CHAPTER II

SOFT REGULAR SEMIGROUPS

Definition : 2.1

An element a of a semigroup S is called von Neumann regular or simply regular if there exists an element $x \in S$ such that $a = axa$. If every element of a semigroup S is regular then S is called a **regular semigroup**.

It is well known that a completely regular semigroup whose idempotents are central can be expressed as union of groups, therefore, the class of regular semigroups forms a core class of semigroups because it still has many similar properties of groups.

In this chapter regular semigroups are classified by considering its soft right ideals and soft left ideals and the following theorems are obtained.

Theorem : 2.2

A semigroup S is a regular semigroup if and only if

$$(R, A) * (L, B) = (R, A) \wedge (L, B)$$

for every soft right ideal (R, A) and soft left ideal (L, B) over S.

Proof :

Since $(R, A) * (L, B) = (H, A \times B)$, where H is a function from $A \times B$ to $P(S)$ defined by

$$H(a, b) = R(a) * L(b)$$

One can easily see that $(R, A) \wedge (L, B) = (G, A \times B)$, where G is a function from $A \times B$ to $P(S)$ such that

$$G(a, b) = R(a) \cap L(b)$$

$A \times B \subseteq A \times B$ and observe that

$$R(a) * L(b) \subseteq R(a) * S \subseteq R(a)$$

and

$$R(a) * L(b) \subseteq S * L(b) \subseteq L(b)$$

Hence, $R(a) * L(b) \subseteq R(a) \cap L(b)$ for all $a \in A, b \in B$, and consequently

$$(H, A \times B) \subseteq (G, A \times B)$$

Conversely, we suppose that $u \in R(a) \cap L(b)$. Since S is regular $u \in S$, there exists $v \in S$ such that $u = uvu$.

Now, since $u \in R(a)$ and $vu \in L(b)$, $u = uvu \in R(a) * L(b)$. This shows that $(R, A) \wedge (L, B) \subseteq (R, A) * (L, B)$.

Therefore, $(G, A \times B) \subseteq (H, A \times B)$ and so, $(R, A) * (L, B) = (R, A) \wedge (L, B)$.

Conversely, suppose that $A = B = S$ and R is a function from A to $P(S)$. Define, $R(u) = uS$, for all $u \in S$ and let L be a function from B to $P(S)$, defined by $L(u) = Su$, for all $u \in S$. Then (R, S) is a soft right ideal and (L, S) is a soft left ideal over S. Thus,

$$u \in R(u) \cap L(u) = R(u)L(u) = uSSu \subseteq uSu.$$

Therefore S is a regular semigroup.

Theorem : 2.3

Let S be a semigroup. Then S is regular if and only if

$$(Q,B) \wedge (L,A) \subseteq (Q,B) * (L,A)$$

for every soft left ideal (L,A) and every soft quasi-ideal (Q,B) over S.

Proof :

Since $(Q,B) * (L,A) = (H, B \times A)$, where H is a function from $B \times A$ to $P(S)$ defined by

$$H(b,a) = Q(b) * L(a)$$

where $b \in B, a \in A$. Also,

$$(Q,B) \wedge (L,A) = (G, B \times A)$$

where G is a function from $B \times A$ to $P(S)$ defined by

$$G(b,a) = Q(b) \cap L(a)$$

where $b \in B, a \in A$. Now $B \times A \subseteq B \times A$. Let $u \in Q(b) \cap L(a)$. Then, $u \in Q(b)$ and $u \in L(a)$. Since $u \in S$ and S is regular, there exists $v \in S$ such that

$$u = uvu \in Q(b) * S * L(a) \subseteq Q(b) * L(a)$$

Hence, $(Q,B) \wedge (L,A) \subseteq (Q,B) * (L,A)$.

Conversely, suppose that $A = B = S$, and Q is a function from B to $P(S)$ defined by $Q(u) = uS$, for all $u \in S$ and L is a function from A to $P(S)$ defined by $L(u) = Su$, for all $u \in S$. Then (Q,S) is a soft quasi-ideal and (L,S) is a soft left ideal over S. Thus,

$$u \in Q(u) \cap L(u) \subseteq Q(u)L(u) = uSSu = uSu$$

This shows that S is a regular semigroup.

The following theorem is another characterization of regular semigroups.

Theorem : 2.4

Let S be a semigroup. Then S is regular if and only if $(Q, B) \wedge (L, A) \subseteq (Q, B) * (L, A)$, for every soft bi-ideal (Q, B) and every soft left ideal (L, A) over S.

Definition : 2.5

A soft semigroup (F, A) over a semigroup S is called a **soft regular semigroup** if for each $\alpha \in A, F(\alpha)$ is regular.

Definition : 2.6

A soft set (F, A) over S is called a **soft set with cover** if

$$\bigcup_{\alpha \in A} F(\alpha) = S.$$

Note :

The following example shows that if S is a regular semigroup then the soft semigroup (F, A) over the semigroup S may not be regular.

Example : 2.7

Let $S = \{a, b, c, d, e\}$ be a semigroup with the binary operation as shown in the following cayley table.

.	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

We can easily verify that the above semigroup is a regular semigroup but not commutative. Consider $S = A$ and $F(a) = \{a, b\}$, $F(b) = \{a, b, c\} = F(c)$, $F(d) = \{a, d\}$, $F(e) = \{a, e\}$. Then (F, S) is a soft semigroup over S but it is not regular because $F(a), F(b)$ and $F(c)$ are not regular subsemigroups of S.

Example : 2.8

Consider a semigroup $S = \{a, b, c, d\}$ with binary operation as defined in the following cayley table.

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	a	d

Clearly, S is not a regular semigroup. Let $A = \{\alpha, \beta\}$ be a set of parameters such that $F(\alpha) = \{a\}$, $F(\beta) = \{\alpha, d\}$. Then (F, A) is a soft regular semigroup over S because $F(\alpha)$ and $F(\beta)$ are regular subsemigroups of S.

Note :

In the above examples, we have demonstrated that the regularity of a semigroup S does not imply the regularity of a soft semigroup over S. Also, the regularity of a soft semigroup over a given semigroup S does not imply the regularity of the semigroup.

Theorem : 2.9

Let (F, A) be a soft regular semigroup over a semigroup S with cover. Then S is a regular semigroup.

Proof :

Let (F, A) be a soft regular semigroup over S. Then $F(\alpha)$ is regular for each $\alpha \in A$. Now, let $x \in S$. Because, $S = \bigcup_{\alpha \in A} F(\alpha)$ there exists some $\beta \in A$, such that $x \in F(\beta)$. Since $F(\beta)$ is regular, there exists $y \in F(\beta)$ such that $x = xyx$.

Since $y \in F(\beta) \subseteq \bigcup_{\alpha \in A} F(\alpha) = S$, S is regular.

It can be easily seen that every soft semigroup (F, A) over S is regular if $x^2 = x$ for all $x \in S$.

Theorem : 2.10

Let (F, A) be a soft semigroup over S (not necessarily) regular. Then the following are equivalent.

1) (F, A) is regular.

2) $(R, B)\hat{o}(L, C) = (R, B) \cap_R (L, C)$ for all soft right ideals (R, B) and soft left ideals (L, C) over $F(\alpha) \neq \emptyset$ for $\alpha \in A$. Whenever $(R, B)\hat{o}(L, C)$ is non-null and non-empty.

Proof :

$(1) \Rightarrow (2)$

By definition, $(R, B)\hat{o}(L, C) = (H, B \cap C)$, where H is a function from $B \cap C$ to $P(F(\alpha))$ defined by

$$H(\alpha) = R(\alpha)L(\alpha) \text{ for } \alpha \in B \cap C.$$

Also, we have $(R, B) \cap_R (L, C) = (G, B \cap C)$, where G is a function from $B \cap C$ to $P(S)$ such that,

$$G(\alpha) = R(\alpha) \cap L(\alpha) \text{ for } \alpha \in B \cap C.$$

Because $B \cap C \subseteq B \cap C$ and also we have,

$$R(\alpha)L(\alpha) \subseteq R(\alpha) * F(\alpha) \subseteq R(\alpha) \quad \text{and}$$

$$R(\alpha)L(\alpha) \subseteq F(\alpha)L(\alpha) \subseteq L(\alpha).$$

This implies that,

$$R(\alpha)L(\alpha) \subseteq R(\alpha) \cap L(\alpha) \text{ for all } \alpha \in B, \alpha \in C.$$

Hence $(H, B \cap C) \subseteq (G, B \cap C)$.

In proving the reverse inclusion, we suppose that $u \in R(\alpha) \cap L(\alpha)$. Since $u \in F(\alpha)$ and $F(\alpha)$ is regular, there exists $v \in F(\alpha)$ such that $u = uvu$.

Now, $u \in R(\alpha), uv \in L(\alpha)$ and hence $u = uvu \in R(\alpha)L(\alpha)$. This shows that $(R, B) \cap_R (L, C) \subseteq (R, B)\hat{o}(L, C)$.

Therefore $(G, B \cap C) \subseteq (H, B \cap C)$. Thus $(R, B)\hat{o}(L, C) = (R, B) \cap_R (L, C)$.

(2) $\Rightarrow$ (1) Suppose that $B = C = F(\alpha)$ and R is a function from $F(\alpha)$ to $P(F(\alpha))$ defined by $R(u) = uF(\alpha)$, for all $u \in F(\alpha)$ and L is a function from $F(\alpha)$ to $P(F(\alpha))$ defined by $L(u) = F(\alpha)u$, for all $u \in F(\alpha)$. Then $(R, F(\alpha))$ is a soft right ideal and $(L, F(\alpha))$ is a soft left ideal over $F(\alpha)$. Thus,

$$u \in R(u) \cap L(u) = R(u)L(u) = uF(\alpha)F(\alpha)u \subseteq uF(\alpha)u$$

Hence $F(\alpha)$ is regular.