Al-Roqi, A.M., **On Generalized Left Derivations in BCI-Algebras,** Applied Mathematics and Information sciences, Vol.8(2014), 1153-1158.
31. Muhiuddin, G., and Al-Roqi, A.M., **Generalizations of Derivations in BCI- Algebras,** Appl. Math. and Information Sciences, Vol. 9 (2015), No.1, 89-94.

32. Neggers, J., Ahn, S.S. and Kim, H.S., **On Q-algebras**, IJMMS, Vol. 27(2001), 749-757.
33. Neggers, J., and Kim, H.S., **On B-algebras**, Math. Vensik, Vol. 54(2002), 21-29.
34. Prabpayak, C., and Leerawat, U., **On derivations of BCC-algebras,** Kasetsart J. (Nat. Sci.), Vol.43 (2009), 398-401.
35. Senapati, T., Bhowmik, M., and Pal, M., **Interval-valued intuitionistic fuzzy BG -subalgebras,** J. Fuzzy Math., Vol. 20(2012), No. 3, 707-720.
36. Senapati, T., Jana, C., Bhowmik, M., Pal, M.,**L-fuzzy G-subalgebras of G-algebras,** Journal of the Egyptian Mathematical Society, Vol.23(2015), 219-223.
37. Senapati, T., Bhowmik, M., and Pal, M., **Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy H -ideals in BCK /BCI -algebras,** Notes on Intuitionistic Fuzzy Sets, Vol.19(2013), No.1, 32-47.
38. Senapati, T., Jana, C., Bhowmik, M., and Pal, M., **On fuzzy G -subalgebras of G -algebras,** Communicated.
39. Senapati, T., Jana, C., Bhowmik, M., and Pal, M., **On L -fuzzy G -subalgebras of G -algebras,** Journal of The Egyptian Mathematical Society, (2014).
40. Walendziak, A., **On BF-algebras**, Math. Slovaca , Vol.57(2007), No. 2, 119- 128.
41. Zadeh, L.A., **Fuzzy sets,** information and control , Vol.8(1965) , 338-353.
42. Zhan, J., and Liu, Y.L., **On f-derivations of BCI-algebras,** International Journal of Mathematics and Mathematical Sciences, Vol.11(2005), 1675–1684.