Operation Approach to β-Open Sets and Applications

By

Shalini R

(19PMA012)

Thesis Submitted to

**Avinashilingam Institute for Home Science and Higher Education for Women,
Coimbatore - 641 043**

In Partial Fulfillment of the Requirement for the Degree of

Master of Science in Mathematics

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Signature of the Head of the Department

Signature of the Supervisor

CERTIFICATE

I certify that the dissertation entitled **"Operation Approach to β-Open Sets and Applications"** submitted for the degree of **Master of Science** by **Shalini R** is the record of research work carried out by her during the period from December 2020 to May 2021 under my guidance and supervision and that this work has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship, or other titles in this university or any other University or Institution of Higher Learning.

K. Sivakamasundari

N. Balamani

Signature of the Head of the department **Signature of the Supervisor**

DECLARATION

I declare that the dissertation entitled **"Operation Approach to β-Open Sets and Applications"** submitted by me for the degree of **Master of Science** is a record of work carried out by me during the period from December 2020 to May 2021 under the guidance of **Dr. N. Balamani,** Assistant Professor (SS), Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore and has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship, or other titles in this University or any other University or Institution of Higher Learning.

R. Shalini

Signature of the Candidate

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CONTENTS

CONTENTS

INTRODUCTION

Introduction

Topology is a branch of mathematics which is good at extracting global qualitative features from complicated geometric structures. General topology known as point set topology is the branch of topology dealing with the basic set theoretic definitions and constructions used in topology. It is the foundation of most other branches of topology, including differential topology, geometric topology, algebraic topology and network topology. The fundamental concepts in point set topology are continuity, compactness and connectedness. The topological structures are modeled suitably in the fields of computer graphics, pattern recognition, artificial intelligence, data mining, information systems, rough set theory, quantum physics etc.

The concepts of closed and open sets play an important role in the study of topological spaces. Stone [16] introduced the regular open sets in topological spaces, as a stronger form of open sets and Levine [8] initiated and studied the concept of semi open sets in topological spaces, as a weaker form of open sets.

Several authors introduced and studied different kinds of open and closed sets in topological spaces. Njastad [11] introduced the concept of α-open sets which are weaker than open sets. Further Levine [9] introduced the notion of generalized closed (g-closed) sets in topological spaces. Abd El-Monsef et al. [1] introduced β-open sets in topological spaces.

Kasahara [15] introduced the notion of an α operation approaches on a class τ of sets and studied the concept of α-continuous functions with α-closed graphs and α-compact spaces. Jankovic [7] introduced the concept of α-closure of a set in X via α-operation and investigated further characterizations of function with α-closed graph. Later, Ogata [6] defined and studied the concept of γ-open sets, and applied it to investigate operation-functions and operation-separation. He also introduced the concept of a family τ_γ of all γ-open sets by using the operation γ on a topological space and investigated the relation between the γ-closures and τ_γ-closures. He further investigated general operator approaches of closed graph of mappings.

Norman Levine [9] initiated the idea of continuous functions. Following Levine different generalizations of continuous functions have been studied by various authors. General topologist have introduced and investigated many different generalizations of continuous functions.

This thesis is devoted to the study of operation approach to β-open sets, separation axioms and continuous mappings using γ-operation in topological spaces.

The following articles are chosen for discussion:

(i) Operation approach to β-open sets and applications by Sanjai Tahiliani [14].

(ii) On Some Separation Axioms via β-γ-open sets by Alias B. Khalaf and Hariwan Z. Ibrahim [2]

Chapter 1 deals with preliminary definitions and results relevant to the study.

In Chapter 2, Operation approach to β-open sets and applications due to Sanjai Tahiliani are studied.

Section 2.2 deals with the concept of γ-operation on β-γ-open sets.

"An operation $\gamma: \beta O(X,\tau) \rightarrow P(X)$ where $P(X)$ is the set of all subsets of X is a mapping satisfying the condition, $V \subseteq V^{\gamma}$ for each $V \in \beta O(X,\tau)$. Here V^{γ} denotes the value of γ at V. We call the mapping γ an operation on $\beta O(X,\tau)$".

"A subset A of X is called an β-γ-open set if for each point $x \in A$, there exists an β-open set U of X containing x such that $U^{\gamma} \subseteq A$. The set of all β-γ-open sets of (X,τ) is denoted by $\beta O(X,\tau)_{\gamma}$".

In section 2.3 β-γ-regular space and β-regular operation using γ-operation are studied and its properties are analyzed.

The concept of closures operators namely β_{γ}-closure and βcl_{γ} are studied and their properties are analyzed in section 2.4.

In the final section of this chapter β-γ-generalized closed sets and its properties are studied.

In Chapter 3, On Some Separation Axioms via β-γ-open sets due to Alias B. Khalaf and Hariwan Z. Ibrahim are studied.

Section 3.2 deals with separation axioms using β-γ-open sets. In this section, seven new types of spaces namely β-γ-T_0, β-γ-$T_0{}'$, β-γ-T_1, β-γ-$T_1{}'$, β-γ-T_2, β-γ-$T_2{}'$ and β-γ-$T_{1/2}$ are studied and their properties are analyzed.

Section 3.3 deals with β-γ-continuous maps in topological spaces. In this section, we studied β-(γ, b)-continuous maps, β-(γ, b)-closed maps, β-(id, b)-closed maps and β-(γ, b)-homeomorphisms.

REVIEW OF LITERATURE

Review of Literature

Abd El-Monsef et al.[1] introduced the concept of β-open sets in topological spaces in the year 1983. In 1979, Kasahara [15] intiated the study of operation approach on topological spaces and studied the concept of α-continuous functions with α-closed graphs and α-compact spaces. Following his work, in 1983, Jankovic [7] introduced α-closure of a set in X via α-operations defined on open sets and studied characterizations of function with α-closed graph. In 1991, Ogata [6] renamed the α-operation as γ-operation and introduced γ-open sets in topological spaces and studied some topological properties related to it. Later, in the year 1992, Rehman and Ahmad [12] investigated the properties of operation on topological spaces with associated topologies.

Since then, several researchers developed many open sets via γ-operation defined on different open sets in topological spaces. Here, a brief survey of some of the articles published on these concepts are dealt.

1. $\boldsymbol{\beta}$-open sets and $\boldsymbol{\beta}$-continuous mappings,

Abd El-Monsef, M. E. El-Deeb, S. N. and Mahmoud, R. A. (1983) [1]

In this article, the authors have introduced a new class of sets called β-open sets which is contained in the class of semi-open and pre-open sets and studied β-continuity and β-open mappings in topological spaces and obtained the relationship of these notions with the existing ones.

2. Operation compact spaces,

Kasahara, S. (1979) [15]

In this article, the author has introduced operations on a topological space and studied the concept of α-closed graph of a function, which generalizes the concepts of closed, strongly-closed and almost-strongly-closed graph of a function, with the help of an operation of topology τ into the power set of X. By using the notion of functions with α-closed graphs, the author has unified several known characterizations of compact spaces, nearly-compact spaces and H-closed spaces.

3. On functions with $\boldsymbol{\alpha}$-closed graphs,

Jankovic, D. S. (1983) [7]

In this article, the author has investigated functions with α-closed graphs and functions with strongly-closed graphs. Also, he has introduced the notion of locally closed functions and studied some results related to locally closed functions and discussed about continuity of almost α-closed functions.

4. **Operations on topological spaces and associated topology,**
   Ogata, H. (1991) [6]

In this article, the author has introduced γ-open sets by using the operation γ on topological spaces and investigated the relation between the γ-closure and τ_γ-closure. Also, he has introduced separation axiom using the operation γ and investigated general operator approaches of closed graphs of mappings.

5. **Operation on Topological spaces-I,**
   Rehamn, F. U. and Ahmad, B. (1992) [12]

   In this article, the authors have introduced the concepts γ-interior, γ-boundary and γ-exterior points of a set in X and discussed some of its properties in topological spaces. They have also studied some properties of (γ, β)- continuous mappings.

6. **On operation approaches of α-open sets and semi open sets,**
   Haruo Maki, Mohammed A.M. Abdel Karim, Akihiro Nishi and Hayao Ogata.(2010) [5]

In this article, the authors have introduced the concept of operation-α-open sets, operation-semi-open sets by using operations from $SO(X, \tau)$ into $P(X)$ and investigated some of its properties. They have also introduced closure operators such as α-closures and semi-closures and studied its properties in topological spaces.

7. **Operation approaches on semi open sets and applications,**
   Krishnan, G. S. S. Ganster, M. and Balachandran, K. (2007) [13]

   In this article, the authors have introduced the class of semi γ-open sets and studied its properties in topological space. Also they have investigated the relation between the semi γ-closure and $SO(X, \tau)_\gamma$-closure and introduced separation axiom using the operation γ and defined a new class of map semi (γ, β)–continuous and studied some of its properties. They have also investigated general operation approaches of closed graphs of mappings.

8. **On operation-preopen sets in topological spaces,**
   Tran Van An, Cuong, D. X. and Maki, H. (2008) [17]

In this article, the authors have introduced new classes of open sets called γ_p- open sets and pre γ_p- open sets by using operation approach on pre-open sets in topological space and studied four kinds of closure operators. Also they have introduced pre γ_p-T_i spaces utilizing the operation γ on pre-open sets and studied some of its properties.

9. Operation approach to $\boldsymbol{\beta}$-open sets and applications,

Tahiliani, S. (2011) [14]

In this article, the author has introduced β-γ-open sets by defining the operation γ on $\beta O(X,\tau)$ and studied its properties in topological spaces. He has also introduced two kinds of closure operators namely β-γ-closure and $\beta O(X,\tau)_\gamma$-closure using γ operation and studied the relationship between them. He has also introduced separation axioms by the β-γ-open sets and β-γ-closed sets and studied its properties. He has defined a new class of map called β-(γ, b)-continuous map and investigated operation approach of β-closed graphs of mappings.

10. Operation approaches on *b*-open sets and applications,

Carpintero, C., Rajesh, N. and Rosas, E. (2012), [3]

In this article, the authors have introduced the concept of an operation γ on a family of b-open sets in a topological space. Using this operation γ, they introduced the concept of b-γ-open sets and studied some of their properties.

11. Operation via-regular open sets,

Carpintero, C., Rajesh, N. and Rosas, E. (2013) [4]

In this article the authors have introduced and studied the notion of γ-regular open sets by using the operation γ, on a topological space. They also introduced the almost (γ, β) continuous functions and investigated some of its important properties.

12. On some separation axioms via $\boldsymbol{\beta}$-$\boldsymbol{\gamma}$-open sets,

Khalaf, A. B. and Ibrahim, H. Z. (2013) [2]

In this article, the authors have introduced β-γ-g.closed sets by using β-γ-open sets and studied its properties in topological spaces. They have also introduced some new separation axioms and characterized some fundamental properties of those spaces.

CHAPTER 1

CHAPTER 1

Preliminaries

The basic definitions and results are presented in this chapter.

1.1 Topological Spaces

Definition 1.1.1

If A is a non-empty subset of (X, τ), then the union of all open sets contained in A is called interior of A and it is denoted by $int(A)$. The intersection of all closed sets containing A is called closure of A and it is denoted by $cl(A)$.

Definition 1.1.2 [16]

A subset A of (X, τ) is called regular open if $A = int(cl(A))$. The complement of regular open set is called regular closed. The intersection of all regular closed sets of X containing A is called regular closure of A and it is denoted by $rcl(A)$.

Definition 1.1.3 [9]

A subset A of (X, τ) is called semi-open if $A \subseteq cl(int(A))$. The complement of semi-open set is called semi-closed. The intersection of all semi-closed sets of X containing A is called semi-closure of A and it is denoted by $scl(A)$.

Definition 1.1.4 [10]

A subset A of (X, τ) is called pre-open if $A \subseteq int(cl(A))$. The complement of pre open set is called pre closed. The intersection of all pre closed sets of X containing A is called pre-closure of A and it is denoted by $pcl(A)$.

Definition 1.1.5 [1]

A subset A of (X, τ) is called β-open if $A \subseteq cl(int(cl(A)))$. The family of all β-open sets is denoted by $\beta O(X, \tau)$. The complement of β-open set is called β-closed. The intersection of all β-closed sets of X containing A is called β-closure of A and it is denoted by $\beta cl(A)$.

Definition 1.1.6

A topological space (X, τ) is said to be a T_0-space (or Kolmogorov) in which every pair of distinct point is topologically distinguishable. That is, for any two different points x and y there is an open set which contains one of these points and not the other.

Definition 1.1.7

A topological space (X, τ) is a T_1-space (or Frechet) if any two distinct points in X are separated.

Definition 1.1.8

A topological space (X, τ) is said to be a T_2-space (or Hausdorff) if the points x and y in a topological space X can be separated by neighbourhoods U of x and V of y such that U and V are disjoint ($U \cap V = \phi$).

Definition 1.1.9 [9]

A topological space (X, τ) is said to be a $T_{1/2}$-space, if every g-closed subset of (X, τ) is closed in (X, τ).

Definition 1.1.10 [15]

If (X, τ) is a topological space, an operation γ on the topology τ is a mapping from τ into the power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V.

Definition 1.1.11 [6]

A subset A of a topological space (X, τ) is called γ-open set if for each $x \in A$, there exists an open set U such that $x \in U$ and $U^\gamma \subset A$. τ_γ denotes the set of all γ-open sets.

Definition 1.1.12 [15]

A topological space (X, τ) is said to be γ-regular if for each $x \in X$ and for each open set V in X containing x, there exists an open set U in X containing x such that $U^\gamma \subseteq V$.

Definition 1.1.13 [15]

An operation γ on τ is said to be γ-regular operation, if for every open neighbourhood U and V of each $x \in X$, there exists an open neighbourhood W of x such that $U^\gamma \cap V^\gamma \supset W^\gamma$.

Definition 1.1.14 [6]

An operation γ is called open-γ-operation, if for every open neighbourhood U of each $x \in X$, there exists a γ-open set S such that $x \in S$ and $U^{\gamma} \supset S$.

Definition 1.1.15 [7]

A point $x \in X$ is in the γ-closure of a set $A \subset X$ if $U^{\gamma} \cap A \neq \phi$ for each open neighbourhood U of x. The γ-closure of a set A is denoted by $cl_{\gamma}(A)$.

Definition 1.1.16 [7]

A subset A of X is said to be γ-closed, if $cl_{\gamma}(A) \subseteq A$. For the family τ_{γ}, we define a set $\tau_{\gamma} - cl(A) = \bigcap\{F: F \supset A, x - F \in \tau_{\gamma}\}$.

Definition 1.1.17 [6]

A topological space (X, τ) is said to be a γ-T_0 space, if for each distinct points $x, y \in X$, there exists an open set U such that either $x \in U$ and $y \notin U^{\gamma}$, or $y \in U$ and $x \notin U^{\gamma}$.

Definition 1.1.18 [6]

A topological space (X, τ) is said to be a γ-T_1 space, if for each distinct points $x, y \in X$, there exists open sets U, V containing x and y respectively such that $y \notin U^{\gamma}$ and $x \notin V^{\gamma}$.

Definition 1.1.19 [6]

A topological space (X, τ) is said to be a γ-T_2 space, if for each distinct points $x, y \in X$, there exists open sets U, V such that $x \in U$, $y \in V$ and $U^{\gamma} \cap V^{\gamma} = \phi$.

Definition 1.1.20 [6]

A subset A of a topological space (X, τ) is said to be γ-$g$closed if $cl_{\gamma}(A) \subset U$ whenever $A \subset U$ and U is γ-open in (X, τ).

Definition 1.1.21 [6]

A space (X, τ) is called a γ-$T_{1/2}$ space if every γ-$g$closed set of (X, τ) is γ-closed.

Definition 1.1.22 [6]

A mapping $f:(X,\tau) \rightarrow (Y,\sigma)$ is said to be (γ,β)-continuous, if for each point x of X and each open set V containing $f(x)$ there exists an open set U such that $x \in U$ and $f(U^{\gamma}) \subseteq V^{\beta}$.

Definition 1.1.23 [6]

A mapping $f:(X,\tau) \rightarrow (Y,\sigma)$ is said to be (γ,β)-closed, if for any γ-closed set A of (X,τ), $f(A)$ is a β-closed set of (Y,σ).

Definition 1.1.24 [6]

A mapping $f:(X,\tau) \rightarrow (Y,\sigma)$ is said to be (γ,β)-homeomorphic, if f is bijective, (γ,β)-continuous and f^{-1} is (β,γ)-continuous.

Definition 1.1.25 [6]

A function $f:(X,\tau) \rightarrow (Y,\sigma)$ is called continuous if $f^{-1}(V)$ is a closed set in (X,τ) for every closed set V in (Y,σ).

Result 1.1.26

Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function. If A_1 and A_2 are subsets of X and Y respectively, then the following results are true.

(1) If $A_1 \subseteq A_2$ then $f(A_1) \subseteq f(A_2)$.

(2) If $A_1 \subseteq A_2$ then $f^{-1}(A_1) \subseteq f^{-1}(A_2)$.

(3) In general, $A \subseteq f^{-1}[f(A)]$. If f is injective then $A = f^{-1}[f(A)]$.

(4) In general, $f[f^{-1}(A)] \subseteq A$. If f is injective then $A = f[f^{-1}(A)]$.

(5) If f is surjective then $[f(A)]^c \subseteq f(A^c)$.

(6) If f is bijective then $[f(A)]^c = f(A^c)$.

CHAPTER 2

CHAPTER 2

Operation Approach on β-open sets

2.1 Introduction

The notion of β-open sets in topological space was introduced by Abd El-Monsef et al [1]. Kasahara [15] defined the concept of an operation on a topological space and introduced the concept of a α-closed graphs of functions in 1979. Following his work, Jankovic [7] developed the concept of α-closed graphs in 1983. Ogata [6] defined and investigated the concept of operation-open sets in topological spaces and operation-separation axioms of topological spaces.

In this chapter, the concept of β-γ-open sets using the operation γ on β-open sets in a topological space (X, τ) and some of its properties are studied. In section 2.3 β-γ regular space and β-regular operation are discussed. In section 2.4 the relationship between βcl_{γ} and $\beta_{\gamma} cl$ in topological space (X, τ) are analyzed and their properties are studied. In the final section β-γ-generalized closed sets are studied and its properties are investigated.

2.2 Operation approaches on β-open sets

Definition 2.2.1

An operation γ: $\beta O(X, \tau) \rightarrow P(X)$ where $P(X)$ is the set of all subsets of X is a mapping satisfying the condition, $V \subseteq V^{\gamma}$ for each $V \in \beta O(X, \tau)$. Here V^{γ} denotes the value of γ at V. We call the mapping γ an operation on $\beta O(X, \tau)$.

Definition 2.2.2

A subset A of X is called an β-γ-open set if for each point $x \in A$, there exists an β-open set U of X containing x such that $U^{\gamma} \subseteq A$. The set of all β-γ-open sets of (X, τ) is denoted by $\beta O(X, \tau)_{\gamma}$.

Example 2.2.3

Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$.

Then $\beta O(X, \tau) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.

Let γ be an operation on $\beta O(X,\tau)$ defined by

$$A^{\gamma} = \begin{cases} A & if\ b \in A \\ cl(A) & if\ b \notin A \end{cases}$$

Then $\beta O(X,\tau)_{\gamma} = \{\phi, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$.

Definition 2.2.4

A subset A of a topological space (X,τ) is called β-γ-closed whenever $X \setminus A$ is β-γ-open.

Proposition 2.2.5

Let γ be an operation on $\beta O(X,\tau)$. Then every β-γ-open set of (X,τ) is β-open in (X,τ), that is $\beta O(X,\tau)_{\gamma} \subseteq \beta O(X,\tau)$.

Proof :

Let $A \in \beta O(X,\tau)_{\gamma}$. Let $x \in A$, then by definition of β_{γ}-open set$A$, for each point $x \in A$, there exists an β-open set U of X containing x such that $U^{\gamma} \subseteq A$. $U \subseteq U^{\gamma}$ (by definition of operation γ). Hence $U \subseteq U^{\gamma} \subseteq A$. Again by definition of β-open set U, $U \subseteq cl(int(cl(U)))$. Since U contains x, $x \in cl(int(cl(U))) \subseteq cl(int(cl(A)))$. That is, $A \subseteq cl(int(cl(A)))$ implies A is β-open. That is $A \in \beta O(X,\tau)$.

Remark 2.2.6

The following example shows that β-open sets need not be β-γ-open.

Example 2.2.7

Let $X = \{a,b,c\}$ and $\tau = \{\phi, X, \{a\}, \{a,b\}\}$.

Let γ be the operation on $\beta O(X,\tau)$ such that

$$A^{\gamma} = \begin{cases} A & if\ b \in A \\ cl(A) & if\ b \notin A \end{cases}$$

Then $\{a\}$ is a β-open set but not a β-γ-open set.

Proposition 2.2.8

Let γ be an operation on $\beta O(X,\tau)$. Then every γ-open set of (X,τ) is β-γ-open.

Proof :

Let A be γ-open set of (X,τ) then for each $x \in A$ there exists an open set U such that $x \in U$ and $U^{\gamma} \subseteq$ A. By definition of operation γ, $U \subseteq U^{\gamma}$(since $U \in \beta O(X,\tau)$. That is $U \subseteq U^{\gamma} \subseteq$ A. Every open set is β-open. Hence A is β- γ-open (for if U is open then $U = int(U) \subseteq intcl(U) \subseteq cl(int(cl(U))$, that is U is β-open.

Remark 2.2.9

The following example shows that β-γ-open sets need not be γ -open sets

Example 2.2.10

Let $X = \{a,b,c\}$ and $\tau = \{\phi,\{a\},\{b\},\{a,b\},X\}$.

Then $\beta O(X,\tau) = \{\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},X\}$.

Let γ be an operation on $\beta O(X,\tau)$ defined by

$$A^{\gamma} = \begin{cases} A & if\ b \in A \\ cl(A) & if\ b \notin A \end{cases}$$

Then $\tau_{\gamma} = \{\phi,\{b\},\{a,b\},X\}$ and $\beta O(X,\tau)_{\gamma} = \{\phi,\{b\},\{a,b\},\{a,c\},\{b,c\},X\}$.

Here $\{a,c\}$ is β-γ-open but not γ -open.

Proposition 2.2.11

Let γ be an operation on $\beta O(X,\tau)$. Let $\{A_{\alpha}\}$ be a collection of β-γ-open sets in (X,τ). Then is $\bigcup_{\alpha \in J} A_{\alpha}$ also a β-γ-open set in (X,τ).

Proof:

Let $\{A_{\alpha}\}$ be a collection of β-γ-open sets in(X,τ). To prove that: $\bigcup_{\alpha \in J} A_{\alpha}$ is also a β-γ-open set in (X,τ). Let $x \in \bigcup_{\alpha \in J} A_{\alpha}$ then $x \in A_{\alpha}$ for some α ∈ J. Since A_{α} is β-γ-open, there exists a β-open set U containing x such that $U^{\gamma} \subseteq A_{\alpha} \subseteq \bigcup_{\alpha \in J} A_{\alpha}$. Therefore $\bigcup_{\alpha \in J} A_{\alpha}$ is a β- γ-open set.

Remark 2.2.12

Intersection of any two β-γ-open sets need not be β-γ-open as observed from the following example.

Example 2.2.13

Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$.

Then $\beta O(X, \tau) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.

Let γ be an operation on $\beta O(X, \tau)$ defined by

$$A^{\gamma} = \begin{cases} A & if\ b \in A \\ cl(A) & if\ b \notin A \end{cases}.$$

Then $\beta O(X, \tau)_{\gamma}$ are $\{\phi, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.

Here $\{a, b\}$ and $\{a, c\}$ are β-γ-open but their intersection $\{a, b\} \cap \{a, c\} = \{a\}$ is not β-γ- open.

2.3 $\boldsymbol{\beta}$-$\boldsymbol{\gamma}$-regular space and $\boldsymbol{\beta}$-regular operation

Definition 2.3.1

A space (X, τ) is said to be a β-γ-regular space if for each $x \in X$ and for each β-open set V containing x, there exists a β-open set U containing x such that $U^{\gamma} \subseteq V$.

Definition 2.3.2

Let γ be an operation on $\beta O(X, \tau)$. Then γ is said to be β-regular if for each $x \in X$ and for every pair of β-open sets U and V containing x, there exists a β-open set W such that $x \in W$ and $W^{\gamma} \subseteq U^{\gamma} \cap V^{\gamma}$.

Theorem 2.3.3

Let (X, τ) be a topological space and an operation on $\beta O(X, \tau)$. Then the following statements are equivalent:

(1) $\beta O(X, \tau) = \beta O(X, \tau)_{\gamma}$.

(2) (X, τ) is a β-γ-regular space.

(3) For every $x \in X$ and every β-open set V of (X, τ) containing x there exists a β-γ-open set W of (X, τ) such that $x \in W$ and $W \subseteq U$.

Proof:

(1) ⇒ (2)

Let $x \in X$ and let V be a β-open set containing x. Then by assumption, V is a β-γ-open set also. By definition of β-γ-open set, that for each $x \in V$, there exists a β-open set U such that $U^{\gamma} \subseteq V$. Therefore (X, τ) is a β-γ-regular space.

(2) ⇒ (3)

Let $x \in X$ and let U be a β-open set containing x. By (2), there is a β-open set W containing x and $W \subseteq W^{\gamma} \subseteq U$ which shows that W is β-γ-open set, satisfying the requirement of (3).

(3) ⇒ (1)

It is always true that β-γ-open set is always β-open. Conversely if U is a β-open set, then by hypothesis, for $x \in U$, there exists a β-γ-open set W of (X, τ) (hence β-open) such that $x \in W$ and $W \subseteq U \subseteq U^{\gamma}$ which shows that U is a β-γ-open set.

Theorem 2.3.4

Let γ be a β-regular operation on $\beta O(X, \tau)$. Then the following statements hold:

(i) If A and B are β-γ-open sets in (X, τ) then $A \cap B$ is also a β-γ-open set in (X, τ).

(ii) $\beta O(X, \tau)_{\gamma}$ forms a topology on X.

Proof:

Let γ be a β-regular operation on $\beta O(X, \tau)$.

(i). If A and B are β-γ-open sets in X, then $A \cap B$ is also an β-γ-open set. For, let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since A and B are β-γ-open sets, there exist β- open sets U and V such that $x \in U$ and $U^{\gamma} \subseteq A$, $x \in V$ and $V^{\gamma} \subseteq B$. Since γ is an β-regular operation, there exists an β-open set W of V such that $W^{\gamma} \subseteq U^{\gamma} \cap V^{\gamma} \subseteq A \cap B$. This implies that $A \cap B$ is β-γ-open set.

(ii)To prove : $\beta O(X,\tau)_\gamma$ forms a topology on X.

(1) ϕ and X are in $\beta O(X,\tau)_\gamma$.

(2) To show : Arbitrary union of β-γ-open sets is β-γ-open.

Let $\{A_\alpha\}$ be a collection of β-γ-open sets in (X,τ). To prove that: $\bigcup_{\alpha\in J} A_\alpha$ is also a β-γ-open set in (X,τ). Let $x \in \bigcup_{\alpha\in J} A_\alpha$ then $x \in A_\alpha$ for some $\alpha \in J$. Since A_α is β-γ-open, there exists a β-open set U containing x such that $U^\gamma \subseteq A_\alpha \subseteq \bigcup_{\alpha\in J} A_\alpha$. Therefore $\bigcup_{\alpha\in J} A_\alpha$ is a β-γ-open set. Hence Arbitrary union of β-γ-open sets is β-γ-open.

(3) To show : Finite intersection of β-γ-open sets is β-γ-open. From (i), If A and B are β-γ-open sets in (X,τ) then $A \cap B$ is β-γ-open. Therefore, finite intersection of β-γ-open sets is β-γ-open.
Then (1), (2) and (3) implies $\beta O(X,\tau)_\gamma$ forms a topology on X.

2.4 βcl_γ and β_γ- cl Operators

Definition 2.4.1

Let γ be an operation on $\beta O(X,\tau)$. The point $x \in X$ is said to be a β-γ-closure point of the set A if $U^\gamma \cap A \neq \phi$ for each β-open set U containing x. $\beta cl_\gamma(A)$ denotes the β-γ-closure of a set A.

Definition 2.4.2

Let γ be an operation on $\beta O(X,\tau)$. Then $\beta O(X,\tau)_\gamma$-$cl(A)$ is defined as the intersection of all β-γ-closed sets containing A.

Theorem 2.4.3

Let (X,τ) be a topological space and A a subset of X. Let γ be an operation on $\beta O(X,\tau)$. Then for each point $y \in X$, $y \in \beta O(X,\tau)_\gamma$-$cl(A)$ if and only if $V \cap A \neq \phi$ for every $V \in \beta O(X,\tau)_\gamma$-$cl(A)$ such that $y \in V$.

Proof:

Let $E = \{y \in X | V \cap A \neq \phi$ for every $V \in \beta O(X,\tau)_\gamma$ and $y \in V\}$. To prove the theorem, it is enough to show that $E = \beta O(X,\tau)_\gamma$-$cl(A)$. Let $x \notin E$. Then there exists a $V \in \beta O(X,\tau)_\gamma$ and $x \in V$ such that $V \cap A \neq \phi$. This implies that $X \backslash V$ is β-γ-closed and $A \subseteq X \backslash V$. Hence $\beta O(X,\tau)_\gamma$-$cl(A)$ (being the intersection of all such β-γ-closed) $\subseteq X \backslash V$. Thus it follows that $x \notin \beta O(X,\tau)_\gamma$-$cl(A)$. Thus $\beta O(X,\tau)_\gamma$-$cl(A) \subseteq E$. Conversely, let $x \notin \beta O(X,\tau)_\gamma$-$cl(A)$. Then there exists a β-γ-closed set F such that $A \subseteq F$ and $x \notin F$. Then we have $x \in X \backslash F$, $X \backslash F \in \beta O(X,\tau)_\gamma$ and $(X \backslash F) \cap A = \phi$. This implies that $x \notin E$.
Hence $E \subseteq \beta O(X,\tau)_\gamma$-$cl(A)$.

Theorem 2.4.4

Let (X,τ) be a topological space, A and B subsets of X and γ an operation on $\beta O(X,\tau)$. Then the following relation holds:

(i) The set β_γ- $cl(A)$ is β-γ-closed and $A \subseteq \beta_\gamma$- $cl(A)$.

(ii) A is β-γ-closed if and only if $A = \beta_\gamma$- $cl(A)$.

(iii) If $A \subseteq B$, then β_γ- $cl(A) \subseteq \beta_\gamma$- $cl(B)$.

(iv) β_γ- $cl(A) \cup \beta_\gamma$- $cl(B) \subseteq \beta_\gamma$- $cl(A \cup B)$.

(v) If γ is β-regular, then β_γ- $cl(A) \cup \beta_\gamma$- $cl(B) = \beta_\gamma$- $cl(A \cup B)$.

(vi) β_γ- $cl(A \cap B) \subseteq \beta_\gamma$- $cl(A) \cap \beta_\gamma$- $cl(B)$

(vii) β_γ- $cl(\beta_\gamma$- $cl(A)) = \beta_\gamma$- $cl(A)$.

Proof:

(i) By definition β_γ- $cl(A) =$ intersection of all β-γ-closed sets containing A, we have $A \subseteq \beta_\gamma$- $cl(A)$. Let $B = \beta_\gamma$- $cl(A)$ then $X \backslash B =$ Union of β-γ-open sets and hence by Proposition 2.2.11, $X \backslash B$ is β-γ-open and hence B is β-γ-closed set.

(ii) If A is β-γ-closed then β_γ- $cl(A)$ being the intersection of such closed sets, is contained in A. This together with (i) implies $A = \beta_\gamma$- $cl(A)$. Conversely by (i) β_γ- $cl(A)$ is a β-γ-closed set and so is A.

(iii) Let $x \in \beta_\gamma\text{-}\,cl(A)$ and let F be any β-γ-closed set containing B and hence A. Thus $x \in F$ and hence $x \in \beta_\gamma\text{-}\,cl(B)$.

(iv) By (iii) $\beta_\gamma\text{-}\,cl(A \cup B) \supseteq \beta_\gamma\text{-}\,cl(A)$ and $\beta_\gamma\text{-}\,cl(A \cup B) \supseteq \beta_\gamma\text{-}\,cl(B)$.
Therefore $\beta_\gamma\text{-}\,cl(A) \cup \beta_\gamma\text{-}\,cl(B) \subseteq \beta_\gamma\text{-}\,cl(A \cup B)$.

(v) Let $x \notin \beta_\gamma\text{-}\,cl(A) \cup \beta_\gamma\text{-}\,cl(B)$. Then by Theorem 2.4.3, there exists two β-γ-open sets U and V containing x such that $U \cap A = \phi$ and $V \cap B = \phi$. By Theorem 2.3.4 (i), $U \cap A$ is β-γ-open in (X, τ) containing x such that $(U \cap V) \cap (A \cup B) = \phi$. Thus we have and $x \notin \beta_\gamma\text{-}\,cl(A \cup B)$. Hence $\beta_\gamma\text{-}\,cl(A \cup B) \subseteq \beta_\gamma\text{-}\,cl(A) \cup \beta_\gamma\text{-}\,cl(B)$. Using (iv), we have the equality.

(vi) As $(A \cap B) \subseteq A$ implies $\beta_\gamma\text{-}\,cl(A \cap B) \subseteq \beta_\gamma\text{-}\,cl(A)$ [by(iii)] and
$(A \cap B) \subseteq B$ implies $\beta_\gamma\text{-}\,cl(A \cap B) \subseteq \beta_\gamma\text{-}\,cl(B)$ [by(iii)].
Therefore $\beta_\gamma\text{-}\,cl(A \cap B) \subseteq \beta_\gamma\text{-}\,cl(A) \cap \beta_\gamma\text{-}\,cl(B)$

(vii) From (i) we get $\beta_\gamma\text{-}\,cl(A) \subseteq \beta_\gamma\text{-}\,cl(\beta_\gamma\text{-}\,cl(A))$. To prove the reverse inequality, let $x \in \beta_\gamma\text{-}\,cl(\beta_\gamma\text{-}\,cl(A))$ and V be any β-γ-open set containing x. We claim that $V \cap A \neq \phi$. Indeed, by Theorem 2.4.3, $V \cap \beta_\gamma\text{-}\,cl(A)$ and so there exists a point z such that $z \in V$ and $z \in \beta_\gamma\text{-}\,cl(A)$. Moreover, by Theorem 2.4.3, for V containing the point z, $V \cap A \neq \phi$. Thus, we have that for any point $x \in V$, $V \cap A \neq \phi$ and so $x \in \beta_\gamma\text{-}\,cl(A)$. Hence we conclude that $\beta_\gamma\text{-}\,cl(\beta_\gamma\text{-}\,cl(A)) \subseteq \beta_\gamma\text{-}\,cl(A)$. Hence we have $\beta_\gamma\text{-}\,cl(\beta_\gamma\text{-}\,cl(A)) = \beta_\gamma\text{-}\,cl(A)$.

Theorem 2.4.5

Let $\gamma : \beta O(X, \tau) \to P(X)$ be an operation on $\beta O(X, \tau)$ and A and B subsets of X. Then the following relations hold:

(i) $\beta cl_\gamma(A)$is a β-closed set in (X, τ) and $A \subseteq \beta cl_\gamma(A)$.

(ii) A is β-γ-closed in (X, τ) if and only if $A = \beta cl_\gamma(A)$ holds.

(iii) If (X, τ) is β-γ-regular, then $\beta cl_\gamma(A) = \beta cl(A)$.

(iv) If $A \subseteq B$, then $\beta cl_\gamma(A) \subseteq \beta cl_\gamma(B)$.

(v) $\beta cl_\gamma(A) \cup \beta cl_\gamma(B) \subseteq \beta cl_\gamma(A \cup B)$ holds for any subsets A and B of X.

(vi) Let γ be a β-regular operation on $\beta O(X,\tau)$ then $\beta cl_\gamma(A\cup B)=\beta cl_\gamma(A)\cup\beta cl_\gamma(B)$ holds for any subsets A and B of X.

(vii) $\beta cl_\gamma(A\cap B)\subseteq\beta cl_\gamma(A)\cap\beta cl_\gamma(B)$ holds.

(viii) If γ is β-open, then $\beta cl_\gamma(A)=\beta_\gamma\text{-}\,cl(A)$ and $\beta cl_\gamma(\beta cl_\gamma(A))=\beta cl_\gamma(A)$.

Proof:

(i) Let $x\in\beta cl(\beta cl_\gamma(A))$. Then $U\cap\beta cl_\gamma(A)\neq\phi$ for every β-open set U containing x. Let $y\in U\cap\beta cl_\gamma(A)$. Then $y\in U$ and $y\in\ \beta cl_\gamma(A)$. Since, U is a β-open set containing y, this implies $U^\gamma\cap A\neq\phi$. And it is true for every β-open set U. Thus, $x\in\beta cl_\gamma(A)$. Hence $\beta cl(\beta cl_\gamma(A))\subseteq\beta cl_\gamma(A)$. This implies that $\beta cl_\gamma(A)$ is a β-closed set. Also, $A\subseteq\beta cl_\gamma(A)$ is clear, for let $x\in A$ and U be an β-open set containing x, if $U^\gamma\cap \mathrm{A}=\phi$, $U\cap A=\phi$, then $A\subseteq X\backslash U$, that is $x\in X\backslash U$, a contradiction.

(ii) Suppose that A is β-γ-closed in (X,τ) then $(X\setminus A)$ is β-γ-open in (X,τ). By (i), It is enough to prove that $\beta cl_\gamma(A)\subseteq A$. If $x\notin A$. Then by definition of β-γ-open, there exists a β-open set U containing x such that $U^\gamma\subseteq X\backslash A$, that is, $U^\gamma\cap A=\phi$. Hence using Definition 2.4.1, we have that $x\notin\beta cl_\gamma(A)$ and so $\beta cl_\gamma(A)\subseteq A$. Hence $A=\beta cl_\gamma(A)$. Conversely, Suppose that $A=\beta cl_\gamma(A)$, then to prove A is β-γ-closed or $X\backslash A$ is β-γ-open. Let $x\ \in\ X\backslash A$, then $x\notin\beta cl_\gamma(A)$. That is there exists a β-open set U containing x such that $U^\gamma\cap \mathrm{A}=\phi$. That is, $U^\gamma\subseteq X\setminus A$, namely $X\backslash A$ is β-γ-open in (X,τ) and so A is β-γ-closed.

(iii) By Definition 2.4.1, we have $\beta cl(A)\subseteq\beta cl_\gamma(A)$. It is enough to prove that $\beta cl_\gamma(A)\subseteq\beta cl(A)$. Let $x\notin\beta cl(A)$. Then, there exists a β-open set U containing x such that $U\cap A=\phi$. Using β-γ-regularity of (X,τ), there exist a β-open set V containing x such that $V^\gamma\subseteq U$ and hence $V^\gamma\cap A=\phi$. Thus we have that $x\notin\beta cl_\gamma(A)$. Therefore, $\beta cl_\gamma(A)\subseteq\beta cl(A)$.

(iv) It is obvious by Definition 2.4.1.

(v) It is obvious that both A and B are contained in $A\cup B$. Thus the result follows from (iv).

(vi) It is enough to show that $\beta cl_\gamma(A\cup B)\subseteq\beta cl_\gamma(A)\cup\beta cl_\gamma(B)$. Let $x\notin\beta cl_\gamma(A)\cup\beta cl_\gamma(B)$. Then, there exist β-open sets U and V such that $x\in U, x\in V$, $U^\gamma\cap A=\ \phi$ and $V^\gamma\cap B=\ \phi$. Since γ is β-regular, by Definition 2.3.2, for the above pair U and V there exists a β-open set W containing x such that $W^\gamma\subseteq U^\gamma\cap V^\gamma$.

Thus we have $W^{\gamma} \cap (A \cup B) \subseteq (U^{\gamma} \cap V^{\gamma}) \cap (A \cup B) \subseteq (U^{\gamma} \cap A) \cup (V^{\gamma} \cap B) = \phi$, that is, $W^{\gamma} \cap (\mathrm{A} \cup \mathrm{B}) = \phi$. Hence, $x \notin \beta cl_{\gamma}(A \cup B)$and so $\beta cl_{\gamma}(A \cup B) \subseteq \beta cl_{\gamma}(A) \cup \beta cl_{\gamma}(B)$.

(vii) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Thus from (iv) the result follows.

(viii) $\beta_{\gamma}\text{-}\,cl(A)$ is the intersection of all β-γ-closed set containing A. We know that, we have $\beta cl_{\gamma}(A) \subseteq \beta_{\gamma}\text{-}\,cl(A)$. Now we prove that $\beta_{\gamma}\text{-}\,cl(A) \subseteq \beta cl_{\gamma}(A)$. Let $x \notin \beta cl_{\gamma}(A)$. Then, there exists a β-open set U containing x such that $U^{\gamma} \cap A = \phi$. Since γ is β-open, there exists a β-γ-open set S such that $x \in S \subseteq U^{\gamma}$. Therefore $S \cap A = \phi$. This implies that $x \notin \beta_{\gamma}\text{-}\,cl(A)$. Therefore $\beta_{\gamma}\text{-}\,cl(A) = \beta cl_{\gamma}(A)$. By Theorem 2.4.4 (vii), $\beta_{\gamma}\text{-}\,cl(\beta_{\gamma}\text{-}\,cl(A)) = \beta_{\gamma}\text{-}\,cl(A)$. Hence we have that $\beta cl_{\gamma}(\beta cl_{\gamma}(A)) = \beta cl_{\gamma}(A)$.

Remark 2.4.6

If γ is not β-regular, then in Theorem 2.4.5(vi) the equality does not hold good as observed from the following example.

Example 2.4.7

Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$.

Then $\beta O(X, \tau) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.

Let $\gamma : \beta O(X, \tau) \rightarrow P(X)$ be an operation defined by $A^{\gamma} = cl(A)$ for any $A \in \beta O(X, \tau)$.

Then $\beta O(X, \tau)_{\gamma} = \{\phi, \{a, c\}, \{b, c\}, X\}$.

Let $A = \{a\}$ and $B = \{b\}$. Then $\beta cl_{\gamma}(A \cup B) = X$ and $\beta cl_{\gamma}(A) = \{a\}$ and $\beta cl_{\gamma}(B) = \{b\}$.

The operation γ is not β-regular.

Theorem 2.4.8

For any subset A of a topological space (X, τ) and any operation $\gamma : \beta O(X, \tau) \rightarrow P(X)$, the following inclusions hold.

(i) $\beta cl(A) \subseteq \beta cl_{\gamma}(A) \subseteq \beta_{\gamma}\text{-}\,cl(A) \subseteq \tau_{\gamma}\text{-}\,cl(A)$.

(ii) $\beta cl(A) \subseteq cl(A) \subseteq cl_{\gamma}(A) \subseteq \tau_{\gamma}\text{-}\,cl(A)$.

Proof:

(i) Let $x \in \beta cl(A)$, then $x \in$ every β-closed sets containing A. Let U be an β-open set containing x, then to prove that $U^{\gamma} \cap A \neq \phi$. If not let, $U^{\gamma} \cap A = \phi$ then $U \cap A = \phi$. That is $A \subseteq X \backslash U$ where, $X \backslash U$ is a β-closed set containing A but not containing x, a contradiction. $\beta cl(A) \subseteq \beta cl_{\gamma}(A)$. Let $x \in \beta cl_{\gamma}(A)$. Then, for every β-open sets U containing x, $U^{\gamma} \cap A \neq \phi$. Let F be a β-γ-closed set containing A, then to prove that $x \in F$. If not let $x \notin F$, then $x \in (X \backslash F)$,which is a β-γ-open set (and hence β-open). Therefore there exists a β-open set V containing x such that $V^{\gamma} \subseteq X \backslash F$ implies $V \subseteq V^{\gamma} \subseteq X \backslash F$. But $X \backslash F$ is disjoint from A and hence V is disjoint from A. But every β-open set containing x has some intersection with A, a contradiction. $\beta cl_{\gamma}(A) \subseteq \beta_{\gamma}\text{-} cl(A)$. Let $x \in \beta_{\gamma}\text{-} cl(A)$, then to Prove That $x \in \tau_{\gamma}\text{-} cl(A)$. Let F be γ-closed set containing A, then, F is also a β-γ-closed set containing A and Hence$x \in F$. $\beta_{\gamma}\text{-}cl(A) \subseteq \tau_{\gamma}\text{-} cl(A)$.

(ii) The implication $\beta cl(A) \subseteq cl(A)$ is trivial. Since every closed set is β-closed. Let $x \in cl(A)$ and let U be an open set of X containing x, such that, $U^{\gamma} \cap A = \phi$. Then $U \cap A = \phi$. This implies $A \subseteq (X \backslash U)$, where $X \backslash U$ is a closed set containing A must contain x. (Since, x belongs to every closed set containing A), which is a contradiction. This implies $x \in cl_{\gamma}(A)$. That is $cl(A) \subseteq cl_{\gamma}(A)$ by Definition of γ-closed set. Next to prove $cl_{\gamma}(A) \subseteq \tau_{\gamma}\text{-} cl(A)$. Let $x \in cl_{\gamma}(A)$. Let F be a γ-closed set containing A, then F is a β-γ-closed set containing A (Theorem 2.1(ii)). That is $x \in \beta_{\gamma}\text{-}cl(A) \subseteq \tau_{\gamma}\text{-} cl(A)$ (by part (i)) $x \in U$, where $U = X \backslash F$ is a γ-open set and disjoint from A.

Theorem 2.4.9

Let V be a topological space A, a subset of X and γ an operation on $\beta O(X, \tau)$. Then the following are equivalent.

(1) A is β-γ-open.

(2) $\beta cl_{\gamma}(X \backslash A) = X \backslash A$.

(3) $\beta_{\gamma}\text{-}cl(X \backslash A) = X \backslash A$.

(4) $X \backslash A$ is β-γ-closed.

Proof:

A is β-γ-open if and only if $X\setminus A$ is β-γ-closed. That is (1) and (4) are equivalent. $X\setminus A$ is β-γ-closed if and only if $\beta cl_{\gamma}(X\setminus A) = X\setminus A$,by Theorem 2.4.4 (ii). Thus (4) and (2) are equivalent. Theorem 2.4.4 (ii), $X\setminus A$ is β-γ-closed iff β_{γ}-$cl(X\setminus A) = X\setminus A$. Hence (4) and (3) are equivalent.

2.5 $\boldsymbol{\beta}$-$\boldsymbol{\gamma}$-generalized closed sets

In this section, the concept of β-γ-generalized closed sets and some properties of this set are analyzed.

Definition 2.5.1

A subset A of the space (X,τ) is said to be β-γ-generalized closed (Briefly. β-γ-g.closed) if $\beta O(X,\tau)_{\gamma}$-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a β-γ-open set in (X,τ). The complement of a β-γ-g.closed set is called a β-γ-g.open set.

Proposition 2.5.2

Every β-γ-closed subset of (X,τ) is also a β-γ-g.closed set.

Proof :

Let A be a β-γ-closed subset of (X,τ) and U be a β-γ-open set containing A. Since A is β-γ-closed, $\beta O(X,\tau)_{\gamma}$-$cl(A) = A$, $\beta O(X,\tau)_{\gamma}$-$cl(A) = A \subseteq U$. Therefore A is β-γ-g.closed.

Remark 2.5.3

The following example shows that a β-γ-g.closed set need not be β-γ-closed.

Example 2.5.4

Consider $X = \{a,b,c\}$ with the topology $\tau = \{\phi,\{a\},\{b\},\{a,b\},\{a,c\},X\}$.

Define an operation γ on $\beta O(X,\tau)$ by

$$A^{\gamma} = \begin{cases} A & if\ A = \{b\} or\ \{a,c\} \\ X & otherwise \end{cases}$$

If we let $A = \{a\}$, since the only β-γ-open supersets of A are $\{a,c\}$ and X, then A is β-γ-g.closed set need not be β-γ-closed.

Theorem 2.5.5

Let (X,τ) be a topological space and γ be an operation on $\beta O(X,\tau)$. Then the following statements are equivalent:

(1) A is β-γ-g.closed in (X,τ).

(2) β_γ-$cl(x) \cap A \neq \phi$ for every $x \in \beta cl_\gamma(A)$.

(3) $\beta cl_\gamma(A) \subseteq \beta_\gamma$-ker$(A)$ holds, where β_γ-ker$(E) = \cap\{V \mid E \subseteq V, V \in \beta O(X,\tau)_\gamma$ for any subset E of (X,τ). That is equal to the intersection of all β-γ-open sets V containing E.

Proof:

(1)⇒(2)

Let A be a β-γ-g.closed set of (X,τ). If, there exists an $x \in \beta cl_\gamma(A)$ such that $(\beta_\gamma$-$cl(x)) \cap A = \phi$, then $A \subseteq X\backslash(\beta_\gamma$-$cl(x))$. A is β-γ-g.closed implies, every $\beta cl_\gamma(A)$ contained in β-γ-open set containing A. Since, β_γ-$cl(x)$ is β-γ-closed, (by Theorem 2.4.4.(i)),we have $\beta cl_\gamma(A) \subseteq X\backslash$ $(\beta_\gamma$-$cl(x))$ as, $X\backslash(\beta_\gamma$-$cl(x))$ is a β-γ-open set containing A. This implies $x \notin \beta cl_\gamma(A)$ which is a contradiction. Therefore, $(\beta_\gamma$-$cl(x)) \cap A \neq \phi$.

(2)⇒(3)

Let $x \in \beta cl_\gamma(A)$. By (2), $(\beta_\gamma$-$cl(x)) \cap A \neq \phi$. Then there exists $z \in (\beta_\gamma$-$cl(x))$ and $z \in A$. If U be any β-γ-open set containing A, then $z \in U$and $z \in (\beta_\gamma$-$cl(x))$. Now we claim $x \in U$, for if $x \notin U$, then $F = X - U$ is a β-γ-closed set containing x, will contain z which is a contradiction to $z \in U$. Thus have $x \in U$, for every a β-γ-open set U containing A. Hence $x \in \beta O(X,\tau)_\gamma$-Ker$(A)$. Therefore $\beta cl_\gamma(A) \subseteq \beta O(X,\tau)_\gamma$-Ker$(A)$.

(3) ⇒ (1)

Let U be any β-γ-open set containing A and $x \in \beta cl_\gamma(A)$. By (3), $x \in \beta cl_\gamma$-Ker(A) which is the intersection of all β-γ-open set containing A. Therefore $x \in U$. That is $\beta cl_\gamma(A) \subseteq U$ implies A is β-γ-g.closed in (X,τ).

Theorem 2.5.6

A subset A of a space X is β-γ-g.closed if and only if $\beta O(X)_{\gamma}$- $cl(A)\backslash A$ does not contain any non-empty β-γ-closed set.

Proof:

Necessity :

Suppose that A is β-γ-g.closed set in X. We prove the result by contradiction. Let F be a β-γ-closed set such that $F \subseteq \beta O(X,\tau)_{\gamma}$- $cl(A)\backslash A$ and $F \neq \phi$. Then $F \subseteq X\backslash A$ which implies $A \subseteq X \setminus F$. Since A is β-γ-g.closed and $X\backslash F$ is β-γ-open, therefore $\beta O(X,\tau)_{\gamma}$-$cl(A) \subseteq X \setminus F$, that is $F \subseteq X \setminus \beta O(X,\tau)_{\gamma}$-$cl(A)$.
Hence $F \subseteq \beta O(X,\tau)_{\gamma}$-$cl(A) \cap (X \setminus \beta O(X,\tau)_{\gamma}$-$cl(A)) = \phi$. This shows that, $F = \phi$ which is a contradiction. Hence $\beta O(X,\tau)_{\gamma}$-$cl(A)\backslash A$ does not contains any non-empty β-γ-closed set in X.

Sufficiency:

Let $A \subseteq U$, where U is β-γ-open in (X,τ). If $\beta O(X,\tau)_{\gamma}$-$cl(A)$ is not contained in U, then $\beta O(X,\tau)_{\gamma}$-$cl(A) \cap (X\backslash U) \neq \phi$. Now, since $\beta O(X,\tau)_{\gamma}$-cl$(A) \cap (X\backslash U) \subseteq \beta O(X,\tau)_{\gamma}$-$cl(A)\backslash A$ and $\beta O(X,\tau)_{\gamma}$-$cl(A) \cap (X\backslash U)$ is a non-empty β-γ-closed set, then we obtain a contradiction and therefore A is β-γ-g.closed.

Corollary 2.5.7

If a subset A of X is β-γ-g.closed set in X, then $\beta O(X,\tau)_{\gamma}$-$cl(A)\backslash A$ does not contain any non-empty γ-closed set in X.

Proof:

Follows from the fact that every γ-open set is β-γ-open.

Remark 2.5.8

The converse of the above corollary is not true in general as it seen from the following example.

Example 2.5.9

Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{c\}, X\}$. Define an operation γ on $\beta O(X, \tau)$ by $\gamma(A) = A$. If we let $A = \{a, c\}$ then A is not β-γ-g.closed, since $A \subseteq \{a, c\} \in \beta O(X, \tau)_\gamma$ and $\beta O(X, \tau)_\gamma$-$cl(A) = X \nsubseteq \{a, c\}$, where $\beta O(X, \tau)_\gamma$-$cl(A) \backslash A = \{b\}$ does not contain any non-empty γ-closed set in X.

Theorem 2.5.10

If A is a β-γ-g.closed set of a space(X, τ), then the following are equivalent:

1. A is β-γ-closed.

2. $\beta O(X, \tau)_\gamma$-$cl(A) \backslash A$ is β-γ-closed.

Proof:

(1) ⇒ (2)

If A is a β-γ-g.closed set which is also β-γ-closed, then by Theorem 2.5.6, $\beta O(X, \tau)_\gamma$-$cl(A) \backslash A = \phi$ which is β-γ-closed.

(2) ⇒ (1)

Let $\beta O(X, \tau)_\gamma$-$cl(A) \backslash A$ be β-γ-closed set and A be β-γ-g.closed. Then by Theorem 3.4, $\beta O(X, \tau)_\gamma$-$cl(A) \backslash A$ does not contain any non-empty β-γ-closed subset. Since $\beta O(X, \tau)_\gamma$-$cl(A) \backslash A$ is β-γ-closed and $\beta O(X, \tau)_\gamma$-$cl(A) \backslash A = \phi$, this shows that A is β-γ-closed.

Theorem 2.5.11

For a space (X, τ), the following are equivalent:

1. Every subset of X is β-γ-g.closed.

2. $\beta O(X, \tau)_\gamma = \beta C(X, \tau)_\gamma$.

Proof:

(1) ⇒ (2)

Let $U \in \beta O(X,\tau)_\gamma$. Then by hypothesis, U is β-γ-g.closed which implies that $\beta O(X,\tau)_\gamma$-$cl(U) \subseteq U$, so, $\beta O(X,\tau)_\gamma$-$cl(U) = U$, therefore $U \in \beta C(X,\tau)_\gamma$. Also let $V \in \beta C(X,\tau)_\gamma$. Then $X \setminus V \in \beta O(X,\tau)_\gamma$, hence by hypothesis $X \setminus V$ is β-γ-g.closed and then $X \setminus V \in \beta O(X,\tau)_\gamma$, thus $V \in \beta C(X,\tau)_\gamma$ according above we have $\beta O(X,\tau)_\gamma = \beta C(X,\tau)_\gamma$.

(2) ⇒ (1)

If A is a subset of a space X such that $A \subseteq U$ where $U \in \beta O(X,\tau)_\gamma$, then $U \in \beta C(X,\tau)_\gamma$ and therefore $\beta O(X,\tau)_\gamma$-$cl(U) \subseteq U$ which shows that A is β-γ-g.closed.

Proposition 2.5.12

If A is γ-open and β-γ-g.closed then A is β-γ-closed.

Proof:

Suppose that A is γ-open and β-γ-g.closed. As every γ-open is β-γ-open and $A \subseteq A$, we have $\beta O(X,\tau)_\gamma$-$cl(A) \subseteq A$, also $A \subseteq \beta O(X,\tau)_\gamma$-$cl(A)$, therefore $\beta O(X,\tau)_\gamma$-$cl(A) = A$. That is A is β-γ-closed.

Theorem 2.5.13

If a subset A of X is β-γ-g.closed and $A \subseteq B \subseteq \beta O(X,\tau)_\gamma$-$cl(A)$, then B is a β-γ-g.closed set in X.

Proof:

Let A be β-γ-g.closed set such that $A \subseteq B \subseteq \beta O(X,\tau)_\gamma$-$cl(A)$. Let U be a β-γ-open set of X such that $B \subseteq U$. Since A is β-γ-g.closed, we have $\beta O(X,\tau)_\gamma$-$cl(A) \subseteq U$. Now $\beta O(X,\tau)_\gamma$-$cl(A \subseteq \beta O(X,\tau)_\gamma$-$cl(B) \subseteq \beta O(X,\tau)_\gamma$-$cl[\beta O(X,\tau)_\gamma$-$cl(A)] = \beta O(X,\tau)_\gamma$-$cl(A) \subseteq U$. That is $\beta O(X,\tau)_\gamma$-$cl(B) \subseteq U$, where U is β-γ-open. Therefore B is a β-γ-g.closed set in X.

Remark 2.5.14

The converse of the above theorem is not true in general as it is seen from the following example.

Example 2.5.15

Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Define an operation γ on $\beta O(X, \tau)$ by $\gamma(A) = A$. Let $A = \{b\}$ and $B = \{b, c\}$. Then A and B are β-γ-g.closed sets in (X, τ). But $A \subseteq B \nsubseteq \beta O(X, \tau)_\gamma$-$cl(A)$.

Proposition 2.5.16

Let γ be an operation on $\beta O(X, \tau)$. Then for each $x \in X$, $\{x\}$ is β-γ-closed or $X \backslash \{x\}$ is β-γ-g.closed in (X, τ).

Proof:

Suppose that $\{x\}$ is not β-γ-closed, then $X \backslash \{x\}$ is not β-γ-open. Let U be any β-γ-open set such that $X \backslash \{x\} \subseteq U$, implies $U = X$. Therefore $\beta O(X, \tau)_\gamma$-$cl(X \backslash \{x\}) \subseteq U$. Hence $X \backslash \{x\}$ is β-γ-g.closed.

Theorem 2.5.17

Let (X, τ) be a topological space and γ an operation on $\beta O(X, \tau)$. If a subset A of X is β-γ-g.closed, then $(\beta cl_\gamma(A) \backslash A)$ does not contain any non-empty β-γ-closed set.

Proof:

Suppose that there exists a non-empty β-γ-closed set F such that $F \subseteq (\beta cl_\gamma(A) \backslash A)$. Then we have $A \subseteq X \setminus F$ and $X \setminus F$ is β-γ-open. It follows from the definition of β-γ-g.closed, $\beta cl_\gamma(A) \subseteq X \setminus F$, that is $F \subseteq (X \setminus \beta cl_\gamma(A))$ and so $F \subseteq \left(\beta cl_\gamma(A) \backslash A\right) \cap (X \setminus \beta cl_\gamma(A))$. Therefore, we have $F = \phi$.

Result 2.5.18

If γ is β-open, then the converse of the above theorem is true.

Proof:

Assume that $\left(\beta cl_{\gamma}(A)\backslash A\right)$ does not contain any non-empty β-γ-closed set. If γ is a β-open operation, then by Theorem 2.4.5(viii) , $\beta cl_{\gamma}(A) = \beta_{\gamma}$-$cl(A)$. If A is not β-γ-g.closed, then there exists a β-γ-open set $U \supseteq A$ with $\beta cl_{\gamma}(A) \not\subset$ U, that is β_{γ}-$cl(A) \not\subset$ U. That is β_{γ}-$cl(A) \cap (X \setminus U) \neq \phi$, and also β_{γ}-$cl(A) \cap (X \setminus U) \subseteq \beta_{\gamma}$-$cl(A) \cap (X \setminus A)$. That is there exists a non-empty β-γ-closed set $\subseteq \beta_{\gamma}$-$cl(A) \cap (X \setminus A)$ contradicting the hypothesis. That is β_{γ}-$cl(A) \subseteq U$ for every a β-γ-open set $U \supseteq A$.

Theorem 2.5.19

Let (X, τ)be a topological space and γ an operation on $\beta O(X, \tau)$.

Then for each $x \in X$, $\{x\}$ is β-γ-closed or $X \backslash\{x\}$ is β-γ-g.closed in (X, τ).

Proof:

Suppose that $\{x\}$ is not β-γ-closed, then $X \backslash\{x\}$ is not β-γ-open. Let U be any β-γ-open set such that $X \backslash\{x\} \subseteq U$. Then $U = X$. Hence, $\beta cl_{\gamma}(X \backslash\{x\}) \subseteq U(= X)$. Therefore, $X \backslash\{x\}$ is a β-γ-g.closed set.

Definition 2.5.20

Let A be a subset of a topological space (X, τ) and γ an operation on $\beta O(X, \tau)$. The union of all β-γ-open sets contained in A is called the β-γ-interior of A and denoted by $\beta O(X, \tau)_{\gamma}$-$int(A)$.

Proposition 2.5.21

A subset A of X is β-γ-g.open if and only if $F \subseteq \beta O(X, \tau)_{\gamma}$-$int(A)$ whenever $F \subseteq A$ and F is β-γ-closed in (X, τ).

CHAPTER 3

CHAPTER 3

Separation Axioms on β-γ-open Sets

3.1 Introduction

Separation axioms are one among the most common, important and interesting concepts. They can be used to define more restricted class of topological spaces. Replacing the sets being separated or doing separation in the separation axioms by different types of sets, several extensions of separation axioms have been introduced by mathematicians from time to time. In this chapter, we studied seven new spaces and its properties. Also we analyzed β-(γ, b)-continuous maps, β-(γ, b)-closed maps, β-(id, b)-closed maps and β-(γ, b)-homeomorphisms.

3.2 β-γ-T_i spaces and β-γ-$T_i{}'$ spaces ($i = 0, \frac{1}{2}, 1, 2$)

Definition 3.2.1

A space (X, τ) is called β-γ-T_0 if for any two distinct points $x, y \in X$, there exists a β-open set U such that either $x \in U$ and $y \notin U^{\gamma}$ or $y \in U$ and $x \notin U^{\gamma}$.

Example 3.2.2

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$.

Then $\beta O(X, \tau\ ) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$.

Define the operation γ on $\beta O(X, \tau)$ as

$$A^{\gamma} = \begin{cases} A & if\ b \in A \\ A \cup \{b\} & if\ b \notin A \end{cases}.$$

Then $\beta O(X, \tau\ )_{\gamma} = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}\}$.

For $a, b \in X$, there exists a β-open set $\{b\}$ such that $b \in \{b\}$ and $a \notin \{b\}^{\gamma} = \{b\}$.

Hence the space (X, τ) is a β-γ-T_0 space.

Definition 3.2.3

A space (X,τ) is called a β-γ-$T_0{}'$ if for any two distinct points $x, y \in X$, there exists β-γ-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

Example 3.2.4

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$.

Then $\beta O(X,\tau) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$.

Define the operation γ on $\beta O(X,\tau)$ as

$$A^{\gamma} = \begin{cases} A & if\ b \in A \\ cl(A) & if\ b \notin A \end{cases}$$

Then $\beta O(X,\tau)_{\gamma} = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$.

For $a, b \in X$, there exists a β-γ-open set $\{a, c\}$ such that $a \in \{a, c\}$ and $b \notin \{a, c\}$.

Hence the space (X,τ) is β-γ-$T_0{}'$ space.

Definition 3.2.5

A space (X,τ) is called β-γ-T_1 if for any two distinct points and $x, y \in X$, there exist two β-open sets U and V containing x and y, respectively, such that $y \notin U^{\gamma}$ and $x \notin V^{\gamma}$.

Example 3.2.6

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$.

Then $\beta O(X,\tau\ ) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Define the operation γ on $\beta O(X,\tau\ )$ as

$$A^{\gamma} = \begin{cases} A & if\ b \in A \\ \{a, c\} & if\ b \notin A \end{cases}$$

Then $\beta O(X,\tau)_{\gamma} = \{X, \phi, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

For $a, b \in X$, there exist β-open sets $\{a\}$ containing a and $\{b\}$ containing b such that

$\{b\} \notin \{a\}^{\gamma} = \{a, c\}$ and $\{a\} \notin \{b\}^{\gamma} = \{b\}$. Hence the space (X,τ) is a β-γ-T_1 space.

Definition 3.2.7

A space is (X, τ) called β-γ-$T_1{}'$ if for any two distinct points $x, y \in X$, there exists two β-γ-open sets U and V containing x and y respectively such that $y \notin U$ and $x \notin V$.

Example 3.2.8

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$.

Then $\beta O(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Define the operation γ on $\beta O(X, \tau\ )$ as

$$A^\gamma = \begin{cases} A & if\ b \in A \\ \{a, c\} & if\ b \notin A \end{cases}$$

Then $\beta O(X, \tau)_\gamma = \{X, \phi, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

For $a, b \in X$, there exist β-open sets $\{a, c\}$ containing a and $\{b\}$ containing b such that

$b \notin \{a, c\}$ and $a \notin \{b\}$. Hence the space (X, τ) is β-γ-$T_1{}'$ space.

Definition 3.2.9

A space (X, τ) is called β-γ-T_2 if for any two distinct points$x, y \in X$, there exists β-open set U, V such that $x \in U,\ y \in V$ and $U^\gamma \cap V^\gamma = \phi$.

Example 3.2.10

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$.

Then $\beta O(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Define the operation γ on $\beta O(X, \tau)$

by $A^\gamma = \begin{cases} A & if\ A = \{b\} or\ \{c\} \\ cl(A) & otherwise \end{cases}$

Then $\beta O(X, \tau)_\gamma = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

For $b, c \in X$, there exist β-γ-open sets $\{b\}$ and $\{c\}$ such that $b \in \{b\}$, $c \in \{c\}$ and

$\{b\}^\gamma \cap \{c\}^\gamma = \{b\} \cap \{c\} = \phi$. Hence the space (X, τ) is a β-γ-T_2 space.

Definition 3.2.11

A space (X,τ) is called β-γ-$T_2{}'$ if for any two distinct points $x,y \in X$, there exist β-γ-open sets U,V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Example 3.2.12

Let $X = \{a,b,c\}$ and $\tau = \{X,\phi,\{a\},\{b\},\{a,b\}\}$.

Then $\beta O(X,\tau) = \{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\}\}$.

Define the operation γ on $\beta O(X,\tau)$ as

$$A^\gamma = \begin{cases} A & if\ b \in A \\ \beta cl(A) & if\ b \notin A \end{cases}.$$

Then $\beta O(X,\tau)_\gamma = \{X,\phi,\{a\},\{b\},\{c\},\{a,c\},\{b,c\}\}$.

For $a,b \in X$, there exist β-γ-open sets $\{a\}$ and $\{b\}$ such that $a \in \{a\}$, $b \in \{b\}$ and

$\{a\} \cap \{b\} = \phi$. Hence the space (X,τ) is β-γ-$T_2{}'$ space.

Theorem 3.2.13

(i) A space (X,τ) is a β-γ-$T_0{}'$ space if and only if for every pair $x,y \in X$ with $x \neq y$, β_γ-$cl(\{x\}) \neq \beta_\gamma$-$cl(\{y\})$.

(ii) Let γ be a β-open operation. A space is (X,τ) a β-γ-T_0 space if and only if for every pair $x,y \in X$ with $x \neq y$, $\beta cl_\gamma(\{x\}) \neq \beta cl_\gamma(\{y\})$.

(iii) Let γ be a β-open operation. A space (X,τ) is β-γ-T_0 if and only if it is β-γ-$T_0{}'$.

Proof:

(i) Let x and y be any two distinct points of a β-γ-$T_0{}'$ space (X,τ). Then, by definition, we assume that there exists a β-γ-open set U such that $x \in U$ and $y \notin U$. Hence $y \in X\backslash U$. Then $X\backslash U$ is a β-γ-closed set containing y we have by definition of β_γ-$cl(\{y\})$ it is contained in $X\backslash U$. Therefore, $x \notin \beta_\gamma$-$cl(\{y\})$ and $x \in \beta_\gamma$-$cl(\{x\})$ and so β_γ-$cl(\{x\}) \neq \beta_\gamma$-$cl(\{y\})$. Conversely assume that for $x,y \in X$, with $x \neq y$. Then we have by hypothesis, β_γ-$cl(\{x\}) \neq \beta_\gamma$-$cl(\{y\})$. Let $z \in \beta_\gamma$-$cl(\{x\})$, but $z \notin \beta_\gamma$-$cl(\{y\})$. We claim that $x \notin \beta_\gamma$-$cl(\{y\})$. For if $x \in \beta_\gamma$-$cl(\{y\})$, then we get β_γ-$cl(\{x\}) \subseteq \beta_\gamma$-$cl(\{y\})$ (by Theorem 2.4.4 (iii) and (vii)) which is a contradiction. This shows that $X\backslash(\beta_{\gamma}\text{-}cl(\{y\})$ is a β-γ-open set containing x but not y. Hence (X,τ) is a β-γ-$T_0{}'$ space.

(ii) Let x and y be any two distinct points of a β-γ-T_0 space (X,τ). Then by definition of β-γ-T_0-space, that there exists a β-open set U such that $x \in U$ and $y \notin U^{\gamma}$. Since γ is a β-open set, there exists a β-γ-open set S such that $x \in S$ and $S \subseteq U^{\gamma}$. Hence, $y \in X\backslash U^{\gamma} \subseteq X\backslash S$. And also $X\backslash S$ is a β-γ-closed set containing y. Therefore $\beta cl_{\gamma}(\{y\})$ which is contained in $X\backslash S$. And $x \notin X\backslash S$ implies x is not in $\beta cl_{\gamma}(\{y\})$ and so $\beta cl_{\gamma}(\{x\}) \neq \beta cl_{\gamma}(\{y\})$. Conversely, suppose that $x \neq y \in X$ with $\beta cl_{\gamma}(\{x\}) \neq \beta cl_{\gamma}(\{y\})$. Then, there exists $z \in \beta cl_{\gamma}(\{x\})$ but $z \notin \beta cl_{\gamma}(\{y\})$. If $x \in \beta cl_{\gamma}(\{y\})$, then we get $\beta cl_{\gamma}(\{x\}) \subseteq \beta cl_{\gamma}(\{y\})$ (by Theorem 2.5 (iii) and (vii)). This implies that $z \in \beta cl_{\gamma}(\{y\})$. This contradiction shows that $x \notin \beta cl_{\gamma}(\{y\})$. So by Definition 2.4.3, there exists a β-open set W such that $x \in W$ and $W^{\gamma} \cap \{y\} = \phi$. Thus, we have that $x \in W$ and $y \notin W^{\gamma}$. Hence (X,τ) is β-γ-T_0.

(iii) This follows from (i) and (ii). The fact that, for any subset A of (X,τ), $\beta_{\gamma}\text{-}cl(A) = \beta cl_{\gamma}(A)$ holds under the assumption that γ is β-open (Theorem 2.4.6(viii)).

Definition 3.2.14

A space (X,τ) is said to be β-γ-$T_{1/2}$ if every β-γ-g.closed set of (X,τ) is β-γ-closed.

In β-γ-$T_{1/2}$ space the collections of β-γ-g.closed sets and β-γ-closed sets coincide.

Theorem 3.2.15

Let (X,τ) be a topological space and γ an operation on $\beta O(X,\tau)$. Then the following properties are equivalent.

(1) A space (X,τ) is β-γ-$T_{1/2}$.

(2) For each $x \in X$, $\{x\}$ is β-γ-closed or β-γ-open.

Proof:

(1) $\Longrightarrow$ (2)

Suppose $\{x\}$ is not β-γ-closed in (X,τ). Then, $X\backslash\{x\}$ is β-γ-g.closed by Theorem 2.5.19.

Since (X,τ) is a β-γ-$T_{1/2}$ space, so by Definition 3.2.14, $X \setminus \{x\}$ is β-γ-closed and so $\{x\}$ is β-γ-open.

(2) ⟹ (1)

Let F be a β-γ-g.closed set in (X,τ). We shall prove that $\beta cl_\gamma(F) = F$. It is sufficient to show that $\beta cl_\gamma(F) \subseteq F$. Assume that there exists a point x such that $x \in \beta cl_\gamma(F) \setminus F$. Then by assumption, $\{x\}$ is β-γ-closed or β-γ-open.

Case 1 :

If $\{x\}$ is a β-γ-closed set and $\{x\} \subseteq \beta cl_\gamma(F) \setminus F$ contradicts the Theorem 2.5.17. Hence $\beta cl_\gamma(F) \setminus F = \phi$.

Case 2 :

If $\{x\}$ is a β-γ-open set. $x \in \beta cl_\gamma(F) \setminus F$ implies $x \in \beta cl_\gamma(F)$ and $x \notin F$. By Theorem 2.4.3, $\{x\}$ is a β-γ-open set containing x, implies $\{x\} \cap F \neq \phi$ which is a contradiction. That is $\beta cl_\gamma(F) \setminus F = \phi$.

Thus in either case, we have $\beta cl_\gamma(F) = F$. Hence F is β-γ-closed.

Theorem 3.2.16

For a topological space (X,τ), let γ be an operation on $\beta O(X,\tau)$.

(i) Then, the following properties are equivalent.

(1) (X,τ) is β-γ-T_1.

(2) For every point $x \in X$, $\{x\}$ is a β-γ-closed set.

(3) (X,τ) is β-γ-$T_1{}'$.

(ii) Every β-γ-T_i' space is β-γ-T_i, where $i \in \{2, 0\}$.

(iii) Every β-γ-T_2 space is β-γ-T_1.

(iv) Every β-γ-T_1 space is β-γ-$T_{1/2}$.

(v) Every β-γ-$T_{1/2}$ space is β-γ-$T_0{}'$.

(vi) Every β-γ-$T_i^{'}$ space is β-γ-$T_{i-1}{'}$ where $i \in \{2,1\}$.

Proof:

(i) : (1) ⟹ (2)

Let $x \in X$ be a point. For each point $y \in X\backslash\{x\}$, then by definition of T_1-space, there exists a β-open set V_y such that, $y \in V_y \subseteq \left(V_y\right)^{\gamma}$ and $x \notin \left(V_y\right)^{\gamma}$. Then $X\backslash\{x\} = \cup\{\left(V_y\right)^{\gamma} | y \in X\backslash\{x\}\}$ and $\left(V_y\right)^{\gamma}$ is a β-γ-open for it is shown that $X \setminus \{x\}$ is β-γ-open in (X, τ).

(2) ⟹ (3)

Let x and y be two distinct points of X. By (2), $X\backslash\{x\}$ and $X\backslash\{y\}$ are required β-γ-open sets such that $y \in X\backslash\{x\},\ x \notin X\backslash\{x\}$ and $x \in X\backslash\{y\},\ x \notin X\backslash\{y\}$.

(3) ⟹ (1)

It is shown that if $x \in U$, where $U \in \beta O(X, \tau)$, then there exists a β-open set V such that $x \in V \subseteq V^{\gamma} \subseteq U$. Using (3), we have (X, τ), that is β-γ-T_1.

(ii) (1) Let space (X, τ) is β-γ-$T_0{'}$, then for $x \neq y$ we get β-γ-open set V such that $x \in U \subseteq U^{\gamma} \subseteq V$ and $y \notin U \subseteq U^{\gamma} \subseteq V$ where U is a β-open set such that $x \in V$ and $y \notin V$ implies $x \in U$ and $y \notin U^{\gamma}$. Hence the space (X, τ) is β-γ-T_0.

(2) Let the space (X, τ) is β-γ-$T_2{'}$, then for $x \neq y$ we get β-γ-open sets V and W such that $x \in U \subseteq U^{\gamma} \subseteq V$ and $y \in S \subseteq S^{\gamma} \subseteq W$ where U and S are β-open sets such that $x \in V$, $y \in W$ and $V \cap W = \phi$ which implies $y \notin U^{\gamma}$, $x \notin S^{\gamma}$ and $U^{\gamma} \cap S^{\gamma} = \phi$. Hence the space (X, τ) is β-γ-T_2.

(iii) Let the space (X, τ) be β-γ-T_2. Then by definition, for $x \neq y$, there exists β-open sets U and V such that $x \in U$ and $y \in V$ and $U^{\gamma} \cap V^{\gamma} = \phi$ which implies $x \notin V^{\gamma}$ and $y \notin U^{\gamma}$. Hence the space (X, τ) is β-γ-T_1.

(iv) Let (X, τ) be β-γ-T_1space. Then by (i), $\{x\}$ is a β-γ-closed set. Now by Theorem 3.2.15, (X, τ) is β-γ-$T_{1/2}$.

(v) Let (X, τ) be β-γ-$T_{1/2}$ space. Let $x \neq y$. Then by Theorem 3.2.15, $\{x\}$ is either β-γ-closed or β-γ-open.

(a) $\{x\}$ is β-γ-closed. Then $X\setminus\{x\}$ is β-γ-open and $y \in X\setminus\{x\}$, $x \notin X\setminus\{x\}$. Therefore the space (X, τ) is β-γ-${T_0}'$.

(b) $\{x\}$ is β-γ-open. Then $x \in \{x\}$ and $y \notin \{x\}$. Therefore the space (X, τ) is β-γ-${T_0}'$.

Hence every β-γ-$T_{1/2}$ space is β-γ-${T_0}'$.

(vi) (i) Let the space (X, τ) be β-γ-T_1'. Then for $x \neq y$, there exists β-γ-open sets U and V containing x and y such that $y \notin U$ and $x \notin V$ which implies $x \in U$ and $y \notin U$ or $y \in V$ and $x \notin V$. Hence the space (X, τ) is β-γ-T_0'.

(ii) Let the space (X, τ) be β-γ-T_2'. . Then for $x \neq y$, there exists β-γ-open sets U and V containing x and y such that $x \in U$, $y \in V$ and $U \cap V = \phi$ which implies $x \notin V$ and $y \notin U$. Hence the space (X, τ) is β-γ-T_1'.

Remark 3.2.17

The following example shows that β-γ-T_1 space need not be a β-γ-T_2 space.

Example 3.2.18

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$.

Then $\beta O(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Define the operation γ on $\beta O(X, \tau)$ as

$$A^{\gamma} = \begin{cases} A & if\ b \in A \\ \{a, c\} & if\ b \notin A \end{cases}.$$

Then $\beta O(X, \tau)_{\gamma} = \{X, \phi, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Hence the space (X, τ) is β-γ-T_1 but not β-γ-T_2.

Remark 3.2.19

The following example shows that β-γ-$T_{1/2}$ space need not be a β-γ-T_1 space.

Example 3.2.20

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$.

Then $\beta O(X,\tau) = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,c\}$.

Define the operation γ on $\beta O(X,\tau)$ by

$$A^{\gamma} = \begin{cases} A & if\ c \notin A \\ cl(A) & if\ c \in A \end{cases}.$$

Then $\beta O(X,\tau)_{\gamma} = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$.

Hence the space (X,τ) is β-γ-$T_{1/2}$ but not β-γ-T_1.

Remark 3.2.21

The following example shows that β-γ -$T_1^{'}$ space need not be a β-γ-$T_2^{'}$ space.

Example 3.2.22

Let $X = \{a,b,c\}$ and $\tau = P(X)$, the power set on X.

Then $\beta O(X,\tau) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}$.

Define the operation γ on $\beta O(X,\tau)$ by $A^{\gamma} = A \cup \{c\}$ if $A = \{a\} or\ \{b\}$, $A^{\gamma} = A \cup \{a\}$ if $A = \{c\}$ and $A^{\gamma} = A$ if $A \neq X$.

Then $\beta O(X,\tau)_{\gamma} = \{\phi, X, \{a,b\}, \{b,c\}, \{a,c\}\}$.

Hence the space (X,τ) is β-γ -$T_1^{'}$ but not β-γ-$T_2^{'}$.

Remark 3.2.23

The following example shows that β-γ-$T_0^{'}$ space need not be a β-γ-$T_{1/2}$ space.

Example 3.2.24

Let $X = \{a,b,c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$.

Then $\beta O(X,\tau) = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,c\}$.

Define the operation γ on $\beta O(X,\tau)$ by

$$A^{\gamma} = \begin{cases} A & if\ b \notin A \\ cl(A) & if\ b \in A \end{cases}.$$

Then $\beta O(X,\tau)_\gamma = \{X, \phi, \{a\}, \{b,c\}, \{a,c\}\}$.

Hence the space (X,τ) is β-γ-T_0' but not β-γ-$T_{1/2}$.

Remark 3.2.25

The following example shows that β-γ-T_0' space need not be a β-γ-T_1'- space.

Example 3.2.26

Let $X = \{a,b,c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$.

Then $\beta O(X,\tau) = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$.

Define the operation γ on $\beta O(X,\tau)$ by

$$A^\gamma = \begin{cases} A & if \ \ A = \{a\} \\ cl(A) & if \ A \neq \{a\} \end{cases}.$$

Then the $\beta O(X,\tau)_\gamma = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$.

Hence the space (X,τ) is β-γ-T_0' but not β-γ-T_1'.

3.3 $\boldsymbol{\beta}$ - $(\boldsymbol{\gamma}, \boldsymbol{b})$-continuous maps

Let $\gamma: \beta O(X,\tau) \to P(X)$ and $b : \beta O(Y,\sigma) \to P(Y)$ be operations on $\beta O(X,\tau)$ and $\beta O(Y,\sigma)$, respectively.

Definition 3.3.1

A mapping $f : (X,\tau) \to (Y,\sigma)$ is said to be β-(γ,b)-continuous if for each $x \in X$ and each β-open set V containing $f(x)$, there exists a β-open set U such that $x \in U$ and $f(U^\gamma) \subseteq V^b$.

Theorem 3.3.2

Let $f : (X,\tau) \to (Y,\tau)$ be a β-(γ,b)-continuous mapping. Then,

(i) $f(\beta cl_\gamma(A)) \subseteq \beta cl_b(f(A))$ holds for every subset A of (X,τ).

(ii) For every β-b-open set B of (Y,σ), $f^{-1}(B)$ is β-γ-open, that is for any $B \in \beta O(Y,\sigma)_b$, $f^{-1}(B) \subseteq \beta O(X,\tau)_\gamma$.

Proof:

(i): Let $y \in f(\beta cl_\gamma(A))$ and let V be any β-open set containing y. Then, there exists a point $x \in \beta cl_\gamma(A)$ and a β-open set U containing x such that $f(x) = y$ and $f(U^\gamma) \subseteq V^b$. We have $U^\gamma \cap A \neq \phi$. Therefore, $\phi \neq f(U^\gamma \cap A) \subseteq f(U^\gamma) \cap f(A) \subseteq V^b \cap f(A)$ and so $y \in \beta cl_b(f(A))$.

(ii): Let B be a β-b-closed set. Then using (i) we have that
$f(\beta cl_\gamma\big(f^{-1}(B)\big) \subseteq \beta cl_b(f\big(f^{-1}(B)\big) \subseteq \beta cl_b(B) = B$.
Thus, $\beta cl_\gamma\big(f^{-1}(B)\big) \subseteq f^{-1}(B)$ and hence $\big(f^{-1}(B)\big) = \beta cl_\gamma\big(f^{-1}(B)\big)$. This implies that $f^{-1}(B)$ is β-γ-closed in (X, τ).

Remark 3.3.3.

In Theorem 3.3.2, the properties of β-(γ, b)-continuity of f, (i) and (ii) are equivalent to each other if one of the following conditions (a) and (b) is satisfied:

(a) $(Y, \sigma)$is a β-b-regular space,

(b) b is a β-open operation.

Proof :

It follows from the proof of Theorem 3.3.2 that we know the following implications: "β-(γ, b)-continuity of f" $\Longrightarrow$(i) $\Longrightarrow$ (ii). Thus, under condition (a), we first show the implication (ii) $\Longrightarrow$$\beta$-($\gamma$, b)-continuity of f. Let $x \in X$a nd let V be a β-open set containing $f(x)$. Since (Y, σ) is a β-b-regular space, $V \in \beta O(Y, \sigma)_b$. Then, by (ii) of Theorem 3.3.2, $x \in f^{-1}(V) \in \beta O(X, \tau)_\gamma$. So, by the definition of β-b-openness of $f^{-1}(V)$, there exists a β-open set Ucontaining x such that $U^\gamma \subseteq f^{-1}(V)$ and so $f(U^\gamma) \subseteq V \subseteq V^b$. Therefore, f is β-(γ, b)-continuous.

Finally, under condition (b), we prove the implication: (ii) $\Longrightarrow$$\beta$-$(\gamma, b)$-continuity of f. Let $x \in X$ and let V be a β-open set containing $f(x)$. Since b is β-open, there exists a β-b-open set U containing $f(x)$ such that $U \subseteq V^b$. By (ii) of Theorem 3.3.2, $x \in f^{-1}(U) \in \beta O(X, \tau)_\gamma$ and so by definition of β-γ-openness of $f^{-1}(U)$, there exists a β-open set Wcontaining x such that $W^\gamma \subseteq f^{-1}(U) \subseteq f^{-1}(V^b)$. Therefore, we have $f(W^\gamma) \subseteq V^b$ and so f is β-(γ, b)-continuous.

Definition 3.3.4

A mapping $f:(X,\tau) \to (Y,\sigma)$ is said to be

(i) β-(γ,b)-closed, if for every β-γ-closed set A of (X,τ), $f(A)$ is β-b-closed in (Y,σ).

(ii) β-(id,b)-closed if for $f(F)$ is β-b-closed in (Y,σ) for every β-closed set F of (X,τ).

Theorem 3.3.5

Suppose f is β-(γ,b)-continuous and f is β-(id,b)-closed. Then the following properties hold.

(i) For every β-γ-g.closed set A of (X,τ), the image $f(A)$ is β-b-g.closed.

(ii) For every β-b-g.closed set B of (Y,σ), $f^{-1}(B)$ is β-γ-g.closed.

Proof :

(i) Let V be a β-b-open set in (Y,σ) such that $f(A) \subseteq V$. Then by Theorem 3.3.2 (ii), $f^{-1}(V)$ is β-γ-open. Since A is β-γ-g.closed and $A \subseteq f^{-1}(V)$, $\beta cl_\gamma(A) \subseteq f^{-1}(V)$ holds and so $f(\beta cl_\gamma(A)) \subseteq V$. Thus, $f(\beta cl_\gamma(A))$ is β-b-closed as $\beta Cl_\gamma(A)$ is β-closed by Theorem 2.4.5 (i) and the assumption that f is β-(id,b)-closed. Therefore $cl_b(f(A)) \subseteq \beta cl_b(f(\beta cl_\gamma(A)))) = f(\beta cl_\gamma(A)) \subseteq V$. Hence $f(A)$ is β-b-g.closed.

(ii) Let U be a β-γ-open in (X,τ) such that $f^{-1}(B) \subseteq U$. Let $F = \beta cl_\gamma\big(f^{-1}(B)\big) \cap (X \backslash U)$. Then by Theorem 2.4.5 (i), F is β-closed in (X,τ). Since f is β-(id,b)-closed, $f(F)$ is β-b-closed in (Y,σ) and $f(F) \subseteq f(\beta cl_\gamma\big(f^{-1}(B)\big) \cap (X \backslash U)) \subseteq \beta cl_\gamma(B) \backslash B$. By Theorem 2.5.17, $f(F) = \phi$ and so $F = \phi$. Hence $\beta cl_\gamma\big(f^{-1}(B)\big) \subseteq U$. Therefore, $f^{-1}(B)$ is β-γ-g.closed in (X,τ).

Theorem 3.3.6

Suppose that $f:(X,\tau) \to (Y,\sigma)$ is a β-(γ,b)-continuous and β-(id,b)-closed. Then the following properties hold.

(i) If f is injective and (Y,σ) is β-b-$T_{1/2}$, then (X,τ) is β-γ-$T_{1/2}$.

(ii) If f is surjective and (X,τ) is β-γ-$T_{1/2}$ then (Y,σ) is β-b-$T_{1/2}$.

Proof :

(i) Let A be a β-γ-g.closed set of (X,τ). Then by Theorem 3.3.5 (i), $f(A)$ is β-b-g.closed. Since (X,τ) is β-γ-$T_{1/2}$, $f(A)$ is β-b-closed. By the Theorem 3.3.5 (ii), $A = f^{-1}(f(A))$ is β-γ-closed. This implies A is β-γ-closed. Hence (X,τ) is β-γ-$T_{1/2}$.

(ii) Let B be a β-b-g.closed set in (Y,σ). By Theorem 3.3.5 (ii), $f^{-1}(B)$ is β-γ-g.closed. Since (X,τ) is β-γ-$T_{1/2}$, so $f^{-1}(B)$ is β-γ-closed. Therefore $B = f(f^{-1}(B))$ is β-b-closed in (Y,σ). Hence (Y,σ) is β-b-$T_{1/2}$.

Definition 3.3.7

Let $f: (X,\tau) \to (Y,\sigma)$ be a function. Then f is said to be β-(γ,b)- homeomorphic, if f is bijective, β-(γ,b)-continuous and f^{-1} is β-(b,γ)-continuous.

Theorem 3.3.8

Suppose that a mapping $f: (X,\tau) \to (Y,\sigma)$ is β-(γ,b)- homeomorphic. If (X,τ) is β-γ-$T_{1/2}$, then (Y,σ) is β-b-$T_{1/2}$.

Proof :

Let $\{y\}$ be a singleton set of (Y,σ). Then there exists a point $x \in X$ such that $y = f(x)$. By Theorem 3.2.15, $\{x\}$ is β-γ-open or β-γ-closed. Therefore by Theorem 3.3.2, $\{y\}$ is β-b-closed or β-b-open. Hence (Y,σ) is β-b-$T_{1/2}$.

Theorem 3.3.9

Let $f: (X,\tau) \to (Y,\sigma)$ be a β-(γ,b)-continuous injection. If (Y,σ) is β-b-T_2 (resp. β-b-T_1), then (X,τ) is β-γ-T_2 (resp. β-γ-T_1).

Proof :

Suppose that (Y,σ) is β-b-T_2. Let x and y be distinct points of X. Then, there exists two β-open sets V and W of Y such that $f(x) \in V$, $f(y) \in W$ and $V^b \cap W^b = \phi$. Since f is β-(γ,b)-continuous, for V and W there exists two β-open sets U and S such that $x \in U$, $y \in S$, $f(U^\gamma) \subseteq V^b$ and $f(S^\gamma) \subseteq W^b$. Therefore we have $U^\gamma \cap S^\gamma = \phi$ and hence (X,τ) is β-γ-T_2 .

Suppose that (Y,σ) is β-b-T_1. Let x and y be distinct points of X. Then, there exists two β-open sets V and W of Y containing $f(x)$ and $f(y)$ such that $f(y) \notin V^b$, $f(x) \notin W^b$. Since f is β-(γ,b)-continuous, for V and W there exists two β-open sets U and S such that $x \in U$, $y \in S$, $f(U^\gamma) \subseteq V^b$ and $f(S^\gamma) \subseteq W^b$. Therefore we have $f(y) \notin U^\gamma$ and $f(x) \notin S^\gamma$ and hence (X,τ) is β-γ-T_1.

SUMMARY AND CONCLUSION

Summary and Conclusion

This dissertation is devoted to the study of an Operation approach to β-open sets and applications in topological spaces.

The concept of β-γ-open sets using the operation γ on β-open sets and some of its properties are studied in topological spaces. β-γ regular space and β-regular operation are discussed. βcl_γ and $\beta_\gamma cl$ operators in topological spaces are analyzed. β-γ-generalized closed sets are studied and its properties are analyzed. As an application of β-γ-open sets seven new spaces are studied and their interrelations are obtained. Using γ-operation, the concepts of β-γ-continuity, β-γ-closed maps and β-γ-homeomorphisms are studied.

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