Pseudo - Metric Spaces

SUBMITTED BY

K. Vijayakumary

A Dissertation submitted to the Bharathiar University, Coimbatore in partial fulfilment of the Requirements for the Degree of Master of Science in Mathematics

MAY 1986

Department of Mathematics

Sri Avinashilingam Home Science College For Women

(Autonomous)

Coimbatore-641 043.

CONTENTS

* * *

ACKNOWLEDGEMENT

The author wishes to express profound gratitude and sincere sentiments of thanks to Mrs. S. SWAROOPARANI,M.Sc., M.Phil., Professor in the Department of Mathematics, Sri Avinashilingam Home Science College for Women, Coimbatore, for guiding in the present study with immense pleasure and great responsibility.

The author also wishes to record profound sense of gratitude to Dr. K.N. MEENAKSHI,M.Sc.,Ph.D., Professor and Head of the Department of Mathematics, Sri Avinashilingam Home Science College for Women, Coimbatore, for her timely help in doing this dissertation.

The author also record her thanks to Dr (Mrs.) RAJAMMAL P. DEVADAS,M.A.,M.S.,Ph.D. (Ohio State), D.Sc., (Madras), Director, Sri Avinashilingam Home Science College for Women, Coimbatore, and Dr. (Mrs) LAKSHMI SANTHA RAJAGOPAL, Principal, Sri Avinashilingam Home Science College for Women, Coimbatore, for all facilities given to her in the course of preparation of this dissertation.

INTRODUCTION

The materials presented in this dissertation is the result of a fruitful study of Pseudo-metric spaces. This study is carried out because a vast number of useful concepts in a wide variety of branches of Mathematics happen to be matrics or Pseudo-matrics.

This dissertation consists of one chapter which has four sections. In Section 1 'Definition and Examples' of Pseudo-metric spaces are given. The large number of examples of Pseudo-metric given in Section 1 emphasizes the extent of protiferation of Pseudo-metrics in Mathematics.

Section II deals with 'Topology Generated by Pseudo-Metrics'.

In Section III 'Properties of Pseudo-Metric Spaces' are presented. In Theorem 3.5 we see that T_o property is precisely what seperates a Pseudo-metric from a metric. We get a characterization of continuity of the function in Theorem 3.6.

From Remark 3.25 we gather that in a Pseudo-metric space totally boundedness does not imply compactness. Hilbert Pseudo-metric is given in Section IV.

In the end bibliography is added, in which the list of books and journals referred for this dissertation is given.

SECTION I

DEFINITION AND EXAMPLES

1.1. <u>Definition</u>

Let X be a set A function $P : X^2 \to R$ is called a Pseudo-metric on X if for all x, y, z in X,

1) $P(x,x) = 0$,

2) $P(x,y) \leq P(y,z) + P(z,x)$ (Triangle inequality)

If P further satisfies the condition

3) $P(x,y) = 0 \implies x = y$,

it is called a Metric on X.

1.2. <u>Note</u>

Some authors follows 'semi-metric' or 'quasi-metric' instead of Pseudo-metric. Pseudo-metric is abreviated as 'ps-metric'.

1.3. <u>Remark</u>

If P is a Ps-metric on X, then for all x, y in X,

1) $P(x,y) \geq 0$;

2) $P(x,y) = P(y,x)$.

Proof:

Putting $z = x$ in the second condition in the definition we get

$$P(x,y) \leq P(y,x)$$

interchanging x and y we have

$$P(y,x) \leq P(x,y)$$

Hence, $$P(x,y) = P(y,x).$$

Putting $x = y$ in the second condition in the definition we get

$$P(x,z) + P(z,x) \geq P(x,x) = 0,$$

which gives

$$2P(x,z) \geq 0 \quad \text{for all } x,z \text{ in } X.$$

Hence $P(x,y) \geq 0$ for all x,y in X.

1.4. Examples

1) Let X be any set. Then

$$P(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric on X. It is called the Discrete Metric on X.

2) $P(x,y) = 0$ for all x,y in X, defines a Ps-metric on X. It is not a metric, if X contains two or more points. It is called the Indiscrete Ps-Metric on X.

3) On R, $P(x,y) = |x-y|$ defines a metric, called the Standard Metric on R.

4) $P(z_1,z_2) = |z_1 - z_2|$ defines what is known as the Standard Metric on C.

5) In the complex theory, another metric is sometimes used instead of the standard one

$$\chi(z_1,z_2) = \frac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\ \sqrt{1+|z_2|^2}}$$

is a metric called the Chordal Distance.

6) Let X be the set of all real sequences (a_i), such that

$$\sum_{i=1}^{\infty} |a_i| < \infty$$

Then,

$$P(a,b) = \sum_{i=1}^{\infty} |a_i - b_i|$$

where $a = (a_1, a_2, \ldots)$ and

$b = (b_1, b_2, \ldots)$, is a metric on X.

7) Let X be the set of all functions $f : R \to R$ such that

$$\int |f| < \infty .$$

Then,

$$\rho(f,g) = \int |f-g| \text{ defines as Ps-metric on } X.$$

Here $\int$ denotes the Lebesgue integral over R.

8) Let X consist of all square-summable real sequenses, that is sequences (a_i) such that $\sum_{i=0}^{\infty} a_i^2$ is convergent. Then

$$\rho(a,b) = \left(\sum_{i=0}^{\infty} (a_i - b_i)^2 \right)^{1/2}$$

defines a metric on X.

9) Let X consist of all square-integrable functions, that is, functions $f : R \to R$ such that $\int f^2 < \infty$.
Then,

$$\rho(f,g) = \left(\int (f-g)^2 \right)^{1/2}$$

defines a Ps-metric on X, in examples (6) to (9), the triangle inequality for ρ is proved by invoking the appropriate form of Minkowski's inequality these Ps-metrics are widely used in functional analysis and measure the ory.

10) Let B denote the set of all functions $f : [0,1] \to R$ which are of bounded variation. If V_f

denote the variation of f, then

$$\rho(f,g) = V_{f-g} \text{ defines a Ps-metric on } \mathbb{B}.$$

This is not a metric. If $f-g$ is a constant, then $\rho(f,g) = 0$.

11) Let X be the set if all continuous functions $f : [0,1] \longrightarrow [0,1]$. Then,

$$\rho(f,g) = \max_{x \in I} |f(x) - g(x)|$$

is a metric.

12) Let X be the set of all Cauchy sequences of real numbers if $a = (a_i)$, $b = (b_i)$,

$$\rho(a,b) = \lim_{n \longrightarrow \infty} |a_n - b_n|$$

defines a Ps-metric.

13) On R^2,

$$\rho_1(a,b) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$$

$$\rho_2(a,b) = |a_1 - b_1| + |a_2 - b_2|$$

$$\rho_3(a,b) = \max \left\{ |a_1 - b_1|, |a_2 - b_2| \right\}$$

where $a = (a_1,a_2)$, $b = (b_1,b_2)$, defines three different metrics.

14) On R^n,

$$\rho_1(a,b) = \left(\sum_{i=1}^{n} (a_i - b_i)^2 \right)^{1/2}$$

$$\rho_2(a,b) = \sum_{i=1}^{n} |a_i - b_i|$$

$$\rho_3(a,b) = \max_{1 \leq i \leq n} \left\{ |a_i - b_i| \right\}$$

where $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$, defines three different metrics. ρ_1 is called the Euclidean (or standard) Metric on R^n. To prove the triangle inequality for ρ_1 one requires the use of Minkowski's inequality.

1.5. <u>Note</u>

Every metric is Ps-metric, but not the converse.

1.6. <u>Remark</u>

Let P be a Ps-metric on X, x, y, z, z_i are any points on X.

1) $P(x,y) \geq |P(x,z) - P(y,z)|$

2) (Generalized triangle inequality)

$$P(x,y) \leq P(y,z_1) + P(z_1,z_2) + \cdots + P(z_n,x).$$

<u>Proof</u>:

1) $P(x,y) + P(y,z) \geq P(z,x)$.

Therefore, $P(x,y) \geq P(z,x) - P(y,z)$

interchanimg x and y,

$$P(y,x) \geq P(z,y) - P(x,z)$$

Hence $P(x,y) \geq |P(x,z) - P(y,z)|$

(2) by induction we prove (2).

SECTION II

TOPOLOGY GENERATED BY PS-METRICS

2.1. Definition

Let P be a Pseudo-metric on X_1 and $\varepsilon > 0$ be any real numbers. The set

$$S(x, \varepsilon : P) = \{ y \,/\, P(x,y) < \varepsilon \}$$

is called the Sphere of Radius ε Centred at x, or ε-sphere at x. The notation $S_p(x, \varepsilon)$ is also used.

2.2. Theorem

Let P be Ps-metric on X. The set of all spheres, namely,

(1) $\mathcal{B} = \{ S(x, \varepsilon) \,/\, x \in X, \;\; \varepsilon > 0 \}$ is a basis.

Proof:

Let x be any point in the intersection of $S(y_1, \varepsilon_1)$ and $S(y_2, \varepsilon_2)$. Then $\varepsilon_1 - P(x,y_1)$ and $\varepsilon_2 - P(x,y_2)$ are both positive numbers; let ε denote the smaller of the two. We will show that $S(x, \varepsilon)$ is contained in the intersection of the two given spheres. Let $t \in S(x, \varepsilon)$. By triangle inequality,

$$\begin{aligned} P(y_1,t) &\leq P(y_1,x) + P(x,t) \\ &< P(y_1,x) + \varepsilon \\ &\leq \varepsilon_1. \end{aligned}$$

Hence t belongs to the first sphere. Similarly, it can be shown that t belongs to the second sphere.

2.3. Definition

The topology generated by the basis

$$\mathbb{B} = \{S(x, \varepsilon) \ / \ x \in X, \ \varepsilon > 0 \}$$

is called the Topology Generated by P.

2.4. Note

The usual notations are,

$$\mathcal{T}(P), \ \mathcal{T}_p, \ \tau_p, \ \text{top}(P), \ \text{etc.}$$

The ordered triple $(X, P, \mathcal{T}_p)$ is called a Pseudo-metric topological space, or briefly, Pseudo-Metric Space. We often write (X,P) or even X instead of thetriple.

2.5. Remark.

The Ps-metric topology $\mathcal{T}_p$ has plenty of bases besides

(1) $\mathbb{B} = \{S(x, \varepsilon) \ / \ x \in X, \ \varepsilon > 0 \}$, the two most commonly used being

(2) $\{S(x, \gamma) \ / \ x \in X, \ 0 < \gamma \in Q \}$

(3) $\{S(x, 1/n) \ / \ x \in X, \ n \in N \}$

More generally, if we let ε run through any set of positive numbers with 0 as a closure point, we get a basis for the $\mathcal{T}_p$.

Hence we have the theorem below.

2.6. Theorem

For any $x \in X$, each of the following is a neighbourhood basis at x in the topology $\mathcal{T}_p$.

(1) $\{S(x, \varepsilon) \,/\, \varepsilon > 0\}$

(2) $\{S(x, r) \,/\, 0 < r \in Q\}$

(3) $\{S(x, 1/n) \,/\, n \in N\}$.

2.7. Examples

1) The indiscrete Ps-metric generates the indiscrete topology. The discrete metric generates the discrete topology.

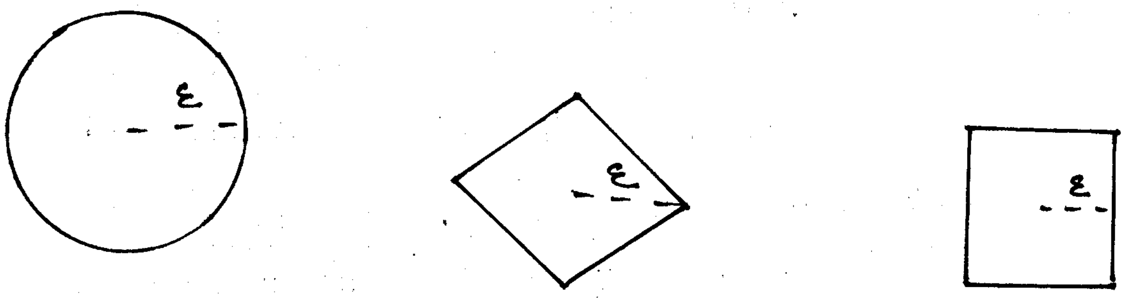
2) For the standard metric on R,

$$S(x, \varepsilon) = \langle x - \varepsilon, x + \varepsilon \rangle$$

Hence it generates the standard topology.

3) For the metric in 1.4 (4), the spheres are circular discs and the topology generated is the standard topology on C.

4) For the three metrics defined in 1.4 (3), the spheres are respectively the circular discs, diamonds and squares, as shown in the figure.



All these three metrics generate the standard topology on R^2. More generally, the three metrics defined in 1.4 (14) generate the standard topology on R^n.

SECTION III

PROPERTIES OF PS-METRIC SPACES

3.1. **Theorem**

Let (X,P) be a Ps-metric spaces, and

$$CS(x, \varepsilon; P) = \{y \ / \ P(x,y) \leq \varepsilon\}.$$

Then,

(1) $CS(x, \varepsilon)$ is a closed nbhd of x;

(2) $\overline{S(x, \varepsilon)} \leq CS(x, \varepsilon) \subseteq S(x, 2\varepsilon)$.

Proof:

1) The set $A = \{y \ / \ P(x,y) > \varepsilon\}$ is the complement of $CS(x, \varepsilon)$. We will show that A is open. Let y be any point of A. Choose $\delta = P(x,y) - \varepsilon$. If $t \in S(y,\delta)$, then

$$P(t,x) \geq |P(x,y) - P(t,y)|$$

$$= \varepsilon + \delta - P(t,y)$$

$$> \varepsilon .$$

which shows that $S(y,\delta) \subseteq A$. Hence A is open.

2) The second inequality of the statement is trivial where first follows from (1).

3.2. **Remark**

In the above theorem $CS(x, \varepsilon)$ is called the closed ε-sphere at x. The notation $CS(x, \varepsilon)$ is not the closure of $S(x, \varepsilon)$. However, if we take the discrete metric, $S(x,1)$ and its closure are both $\{x\}$, but $CS(x, 1)$ is the whole set X. Thus, despite our intuition, the first inequality in 3.1 (2) cannot be improved to an equality. Those Ps-metrics in which equality occurs are sometimes called 'round'.

3.3. Definition

A topological space X is said to be a T_0 space if it satisfies any of the following equavalent conditions.

(1) (To axiom) $x \neq y$ iff there exists a nbhd of one of the points and not containing the other.

(2) $x \neq y$ iff there exists a subbasis nbhd of one point which misses the other.

(3) $\bar{x} = \bar{y}$ iff $x = y$.

3.4. Definition

A topological space X is said to be regular if it satisfies any of the following equivalent conditions.

(1) (Regularity axiom) Given any point x and a nonempty closed set F not containing x, there exist disjoint nbhds of x and F.

(2) Given any point x and a neighbourhood N of x, there exists a neighbourhood M of x such that $\bar{M} \subseteq N$.

(3) For every x, $\mathcal{N}_x$ is a neighbourhood basis at x.

(4) Each point of X has a local basis of closed neighbourhoods.

(5) Each point of X has a local subbasis of closed neighbourhoods.

A space which is regular and T_o is called a T_3 space.

3.5. Theorem

(1) A Ps-metric is T_o iff it is a metric.

(2) Every PS-metric space is regular. Hence, every metric space is T_3.

Proof:

(1) Let P be a metric, then,

$$x \neq y \implies P(x,y) = \delta > 0$$

Hence $S(x,\delta)$ contains x but not y. Similary, the reverse argument holds.

(2) The closed ε-spheres at x form a closed neighbourhood basis at x. Therefore, a PS-metric space is regular. Therefore, a metric space is T_3.

3.6. Theorem

For any function $f : (X,P) \to (Y,q)$, the following statements are equivalent.

(1) f is continuous.

(2) For every $x \in X$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$f(S(x,\delta)) \subseteq S(fx,\varepsilon).$$

(3) For every $x \in X$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$S(x,\delta) \subseteq f^{-1}[S(fx,\varepsilon)]$$

(4) For every $x \in X$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$P(x,y) < \delta \implies q(fx, f_y) < \varepsilon.$$

Proof:

(1) ==> (3).

Since f is continuous, $f^{-1}(S(f_x,\varepsilon))$ is an open neighbourhood of x and hence must contain a spherical neighbourhood centred at x.

(3) ==> (1).

The condition implies that the preimage of every basic open neighbourhood of $f(x)$ is a neighbourhood of x.

(2) <==> (3)

As we have $f(A) \subseteq B \iff A \subseteq f^{-1}(B)$ where $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.

(2) $\Longrightarrow$ (4)

Since each is restatement of the other.

3.7. <u>Theorem</u>

A function $f : (X,P) \longrightarrow (Y,q)$ satisfies the following any equivalent conditions.

(1) For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$f(S(x,\delta)) \subseteq S(f_x, \varepsilon), \text{ for all } x \text{ in } X.$$

(2) For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$S(x,\delta) \subseteq f^{-1}(S(f_x, \varepsilon)) \text{ for all } x \text{ in } X.$$

(3) For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$P(x,y) < \delta \Longrightarrow q(f_x, f_y) < \varepsilon.$$

The equivalent conditions proved already.

3.8. <u>Definition</u>

The function defined above is called p-q uniformly continuous (or uniformly continuous).

3.9. <u>Theorem</u>

If P and q are Ps-metrics on X, we write $p \lesssim q$ if any of the following equivalent conditions holds.

(1) $p(x,y) \leq q(x,y)$ for all x,y in X

(2) $S(x, \varepsilon; q) \subseteq S(x, \varepsilon; p)$, for all x in X and all $\varepsilon > 0$.

Proof:

Let us prove (1) assuming (2) is true since (1) $\Longrightarrow$ (2) trivially. Let x,y be any points, and let $q(x,y) = a$. Then,

$$y \in S(x, a+\varepsilon; q) \text{ , for all } \varepsilon > 0$$

Hence, $y \in S(x, a+\varepsilon; p)$ also.

that is, $p(x,y) < a + \varepsilon$.

This is true for all positive ε,

$$p(x,y) \leq a$$

$$\Longrightarrow \quad p(x,y) \leq q(x,y).$$

3.10. Theorem

If p and q are Ps-metric on X, any of the following equivalent conditions holds.

(1) $\mathcal{T}(p) \subseteq \mathcal{T}(q)$.

(2) For every $x \in X$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$S(x, \delta; q) \subseteq S(x, \varepsilon; p).$$

Notationally we write $p \underset{t}{\leq} q$.

3.11. <u>Definition</u>

In the above theorem p is said to be topologically weaker than q.

3.12 <u>Theorem</u>

(1) The relations $\leq$ and $\overset{t}{\leq}$ are both reflexive orders.

(2) $p \leq q \;\not\Longrightarrow\; p \overset{t}{\leq} q$

3.13. <u>Remark</u>

As a reflexive order, each of $\leq$ and $\overset{t}{\leq}$ determines an equivalence relation. The first one is trivial (since $p \leq q$ and $q \leq p \Longrightarrow p = q$). The equivalence relation determined by $\overset{t}{\leq}$ is denoted by $\overset{t}{\sim}$ and called the topological equivalence, abbreviated to 'Top-Equivalence' or to 't-Equivalence'. Note that if k is any positive constant, p and kP are topologically equivalent Ps-metrics.

3.14. <u>Definition</u>

Let (X,P) be a Ps-metric space, and $A \subseteq X$. Then

(1) $$\delta_p(A) = \sup\{P(x,y) \;/\; x,y \in A\}$$
$$= \sup\{p(A \times A)\}$$

is called the p-diameter of A, or simply Diameter of A. We also write $\delta(A)$ instead of $\delta_p(A)$.

3.15. Note:

The supremum is taken in the extended real system, which means that every subset of X has a diameter. This means that $\delta(\phi) = -\infty$ and that for some subsets, the diameter may be ∞.

3.16. Theorem

Let (X,P) be any Ps-metric space, A,B any subsets of X, and x any point of X.

(1) $\delta(A) \geq 0$, for all nonempty A.

(2) $A \subseteq B \implies \delta(A) \leq \delta(B)$.

(3) $\delta(\bar{A}) = \delta(A)$.

(4) $\delta(A) = 0 \iff a \subseteq A \subseteq \bar{a}$, for some point a.

(5) $\delta(A) < \infty$, if A is finite.

(6) $\delta(S(x,\varepsilon)) \leq 2\varepsilon$; $\delta(CS(x,\varepsilon)) \leq 2\varepsilon$.

Equality need not hold in either case.

(7) $\delta(A \cup B) \leq P(a,b) + \delta(A) + \delta(B)$, where a,b are any points of A,B respectively. Hence, if A and B intersect,

$$\delta(A \cup B) \leq \delta(A) + \delta(B).$$

(8) More generally,

$$\delta(A_1 \cup A_2 \cup \ldots \cup A_n) \leq M + \delta(A_1) + \delta(A_2) + \ldots + \delta(A_n)$$

where M is the diameter of a finite set of points taken one each from the s ets A_i.

3.17. Theorem

Let (X,P) be a Ps-metric space, and $B \subseteq X$. Then B satisfies any of the following conditions.

(1) $\delta(B) < \infty$

(2) $P(B \times B)$ is bounded in R.

(3) There exists a real number M such that for all a,b in B

$$P(a,b) \leq M.$$

(4) B is contained in some sphere.

3.18. Definition

Let (X,P) be a Ps-metric space, and $B \subseteq X$. Then B is said to be P-bounded (or bounded) if it satisfies any of the above equivalent conditions.

3.19. Examples

The empty set, more generally every finite set, is bounded no matter which Ps-metric is used. Closure of any finite set (in particular, every point closure) is bounded, irrespective of which Ps-metric is used. In the discrete metric, as well as in indiscrete Ps-metric, every set is bounded.

3.20. Remark

Very often, a set is defined to be bounded if its diameter is finite. This would entail an unsatisfactory situation because the empty set would not be bounded according to such a definition.

3.21. Theorem

Let (X,P) be a Ps-metric space, and $B \subseteq X$. Then B satisfies any of the following equivalent conditions.

(1) For every $\varepsilon > 0$, there exists a finite set $\{a_1, a_2, \ldots, a_n\}$ such that

$$B \subseteq \cup \{S(a_i, \varepsilon) \;/\; 1 \leq i \leq n\}.$$

(2) For every $\varepsilon > 0$, there exist finitely many subsets $A_1, A_2, \ldots, A_n$ such that

$$\delta(A_i) \leq \varepsilon, \text{ for all } 1 \leq i \leq n,$$

and
$$B \subseteq \cup \{A_i \;/\; 1 \leq i \leq n\}.$$

Proof:

(1) ==> (2) trivially.

To prove (2) ==> (1),

Choose a point a_i from each A_i. Then B is covered by the spheres $S(a_i, 2\varepsilon)$.

3.22. Definition

Let (X,P) be a Ps-metric space, and $B \subseteq X$. If B satisfies any of the above equivalent conditions B is said to be totally p-bounded (or totally bounded).

3.23. Remark

If p is a Ps-metric on X, and $A \subseteq X$, then the restriction of p to $A \times A$ is a Ps-metric on A which is denoted by $PA \times A$, or more simply by PA or even by p. It is easy to see that in all the three definitions - diameter, boundedness and total boundedness, we could have used, instead of p, its restriction to the set concerned.

3.24. Theorem

In a Ps-metric space, compact

$\Longrightarrow$ totally bounded $\Longrightarrow$ bounded.

Proof:

Let A be a compact subset of a Ps-metric space X. The set of all ε-spheres is an open cover for A, hence it must contain a finite subcover. Therefore A is totally bounded.

Let B be a totally bounded subset of X. Then there are finitely many points $a_1, a_2, \ldots, a_n$ such that B is covered by the spheres $S(a_i, 1)$. Then B is contained in the sphere $S(a_1, M+1)$, where M is the diameter of the set $\{a_1, a_2, \ldots, a_n\}$. Thus, B is bounded.

3.25. Remark:

The implications in the above theorem cannot be reversed. If X is infinite and is given the discrete metric, then X is bounded, but it is not totally bounded because it cannot be covered by finitely many 1/2-spheres. The interval $\langle 0, 1 \rangle$ shows that total boundedness need not imply compactness.

3.26. Corollary

In a metric space, every compact set is closed and totally bounded.

3.27. Theorem

Let (X, P) be a Ps-metric space.

(1) If B is bounded, and $A \subseteq \bar{B}$, then A is bounded.

(2) Finite union of bounded sets is bounded.

Both statements are valid if 'totally bounded' replaces 'bounded'.

Proof:

Both statements for boundedness follow easily from the properties in Theorem 3.16. As for total boundedness,

(2) being trivially easy, we will only prove (1).

Let $\varepsilon > 0$ be given. Then there is a finite set $\{a_i\}$ such that

$$B \subseteq \cup S(a_i, \varepsilon/2).$$

Hence,
$$A \subseteq \bar{B}$$
$$\subseteq \cup \overline{S(a_i, \varepsilon/2)}$$
$$\subseteq \cup S(a_i, \varepsilon).$$

3.28. Remark

If (X,P) is a Ps-metric space, and the set X is bounded, it is conventional to say that p is bounded. Specifically, if the diameter of X is M, then as a function p is bounded by $\underline{M}$. But we write $P \leq M$ instead of the more accurate $P \leq \underline{M}$, and say that p is bounded by M.

3.29. Theorem

Let (X,P) be a Ps-metric space. Define,

(1) $\rho = p/1+p$.

(2) $\pi = p \wedge \underline{1}$.

Then, ρ and π are both Ps-metrics bounded by 1 and both are t-equivalent to p.

<u>Proof:</u>

$\rho(x,x) = 0 = \pi(x,x)$, $\rho < 1$, and $\pi \leq 1$ are obvious. We will verify the triangle in equality. The inequality

$$\pi(x,y) \leq \pi(y,z) + \pi(z,x)$$

is trivial in case the right-hand side is ≥ 1. If it is less than 1, each term has to be less than 1; and then

$$\begin{aligned} \pi(y,z) + \pi(z,x) &= p(y,z) + p(z,x) \\ &\leq p(x,y) \\ &\leq \pi(x,y) \end{aligned}$$

The proof of the triangle inequality for ρ depends on the easily verifiable inequalities

$$(3) \quad a \leq b \iff a/1+a \leq b/1+b \quad ;$$

$$a < b \iff a/1+a < b/1+b$$

valid for all nonnegative real numbers a,b. Writing p_1, p_2, p_3 for $p(x,y)$, $p(y,z)$, $p(z,x)$ respectively, we have $p_1 \leq p_2 + p_3$. Hence,

$$\begin{aligned} p_1/1+p_1 &\leq p_2+p_3/1+p_2+p_3 \\ &= (p_2/1+p_2+p_3) + (p_3/1+p_2+p_3) \\ &\leq (p_2/1+p_2) + (p_3/1+p_3). \end{aligned}$$

This proves the triangle inequality for ρ . Thus, $\rho_{A,}$ and π are Ps-metrics bounded by 1. To prove that they are t-equivalent with p, we will show that

$$\mathcal{T}_{\rho} \subseteq \mathcal{T}_{\pi} \quad \rho,$$

$$\subseteq \mathcal{T}_{p}$$

$$\subseteq \mathcal{T}_{p}$$

Of these, the first two inequalities follow directly from $\rho \leq \pi \leq p$. Lastely, by virtue of (3) for all x,y

$$(4) \qquad \rho(x,y) < (\varepsilon/1+\varepsilon)$$

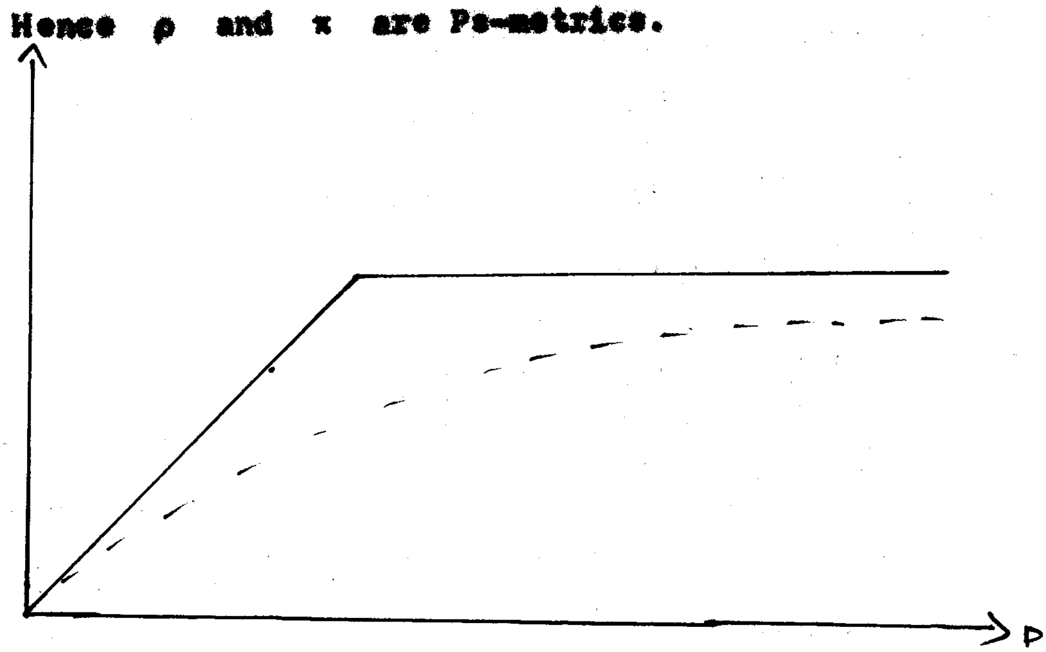
$$\Longrightarrow \qquad (p(x,y)/1+p(x,y)) < (\varepsilon/1+\varepsilon)$$

$$\Longrightarrow \qquad p(x,y) < \varepsilon .$$

This shows that $S(x,(\varepsilon/1+\varepsilon); \rho)$ $S(x, \varepsilon; p)$, for every x, proving that

$$\mathcal{T}p \subseteq \mathcal{T}\rho \,.$$

Hence ρ and π are Ps-metrics.



Graphs of $\rho = (p/(1+p))$ (dotted line)

$\pi = p \wedge 1$ (solid line).

3.30. **Remark**

The above theorem combined with the fact that P and kP ($k > 0$ is any constant) are t-equivalent, implies that any arbitrary Ps-metric can be replaced by a t-equivalent Ps-metric bounded by any pre-assigned positive number.

3.31. **Theorem**

Let $\{f_i : X \longrightarrow (Y_i, q_i)\}$ be a family of functions from a set X to Ps-metric spaces, each q_i being bounded. Then

(1) $P(a,b) = \sup_{i \in I} \{q_i(f_i a, f_i b)\}$, for all a,b in X

or equivalently,

(2) $p = V\{q_i \, o(f_i \times f_i)\}$ defines a Ps-metric on X.

Proof:

For notational simplicity, we will write a_i, b_i for $f_i a$, $f_i b$ respectively. First observe that the set a $\{q_i(a_i, b_i)\}_i$ is bounded and therefore has a supremum. Hence, $P(a,b)$ is a uniquely determined finite number for every a,b.

$P(a,a) = 0$ being obvious, it remains only to prove the triangle inequality. For all indices i,

$$q_i(a_i,b_i) \leq q_i(b_i,c_i) + q_i(c_i,a_i)$$

$$\leq p(b,c) + p(c,a).$$

Taking the supremum over all i, we get

$$p(a,b) \leq p(b,c) + p(c,a).$$

3.32. <u>Definition</u>

The above defined Ps-metric on X is called the initial Pseudo-Metric of the Family $\{ f_i : X \to (Y_i,q_i) \}$.

3.33. <u>Remark</u>

The initial Ps-metric is bounded. The boundendness condition on q_i is hypothesized solely for ensuring that the set $\{q_i(a_i,b_i)\}$ is always bounded and its supremum exists finitely. If the family is finite, this condition is not required. It will be understood implicitly that whenever the initial Ps-metric of a finite family is involved, the q_is are not assumed bounded.

3.34. <u>Result</u>

If p is the initial Ps-metric defined in 3.31, Observe that $q_i(a_i,b_i) \leq p(a,b)$, for all a,b in X. This means each f_i 'reduces' distances such functions are therefore called contractive functions and they play an important role especially in differential and integral equations.

3.35. <u>Definition</u>

Let $f : (X,p) \longrightarrow (Y,q)$ be a function. f is said to be contractive if for all a,b in X,

(1) $q(f_a, f_b) \leq p(a,b)$

f is said to be distance-preserving if for all a,b in X,

(2) $q(f_a, f_b) = p(a,b)$.

3.36. <u>Note</u>

Conditions (1) and (2) in the above definition could also be written in the functional rotation as

$q_o(f \times f) \leq p$ and

$q_o(f \times f) = p$ respectively.

3.37. <u>Theorem</u>

Let $f : X \longrightarrow Y$, where X and Y are Ps-metric spaces. Then, f is distance preserving

$\Longrightarrow$ f is contractive

$\Longrightarrow$ f is uniformly continuous

$\Longrightarrow$ f is continuous.

<u>Proof:</u>

All implications follow easily from the definitions of all the four concepts.

3.38. Theorem

The initial Ps-metric of a family $\{f_i\}$ is the smallest Ps-metric (with respect to order relation $\leq$) on X for which every f_i is contractive.

3.39. Theorem

If p is the initial Ps-metric of $\{f_i : X \rightarrow (Y_i, q_i)\}$, then

$$IT\{f_i\} \subseteq \mathcal{T}_p.$$

Equality occurs if the family is finite.

Proof:

When X is given the initial Ps-metric, each f_i is contractive, and hence continuous. From this follows the first part of the theorem. For the second part, if I is finite, then

$$\bigcap_{i \in I} f_i^{-1}(S(f_i x, \varepsilon; q_i)) \subseteq S(x, \varepsilon; p)$$

which shows that

$$\mathcal{T}_p \subseteq IT\{f_i\}.$$

3.40 Theorem.

Every Ps-metric space is first-countable. But not the converse.

3.41. **Theorem**

In a Ps-metric space,

(1) compact (c) iff countably compact (C C) iff sequentially compact (S.C).

(2) totally bounded (T.B) implies second countable (C_{11})

(3) second countable iff seperable (S) iff Lindelof (L).

Proof:

Using the obvious abbreviations for the various properties, we will prove the following chain of implications.

$$C \Longrightarrow C.\ C \Longrightarrow S.C \Longrightarrow T.B \Longrightarrow S \Longrightarrow C_{11} \Longrightarrow L \Longrightarrow S.$$

The first implication holds in any space; the second is true because every Ps-metric space is first-countable. To prove the third, suppose that X is not totally bounded. Then there exists an $\varepsilon > 0$ such that no finite set of ε-spheres can cover X. Choose a sequence inductively thus; let x_1 in X be arbitrary; and after having chosen $x_1, x_2, \ldots, x_k$, let x_{k+1} be any point not in

$$\bigcup_{i=1}^{k} S(x_i, \varepsilon)$$

This defines an infinite sequence which cannot have a convergent subsequence because the distance between any two points in the sequence is greater than or equal to . Thus X cannot be sequentially compact.

(T.B ==> S).

If X is totally bounded, for each $n \in N$ there is a finite set A_n such that

$$\bigcup_{a \in A_n} S(a, 1/n) = X.$$

The set

$$A = \bigcup_{n=1}^{\infty} A_n$$ is countable and is easily seen to be dense. ($S \Longrightarrow C_{II}$).

Suppose that, X is separable and let D be a countable dense subset. Then,

$$\{S(d, 1/n) \;/\; d \in D,\; n \in N\}$$ is a countable basis.

($C_{II} \Longrightarrow L$). True in any case .

(L ==> S).

Let X be a Lindelof space. For each $n \in N$, the open cover

$$\{S(x, 1/n) \;/\; x \in X\}$$ for X.

contains a countable subcover. Let D_n denote the set of centres of the countable subcover. Then

$$D = \bigcup_{n=1}^{\infty} D_n$$ is a countable dense subset.

The implications proved above obviously imply (2) and (3). To complete the proof of (1), it remains to show that sequential compactness implies compactness. If X is sequentially compact, it is also countably compact. We have just proved that sequential compactness implies L. Thus, X is both L and CC and hence is compact.

SECTION IV

HILBERT PS-METRIC

4.1. Theorem

Let $\{f_i : X \to (Y_i, q_i)\}$, in Z_+, be a countable family, each q_i being a bounded Ps-metric. Let (C_i) be a sequence of positive constants such that $\Sigma C_i = 1$. For any constant $k \geq 1$, define

$$h_k(a,b) = \sqrt{\Sigma\, C_i^k\, q_i^k\,(f_i a,\, f_i b)}$$

for all a, b in X. (Σ indicates summation from 0 to ∞ unless otherwise noted). Then, h_k is a bounded Ps-metric and generates the initial topology of $\{f_i\}$. Further, each f_i is uniformly continuously, when X is given this Ps-metric.

Proof:

Let $h = h_k$, and $a_i = f_i(a)$.

$h(a,a) = 0$ is obvious, and the triangle inequality comes directly on applying Minkowski's inequality to the sequences (α_i) and (β_i), where

$$\alpha_i = c_i\, q_i\,(b_i,\, d_i)$$

and $$\beta_i = c_i\, q_i\,(d_i,\, a_i).$$

Thus h isa Ps-metric, that it is bounded is obvious from the definition.

Let denote the initial topology of $\{f_i\}$. For any index j, it is

$$h(a,b) < e_j \varepsilon$$

$$\Longrightarrow \qquad q_j(a_j,b_j) < \varepsilon \text{ for all } a,b \text{ in } X.$$

This shows that f_i is uniformly continuous, and hence it is continuous, when X carries the topology $\mathcal{T}_h$. This further implies that

$$\mathcal{T} \subseteq \mathcal{T}_h .$$

To prove the oppsositive inequality, given any sphere $S(a, \varepsilon; h)$, choose an integer N such that

$$\sum_{i=N+1}^{\infty} e_i^k < \frac{\varepsilon^k}{2}$$

Let $A = \bigcap_{i=0}^{N} f_i^{-1} S(a_i, \delta; q_i)$, where $\delta = \varepsilon/2$. Then for all b in A,

$$h^k(a,b) = \sum_{i=0}^{N} e_i^k q_i^k (a_i,b_i) + \sum_{N+1}^{\infty} e_i^k q_i^k (a_i,b_i)$$

$$< \delta^k \sum_{0}^{N} e_i^k + \sum_{N+1}^{\infty} e_i^k$$

$$< \delta^k + \varepsilon^k/2$$

$$\leq \varepsilon^k .$$

Hence, $h(a,b) < \varepsilon$. This shows that

$$A \subseteq S(a, \varepsilon ; h) \text{ and}$$

consequently

$$\mathcal{T}_h \subseteq \mathcal{T} .$$

4.2. Definition

Above defined h_k is called a Hilbert Ps-Metric.

4.3. Remark:

There are four special cases of the initial Ps-metric construction.

1) The initial Ps-metric of a singleton $f : X \longrightarrow (y,q)$ is called the Preimage Ps-metric of q by f. It is given by the formula

$$p = q \circ (f \times f).$$

2) The preimage Ps-metric of the inclusion function

$i : A \subseteq A(X,P)$ is the induced Ps-metric, and is precisely the Ps-metric P_A defined earlier.

3) If $\{p_i\}$ is a family of Ps-metric on a set X, then

$p = V_{p_i}$ is supremum Ps-metric, which could also be defined as the initial Ps-metric of the family of identity functions.

(4) If $\{(X_i, q_i)\}$ is a family of Ps-metric spaces, and $X = \pi X_i$, the initial Ps-metric of the family of projections is called the product Ps-metric, sometimes also called the uniform Ps-metric or sup Ps-metric.

4.4. <u>Theorem</u>

The product Ps-metric is the smallest Ps-metric such that each projection is contractive.

Now we will consider three cases of the product.

(1) <u>Finite</u>:

If $\{(X_i, q_i)\}$, $i \leq i \leq n$, is a finite family of Ps-metric spaces, we take $e_i = 1$ in defining the Hilbert Ps-metrics

$$h_k(a,b) = \left(\sum_{i=1}^{n} q_i^k (a_i, b_i) \right)^{1/k}$$

When $k = 2$, the Hilbert Ps-metric is usually called the Euclidean Ps-metric.

(2) <u>Countably Finite</u>:

If the family is countably infinite, we have at our disposal the product Ps-metric as well as all the Hilbert Ps-metrics. The latter are mostly found to be more convenient since they generate the product topology.

(3) Uncountable:

If the family is uncountable, the Hilbert Ps-metrics are not available and there may not be any Ps-metric that generates the product topology. The topology generated by theproduct Ps-metric is generally too large, even larger than the box topology. Even so, it remains a useful Ps-metric, especially in function spaces.

BIBLIOGRAPHY

[1] MANGESH G. MURDESHWAR : 'General Topology', Wiley Eastern Limited, New Delhi, 1983.

[2] KLEE, V.L. Jr. : 'Some Characterization of Compactness', American Mathematics Monthly 58 (1951), 389-393.

[3] VAUGHAN, H.E. : 'On a Class of Metrics Defining a Metric Space'. Bull. Amer. Math. Soc. 44 (1938), 557-561.

[4] SIMMONS, GEORGE : 'Introduction to Topology and Modern Analysis', McGraw-Hill, 1963.

[5] WILANSKY, ALBERT : 'Topology for Analysis', Ginn, 1970.

* * *