

CONCLUSION

Mathematically, the (m, N) policy vacation queueing models are the generalizations of N -policy and the classical 1-policy queueing models. The system size of every (m, N) policy model is decomposed into two random variables, one of which is the system size of the corresponding $M^X/G/1$ model without (m, N) policy and the other random variable gives the PGF of the conditional system size distribution $\left(\frac{I_{(m,N)}(z)}{I_{(m,N)}(1)}\right)$ during the server idle period.

Thus deriving the expression for $I_{(m, N)}(z)$ is the primary task for the (m, N) policy queueing models. Lee et al. (2003) have defined β_n , ψ_n and ϕ_n for the non-vacation, single and multiple vacation $M^X/G/1$ reliable single server queueing systems independently and obtained $I_{(m, N)}(z)$ in terms of these functions. In chapters II to V of the present work, the author has made an attempt to derive the expression for $I_{(m, N)}(z)$ by re-defining β_n , ψ_n and ϕ_n suitably for J -vacation policy, $\langle p, J \rangle$ vacation policy, Bernoulli schedule single vacation policy and has shown that the corresponding results of the single, multiple and non-vacation queueing models can be deduced as special cases of these variant vacation policies.

The present work also concentrates on the following features which are not considered in the literature of (m, N) policy models.

- The server operates two phases of services whereby, the first phase is a single essential phase and the second phase being an optional phase and contains multi- optional service facilities.
- The server is subject to breakdowns while serving the customers and the service interrupted customers thus may resume the service from where they were interrupted or may repeat their service from the beginning or quit the system without completing the service.
- The dissatisfied customers may demand for re-services after completing their regular service.

- The server can take optional vacation between any two consecutive services during busy period.

Chapters VI to VIII deal with the 1-policy queueing systems with distinct features :

Chapter VI considers two phase service channels wherein each customer undergoes two stages of services one after another and the server operates, Multiple Adapted Vacation policy (MAV) during idle period and optional Bernoulli Schedule Vacation policy during busy period. The dissatisfied customers are allowed to demand re-services after completing the regular service. The server is subject to breakdowns and whenever the system breaks down, the repair time is delayed and the service interrupted customers will repeat their service from the beginning as soon as the server is fixed. The MAV policy generalizes the other vacation policies. It is shown that the results including that of $\langle p, J \rangle$ vacation policy can be derived from the results of MAV models.

Most of the existing queueing models allow the customers to take infinite number of feedbacks which are hardly possible in everyday life. In chapter VII, a repairable $M^X/G/1$ queueing model with finite number of immediate feedbacks (at most m) is considered. The system also contains multi-optional service facilities . A corresponding queueing model with infinite number of feedbacks is also analysed in chapter VIII to justify that as $m \rightarrow \infty$ the results of both the models coincide with each other. The cases of repeat or resumption of interrupted service when the server is fixed are considered as two different cases in chapters VII and VIII and it is shown that if the service time is exponential, then the two types of behaviours of the interrupted customers will lead to the same result.

All the models analysed are unique and general in nature. The analytical treatment of all models are done by Supplementary Variable Technique. The PGFs of the queue length distribution are presented in a compact form. The partial generating functions of the system size when the system is in different states are calculated and the formulae for the system

size probabilities and the expected number of customers waiting in the system when the system is in different states are also derived. Moreover, expected cycle length, expected idle time and expected busy period are also calculated.

A cost model is introduced for the models of chapters II to V, and a procedure to find the optimal values of (m, N) that minimize the average cost is also developed. The results of various queueing models are derived as particular cases of the models presented in the work.

The decomposition property of vacation queues is established for all the models and the PGFs of the departure point system size distribution are also derived .

The numerical computation of performance measures for batch arrival models (especially for (m, N) policy) are not easy in general. The measures for all the batch arrival models are calculated using C++ programs.

FUTURE ENHANCEMENT

The following are some of the possible extensions suggested for future research:

The distributions for Idle period, busy period and waiting time may be derived using the results obtained in the present model.

The (m, N) policy may be analysed for the Multiple Adapted Vacation policy queueing systems and the expression for $I_{(m, N)}(z)$ may be obtained.

The joint optimal policy for (ρ, J, m, N) may also be calculated for $\langle \rho, J \rangle$ vacation models.

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APPENDIX – 1

The following are the identities and the results used in the analytical solutions. Let m and N be non negative integers.

A. Identities

1.
$$\sum_{n=2}^{\infty} z^n \left(\sum_{k=1}^{n-1} P_{n-k}^*(\theta) g_k \right) = \left(\sum_{k=1}^{\infty} g_k z^k \right) \left(\sum_{n=1}^{\infty} P_n^*(\theta) z^n \right) = X(z) P^*(z, \theta)$$
2.
$$\sum_{n=m+1}^{\infty} z^n \left(\sum_{k=1}^{n-m} SE_{n-k}^*(\theta) g_k \right) = \left(\sum_{k=1}^{\infty} g_k z^k \right) \left(\sum_{k=m}^{\infty} SE_n^*(\theta) z^n \right) = X(z) SE^*(z, \theta)$$
3.
$$\sum_{n=1}^{\infty} z^n \left(\sum_{k=1}^n Q_{n-k}^*(\theta) g_k \right) = \left(\sum_{k=1}^{\infty} g_k z^k \right) \left(\sum_{n=0}^{\infty} Q_n^*(\theta) z^n \right) = X(z) Q^*(z, \theta)$$

$m = 1$ and 0 in (2) give the results in (1) and (3) respectively.

4.
$$\sum_{n=1}^{m-1} z^n \left(\sum_{k=1}^n PI_{n-k} g_k \right) + \sum_{n=m}^{\infty} z^n \left(\sum_{k=n-m+1}^{\infty} PI_{n-k} g_k \right) = \left(\sum_{k=1}^{\infty} g_k z^k \right) \left(\sum_{n=0}^{m-1} PI_n z^n \right) = X(z) PI(z)$$
5.
$$\left(\sum_{k=1}^{\infty} g_k z^k \right) \left(\sum_{n=m}^{N-1} U_n z^n \right) = \sum_{n=m+1}^{N-1} z^n \left(\sum_{k=1}^{n-m} U_{n-k} g_k \right) + \sum_{n=N}^{\infty} z^n \left(\sum_{k=n-N+1}^{n-m} U_{n-k} g_k \right)$$
6.
$$\left(\sum_{k=0}^{\infty} h_k z^k \right) \left(\sum_{n=m}^{\infty} \zeta_n z^n \right) = \sum_{n=m}^{\infty} z^n \left(\sum_{i=m}^n \zeta_i h_{n-i} \right)$$
7.
$$\left(\sum_{n=0}^{\infty} \alpha_n z^n \right) \left(\sum_{n=0}^{m-1} \delta_n z^n \right) = \sum_{n=0}^{m-1} z^n \left(\sum_{i=0}^n \alpha_i \delta_{n-i} \right) + \sum_{n=m}^{\infty} z^n \left(\sum_{i=0}^{m-1} \delta_i \alpha_{n-i} \right)$$
8.
$$\sum_{r=m}^n \delta_r \left(\sum_{k=0}^{n-r} h_k \pi_{n-r-k} \right) = \sum_{k=m}^n \pi_{n-k} \left(\sum_{i=m}^k \delta_i h_{k-i} \right)$$
9.
$$\sum_{k=1}^n g_k \left(\sum_{i=0}^{n-k} \alpha_i \pi_{n-k-i} \right) = \sum_{k=0}^{n-1} \alpha_k \left(\sum_{i=1}^{n-k} g_i \pi_{n-k-i} \right)$$

(8) and (9) give the re-arrangement of the summation.

B. Results using L' Hospital Rule

If $f(1) = g(1) = 0$, then $\frac{d}{dz} \left(\frac{f(z)}{g(z)} \right)_{z=1} = \frac{g'(1)f''(1) - f'(1)g''(1)}{2(g'(1))^2}$, where the

dashes represent the derivatives of the functions.

Let S, D, R, H be any random variables with LST $S^*(\theta), D^*(\theta), R^*(\theta)$ and $H^*(\theta)$ respectively and let $w_X(z) = \lambda (1 - X(z))$, $g_a(w_X(z)) = a + w_X(z)$ and $h_a(w_X(z)) = w_X(z) + a (1 - R^*(w_X(z)))$, then

$$\begin{aligned} \lim_{z \rightarrow 1} \frac{1 - S^*(g_a(w_X(z)))}{g_a(w_X(z))} &= \frac{1 - S^*(a)}{a} \\ \frac{d}{dz} \left(\frac{1 - S^*(g_a(w_X(z)))}{g_a(w_X(z))} \right)_{z=1} &= \lambda E(X) \frac{S^{*'}(a)}{a} + \frac{1 - S^*(a)}{a^2} \\ \lim_{z \rightarrow 1} \frac{-w_X(z)}{D(z)} &= \frac{\lambda E(X)}{D'(1)} \\ \frac{d}{dz} \left(\frac{-w_X(z)}{D(z)} \right) &= \frac{\lambda E(X(X-1))D'(1) + \lambda E(X)(-D''(1))}{2(D'(1))^2} \end{aligned}$$

(for any denominator $D(z)$ with $D(1) = 0$).

$$\begin{aligned} \lim_{z \rightarrow 1} \left(\frac{1 - S^*(h_a(w_X(z)))}{h_a(w_X(z))} \right) &= E(S) (1 + a E(R)) \\ \frac{d}{dz} \left(\frac{1 - S^*(h_a(w_X(z)))}{h_a(w_X(z))} \right)_{z=1} &= \frac{\lambda E(X)E(S^2)}{2} (1 + a E(R)) \\ \lim_{z \rightarrow 1} \left(\frac{-w_X(z)}{z - H^*(w_X(z))} \right) &= \frac{\lambda E(X)}{1 - \rho_H} \\ \frac{d}{dz} \left(\frac{-w_X(z)}{z - H^*(w_X(z))} \right)_{z=1} &= \frac{\lambda E(X(X-1)) + (\lambda E(X))^3 E(H^2)}{2(1 - \rho_H)^2} \end{aligned}$$

where $\rho_H = \left(\frac{d}{dz} H^*(w_X(z)) \right)_{z=1} = \lambda E(X) E(H)$

$$\begin{aligned} \lim_{z \rightarrow 1} \left(\frac{1 - R^*(w_X(z))D^*(w_X(z))}{w_X(z)} \right)_{z=1} &= E(D) + E(R) \\ \frac{d}{dz} \left(\frac{1 - R^*(w_X(z))D^*(w_X(z))}{w_X(z)} \right)_{z=1} &= \frac{\lambda E(X)}{2} (E(D^2) + E(R^2) + 2 E(R) E(D)) \\ \lim_{z \rightarrow 1} \frac{z-1}{z - H^*(w_X(z))} &= \frac{1}{1 - \rho_H} \\ \frac{d}{dz} \left(\frac{z-1}{z - H^*(w_X(z))} \right) &= \frac{\lambda E(X(X-1))E(H) + (\lambda E(X))^2 E(H^2)}{2(1 - \rho_H)^2} \end{aligned}$$

APPENDIX – 2

VARIOUS DISTRIBUTIONS USED IN NUMERICAL ANALYSIS

Discrete Distributions :

Name	Distribution	Probability mass function
Geometric distribution ($X \sim \text{Geo}(p)$)	$(1 - p) p^{k-1}, k \geq 1$	$\frac{(1 - p) z}{1 - pz}$
Binomial (N, p)	$NC_k p^k q^{N-k} \quad k = 0, 1, \dots, N$	$pz + q$

Continuous Distributions :

Name	Density function (f(x))	LST
k-stage hyper exponential distribution when $k = 1$	$\sum_{i=1}^k a_i \mu_i e^{-\mu_i x}, 0 \leq x \leq 1$ $\sum_{i=1}^{\infty} a_i = 1$ $\mu e^{-\mu x}$ (exponential distribution)	$\sum_{i=1}^k \frac{a_i \mu_i}{\mu_i + \theta}$ $\frac{\mu}{\mu + \theta}$
Uniform distribution	$1, 0 \leq x \leq 1$ $0, \text{ otherwise}$	$\frac{e^{-as} - e^{-bs}}{b - s}$
Gamma distribution $G(k, \lambda)$	$\frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma k}, x > 0$ $0, \quad x < 0$	$\left(\frac{\lambda}{\lambda + \theta} \right)^k$
Erlang-k-distribution E_k	$\frac{(\lambda k)^k x^{k-1} e^{-\lambda k x}}{\Gamma k}, x > 0$	$\left(\frac{k\lambda}{k\lambda + \theta} \right)^k$

APPENDIX – 3

Applications – Examples

Example – 1

The following quality control problem (constructed by Kella (1989)) satisfies the criteria of the (m, N) policy of the queueing systems analysed in chapters II to V :

A manufacturing plant, produces certain items, that are occasionally defective. The good items produced are marketed, while the defective ones are kept in storage until they can be reworked to meet specifications. One of the machines in the plant may be converted as needed (at some cost), from production mode to a repair mode in order to perform the rework. Instead of starting the rework process as soon as the defective items are produced, an appropriate cut off number N is suggested to start the rework. Since setup time may be required before starting the process, the special machine is released from production mode when the number of items accrued to at least m ($m \leq N$) so that during the setup operation more defective items may be accumulated and the number may raise to N at the end of the setup period. Even if N defective items are not gathered at the end of the setup work, the special machine continue to stay in the repair mode till the number of items raised to at least N . Because of the cost involved in switching modes, after conversion to repair mode, the machine will rework all the defective items (including the new arrivals) exhaustively, and then switch back to the production mode.

In this example, the defective items may be interpreted as the arriving units and the special machine as the server. The server is said to be on a series of vacations, when it is in the production mode. Thus the vacation time is the time required to produce the products. At the end of each vacation, the management takes decision whether to keep the machine in productive mode or change it to the repair mode ((i.e.) to take another vacation). The defective items are reworked one by one. Locating the kinds of defect may be

considered as first phase service. The second phase consists of multi-optional facilities and the items may be treated accordingly.

Some unpredictable interruptions may occur during the re-work period and the special machine may itself undergo repair process and the rework may be resumed or repeated when the machine is fixed.

At the end of each rework the machine may need time to rework the next item. This can be referred as vacation between services.

If the rework is not properly done, it may be repeated again and this may correspond to feedback service.

Example – 2

Operational states of a Heterogeneous Sensor Network (HSN) node particularly in Vehicle Tracking System can be considered as an example for the model of chapter II. HSN is used in Wireless Sensor Networks (WSN) to improve sensor network performance in terms of energy consumption. A HSN model consists of two physically different types of sensor nodes. A small number of powerful high-end sensors (H-sensors) and a large number of low-end sensors (L-sensors) uniformly distributed in the field. After deployment, clusters are formed and H-sensor in each cluster serves as cluster head (CH). In some of the existing Vehicle Tracking Systems, data packets are directly sent from L-sensors to the Base station (BS). So the Base station should always be listening to the arrival of data, which necessitates investment on high end servers. Instead, if a cluster head (CH) collects data from multiple L-sensors and sends data to the BS using (m, N)-policy as proposed by our model, the contention in the data cables can be reduced and the sensor nodes may be efficiently used. In the model proposed, the data packets arrive in batches to form a queue and are processed one by one. (Batch arrival and single service).

The two major operational states of a sensor node (L-sensor and H-sensor node) are sleep state and active state. In sleep state, a node cannot interact with the external world. In active state, a sensor node may be in idle

mode or it may generate data or transmit and /or receive data packets. An **H**-sensor node in a cluster during its period of active time, usually remains in IDLE state. When **m** number of data packets accumulate, the sensors are kept in the ready state to operate (setup) and switches to busy state when the node's buffer is filled at least with threshold number of packets (**N**). During busy state, the sensor node will be processing the data and it can be considered as First Essential State (FES). The data can be transmitted as received or can be manipulated for further actions. The different types of manipulation can be considered as Second Optional Services (SOS). The node switches back from BUSY state to vacation state when there are no packets in the buffer. In vacation state the sensor node can be used for some other purposes like comparing the data with the previous data so that if there is no variation in the data stored, that data need not be saved.

After comparing one set of data, if the node finds **m** new data packets in the queue then the node will switch to operate state (set up operations). The nodes thus repeat the process of comparing the data with the previous data until **m** number of new packets arrive or the number of consecutive vacations taken by the node reaches a fixed number **J**, whichever occurs earlier. Unpredictable breakdowns may occur while different **H**-sensors send data to BS at the same time and stop the transmissions. The remaining data can be sent to the BS as soon as the node is fixed.