Chapter II

CHAPTER – II

CONNECTEDNESS IN L-FUZZY TOPOLOGICAL SPACES

In this chapter we discuss the concepts of P-connectedness and P2-connectedness in L-fuzzy topological spaces introduced by Bai [5] and Li, Fang and Zhao [23], respectively. Both P-connectedness and P2-connectedness are defined using the notion of preclosed sets and they preserve some fundamental properties of connected sets in general topological spaces. It has been proved that the preclosure of a P-connected set (resp. P2-connected set) is P-connected (resp. P2-connected) and union of P-connected (resp. P2-connected) sets is P-connected (resp. P2-connected) provided their intersection is a non-null set. Moreover, the image of a P-connected space (resp. P2-connected space) under a P-irresolute order homomorphism is P-connected (resp. P2-connected). Some interesting characterizations of P-connectedness (resp. P2-connectedness) are proved and the famous K.Fan's theorem has been extended to both P-connectedness and P2-connectedness. Relations among connectedness, P-connectedness and P2-connectedness are discussed. It has been shown that every P2-connected L-fuzzy set is P-connected. An interesting example is given to show that the converse is not true. But if $1 \in M(L)$, then every P-connected fuzzy set is also P2-connected. Also, it has been proved that if D is a crisp set, then D is P-connected iff it is P2-connected. First let us give the preliminary definitions and results needed for out discussion.

Section 2.1

Preliminary definitions and results

Definition : 2.1.1

A **poset** is a set X with a binary relation $x \leq y$ for $x, y \in X$, which satisfies

P1. For all x, $x \leq x$ (Reflexivity)

P2. If $x \leq y$ and $y \leq x$ then $x = y$ (Anti symmetry)

P3. If $x \leq y$ and $y \leq z$ then $x \leq z$ (Transitivity)

Definition : 2.1.2

By a **smallest element** of a subset Y of a poset X, we mean an element $a \in Y$ such that $a \leq y$ for all $y \in Y$.

Definition : 2.1.3

By a **largest element** of a subset Y of a poset X, we mean an element $b \in Y$ such that $b \geq y$ for all $y \in Y$.

Definition : 2.1.4

The (unique) smallest and largest elements of the whole poset X, when they exist, are called the **universal bounds of X**, and are denoted by 0 and 1, respectively. 0 is referred to as the **zero element** of X.

Definition : 2.1.5

A **lattice** is defined as a poset L in which any two elements x and y have a greatest lower bound (g.l.b) (or) "meet" $x \wedge y$ and a least upper bound (l.u.b) (or) "join" $x \vee y$.

Definition : 2.1.6

In a lattice L if

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ are satisfied for every $x, y, z \in L$, then L is said to be a **distributive lattice**.

Definition : 2.1.7

In a distributive lattice L if every subset X has a g.l.b, **inf X**, and a l.u.b, **sup X**, then L is called a **completely distributive lattice**.

Definition : 2.1.8

By a **complement** of an element x in a lattice we mean an element $y \in L$ such that

$x \wedge y = 0$ and $x \vee y = 1$

where 0 and 1 are the smallest and largest elements in L.

The lattice L is called **complemented** if all its elements have complements.

Theorem : 2.1.9

In a complemented distributive lattice each element has one and only one complement and the unique complement of x is denoted by x′ (hence $x \wedge x' = 0$ and $x \vee x' = 1$).

Properties

(1) $x \wedge 0 = 0$, $x \vee 0 = x$

(2) $x \wedge 1 = x$, $x \vee 1 = 1$

(3) $(x')' = x$ (involution law)

(4) $(x \wedge y)' = x' \vee y'$, $(x \vee y)' = x' \wedge y'$ (De Morgan's law)

(5) $x \wedge a = 0 \Leftrightarrow x \leq a'$

(6) $x \leq y \Rightarrow y' \leq x'$ (inverts order)

Remark : 2.1.10

In view of properties (3) and (6) " ′ " is referred to as "**order reversing involution**".

Notation : 2.1.11

Throughout this dissertation by $L = L(\vee, \wedge, ')$ we mean a completely distributive lattice with order reversing involution " ′ ". 0 and 1 denote the smallest element and largest element in L, respectively.

Definition : 2.1.12

If $f : L_1 \rightarrow L_2$ where L_1 and L_2 are completely distributive lattices, then $f^{-1} : L_2 \rightarrow L_1$ is the **inverse of f** defined by

$$\mathbf{f^{-1}(B)} = \vee\{A \in L_1 / f(A) \leq B\} \text{ for any } B \in L_2.$$

Definition : 2.1.13

If $f : L_1 \rightarrow L_2$ where L_1 and L_2 are completely distributive lattices, then

$$\mathbf{f(A)} = \wedge\{B \in L_2 / f(A) \leq B\} \text{ for any } A \in L_1$$

Definition : 2.1.14

Let L be a completely distributive lattice and X be a nonempty set. A mapping $D : X \rightarrow L$ is called an **L-fuzzy set**. The set of all **L-fuzzy sets** on X is denoted by $\mathbf{L^X}$.

The smallest element and largest element in L^X are denoted by $\bar{0}$ and $\bar{1}$.

$\bar{0}$ is referred to as the **null set**.

Definition : 2.1.15

If $D : X \rightarrow L$ is an L-fuzzy set then its **complement** D' is also an L-fuzzy set defined by $D'(x) = (D(x))'$ for every $x \in X$.

Definition : 2.1.16

If A is an ordinary subset of X, i.e. if $A \subset X$, then A can be considered as an L-fuzzy set by identifying A with its characteristic function. Hence A is identified with the mapping from $X \rightarrow L$ given by

$$\begin{aligned} A(x) &= 0 \quad \text{if } x \in A \\ &= 1 \quad \text{if } x \notin A \end{aligned}$$

where 0 and 1 are the smallest and the largest elements in L. With this identification, when we consider $A \subset X$ as an L-fuzzy set we call A as a **crisp set**.

Definition : 2.1.17

Let L be a completely distributive lattice and $a \in L$, a is called an **irreducible element** of L, if the equality $p \vee q = a$ implies $p = a$ (or) $q = a$, where $p \in L$ and $q \in L$ implies $p = a$ (or) $q = a$, where $p \in L$ and $q \in L$.

Definition : 2.1.18

An element $p \in L$ is called a **prime element** if for $b, c \in L$, $p \geq b \wedge c$ implies $p \geq b$ or $p \geq c$.

Definition : 2.1.19

An element $p \in L$ is called **union-irreducible** or a **co-prime element** if p' is a prime element i.e. if for $b, c \in L$, $p \leq a \vee b$ implies $p \leq b$ or $p \leq c$.

Notation : 2.1.20

- The set of all non zero union-irreducible (or co-prime) elements in a lattice L is denoted by **M(L)**.
- The set of all non zero union-irreducible (or co-prime) elements in L^X is denoted by **$M^*(L^X)$**.
- For any L-fuzzy set D, $M^*(D)$ denotes the set of all non zero union-irreducible (or co-prime) elements contained in D.

Definition : 2.1.21

For each $\alpha \in L$, a **fuzzy point x_α** in X is a L-fuzzy set of X defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

In this case, x and α are called the **support** and the **value** of x_α, respectively.

Definition : 2.1.22

If $\alpha \in M(L)$ the fuzzy point x_α is called a **molecule**.

Definition : 2.1.23

An **L-fuzzy topological space**, briefly, written as **L-fts** is a pair (L^X, δ) where δ is a subfamily of L^X which contains $\bar{0}$, $\bar{1}$ and is closed for arbitrary suprema and finite infima. δ is called an **L-fuzzy topology** on X. Each member of δ is called an **open L-fuzzy set** and its complement is called a **closed L-fuzzy set**.

Definition : 2.1.24

Let (L^X, δ) be an L-fts. The **closure** and **interior** of an L-fuzzy set $A \in L^X$ are defined respectively, as

$$\textbf{cl(A)} = \bigwedge\{B : B \geq A, B' \in \delta\}$$

$$\textbf{int(A)} = \bigvee\{B : B \leq A, B \in \delta\}$$

It is easily seen that cl A is the smallest closed L-fuzzy set larger than A and int A is the largest open L-fuzzy set smaller than A.

Definition : 2.1.25

An L-fuzzy set A in X is called a **semi-open set** if there exists an open L-fuzzy set B such that $B \leq A \leq cl(B)$, or equivalently, $A \leq cl(int(A))$.

The complement of an L-fuzzy set which is semi-open is called a **semi-closed set**.

Definition : 2.1.26

The intersection of all semi-closed sets containing the L-fuzzy set A is called the **semi-closure** of A and is denoted by **scl(A)**.

Definition : 2.1.27

The union of all semi-open sets contained in an L-fuzzy set A in X is called the **semi-interior** of A and is denoted by **sint(A)**.

.**Definition : 2.1.28**

$A \in L^X$ is called a preopen set iff $A \le \text{int cl } A$ and $A \in L^X$ is called a **preclosed set** iff $A \ge cl(int(A))$.

The pre-closure of the L-fuzzy set A is the intersection of all preclosed sets, each containing A. it will be denoted by **pcl(A)**.

Definition : 2.1.29

Let L_1 and L_2 be fuzzy lattices. A mapping $f : L_1 \to L_2$ is called an **order-homomorphism** if the following conditions hold :

(1) $f(0) = 0$

(2) $f(\vee A_i) = \vee f(A_i)$ for $\{A_i\} \subset L_1$

(3) $f^{-1}(B') = (f^{-1}(B))'$ for each $B \in L_2$.

Definition : 2.1.30

Let (L_1^X, δ) and $(L_2^Y, \mathcal{T})$ be two L-fts's. A mapping $f : (L_1^X, \delta) \to (L_2^Y, \mathcal{T})$ is called a **P-irresolute** mapping if $f^{-1}(B)$ is a preopen set of L_1^X for each preopen set B of L_2^Y.

Theorem : 2.1.31

An order-homomorphism $f : (L_1^X, \delta) \to (L_2^Y, \mathcal{T})$ is P-irresolute iff $pcl(f^{-1}(B)) \le f^{-1}(pcl\ B)$, for each $B \in L_2^Y$.

Definition : 2.1.32

Let (L^X, δ) be an L-fts, $M^*(L^X = \{x_\alpha / \alpha \in M(L)\}$, $x_\lambda \in M^*(L^X)$ and P a preclosed set in L^X. P is called a **preclosed remote-neighbourhood**, or briefly, **PCRN**, of x_λ, if $x_\lambda \in P$. The set of all PCRNs of x_λ will be denoted by **$\delta(x_\lambda)$**.

Section 2.2

P-connectedness in L-fuzzy topological spaces

In this section we give the definition of P-connected sets in L-fts's introduced by Bai [5] and discuss some fundamental properties of P-connected sets which are analogous to the corresponding properties in general topological spaces. The famous K.Fan's theorem on connectedness is also extended to P-connectedness in L-fts.

Definition : 2.2.1

Let (L^X, δ) be an L-fts and A, B $\in L^X$. Then A and B are said to **P-separated** if $pcl(A) \wedge B = A \wedge pcl(B) = \bar{0}$.

Theorem : 2.2.2

Let (L^X, δ) be an L-fts and A, B $\in L^X$. If A and B are P-separated and $C \leq A$, $D \leq B$, then C and D are also P-separated.

Proof

Given A and B are P-separated and $C \leq A$ and $D \leq B$

Consider $pcl(C) \wedge D \leq pcl(A) \wedge B$

$= \bar{0}$ as A and B are P-separated

$\therefore pcl(C) \wedge D = \bar{0}$

Similarly, $C \wedge pcl(D) \leq A \wedge pcl(B) = \bar{0}$

Hence $C \wedge pcl(D) = \bar{0}$

$\therefore$ C and D are P-separated.

Definition : 2.2.3

Let (L^X, δ) be an L-fts and $A \in L^X$. A is called a **P-connected set** if A cannot be represented as a union of two P-separated non-null sets. Specifically, when $A = \bar{1}$ is P-connected, we call (L^X, δ) a **P-connected space**.

Theorem : 2.2.4

Let (L^X, δ) be an L-fts. Then the following are equivalent :

(1) (L^X, δ) is not a P-connected space

(2) There exist two non-null preclosed sets A and B such that $A \vee B = \bar{1}$ and $A \wedge B = \bar{0}$.

(3) There exist two-non-null preopen sets A and B such that $A \vee B = \bar{1}$ and $A \wedge B = \bar{0}$.

Proof

(1) ⇒ (2)

Since (L^X, δ) is not a P-connected space, there exist non-null P-separated sets C and D such that $C \vee D = \bar{1}$

Take $A = \mathrm{pcl}(C)$ and $B = \mathrm{pcl}(D)$.

Then A and B are non-null preclosed sets.

$$\begin{aligned} \text{Consider } A \vee B &= \mathrm{pcl}(C) \vee \mathrm{pcl}(D) \\ &\geq C \vee D \\ &= \bar{1} \end{aligned}$$

$$\therefore A \vee B = \bar{1}$$

$$\begin{aligned} \text{Consider } A \wedge B &= \mathrm{pcl}(C) \wedge \mathrm{pcl}(D) \\ &= \mathrm{pcl}(C) \wedge \mathrm{pcl}(D) \wedge \bar{1} \\ &= \mathrm{pcl}(C) \wedge \mathrm{pcl}(D) \wedge (C \vee D) \\ &= (\mathrm{pcl}(C) \wedge \mathrm{pcl}(D) \wedge C) \vee (\mathrm{pcl}(C) \wedge \mathrm{pcl}(D) \wedge D) \\ &= (C \wedge \mathrm{pcl}(D)) \vee (\mathrm{pcl}(C) \wedge D) \end{aligned}$$

$= \bar{0} \vee \bar{0}$ since C and D are P-separated

$= \bar{0}$

Hence (2).

(2) ⇒ (3)

By (2), there exist non-null preclosed sets A and B such that $A \vee B = \bar{1}$ and $A \wedge B = \bar{0}$.

Let $C = A'$ and $D = B'$

Then C and D are non-null preopen sets.

Also, $C \vee D = A' \vee B'$

$= (A \wedge B)'$

$= (\bar{0})'$

$= \bar{1}$

and $C \wedge D = A' \wedge B'$

$= (A \vee B)'$

$= (\bar{1})'$

$= \bar{0}$

Hence (3).

(3) ⇒ (1)

By (3), there exist non-null preopen sets A and B such that $A \vee B = \bar{1}$ and $A \wedge B = \bar{0}$.

Let $C = A'$ and $D = B'$

Then $C \vee D = A' \vee B'$

$= (A \wedge B)'$

$= (\bar{0})'$

$= \bar{1}$

Also, $\mathrm{pcl}(C) \wedge D = \mathrm{pcl}(A') \wedge B'$

$= A' \wedge B'$ since A' is preclosed

$= (A \vee B)'$

$= (\bar{1})'$

$= \bar{0}$

and $C \wedge \text{pcl}\, D = A' \wedge \text{pcl}(B')$

$= A' \wedge B'$ since B' is preclosed

$= (A \vee B)'$

$= (\bar{1})'$

$= \bar{0}$

Hence C and D are P-separated and $C \vee D = \bar{1}$

$\therefore$ (L^X, δ) is not a P-connected space.

Hence (1).

Theorem : 2.2.5

Let A be a non-null P-connected set in L-fts (L^X, δ) and C and D be two P-separated sets in (L^X, δ). If $A \leq C \vee D$, then $A \leq C$ or $A \leq D$.

Proof

Since $A \leq C \vee D$, $A = A \wedge (C \vee D)$

$= (A \wedge C) \vee (A \wedge D)$

By Theorem 2.2.2, since C and D are P-separated, $A \wedge C \leq C$ and $A \wedge D \leq D$. we get $A \wedge C$ and $A \wedge D$ are P-separated.

Since A is P-connected and

$A = (A \wedge C) \vee (A \wedge D)$, we must have

$A \wedge C = \bar{0}$ or $A \wedge D = \bar{0}$

Suppose $A \wedge C = \bar{0}$, then

$A = A \wedge (C \wedge D)$

$= (A \wedge C) \vee (A \wedge D)$

$= \bar{0} \vee (A \wedge D)$

$= A \wedge D$

Hence $A \leq D$.

Similarly, $A \wedge D = \bar{0}$ implies $A \leq C$.

Hence the result.

Theorem : 2.2.6

Let A be a P-connected set in an L-fts (L^X, δ). If $A \leq B \leq pcl(A)$ then B is also P-connected in (L^X, δ).

Proof

Given $A \leq B \leq pcl(A)$

Suppose $B = C \vee D$, where C and D are P-separated.

Let $F = A \wedge C$, $G = A \wedge D$.

Since $F \leq C$ and $G \leq D$, by Theorem 2.2.2 F and G are P-separated.

Consider
$$\begin{aligned} F \vee G &= (A \wedge C) \vee (A \wedge D) \\ &= A \wedge (C \vee D) \\ &= A \wedge B \\ &= A \text{ as } A \leq B. \end{aligned}$$

Since A is P-connected, either $F = \bar{0}$ or $G = \bar{0}$. Suppose $F = \bar{0}$

Then
$$\begin{aligned} A &= F \vee G \\ &= \bar{0} \vee G \\ &= G \\ &= A \wedge D. \end{aligned}$$

Hence $A \leq D$.

$\therefore B \leq pcl(A) \leq pcl(D)$

Since $B = C \vee D$, $C \leq B$.

$\therefore C = C \wedge B \leq C \wedge pcl(A) \leq C \wedge pcl(D) = \bar{0}$ since C and D are P-separated.

Hence $C = \bar{0}$

Similarly, if $G = \bar{0}$, we get $D = \bar{0}$

Hence $B = C \vee D \Rightarrow$ either $C = \bar{0}$ or $D = \bar{0}$

$\therefore$ B cannot be represented as a union of two non-null P-separated sets.

$\therefore$ B is connected.

Theorem : 2.2.7

Let $\{A_t : t \in T\}$ be a family of P-connected sets in an L-fts (L^X, δ). Suppose there is an $s \in T$ such that A_t and A_s are not P-separated for each $t \neq s$, then $\vee_{t \in T} A_t$ is P-connected.

Proof

Let $A = \bigvee_{t \in T} A_t$

Suppose there exist P-separated sets B and C such that $A = B \vee C$

Since $A_s \leq \bigvee_{t \in T} A_t = A = B \vee C$ and

A_s is P-connected, by Theorem 2.2.5 we get $A_s \leq B$ or $A_s \leq C$.

Suppose $A_s \leq B$.

Similarly, for any $t \neq s$ also, we get

$A_t \leq B$ or $A_t \leq C$

Claim

$A_t \leq B$

Suppose not, then $A_t \leq C$

As B and C are P-separated and $A_s \leq B$,

By Theorem 2.2.2 we get A_s and A_t are P-separated which is a contradiction.

Hence $A_t \not\leq C$

$\therefore A_t \leq B$

Hence the claim.

Hence $A_s \leq B$ and $\forall\, t \neq s$, $A_t \leq B$

$\therefore \bigvee_{t \in T} A_t \leq B$ i.e., $A \leq B$.

Consider $C = C \wedge A$

$\leq C \wedge B$

$\leq C \wedge \mathrm{pcl}(B)$

$= \overline{0}$ as B and C are P-separated.

$\therefore C = \overline{0}$

Similarly if we assume $A_s \leq C$, we get $B = \overline{0}$

Hence A cannot be expressed as the union of two non-null P-separated sets

$\therefore$ A is P-connected.

Corollary : 2.2.8

Let $\{A_t : t \in T\}$ be a family of P-connected sets in an L-fts (L^X, δ). If $\wedge_{t\in T} A_t \neq \overline{0}$, then $\vee_{t\in T} A_t$ is P-connected.

Proof

$\bigwedge\limits_{t\in T} A_t \neq \overline{0}$

$\Rightarrow A_t \neq \overline{0} \quad \forall t \in T.$

$\Rightarrow \mathrm{pcl}(A_t) \neq \overline{0} \ \ \forall t \in T.$

$\Rightarrow$ For any $s \in T$, $\mathrm{pcl}(A_t) \wedge A_s \neq \overline{0}$ and

$A_t \wedge \mathrm{pcl}(A_s) \neq \overline{0} \ \ \forall t \in T.$

$\Rightarrow A_s$ and A_t are not P-separated.

Hence by Theorem 2.2.7, we get $\bigvee\limits_{t\in T} A_t$ is P-connected.

Theorem : 2.2.9

Let $f : L_1^X \rightarrow L_2^Y$ be a P-irresolute order-homomorphism. If A is P-connected in L_1^X, then f(A) is P-connected in L_2^Y.

Proof

Suppose there exist P-separated sets B and C

Such that $f(A) = B \vee C$

Since $A \leq f^{-1}(f(A))$, we get

$$A \leq f^{-1}(B \vee C)$$
$$= f^{-1}(B) \vee f^{-1}(C)$$

Since f is a P-irresolute order-homomorphism by Theorem 2.1.31, we get

$$\mathrm{pcl}(f^{-1}(B)) \wedge f^{-1}(C)$$
$$\leq f^{-1}(\mathrm{pcl}(B)) \wedge f^{-1}(C)$$
$$= f^{-1}(\mathrm{pcl}(B) \wedge C)$$
$$= f^{-1}(\bar{0}) \text{ as B and c are P-separated}$$
$$= \bar{0}.$$

Similarly $f^{-1}(B) \wedge \mathrm{pcl}\,(f^{-1}(C)) = \bar{0}$

$\therefore$ $f^{-1}(B)$ and $f^{-1}(C)$ are P-separated.

Since A is P-connected, by Theorem 2.2.5, we get

$A \leq f^{-1}(B)$ or $A \leq f^{-1}(C)$

Suppose $A \leq f^{-1}(B)$

Then $f(A) \leq f(f^{-1}(B))$

$\leq B$.

$$\therefore \quad C = C \wedge f(A)$$
$$\leq C \wedge B$$
$$\leq C \wedge \mathrm{pcl}(B)$$
$$= \bar{0} \text{ as B and C are P-separated}$$
$$\therefore C = \bar{0}$$

Similarly, if $A \leq f^{-1}(C)$, we get $B = \bar{0}$

$\therefore$ f(A) cannot be represented as union of non-null P-separated sets.

$\therefore$ f(A) is P-connected.

Corollary : 2.2.10

Let f : $L_1^X \to L_2^Y$ be a P-irresolute order-homomorphism and onto. If L_1^X, is a P-connected space, then so is L_2^Y.

Proof

This is immediate from Theorem 2.2.9.

Extension of K.Fan's theorem to P-connectedness

Theorem : 2.2.11

Let (L^X, δ) be an L-fts and $A \in L^X$. Then A is P-connected iff for each pair a, b of points of $M^*(A)$ and each mapping

$$P : M^*(A) \to U\{\delta(x) : x \in M^*(A)\},$$

where $P(x) \in \delta(x)$ for each $x \in M^*(A)$, there exists in $M^*(A)$ a finite number of points $x_1 = a, x_2, \ldots, x_n = b \ni A \not\leq P(x_i) \vee P(x_i + 1)$, $i = 1, 2 \ldots n - 1$

$M^*(A)$ denotes the set of all point of A, $\delta(x)$ denotes the set of all PCRNS of x for each $x \in M^*(A)$.

Proof

Sufficiency

Assume the given condition. We have to prove A is P-connected.

Suppose not, then there exist $B, C \in L^X$ and $B \neq \bar{0}$, $C \neq \bar{0}$ such that $pcl(B) \wedge C = B \wedge pcl(C) = \bar{0}$ and $A = B \vee C$.

Consider the mapping

$$P : M^*(A) \to U\{\delta(x) : x \in M^*(A)\},$$

defined by

$$P(x) = \begin{cases} pcl(C), & \text{if } x \leq B \\ pcl(B), & \text{if } x \leq C \end{cases}$$

Claim

$x \not\leq P(x)$

If $x \leq B$, then $B(x) \neq 0 \Rightarrow (pcl(C))\,(x) = 0$ as $B \wedge pcl(C) = \bar{0}$

$\Rightarrow x \not\leq pcl(C)$

$\Rightarrow x \not\leq P(x)$, since $x \leq B \Rightarrow P(x) = pcl(C)$

If $x \leq C$, then $C(x) \neq 0 \Rightarrow pcl(B(x)) = 0$ as $pcl(B) \wedge C = \bar{0}$

$\Rightarrow x \not\leq pcl(B)$

$\Rightarrow x \not\leq P(x)$, since $x \leq C \Rightarrow P(x) = pcl(B)$

Hence the claim.

Since $P(x)$ is a preclosed set and $x \not\leq P(x)$ we get $P(x) \in \delta(x)$ for each $x \in M^*(A)$.

Take the point a out of B and the point b out of C. Then $a, b \in M^*(A)$. Since for arbitrary finite points $x_1 = a, x_2, \ldots, x_n = b$, either $x_i \leq B$ or $x_i \leq C$ $(I = 1, 2, \ldots, n)$, we must have either $P(x_1) = pcl(C)$ or $P(x_i) = pcl(B)$. Since $x_1 = a \leq B$ and $x_n = b \leq C$, we get $P(x_1) = pcl(C)$ and $P(x_n) = pcl(B)$.

Hence there exists $1 \leq j \leq n - 1$ such that

$P(x_j) = pcl(C)$ and $P(x_{j+1}) = pcl(B)$

This shows that

$A = B \vee C \leq pcl(B) \vee pcl(C) = P(x_j) \vee P(x_{j+1})$

Hence condition (i) is not satisfied which is a contradiction.

Therefore A must be P-connected.

Thus sufficiency is proved.

Necessity

Conversely, assume that A is P-connected. Suppose that the condition of the theorem does not hold. That is, there are points $a, b \in M^*(A)$ and there is a mapping

$P : M^*(A) \rightarrow U\{\delta(x) : x \in M^*(A)\}$, where $P(x) \in \delta(x)$ for each $x \in M^*(A)$ such that

(i) does not hold for arbitrary finite points $x_1, x_2, \ldots, x_n \in M^*(A)$. For the sake of convenience, we follow the agreement that for arbitrary $a, b \in M^*(A)$, a and b are joined if there are finite points $x_1, \ldots x_n \in M^*(A)$ such that (i) holds. Otherwise, a and b are not joined.

Let $\mu = \{x \in M^*(A) : \text{a and x are joined}\}$,

$\nu = \{x \in M^*(A) : \text{a and x are not joined}\}$

$B = V\mu$

$C = V\nu$

Obviously, a and a are joined so $a \in \mu$ and $a \leq B$.

By assumption, a and b are not joined, and so $b \in \nu$ and $b \leq C$. Hence $B \neq \bar{0}$, $C \neq \bar{0}$.

Since for each $x \in M^*(A)$ either $x \in \mu$, or $x \in \nu$, $A = B \vee C$.

Now we need only prove

$\text{pcl}(B) \wedge C = B \wedge \text{pcl}(C) = \bar{0}$

Suppose that $\text{pcl}(B) \wedge C \neq \bar{0}$.

There is an x such that $(\text{pcl}(B) \wedge C)(x) \neq 0$.

Take $x \leq \text{pcl}(B) \wedge C$

Then $x \leq \text{pcl}(B)$ and $x \leq C$

$x \leq \text{pcl}(B) \Rightarrow B \nleq P(x)$ since $P(x) \in \delta(x)$

$\Rightarrow$ there exists $y \in \mu$ such that $y \nleq P(x)$

Since $P(y) \in \delta(y)$, $y \nleq P(y)$

Hence $y \nleq P(x) \vee P(y)$ and $y \leq B \leq A$.

Thus, $A \nleq P(x) \vee P(y)$, y and a are joined, a and x are joined.

Since $x \leq C$, we have $C \nleq P(x)$, and so there is $z \in \nu$ such that $z \nleq P(x)$.

Hence, as above we get $z \nleq P(x) \vee P(z)$ and $z \leq C \leq A$.

Thus, $A \nleq P(x) \vee P(z)$. Since x and a are joined, a and z are joined.

This contradicts that $z \in \nu$.

Thus, $\text{pcl}(B) \wedge C = \bar{0}$. In a similar way we can prove the $B \wedge \text{pcl}\, C = \bar{0}$.

Hence B and C are P-separated.

$\therefore$ A = B $\vee$ C is not connected which is a contradiction.

Hence the condition of the theorem holds.

Section 2.3

P2-connectedness in L-fuzzy topological spaces

In this section we discuss the results on P2-connectedness due to Li, Fang and Zhao [23]. First let us give the definition of connectedness introduced by Wang and Shi [38] and a property of L-fuzzy sets which we will be using for our discussion.

Definition : 2.3.1 (Wang and Shi [38])

In an L-fts (L^X, δ) an L-fuzzy set D is called **connected** if there do not exist closed sets A, B such that

$D \not\leq A, D \not\leq B, D' \vee A \vee B = \bar{1}, D \wedge A \wedge B = \bar{0}.$

Lemma : 2.3.2

Let A, B $\in L^X$ and A $\not\leq$ B. If $1 \in M(L)$, then $A' \vee B \neq \bar{1}$.

Definition : 2.3.3

Let (L^X, δ) be an L-fts, $D \in L^X$. D is called **P2-connected** if there do not exist preclosed sets A, B such that

$D \not\leq A, D \not\leq B, D' \vee A \vee B = \bar{1}, D \wedge A \wedge B = \bar{0}.$

Now, let us give an example of a P2-connected set.

Example : 2.3.4

P2-Connected Set

Let $X_1 \cap X_2 = \phi$, $X = X_1 \cup X_2$, $L = [0, 1]$. Define fuzzy set $(C_a, C_b) \in [0, 1]^X$ as follows :

$$(C_a, C_b)(x) = \begin{cases} a, & x \in X_1 \\ b, & x \in X_2 \end{cases}$$

Take

$\nu = \{(C_{0.4}, C_1), (C_1, C_{0.4}), (C_{0.5}, C_0), (C_0, C_{0.5}), (C_{0.7}, C_0), (C_0, C_{0.7})\}$

Let δ be a [0, 1]-topology generated by ν on X. Now we prove that $(C_{0.5}, C_{0.5})$ is P2-connected.

Suppose that $(C_{0.5}, C_{0.5})$ is not P2-connected. Then there exist two preclosed sets A, B such that

$(C_{0.5}, C_{0.5}) \not\leq A$, $(C_{0.5}, C_{0.5}) \not\leq B$, $(C_{0.5}, C_{0.5})'$, $A \vee B = \bar{1}$, $(C_{0.5}, C_{0.5}) \wedge A \wedge B = \bar{0}$

This implies that

$(C_{0.5}, C_{0.5}) \not\leq A$, $(C_{0.5}, C_{0.5}) \not\leq B$, $A \vee B = \bar{1}$, $A \wedge B = \bar{0}$ obviously, A (or B) satisfying $A \wedge B = \bar{0}$ must be in $\{(C_a, C_0), (C_0, C_a) / a \in [0.6, 0.7]\}$.

But any two preclosed sets in $\{(C_a, C_0)\ (C_0, C_a) / a \in [0.6, 0.7)\}$ don't satisfy $A \vee B = \bar{1}$. Therefore $(C_{0.5}, C_{0.5})$ is P2-connected.

Characterization theorems

Theorem : 2.3.5

Let (L^X, δ) be an L-fts, $D \in L^X$. If $1 \in M(L)$, then the following conditions are equivalent :

1. D is P2-connected.
2. There do not exist preclosed sets A, B such that

   $D \wedge A \neq \bar{0}$, $D \wedge B \neq \bar{0}$, $D' \vee A \vee B = \bar{1}$, $D \wedge A \wedge B = \bar{0}$
3. There do not exist preopen sets U, V such that

   $D \not\leq U$, $D \not\leq V$, $D' \vee U \vee V = \bar{1}$, $D \wedge U \wedge V = \bar{0}$

Proof

(1) ⇒ (2)

Suppose that there exists preclosed sets A, B such that $D \wedge A \neq \bar{0}$, $D \wedge B \neq \bar{0}$, $D' \vee A \vee B = \bar{1}$, $D \wedge A \wedge B = \bar{0}$ (i)

Claim

$D \nleq A$

Suppose $D \leq A$, then $D \wedge A \wedge B = D \wedge B$

(i.e.,) $\bar{0} = D \wedge B$

This is a contradiction

$\therefore D \nleq A$ (ii)

Hence the claim

Similarly if $D \leq B$, then $D \wedge A \wedge B = D \wedge A$

(i.e.,) $\bar{0} = D \wedge A$

which is again a contradiction

$\therefore D \nleq B$ (iii)

$\therefore$ From (i), (ii) and (iii) we get D is not P2-connected.

This contradicts (1)

$\therefore$ Hence (2) is true.

(2) ⇒ (3)

Suppose there exists preopen sets U, V such that

$D \nleq U$, $D \nleq V$, $D' \vee U \vee V = \bar{1}$, $D \wedge U \wedge V = \bar{0}$.

Let $A = U'$, $B = V'$. Then A and B are preclosed sets.

Now $D' \vee U \vee V = \bar{1}$

$\Rightarrow D \wedge U' \vee V' = \bar{0}$

$\Rightarrow D \wedge A \wedge B = \bar{0}$

and $D \wedge U \vee V = \bar{0}$

$\Rightarrow D' \vee U' \vee V' = \bar{1}$

$\Rightarrow D' \vee A \vee B = \bar{1}$

To prove : $D \wedge A \neq \bar{0}$

Suppose not, then $D \wedge A = \bar{0}$

$\Rightarrow D' \vee A' = \bar{1}$

$\Rightarrow D' \vee U = \bar{1}$

$\Rightarrow D \leq U$ (by lemma 2.3,2)

which contradicts $D \not\leq U$

$\therefore D \wedge A \neq \bar{0}$

Similarly, $D \wedge B \neq \bar{0}$

$\therefore$ There exists preclosed sets A, B such that

$D \wedge A \neq \bar{0}$, $D \wedge B \neq \bar{0}$ $D' \vee A \vee B = \bar{1}$, $D \wedge A \wedge B = \bar{0}$

This contradictions (2)

Hence (3).

(3) $\Rightarrow$ (4) is analogous to (1) $\Rightarrow$ (2)

(4) $\Rightarrow$ (1) is analogues to (2) $\Rightarrow$ (3).

Theorem : 2.3.6

Let (L^X, δ) be an L-fts and $D \in L^X$. Then D is P2-connected iff for any two co-prime elements a, $b \leq D$, there exists a P2-connected set E such that a, $b \leq E \leq D$.

Proof

Assume that D is P2-connected. Let a, b be two co-prime elements less than or equal to D

Take E = D

Then there exists a P2-connected set E such that a, $b \leq E = D$.

Conversely, assume the given condition.

Suppose D is not P2-connected.

Then there exist two preclosed sets A, B in (L^X, δ) such that

$D \not\leq A$, $D \not\leq B$, $D' \vee A \vee B = \bar{1}$, $D \wedge A \wedge B = \bar{0}$

Take two co-prime elements a, b $\leq$ D such that a $\nleq$ A and b $\nleq$ B.

By assumption there exists a P2-connected set E such that

a, b $\leq$ E $\leq$ D.

$\therefore$ E $\nleq$ A, E $\nleq$ B, $E' \vee A \vee B = \bar{1}$, $E \wedge A \wedge B = \bar{0}$

$\therefore$ E is not P2-connected, which is a contradiction.

$\therefore$ D is P2-connected.

Properties of P2-connected sets

Theorem : 2.3.7

Let D be a P2-connected set in (L^X, δ). If D $\leq$ E $\leq$ pcl(D), then E is also P2-connected set.

Proof

Suppose E is not P2-connected.

The by definition there exists preclosed sets A, B such that

E $\nleq$ A, E $\nleq$ B, $E' \vee A \vee B = \bar{1}$, $E \wedge A \wedge B = \bar{0}$

Since D $\leq$ E, D′ $\geq$ E′. So we get

$D' \vee A \vee B = \bar{1}$ (1)

Also D $\leq$ E and $E \wedge A \wedge B = \bar{0}$

$\Rightarrow D \wedge A \wedge B = \bar{0}$ (2)

Claim

D $\nleq$ A

Suppose not, D $\leq$ A

Since A is a preclosed set containing D, pcl(D) $\leq$ A

Given E $\leq$ pcl(D)

$\therefore$ E $\leq$ A which contradicts E $\nleq$ A.

$\therefore$ D $\nleq$ A (3)

Hence the claim.

Similarly we can prove D $\nleq$ B (4)

∴ By (1), (2), (3) and (4) we get, D is not P2-connected.

This is a contradiction.

Hence E is P2-connected.

Lemma : 2.3.8

Let (L^X, δ) be an L-fts and $D, E \in L^X$. Then D and E are P-separated iff there exists two preclosed sets A, B such that $D \leq A$, $E \leq B$ and $(D \vee E) \wedge A \wedge B = \bar{0}$.

Proof

Assume that there exists preclosed sets A, B such that

$D \leq A$, $E \leq B$ and $(D \vee E) \wedge A \wedge B = \bar{0}$.

Since $E \leq B$ and B is preclosed we get $pcl(E) \leq B$.

Since $D \leq A$ and A is preclosed we get $pcl(D) \leq A$.

Consider $(D \wedge pcl(E)) \vee (pcl(D) \wedge E)$

$$\leq (D \wedge B) \vee (A \wedge E)$$
$$= (D \vee A) \wedge (D \vee E) \wedge (B \vee A) \wedge (B \vee E)$$
$$= A \wedge (D \vee E) \wedge (B \vee A) \wedge B \ (\because D \leq A \text{ and } E \leq B).$$
$$= (D \vee E) \wedge A \wedge B \wedge (A \vee B)$$
$$= (D \vee E) \wedge A \wedge B \ (\because (A \wedge B) \leq A \vee B)$$
$$= \bar{0}$$

∴ $D \wedge pcl(E) = \bar{0}$ and $pcl(D) \wedge E = \bar{0}$

∴ D and E are P-separated.

Conversely, assume that D and E are P-separated. Then

$pcl(D) \wedge E = D \wedge pcl(E) = \bar{0}$. Take $A = pcl(D)$ and $B = pcl(E)$.

Then A and B are preclosed sets.

Since $D \leq pcl(D)$ and $E \leq pcl(E)$, we get $D \leq A$ and $E \leq B$

Consider

$$(D \vee E) \wedge A \wedge B = (D \vee E) \wedge (pcl(D) \wedge pcl(E))$$
$$= (D \wedge pcl(D) \wedge pcl(E)) \vee (E \wedge pcl(D) \wedge pcl(E))$$

$= (D \wedge \mathrm{pcl}(E)) \vee (\mathrm{pcl}(D) \wedge E)$

$= \bar{0} \vee \bar{0}$

$= \bar{0}$

Theorem : 2.3.9

Let D and E be two P2-connected L-fuzzy sets in an L-fts (L^X, δ). If D, E are not P-separated, then $D \vee E$ is P2-connected.

Proof

Suppose $D \vee E$ is not P2-connected. Then there exist two preclosed sets A, B such that

$D \vee E \not\leq A$, $D \vee E \not\leq B$, $(D \vee E)' \vee A \vee B = \bar{1}$, $(D \vee E) \wedge A \wedge B = \bar{0}$

Consider $(D \vee E)' \vee A \vee B = \bar{1}$

$\Rightarrow (D' \wedge E') \vee A \vee B = \bar{1}$

$\Rightarrow (D' \vee A \vee B) \wedge (E' \vee A \vee B) = \bar{1}$

$\Rightarrow D' \vee A \vee B = \bar{1}$ and $E' \vee A \vee B) = \bar{1}$

Consider $(D \vee E) \wedge A \wedge B = \bar{0}$

$\Rightarrow (D \wedge A \wedge B) \vee (E \wedge A \wedge B) = \bar{0}$

$\Rightarrow D \wedge A \wedge B = \bar{0}$ and $E \wedge A \wedge B = \bar{0}$

Since $D \vee E \not\leq A$, either $D \not\leq A$ or $E \not\leq A$

Suppose $D \not\leq A$

Then $D \leq B$ since D is P2-connected.

Since $D \vee E \not\leq B$, either $D \not\leq B$ or $E \not\leq B$.

Since $D \leq B$ we must have $E \not\leq B$

Since E is P2-connected we must have $E \leq A$.

Hence there exists preclosed sets A and B such that

$D \leq B$, $E \leq A$ and $(D \vee E) \wedge A \wedge B = \bar{0}$.

∴ By Lemma 2.3.8 D and E are P-separated which contradicts that D and E are not P-separated.

∴ $D \vee E$ is P2-connected.

Theorem : 2.3.10

Let $\{D_t \ / \ t \in \Omega\}$ be a family of P2-connected L-fuzzy sets. If there is an $s \in \Omega$ such that for each $t \in \Omega - \{s\}$, D_t and D_s are not P-separated, then $\bigvee_{t\in\Omega} D_t$ is P2-connected.

Proof

Suppose $\bigvee_{t\in\Omega} D_t$ is not P2-connected. Then there exists preclosed sets A, B such that

$$\bigvee_{t\in\Omega} D_t \not\leq A, \quad \bigvee_{t\in\Omega} D_t \not\leq B, \quad (\bigvee_{t\in\Omega} D_t)' \vee A \vee B = \bar{1} \ (\bigvee_{t\in\Omega} D_t)' \wedge A \wedge B = \bar{0}$$

Now $\bigvee_{t\in\Omega} D_t \not\leq A$.

$\Rightarrow$ there exists $x \ni (\bigvee_{t\in\Omega} D_t)(x) > A(x)$

$\Rightarrow \sup \{D_t(x) \ / \ t \in \Omega\} > A(x)$

$\Rightarrow$ By supremum property there exists $t_1 \in \Omega$ such that $D_{t_1}(x) > A(x)$

Similarly since $\bigvee_{t\in\Omega} D_t \not\leq B$, there exists $t_2 \in \Omega$ and $y \in X$ such that $D_{t_2}(y) > B(y)$

Hence $(D_{t_1} \vee D_{t_2} \vee D_s)(x) \geq D_{t_1}(x)$

$$> A(x)$$

and $(D_{t_1} \vee D_{t_2} \vee D_s)(y) \geq D_{t_2}(y)$

$$> B(y)$$

Hence there exists $t_1, t_2 \in \Omega$ such that

$$D_{t_1} \vee D_{t_2} \vee D_s \not\leq A, \ D_{t_1} \vee D_{t_2} \vee D_s \not\leq B \qquad (1)$$

Since $(D_{t_1} \vee D_{t_2} \vee D_s)' \geq (\bigvee_{t\in\Omega} D_t)'$ and

$D_{t_1} \vee D_{t_2} \vee D_s \leq \bigvee_{t\in\Omega} D_t$, we get

$$(D_{t_1} \vee D_{t_2} \vee D_s)' \vee A \vee B \geq (\bigvee_{t\in\Omega} D_t)' \vee A \vee B$$

$$= \bar{1}$$

and

$(D_{t_1} \vee D_{t_2} \vee D_s) \wedge A \wedge B \leq (\bigvee_{t \in \Omega} D_t)' \wedge A \wedge B$

$= \bar{0}$

$\therefore (D_{t_1} \vee D_{t_2} \vee D_s)' \vee A \vee B = \bar{1}$ (2)

and

$(D_{t_1} \vee D_{t_2} \vee D_s) \wedge A \wedge B = \bar{0}$ (3)

From (1), (2) and (3) we get

$D_{t_1} \vee D_{t_2} \vee D_s$ is not P2-connected (4)

Since D_{t_1} and D_s are not P-separated and D_{t_1} D_s are P2-connected we get by Theorem 2.3.9.

$$\left.\begin{array}{l} D_{t_1} \vee D_s \text{ is P2-connected} \\ \text{Similarly } D_{t_2} \vee D_s \text{ is P2-connected} \end{array}\right\} \quad (5)$$

Claim

$D_{t_1} \vee D_s$ and $D_{t_2} \vee D_s$ are not P-separated. Suppose not, then

$pcl(D_{t_1} \vee D_s) \wedge (D_{t_2} \vee D_s) = \bar{0}$ and $(D_{t_1} \vee D_s) \wedge pcl(D_{t_2} \vee D_s) = \bar{0}$

$\Rightarrow (pcl(D_{t_1}) \vee pcl(D_s)) \wedge (D_{t_2} \vee D_s) = \bar{0}$ and

$(D_{t_1} \vee D_s) \wedge (pcl(D_{t_2}) \vee pcl(D_s)) = \bar{0}$

$\Rightarrow (pcl(D_{t_1}) \wedge D_s) \vee \ldots = \bar{0}$ and $(pcl(D_s) \vee (D_{t_1}) \vee \ldots = \bar{0}$

$\Rightarrow pcl(D_{t_1}) \wedge D_s = \bar{0}$ and $pcl\, D_s \wedge D_{t_1} = \bar{0}$

D_{t_1} and D_s are P-separated which contradicts that D_{t_1} and D_s are not P-separated.

Hence the claim.

By (5) and claim, using Theorem 2.3.9 we get

$D_{t_1} \vee D_{t_2} \vee D_s$ is P2-connected (6)

(4) and (6) contradicts each other.

$\therefore \bigvee_{t \in \Omega} D_t$ is P2-connected.

Corollary : 2.3.11

Let $\{D_t \,/\, t \in \Omega\}$ be a family of P2-connected L-fuzzy sets. If $\bigwedge_{t\in\Omega} D_t \neq \bar{0}$ then $\bigvee_{t\in\Omega} D_t$ is P2-connected.

Proof

$\bigwedge_{t\in\Omega} D_t \neq \bar{0}$

$\Rightarrow D_t \neq \bar{0} \ \forall\, t \in \Omega$

$\Rightarrow \text{pcl } D_t \neq \bar{0} \ \forall\, t \in \Omega$

$\Rightarrow$ For any $s \in \Omega$, $\text{pcl}(D_t) \wedge D_s \neq \bar{0}$ and $D_t \wedge \text{pcl}(D_s) \neq \bar{0} \ \forall\, t \in \Omega$

$\Rightarrow D_t$ and D_s are not P-separated.

Hence by Theorem 2.3.10, we get

$\bigvee_{t\in\Omega} D_t$ is P2-connected.

The following theorem shows that P2-connectedness is preserved by a P-irresolute mapping.

Theorem : 2.3.12

Let $f : (L^X, \delta) \to (L^Y, T)$ be a P-irresolute mapping. If D is P2-connected in (L^X, δ), then so is f(D) in (L^Y, T).

Proof

Suppose that f(D) is not P2-connected in (L^Y, T). Then there exists two preclosed sets A, B in (L^Y, T) such that

$f(D) \nleq A,\ f(D) \nleq B,\ f(D)' \vee A \vee B = \bar{1},\ f(D) \wedge A \wedge B = \bar{0}$

Since $D \leq f^{-1}(f(D))$ we get

$$D' \geq f^{-1}(f(D))'$$

$$= f^{-1}(f(D)')$$

Consider $f(D) \not\leq A$

$\Rightarrow f^{-1}(f(D)) \not\leq f^{-1}(A)$

$\Rightarrow D \not\leq f^{-1}(A)$ (1)

Similarly $f(D) \not\leq B$

$\Rightarrow D \not\leq f^{-1}(B)$ (2)

Consider, $f(D)' \vee A \vee B = \bar{1}$

$f^{-1}(f(D)') \vee f^{-1}(A) \vee f^{-1}(B) = f^{-1}(\bar{1}) = \bar{1}$

$\Rightarrow D' \vee f^{-1}(A) \vee f^{-1}(B) \geq f^{-1}(f(D)') \vee f^{-1}(A) \vee f^{-1}(B)$

$= \bar{1}$

$\Rightarrow D' \vee f^{-1}(A) \vee f^{-1}(B) = \bar{1}$ (3)

Consider

$f(D) \wedge A \wedge B = \bar{0}$

$\Rightarrow f^{-1}(f(D)) \wedge f^{-1}(A) \wedge f^{-1}(B) = f^{-1}(\bar{0})$

$= \bar{0}$

$\Rightarrow D \wedge f^{-1}(A) \wedge f^{-1}(B) \leq f^{-1}(f(D)) \wedge f^{-1}(A) \wedge f^{-1}(B)$

$= \bar{0}$ (4)

Since $f : (L^X, \delta) \to (L^Y, T)$ is a P irresolute mapping, we get that $f^{-1}(A)$ and $f^{-1}(B)$ are preclosed sets in (L^X, T) (5)

From (1), (2), (3), (4) and (5) we get that D is not P2-connected.

This is a contradiction.

Hence f(D) is P2-connected in (L^Y, T).

Extension of K. Fan's Theorem to P2-Connectedness

First let us give the definition needed for K. Fan's theorem and discuss an example.

Definition : 2.3.13

Let (L^X, δ) be an L-fts, $D \in L^X$ and F denote the set of all preclosed sets in (L^X, δ). A mapping $P : M^*(D) \to \mathcal{F}$ is called a **pre-remote neighbourhood mapping** on D, if for each $e \in M^*(D)$, we have $e \not\leq P(e)$.

Example : 2.3.14

Pre-remote neighbourhood mapping

Let $X_1 \cap X_2 = \phi$, $X = X_1 \cup X_2$, $L = [0, 1]$. Define the fuzzy set $(C_a, C_b) \in [0, 1]^X$ as follows :

$$(C_a, C_b)(x) = \begin{cases} a, & x \in X_1 \\ b, & x \in X_2 \end{cases}$$

Let $\delta = \{\bar{0}, \bar{1}\ (C_{0.5}, C_{0.5})\}$

Then δ is a [0, 1] fuzzy topology on X

For every $e \in M(L^X)$, define

$$P(e) = \begin{cases} (C_{0.5}, C_{0.5}), & \text{if } e \not\leq (C_{0.5}, C_{0.5}); \\ \bar{0} & \text{if } e \not\leq (C_{0.5}, C_{0.5}); \end{cases}$$

Take $e \in M(L^X)$

If $e \not\leq (C_{0.5}, C_{0.5})$, by definition, $P(e) = (C_{0.5}, C_{0.5})$

$\therefore\ e \not\leq P(e)$

If $e \leq (C_{0.5}, C_{0.5})$, by definition $P(e) = \bar{0}$

Since $e \neq \bar{0}$; there exists x such that $e(x) \neq 0$

Since $P(e)(x) = \bar{0}(x) = 0$, we get that

$e(x) > P(e)(x)$

$\therefore\ e \not\leq P(e)$

$\therefore$ P is a **pre-remote neighbourhood mapping**.

Theorem : 2.3.15

Let (L^X, δ) be an L-fts and $D \in L^X$. Then D is P2-connected iff for any two co-prime elements $a, b \in M^*(D)$ and any pre-remote neighbourhood mapping

$P : M^*(D) \to \mathcal{F}$, there exists finitely many co-prime elements $e_1 = a, e_2, \ldots, e_n = b$ in D such that

$D' \vee P(e_i) \vee P(e_i + 1) \neq \bar{1}$ $\qquad i = 1, 2, \ldots, n-1$

Proof

First assume the given condition suppose D is not P2-connected.

Then there exist two preclosed sets A, B such that

$D \not\leq A, D \not\leq B, D' \vee A \vee B = \bar{1}, D \wedge A \wedge B = \bar{0}$

Define a pre-remote neighbourhood mapping $P : M^*(D) \to \mathcal{F}$ such that

$$\forall\, e \in M^*(D),\ P(e) = \begin{cases} A, & \text{if } e \leq B \\ B, & \text{if } e \not\leq B \end{cases}$$

Take $a, b \in M^*(D)$ such that $a \not\leq A$ and $b \not\leq B$.

Take arbitrary finitely many co-prime elements $e_1 = a, e_2, \ldots, e_n = b$ in D

$b \not\leq B \Rightarrow P(b) = B.$

$\Rightarrow P(e_n) = B$

$P(e_n - 1) = A$ or B.

If $P(e_n - 1) = A$, then $D' \vee P(e_n - 1) \vee P(e_n)$

$= D' \vee A \vee B$

$= \bar{1}$

If $P(e_n - 1) = B$, then if $P(e_n - 2) = A$ we get

$D' \vee P(e_n - 2) \vee P(e_n - 1) = D' \vee A \vee B$

$= \bar{1}$

If all $P(e_2), P(e_3) \ldots P(e_n) = B$ then $D' \vee P(e_1) \vee P(e_2) = D' \vee P(a) \vee B$

Since P is a pre-remote neighbourhood mapping, $a \not\leq P(a)$.

As $a \not\leq A$, we must have $P(a) = A$

$\therefore D' \vee P(e_1) \vee P(e_2) = D' \vee A \vee B = \bar{1}$

Thus there exists $i \ni D' \vee P(e_i) \vee P(e_{i+1}) = \bar{1}$.

This contradicts the condition of the theorem.

Hence D is P2-connected.

Conversely, assume that D is P2-connected suppose that there exists two co-prime elements $a, b \in M^*(D)$ and a pre-remote neighbourhood mapping $P : M^*(D) \to \mathscr{F} \ni$ for arbitrary finitely many co-prime elements $e_1 = a, e_2, \ldots, e_n = b$ in D the following fact is true :

$D' \vee P(e_i) \vee P(e_{i+1}) \neq \bar{1}, \quad i = 1, 2, \ldots, n-1$

In this case, we say that a and b cannot be linked. Let

$\mathscr{A} = \{e \in M^*(D) /$ a and e can be linked$\}$

$\mathscr{B} = \{e \in M^*(D) /$ a and e cannot be linked$\}$

Then $\forall\, e \in \mathscr{A}$ and $\forall\, d \in \mathscr{B}$, we have that

$D' \vee P(c) \vee P(d) = \bar{1}$

Put

$A = \wedge \{P(c) / c \in \mathscr{A}\}$, $B = \wedge \{P(d) / d \in \mathscr{B}\}$,

Obviously we have that

$D \not\leq A$, $D \not\leq B$, $D' \vee A \vee B = \bar{1}$, $D \wedge A \wedge B = \bar{0}$.

This shows that D is not P2-connected Which is a contradiction.

Hence the condition is true.

Relations among a few kinds of connectedness

Now let us discuss the relations among connectedness, P-connectedness and P2-connectedness in L-fuzzy topological spaces.

Theorem : 2.3.16

A P2-connected L-fuzzy set must be connected.

Proof

Let D be a P2-connected L-fuzzy set.

Suppose D is not connected.

Then there exists closed sets A, B such that

$D \nleq A$, $D \nleq B$, $D' \vee A \vee B = \bar{1}$, $D \wedge A \wedge B = \bar{0}$.

Since closed set is a preclosed set we get that there exists preclosed sets A, B satisfying the above conditions.

Hence by definition, D is not P2-connected.

This is a contradiction.

$\therefore$ D is connected.

Remark : 2.3.17

In general a connected L-fuzzy set need not be P2-connected. This can be seen from the following example.

Example : 2.3.18

A connected L-fuzzy set which is not P2-connected

Let $X_1 \cap X_2 = \phi$, $X = X_1 \cup X_2$, $L = [0, 1]$. Define the fuzzy set $(C_a, C_b) \in [0, 1]^X$ as follows

$$(C_a, C_b)(x) = \begin{cases} a, & x \in X_1 \\ b, & x \in X_2 \end{cases}$$

Take

$\nu = \{(C_{0.6}, C_0) ; (C_0, C_{0.6}), (C_{0.5}, C_1), (C_1, C_{0.5}), (C_{0.3}, C_1), (C_1, C_{0.3})\}$

Let δ be the [0.1]-topology generated by ν on X. It is easy to see that $\bar{1}$ is connected. But $\bar{1}$ is not P2-connected because there exist two preclosed sets (C_1, C_0) and (C_0, C_1) such that $\bar{1} \nleq (C_1, C_0)$ $\bar{1} \nleq (C_0, C_1)$,

$\bar{1}' \vee (C_1, C_0) \vee (C_0, C_1) = \bar{1}$, $\bar{1} \wedge (C_1, C_0) \wedge (C_0, C_1) = \bar{0}$

In order to discuss the relation between P-connectedness and P2-connectedness first we present two characterizations of P-connectedness.

Characterizations of P-connectedness

Theorem : 2.3.19

Let (L^X, δ) be an L-fts, $D \in L^X$. Then the following conditions are equivalent :

(1) D is P-connected

(2) There do not exist preclosed set A and B such that

$D \wedge A \neq \bar{0}$, $D \wedge B \neq \bar{0}$, $D \leq A \vee B$, $D \wedge A \wedge B = \bar{0}$

(3) There do not exist preclosed sets A, B such that

$D \not\leq A$, $D \not\leq B$, $D \leq A \vee B$, $D \wedge A \wedge B = \bar{0}$

Proof

(1) ⇒ (2)

Suppose that there exist preclosed sets A and B such that

$D \wedge A \neq \bar{0}$, $D \wedge B \neq \bar{0}$, $D \leq A \vee B$, $D \wedge A \wedge B = \bar{0}$

Let $A_1 = D \wedge A$ and $B_1 = D \wedge B$

Then $A_1 \neq \bar{0}$, $B_1 \neq \bar{0}$

Consider $A_1 \vee B_1 = (D \wedge A) \vee (D \wedge B)$

$= D \wedge (A \vee B)$

$= D$

$\therefore D = A_1 \vee B_1$

Consider

$(A_1 \wedge pcl(B_1)) \vee (pcl(A_1) \wedge (B_1)$

(i.e.) $= (D \wedge A \wedge pcl(B_1)) \vee (pcl(A_1) \wedge D \wedge B)$

Since $A_1 = D \wedge A \leq A$, a preclosed set

We get $pcl(A_1) \leq pcl(A) = A$

$\therefore pcl(A_1) \leq A$

Similarly $pcl(B_1) \leq B$

$\therefore (A_1 \wedge pcl(B_1)) \vee (pcl(A_1) \wedge B_1)$

$\leq (D \wedge A \wedge B) \vee (A \wedge D \wedge B)$

$= D \wedge A \wedge B$

$= \bar{0}$

$\therefore A_1 \wedge pcl(B_1) = \bar{0}$ and $pcl(A_1) \wedge B_1 = \bar{0}$

$\therefore$ A_1, B_1 are P-separated

As $D = A_1 \vee B_1$, D is not P-connected.

This is a contradiction.

Hence (2) is true.

(2) ⇒ (1)

Suppose D is not P-connected. Then there exist two non null sets A and B such that $D = A \vee B$ and A, B are P-separated.

$\therefore pcl(A) \wedge B = A \wedge pcl(B) = \bar{0}$

Let $A_1 = pcl(A)$ and $B_1 = pcl(B)$

Consider

$D \wedge A_1 = (A \vee B) \wedge pcl(A)$

$= (A \wedge pcl\,(A)) \vee (B \wedge pcl(A))$

$= A \vee \bar{0}$

$= A \neq \bar{0}$

Similarly,

$D \wedge B_1 = B \neq \bar{0}$

$D = A \vee B$

$\leq pcl(A) \vee pcl(B)$

$= A_1 \vee B_1$

$D \leq A_1 \vee B_1$

Consider

$D \wedge A_1 \wedge B_1 = D \wedge pcl(A) \wedge pcl(B)$

$= (A \vee B) \wedge (pcl(A) \wedge pcl(B))$

$$= (A \wedge pcl(A) \wedge pcl(B)) \vee (B \wedge pcl(A) \wedge pcl(B))$$

$$= (A \wedge pcl(B)) \vee (B \wedge pcl(A))$$

$$= \bar{0} \vee \bar{0}$$

$$= \bar{0}$$

This contradicts (2)

Hence D is P-connected

(2) ⇒ (3) is obvious.

Theorem : 2.3.20

Let $1 \in M(L)$. If D is P-connected then it is also P2-connected.

Proof

Suppose that D is not P2-connected. Then there exist preclosed sets A and B such that

$D \wedge A \neq \bar{0}$, $D \wedge B \neq \bar{0}$, $D' \vee A \vee B = \bar{1}$, $D \wedge A \wedge B = \bar{0}$

Since $D' \vee A \vee B = \bar{1}$, by Lemma 2.3.2 $D \leq A \vee B$.

By (2) ⇒ (1) of Theorem 2.3.19 we get D is not P-connected.

This is a contradiction

∴ D is P2-connected.

Theorem : 2.3.21

Let D be a crisp set in (L^X, δ). Then D is P-connected if and only if it is P2-connected.

Proof

Since D is a crisp set we get $D(x) = 0$ (or) $D(x) = 1$

Claim

$D' \vee A \vee B = \bar{1} \Leftrightarrow D \leq A \vee B$

Assume

$D' \vee A \vee B = \bar{1}$

Take $x \in X$. Then $D'(x) = 1$ (or) $D'(x) = 0$.

$D'(x) = 1 \Rightarrow D(x) = 0 \leq (A \vee B)(x)$

$D'(x) = 0 \Rightarrow D(x) = 1$ and $(A \vee B)(x) = 1$

$\Rightarrow D(x) = (A \vee B)(x)$

$\therefore \quad D(x) \leq (A \vee B)(x) \qquad \forall x$

(i.e.) $\quad D \leq A \vee B$

Conversely, assume $D \leq A \vee B$

Take $x \in X$. Then $D(x) = 0$ or $D(x) = 1$

$D(x) = 0 \Rightarrow D'(x) = 1$

$\Rightarrow (D' \vee A \vee B)(x) = 1$

Suppose $D(x) = 1$

Then as $D \leq A \vee B$ we get

$D(x) \leq (A \vee B)(x)$

$\Rightarrow 1 \leq (A \vee B)(x)$

$\Rightarrow (A \vee B)(x) = 1$

$\Rightarrow (D' \vee A \vee B)(x) = \bar{1}$

$\therefore D' \vee A \vee B = \bar{1}$

Hence the claim.

D is P-connected $D \wedge A \neq \bar{0}$, $D \wedge B \neq \bar{0}$,

$\Leftrightarrow$ There do not exist preclosed sets, A, B such that $D \wedge A \neq \bar{0}$, $D \wedge B \neq \bar{0}$, $D \leq A \vee B$, $D \wedge A \wedge B = \bar{0}$ (by Theorem 2.3.19)

$\Leftrightarrow$ There do not exist preclosed sets A, B such that $D \wedge A \neq \bar{0}$, $D \wedge B \neq \bar{0}$, $D' \vee A \vee B = \bar{1}$, $D \wedge A \wedge B = \bar{0}$ (since $D \leq A \vee B \Leftrightarrow D' \vee A \vee B = \bar{1}$ by the claim)

$\Leftrightarrow$ D is P2-connected (by Theorem 2.3.5).