# Some interesting problems from Topology

By

Indirani .P

A thesis Submitted to the Avinashilingam Institute
for Home Science and Higher Education for Women
(Deemed University), Coimbatore - 641 043
in partial fulfilment of the requirements
for the degree of Master of Science
in Mathematics

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# Certificate

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IN PARTIAL FULFILMENT OF THE REQUIREMENTS
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CERTIFIED AS BONAFIDE RESEARCH WORK

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**and the Head of the Department**
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DEDICATED TO MY PARENTS.

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# Introduction

# INTRODUCTION

This thesis is an attempt to study the following topics

(1) Construction of new topologies from given topologies having some interesting properties.

(2) Discussion of some famous counter examples from topology.

Regarding question (1) we consider a topological space $(S, \tau)$ and an ideal **I** of subsets S. The problem is to construct the smallest topology $\tau^*$ containing $\tau$ such that all the members of I are closed in $\tau^*$. This question is examined in detail in the following papers.

1. "New topology from old via ideals" by Dragan Jankovic and T.R.Hamlett [ 4 ].
2. "A topology formed from a given topology and ideal " by P.Samuels [ 7 ].
3. "Ideally equivalent topologies and semitopological properties" by David A.Rose and T.R.Hamlett [ 3 ].

In this thesis we discuss the first paper in detail and mention the important results contained in the other two papers.

Given a topological space $(X, \tau)$ and an ideal **I** of subsets of X. The collection of subsets $\mathbf{B(I,\tau)} = \{ V-I \; ; \; V \in \tau \; , \; I \in \mathbf{I} \}$ defines a base for a topology $\tau^*$. This topology $\tau^*$ has got many interesting properties. $\tau^*$ is the smallest topological containing $\tau$ in which every member of **I** is closed. The study of $\tau^*$ corresponding to various ideals $\mathbf{I}_f$, $\mathbf{I}_c$, $\mathbf{I}_s$, $\mathbf{I}_n$ and $\mathbf{I}_{cd}$ where

$I_f \rightarrow$ the ideal of finite subsets of X.

$I_c \rightarrow$ the ideal of countable subsets of X.

$I_s \rightarrow$ the ideal of scattered sets in $(X, \tau)$.

$I_n \rightarrow$ the ideal of nowhere dense sets in $(X, \tau)$.

$I_{cd} \rightarrow$ the ideal closed discrete sets in $(X, \tau)$.

give rise to many interesting examples and results.

**Definition**

Let $(X, \tau)$ be a space with $\mathbf{I}$ an ideal on X. We say the topology $\tau$ is compatible with ideal $\mathbf{I}$, denoted $\tau \sim \mathbf{I}$ if the following condition holds : for everey $A \subseteq X$ ; if for every $x \in A$ there exists a neighbourhood U of x such that $U \cap A \in \mathbf{I}$ then $A \in \mathbf{I}$.

It is interesting to note that a space $(X, \tau)$ is hereditarily Lindelof iff $\tau \sim I_c$.

The study of ideal topologies lead to a simple proof of the following famous theorem.

**Theorem (Cantor - Bendixson)**

A second countable space can be represented as the union of two sets one of which is perfect and the other countable.

Another important result on ideal topologies is as follows.

**Theorem**

Let $(X, \tau)$ be a space with an ideal $\mathbf{I}$ on X such that $\mathbf{I}$ and $\{x\} \in \mathbf{I}$. If a set $A \subseteq X$ is closed with respect to $\tau^*$, then A is the union of a set which is perfect with respect to $\tau$ and a set in **I**.

The concept of compatibility is characterised as follows.

**Theorem**

Let $(X, \tau)$ be a space and **I** an ideal on X. The following are equivalent.

(a) $\tau \sim \mathbf{I}$ and $\{x\} \in \mathbf{I}$ for each $x \in X$.

(b) Scattered sets in $(X, \tau^*)$ are in **I**.

(c)± Discrete sets in $(X, \tau^*)$ are in **I**.

**Definition**

Spaces in which compact sets are finite have been called cf - spaces or pseudofinite or anticompact.

The study of ideal topologies lead to the following results on anticompact spaces.

**Theorem**

Let $(X, \tau)$ be a Hausdorff space. Then $(X, \tau^*(I_c \cap I_n \cap I_k))$ is anticompact where the ideal of relatively compact sets are denoted by $\mathbf{I}_k$ defined as $\mathbf{I}_k = \{A \subseteq X \;;\; cl(A) \text{ is compact in } (X, \tau)\}$.

**Corollary**

Let $(X, \tau)$ be a dense-in-itself Hausdorff space. Then $(X, \tau^*(\mathbf{I}_n))$ is anticompact.

Second chapter is devoted to the discussion of four interesting examples.

The first one is called "Prime ideal topology", which is defined on the collection of all prime ideals of the ring of integers. This space is $T_0$, compact, hyperconnected, second countable and locally connected. Further this space does not satisfy the $T_1$, $T_4$ axioms.

In the second example we consider the set X of all integers greater than or equal to 2 and consider the family $\{U_n \;/\; n \geq 2\}$ where $U_n = \{x \in Z^+/x \text{ divides } n\}$. The topology $\tau$ is generated by the family $U_n$. It is shown here that $(X, \tau)$ is $T_0$, ultraconnected, locally connected, locally compact, scattered, separable and second countable. This space does not satisfy the $T_1$-axiom. Also this space is not countably compact and not strongly locally compact.

In the third example we consider a topological space $(X, \tau)$ such that $(X, \tau)$ is compact, $T_1$ and not Hausdorff. The important point about this example is that $\tau$ is a maximal compact non-Hausdorff topology on X.

In the last example we construct a Minimal Hausdorff topology.

The examples studied in this chapter are taken from the book "Counter examples in Topology" by Lynn Arthur Steen and J.Arthur Seebach, Jr.

# Chapter - I

# CHAPTER - I

## SECTION - 1.1. DEFINITION OF IDEAL TOPOLOGY.

### Introduction

Given a topological space $(S, \tau)$ and an ideal $I$ of subsets of $S$, we shall construct a topology $\tau^*$ such that $\tau^*$ contains $\tau$ and every members of $I$ is closed in $\tau^*$. To achieve this we associate with every set $A$, a set $A^*$ such that $A \cup A^*$ satisfies the four Kuratowski axioms and defines a topology $\tau^*$.

We shall start with the preliminary definitions.

**Definition 1.1.1.**

A nonempty collection $I$ of subsets of a set $X$ is said to be an ideal on $X$, if it satisfies the following two conditions.

1. $A \in I$ and $B \subseteq A \Rightarrow B \in I$
2. $A \in I$ and $B \in I \Rightarrow A \cup B \in I$

**Examples**

$I_f$ - the ideal of finit subsets of $X = \{A \subseteq X \ / \ A \text{ is finite}\}$.

$I_c$ - the ideal of countable subsets of $X = \{A \subseteq X \ / \ A \text{ is countable}\}$.

A set $A$ is discrete if $A \cap A^d = \phi$ where $A^d$ is the derived set of $A$.

$I_{cd}$ - the ideal of closed discrete sets in ( X, $\tau$ )

$$= \{A \subseteq X : A^d = \phi\}$$

In any space X, a set $B \subset Y$ is called nowhere dense if its closure has no interior.

$I_n$ - the ideal of nowhere dense sets in (X, $\tau$ )

$$= \{A \subseteq X : \text{Int}(\text{cl}(A)) = \phi\}$$

Note :

N(x) will denote the open neighbourhood system at x (i.e)

$N(x) = \{U \in \tau : x \in U\}$.

**Definition 1.1.2.**

Let (X, $\tau$ ) be a space with an ideal **I** on X. Then $A^*(I, \tau) = \{x \in X : A \cap U \notin I \text{ for every } U \in N(x)\}$ is called the local function of A with respect to **I** and $\tau$ . Also cl(A) is defined as $\text{cl}(A) = \{x \in X : A \cap U \neq \phi \text{ for every } U \in N(x)\}$.

**Theorem 1.1.3.**

Let (X, $\tau$ ) be a space with **I** an ideal on X and let A and B be subsets of X. Then

a. $A \subseteq B \Rightarrow A^* \subseteq B^*$

b. $A^* = \text{cl}(A^*) \subseteq \text{cl}(A)$ ($A^*$ is a closed subset of cl(A)).

c. $(A^*)^* \subseteq A^*$

d. $(A \cup B)^* = A^* \cup B^*$

e. $A^* - B^* = (A-B)^* - B^* \subseteq (A-B)^*$

f. $U \in \tau \Rightarrow U \cap A^* = U \cap (U \cap A)^* \subseteq (U \cap A)^*$ and

g. $I \in \mathrm{I} \Rightarrow (A \cup I)^* = A^* = (A - I)^*$

**Proof :-**

a. $A \subseteq B \Rightarrow A^* \subseteq B^*$

$x \in A^* \Rightarrow U \cap A \notin \mathrm{I}$ for every $U \in N(x)$,

$\Rightarrow U \cap B \notin \mathrm{I}$ for every $U \in N(x)$, since $A \subseteq B$.

$\Rightarrow x \in B^*$

$\therefore$ $A^* \subseteq B^*$ whenever $A \subseteq B$

b. $A^* = cl(A^*)$.

Let $x \in cl(A^*) \Rightarrow U \cap A^* \neq \phi$ for every $U \in N(x)$

$\Rightarrow$ there exists $z \in U \cap A^*$, for every $U \in N(x)$.

$\Rightarrow z \in U$ and $z \in A^*$ for every $U \in N(x)$

$\Rightarrow U$ is a neighbourhood of $z$ and $z \in A^*$, for every $U \in N(x)$.

$z \in A^* \Rightarrow N(z) \cap A \notin \mathrm{I}$, for every $N(z)$.

But $U$ is also a neighbourhood of $z$ for every $U \in N(x)$.

So $U \cap A \notin \mathrm{I}$ for every $U \in N(x)$.

$\Rightarrow x \in A^*$.

$\therefore$ $cl\,(A^*) \subset A^*$.

We know that $A^* \subset cl(A^*)$

So $A^* = cl(A^*)$.

To prove that $cl(A^*) \subseteq cl(A)$.

$x \in cl(A^*) \Rightarrow U \cap A^* \neq \phi$ for every $U \in N(x)$.

$\Rightarrow$ there exists $z \in U \cap A^*$ for every $U \in N(x)$.

$\Rightarrow U$ is a neighbourhood of $z$ and $z \in A^*$ for every

$U \in N(x)$.

$z \in A^* \Rightarrow U \cap A \notin I$ for every $U \in N(z)$.

$\Rightarrow U \cap A \neq \phi$ for every $U \in N(x)$.

$\Rightarrow x \in cl(A)$.

$\Rightarrow cl(A^*) \subset cl(A)$.

c. $(A^*)^* \subseteq A^*$.

$x \in (A^*)^* \Rightarrow U \cap A^* \notin I$ for every $U \in N(x)$.

But we know $A^* = cl(A^*) \subseteq cl(A)$.

$\Rightarrow (A^*)^* \subseteq cl(A^*)$.

$\therefore$ $x \in (A^*)^* \Rightarrow x \in cl(A^*)$.

$\Rightarrow U \cap A^* \neq \phi$ for every $U \in N(x)$.

Take $z \in U \cap A^*$ for every $U \in N(x)$.

$\Rightarrow U$ is a neighbourhood of $z$ and $z \in A^*$ for every $U \in N(z)$.

$\Rightarrow U \cap A \notin I$ for every $U \in N(x)$.

$\Rightarrow x \in A^*$

$\therefore$ $(A^*)^* \subseteq A^*$.

d. $(A \cup B)^* = A^* \cup B^*$.

$x \in (A \cup B)^* \Rightarrow U \cap (A \cup B) \notin I$ for every $U \in N(x)$.

$\Rightarrow [(U \cap A) \cup (U \cap B)] \notin I$ for every $U \in N(x)$.

$\Rightarrow U \cap A \notin I$ or $U \cap B \notin I$ for every $U \in N(x)$.

$\Rightarrow x \in A^*$ or $x \in B^*$.

$\Rightarrow x \in A^* \cup B^*$.

$\therefore$ $(AUB)^* \subseteq A^* \cup B^*$.

Also $A^* \subset (A \cup B)^*$ and $B^* \subset (A \cup B)^*$ since $A \subset (A \cup B)$ and $B \subset (A \cup B)$.

$\therefore$ $A^* \cup B^* \subset (A \cup B)^*$.

$\therefore$ $(A \cup B)^* = A^* \cup B^*$.

e. $A^* - B^* = (A - B)^* - B^* \subseteq (A - B)^*$

$x \in A^* - B^* \Rightarrow x \in A^*$ and $x \notin B^*$.

$\Rightarrow U \cap A \notin I$ for every $U \in N(x)$ and there exists $U_t \in N(x)$ such that $U_t \cap B \in I$. - - - - - - - - - - - - - - - (1)

To prove that $x \in (A - B)^* - B^*$.

(i,e) to prove that $x \in (A-B)^*$ and $x \notin B^*$.

(i.e) to prove that $x \in (A - B)^*$.

(i.e) to prove that $U \cap (A - B) \notin I$ for every $U \in N(x)$.

suppose, $U \cap (A - B) \in I$ for some $U \in N(x)$.

$\Rightarrow U \cap A \cap X - B \in I$ for some $U \in N(x)$. - - (2)

Let $U' = U \cap U_t$ $\therefore$ $U' \cap B \in I$ for some $U' \in N(x)$. - - - - - - (3)

since $U' \in N(x)$.

$\Rightarrow U' \cap A \cap (x - B) \in I$ for some $U' \in N(x)$. - - - - (4)

from (2).

$\Rightarrow U' \cap A \in I$ for some $U' \in N(x)$ from (3) and (4).

which is a contradiction to $x \in A^*$.

Also we know that $x \notin B^*$.

$\therefore$ $x \in (A - B)^* - B^*$

$\Rightarrow A^* - B^* \subset (A - B)^* - B^*$.

conversely,

Take $x \in (A - B)^* - B^*$.

$\Rightarrow$ $x \in (A - B)^*$ and $x \notin B^*$.

$$U \cap (A - B) \notin I \text{ for every } U \in N(x). \quad ------- \quad (1)$$

and there exists $U_r \in N(x)$ such that $U_r \cap B \in I$.

Now to prove that $x \in A^* - B^*$.

(i.e) to prove $x \in A^*$ and $x \notin B^*$.

(i.e) to prove $U \cap A \notin I$ for every $U \in N(x)$.

Assume $U \cap A \in I$ for some $U \in N(x)$.

$\Rightarrow U \cap (A - B) \in I$ for some $U \in N(x)$.

Which is a contradiction to (1).

$\therefore x \in A^*$. Also $x \notin B^*$.

$\therefore x \in A^* - B^*$.

$\therefore (A - B)^* - B^* \subset A^* - B^*$.

$\therefore A^* - B^* = (A - B)^* - B^*$.

Since $(A - B)^* - B^* \subset (A - B)^*$.

f. $U \in \tau \Rightarrow U \cap A^* = U \cap (U \cap A)^* \subseteq (U \cap A)^*$.

To prove that $U \cap A^* = U \cap (U \cap A)^*$.

$x \in U \cap (U \cap A)^* \Rightarrow x \in U$ and $V \cap (U \cap A) \notin I$ for every $V \in N(x)$.

$\Rightarrow x \in U$ and $(V \cap U) \cap A \notin I$ for every $V \in N(x)$.

$\Rightarrow x \in U$ and $W \cap A \notin I$ for every $W \in N(x)$, where $W = U \cap V \in N(x)$

$\Rightarrow x \in U$ and $x \in A^*$.

$\Rightarrow x \in U \cup A^*$.

$\Rightarrow U \cap (U \cap A)^* \subset U \cap A^*$.

Now to prove that $U \cap A^* \subseteq U \cap (U \cap A)^*$.

$x \in U \cap A^* \Rightarrow x \in U$ and $x \in A^*$.

$\Rightarrow x \in A^* \Rightarrow V \cap A \notin I$ for every $V \in N(x)$.

U is a neighbourhood of x and V is a neighbourhood of x. Therefore $V \cap U$ is also a neighbourhood of x.

$\therefore (V \cap U) \cap A \notin I$ for every $V \in N(x)$ and for every $U \in N(x)$.

$\therefore V \cap (U \cap A) \notin I$ for every $V \in N(x)$ and for every $U \in N(x)$.

$\therefore x \in (U \cap A)^*$.

$\therefore U \cap A^* \subseteq U \cap (U \cap A)^*$.

$\therefore U \cap A^* = U \cap (U \cap A)^*$.

g. $I \in I \Rightarrow (A \cup I)^* = A^* = (A - I)^*$.

$I^* = \{x \in X : I \cap U \notin I$ for every $U \in N(x)\} = \phi$ .

$$\Rightarrow (A \cup I)^* = A^* \cup I^* = A^* \cup \phi = A^*.$$

Now to prove that $A^* = (A - I)^*$.

Always $(A - I)^* \subseteq A^*$.

Now to prove that $A^* \subseteq (A - I)^*$.

$$x \in A^* \Rightarrow U \cap A \notin I \text{ for every } U \in N(x) \quad - (I).$$

Now to prove that $x \in (A - I)^*$. Assume

$U_1 \cap (A - I) \in I$ for some $U_1 \in N(x)$.

$U_1 \cap (A - I) = B \in I$.

$U_1 \cap A \subseteq B \cup I \in I$.

$$\Rightarrow U_1 \cap A \in I.$$

Which is a contradiction to - (I)

$$\Rightarrow (A - I)^* \subseteq A^*.$$

$\therefore A^* = (A - I)^*$.

**Definition 1.1.4.**

Let P(X) be the class of all subsets of X. If $(\ \ )^c: P(X) \rightarrow P(X)$ is a function satisfing

1. $\phi^{c} = \phi$ .

2. $A \in P(X) \Rightarrow A \subseteq A^{c}$.

3. $A \in P(X),\ B \in P(X) \Rightarrow (A \cup B)^{c} = A^{c} \cup B^{c}$.

and 4. $A \in P(X) \Rightarrow (A^{c})^{c} = A^{c}$.

then $(\ \ )^{c}$ is called a Kuratowski closure operator and $\{A \in P(X) : A = A^{c}\}$ is the collection of all closed sets for a topology on X.

**Theorem 1.1.5.**

$d : P(X) \rightarrow P(X)$ is a function satisfing

1. $d(\phi) = \phi$

2. $d(A \cup B) = d(A) \cup d(B)$

3. $d(d(A)) \subseteq d(A)$

then $(\ \ )^{c} : P(X) \longrightarrow P(X)$ defined by $A^{c} = A \cup d(A)$ is a Kuratowski closure operator on P(X).

**Proof :-**

$A^{c} = A \cup d(A)$

So, (1) $A \subset A^{c}$ is satisfied.

(2) $\phi^{c} = \phi \cup d(\phi) = \phi \cup \phi = \phi$ .

(3) $(A \cup B)^{c} = (A \cup B) \cup d(A \cup B)$

$= (A \cup d(A)) \cup (B \cup d(B))$ $A \in P(X),\ B \in P(X)$.

$= A^{c} \cup B^{c}$.

(4) $(A^{c})^{c} = A^{c} \cup d(A^{c})$

$= A \cup d(A) \cup d(A \cup d(A))$.

$\subset A \cup d(A) \cup d(A) \cup d(A)$.

$$= A \cup d(A) = A^c.$$

$\therefore (A^c)^c \subset A^c.$

But $A \subset A^c$ for every $A$.

$$A^c \subset (A^c)^c.$$

$\therefore (A^c)^c = A^c.$

$\therefore A^c$ is a Kuratowski closure operator on $P(X)$.

**Theorem 1.1.6.**

The function $* : P(X) \rightarrow P(X)$ defined by $cl^*(A) = A \cup A^*$ is a closure operator.

**Proof :-**

To check $cl^*$ satisfies the four conditions of Kuratowski.

1. $cl^*(\phi) = \phi \cup \phi^* = \phi \cup \phi = \phi$ .

2. $cl^*(A \cup B) = (A \cup B) \cup (A \cup B)^*.$

$$= (A \cup B) \cup (A^* \cup B^*)$$

$$= (A \cup A^*) \cup (B \cup B^*)$$

$$= cl^*(A) \cup cl^*(B)$$

3. We know that $cl^*(A) = A \cup A^*.$

$\therefore A \subset cl^*(A).$

4. We know that $A \subset cl^*(A).$

$\therefore cl^*(A) \subset cl^*\, cl^*(A).$ Now to prove that $cl^*cl^*(A) \subset cl^*(A).$

$$cl^*cl^*(A) = cl^*(cl^*(A)) = cl^*(A \cup A^*)$$

$$= cl^*(A) \cup cl^*(A^*).$$

$$= (A \cup A^*) \cup (A^* \cup A^{**})$$

$$= A \cup A^* \cup A^{**}$$

$$\subset A \cup A^* \cup A^*$$

$= A \cup A^* = cl^*(A).$

$\therefore\ cl^*cl^*(A) \subset cl^*(A).$

$\therefore\ cl^*cl^*(A) = cl^*(A).$

$\therefore\ cl^*(A) = A \cup A^*$ is a Kuratowski closure operator on $P(X)$.

**Definition 1.1.7.**

We will define $\tau^*$ in terms of the closure operator $cl^*(A) = A \cup A^*$ by

$$\tau^*(I) = \{U \subseteq X : cl^*(X-U) = X - U\}$$

**Special cases :**

If $I = \{\phi\}$, then $A^* = cl(A)$. Hence in this case $cl^*(A) = cl(A)$ and hence $\tau^* = \tau$.

If $I = P(X)$ then $A^* = \phi$ for every $A \subseteq X$ and hence $\tau^*(I)$ is the discrete topology.

If $I = I_{cd}$ we have $\tau^*(I) = \tau$.

**Example 1.1.8.**

Let $\psi$ be the indiscrete topology on any set $X$ and $I_f$ be the ideal of finite sets.

Then $x \in A^* \Rightarrow X \cap A \notin I_f$, since $X$ is the only neighbourhood of $x$.

If $A$ is finite then $A^* = \phi$

$\therefore\ cl^*(A) = A \cup A^* = A$, when $A$ is finite.

If A is infinite then $A^* = X$.

$$\therefore\ cl^*(A) = A \cup A^* = A \cup X = X, \text{ when } A \text{ is infinite.}$$

$$\therefore\ cl^*(A) = \begin{cases} X & \text{if } A \notin I_f \\ A & \text{if } A \in I_f. \end{cases}$$

$$\therefore\ \psi^*(I_f) = \{U \,/\, cl^*(X - U) = X - U\}$$
$$= \{U \,/\, X - U \text{ is finite}\}.$$

This means that $\psi^*(I_f)$ is the well-known cofinite topology.

**Example 1.1.9.**

Let $\psi$ be the indiscrete topology on any set X and $I_c$ be the ideal of countable sets. Then

$x \in A^* \Rightarrow X \cap A \notin I_c$, since X is the only neighbourhood of x.

If A is countable then $A^* = \phi$,

$$\therefore\ cl^*(A) = A \cup A^* = A \cup \phi = A, \text{ when } A \in I_c.$$

If A is uncountable then $A^* = X$.

$$\therefore\ cl^*(A) = A \cup A^* = A \cup X = X, \text{ when } A \notin I_c.$$

$$\therefore\ cl^*(A) = \begin{cases} A & \text{if } A \in I_c \\ X & \text{if } A \notin I_c. \end{cases}$$

$$\therefore\ \psi^*(I_c) = \{U/cl^*(X - U) = X - U\}$$
$$= \{U/X - U \in I_c\}.$$

This means that $\psi^*(I_c)$ is the cocountable topology.

## SECTION 1.2 : COMPARISION OF IDEAL TOPOLOGY

### Introduction

In this section we shall discuss what happens when we change the ideals (i.e) given ideals I and J what is the relationship between $A^*(I)$ and $A^*(J)$.

**Theorem 1.2.1.**

Let $(X, \tau)$ be a space with I and J ideals on X. Then $I \subseteq J \Rightarrow A^*(J) \subseteq A^*(I)$ where A is a subset of X.

**Proof :-**

$x \in A^*(J) \Rightarrow U \cap A \notin J$ for every $U \in N(x)$

$\Rightarrow U \cap A \notin I$ for every $U \in N(x)$ $\therefore I \subseteq J$

$\Rightarrow x \in A^*(I)$

$\therefore A^*(J) \subseteq A^*(I)$ whenever $I \subseteq J$.

**Theorem 1.2.2.**

Let $(X, \tau)$ be a space with I and J ideals on X, and $A \subseteq X$. Then

a. $A^*(I \cap J) = A^*(I) \cup A^*(J)$

b. $A^*(I \cup J, \tau) = A^*(I, \tau^*(J)) \cap A^*(J, \tau^*(I))$

**Proof :-**

Take $x \in A^*(I \cap J)$

$\Rightarrow U \cap A \notin I \cap J$ for every $U \in N(x)$

$\Rightarrow U \cap A \notin I$ for every $U \in N(x)$ or $U \cap A \notin J$ for every $U \in N(x)$

$$\Rightarrow x \in A^*(\mathbf{I}) \text{ or } x \in A^*(\mathbf{J})$$

$$\Rightarrow x \in A^*(\mathbf{I}) \cup A^*(\mathbf{J}).$$

$$\therefore A^*(\mathbf{I} \cap \mathbf{J}) \subset A^*(\mathbf{I}) \cup A^*(\mathbf{J})$$

Similarly we can prove

$$A^*(\mathbf{I}) \cup A^*(\mathbf{J}) \subset A^*(\mathbf{I} \cap \mathbf{J})$$

$$\therefore A^*(\mathbf{I} \cap \mathbf{J}) = A^*(\mathbf{I}) \cup A^*(\mathbf{J}).$$

b. Assume $x \notin A^*(\mathbf{I} \cup \mathbf{J}, \tau)$. Then there exists a $U \in N(x)$ such that $U \cap A \in \mathbf{I} \cup \mathbf{J}$. Let $I \in \mathbf{I}$ and $J \in \mathbf{J}$ such that $U \cap A = I \cup J$. Because of the heredity of $\mathbf{J}$, we may assume $I \cap J = \phi$. Thus we have $(U \cap A) - I = J$ and $(U \cap A) - J = I$.

$$\Rightarrow (U - I) \cap A = J \in \mathbf{J} \text{ and } (U - J) \cap A = I \in \mathbf{I}$$

$$\Rightarrow x \notin A^*(\mathbf{J}, \tau^*(\mathbf{I})) \text{ or } x \notin A^*(\mathbf{I}, \tau^*(\mathbf{J})))$$

Therefore $A^*(\mathbf{J}, \tau^*(\mathbf{I})) \cap A^*(\mathbf{I}, \tau^*(\mathbf{J})) \subseteq A^*(\mathbf{I} \cup \mathbf{J}, \tau)$

Now assume $x \notin A^*(\mathbf{I}, \tau^*(\mathbf{J}))$. This implies that there exists $U \in N(x)$ and $J \in \mathbf{J}$ such that $(U - J) \cap A \in \mathbf{I}$. We may assume $J \subseteq A$. Now define $I = (U - J) \cap A$ and we have

$U \cap A = I \cup J \in \mathbf{I} \cup \mathbf{J} \Rightarrow x \notin A^*(\mathbf{I} \cup \mathbf{J}, \tau)$

So, $A^*(\mathbf{I} \cup \mathbf{J}, \tau) \subseteq A^*(\mathbf{I}, \tau^*(\mathbf{J}))$.

Similarly, we have that $A^*(\mathbf{I} \cup \mathbf{J}, \tau) \subseteq A^*(\mathbf{J}, \tau^*(\mathbf{I}))$

Hence $A^*(\mathbf{I} \cup \mathbf{J}, \tau) \subseteq A^*(\mathbf{I}, \tau^*(\mathbf{J})) \cap A^*(\mathbf{J}, \tau^*(\mathbf{I}))$

$$\therefore A^*(\mathbf{I} \cup \mathbf{J}, \tau) = A^*(\mathbf{I}, \tau^*(\mathbf{J})) \cap A^*(\mathbf{J}, \tau^*(\mathbf{I})).$$

**Theorem 1.2.3.**

Let $(x, \tau)$ be a space with $\mathbf{I}$ and $\mathbf{J}$ ideals on $X$. Then

a. $\tau^*(I \cup J) = (\tau^*(J))^*(I) = (\tau^*(I))^*(J)$.

b. $\tau^*(I \cup J) = \tau^*(I) \cup \tau^*(J)$ .

c. $\tau^*(I \cap J) = \tau^*(I) \cap \tau^*(J)$.

**Proof :-**

The basis element of $\tau^*$ with respect to $I \cup J$ is of the form $V - B$, where $B \in I \cup J$, and $V \in \tau$.

The basis element of $[\tau^*(J)]^*(I)$ is of the form $V - (I \cup J)$. Since the basis element of $(\tau^*(J))^*$ with respect to $I$ is of the form $W - I = (V - J) - I$ where $V \in \tau$ , $J \in J$, $I \in I$ and $W \in \tau^*$..

$= V - B$ where $B \in I \cup J$.

Also any basis element of $(\tau^*(I))^*$ with respect to $J$ is of the form $W - J$, where $W \in \tau^*$ and $J \in J$.

$= (V-I) - J = V - B$ where $V \in \tau$ , $B \in I \cup J$.

So, $\tau^*(I \cup J) = (\tau^*(J))^*(I)$

$= (\tau^*(I))^*(J)$

b. $\tau^*(I \cup J) = \tau^*(I) \cup \tau^*(J)$

The basis element of $\tau^*(I) \cup \tau^*(J)$ is of the form $(W - I) \cup (V - J) = V - B$ where $V = W \cup U \in \tau$, $B \in I \cup J$.

Which is the basis element of $\tau^*$ with respect to $I \cup J$.

c. $\tau^*(I \cap J) = \tau^*(I) \cap \tau^*(J)$.

Basis element of $\tau^*$ with respect to $I \cap J$ is of the form $V - B$ where $B \in I \cap J$, $V \in \tau$.

Basis element of $\tau^*(I) \cap \tau^*(J)$ is the form $(U-I) \cap (W-J) = V-B$ where $V = U \cap W \in \tau$ and $B \in I \cup J$ $\therefore$ $\tau^*(I \cap J) = \tau^*(I) \cap \tau^*(J)$.

## SECTION 1.3. BASE FOR IDEAL TOPOLOGY.

**Introduction .**

In this section we shall discuss some simple examples and construct a base for $\tau^*$.

We shall define

$B(\mathbf{I}, \tau) = \{V - I \;:\; V \in \tau,\ I \in \mathbf{I}\}$ and we shall prove this forms a basis for $\tau^*$.

**Theorem 1.3.1.**

Every member of **I** is closed in $\tau^*$.

**Proof :-**

To prove that every member of **I** is closed in $\tau^*$.

(i.e) to prove that $cl^*(I) = I$, for every $I \in \mathbf{I}$.

We know that $cl^*(I) = I \cup I^*$.

But $I^* = \{x \;/\; U \cap I \in \mathbf{I}, \text{ for every } U \in N(x)\} = \phi$

$\therefore\ cl^*(I) = I \cup I^* = I \cup \phi = I.$

So, every member of **I** is closed in $\tau^*$.

**Theorem 1.3.2.**

Let $(X, \tau)$ be a space and **I** an ideal on X. Then $B(\mathbf{I}, \tau)$ is a basis for $\tau^*$.

**Proof :-**

Let $(X, \tau)$ be a space and **I**, an ideal on X. We know that A is $\tau^*$ - closed iff $A^* \subseteq A$. Now we have $U \in \tau^*$ iff $X - U$ is $\tau^*$ - closed iff $(X - U)^* \subseteq (X - U)$ iff $U \subseteq X - (X - U)^*$. Therefore $x \in U \Rightarrow x \notin (X - U)^* \Rightarrow$ there exists a $V \in N(x)$ such that $V \cap (X - U) \in \mathbf{I}$. Now let $I = V \cap (X - U)$

and we have $x \in V - I \subseteq U$, where $V \in \tau$ and $I \in \mathbf{I}$.

$$\therefore U = \bigcup_{V \in \tau} (V - I).$$

$\therefore$ U is $\tau^*$ - closed.

We shall prove that $\mathbf{B}(\mathbf{I}, \tau)$ is closed under finite intersection.

$\bigcap_{i=1}^{n} (V_i - I_i) = \bigcap_{i=1}^{n} V_i - \bigcup_{i=1}^{n} I_i = V - I \in \mathbf{B}(\mathbf{I}, \tau)$ where

$V = \bigcap_{i=1}^{n} V_i \in \tau$ and $I = \bigcup_{i=1}^{n} I_i \in \mathbf{I}$.

Hence $\mathbf{B}(\mathbf{I}, \tau)$ is a base for $\tau^*$.

**Theorem 1.3.3.**

Let $(X, \tau)$ be a space and $\mathbf{I}$ an ideal on X. Then $\tau^*(\mathbf{I}) = \tau \cup \psi^*(\mathbf{I})$ where $\psi$ denotes the indiscrete topology on X.

**Proof :-**

The basis of element of $\psi^*(\mathbf{I})$ is of the form $X - I$ where $I \in \mathbf{I}$. $\therefore$ the basis element of $\tau \cup \psi^*(\mathbf{I})$ is of the form $V \cup (X - I) = V - I$ where $V \in \tau$ and $I \in \mathbf{I}$, which is the basis element of $\tau^*(\mathbf{I})$.

**Theorem 1.3.4.**

Let $(x, \tau)$ be a space and $\mathbf{I}$ an ideal of X. Then $A^*(\mathbf{I}, \tau) = A^*(\mathbf{I}, \tau^*)$. Hence $\tau^* = \tau^{**}$.

**Proof : -**

Take $\mathbf{I} = \mathbf{J}$ in 1.2.2. (b) we get $A^*(\mathbf{I}, \tau) = A^*(\mathbf{I}, \tau^*)$. Hence $\tau^* = \tau^{**}$.

**Example 1.3.5.**

Let $(X, \tau)$ be a space. Then it is easy to check that $I_{cd} = \{A \subseteq X : A^d = \phi\}$ is an ideal on X and that $A^d \subseteq A^*$. Therefore $\tau^* = \tau$. Also note that $I_{cd}$ is the largest ideal on X. with the property $\tau^* = \tau$.

**Example 1.3.6.**

Let $(X, \tau)$ be a space and $A \subseteq X$. Define $I(A) = \{B \subseteq X : B \subseteq A\}$. Now to prove that $I(A)$ is an ideal on X.

Let $B \in I(A)$ and $C \subseteq B$ then to prove that $C \in I(A)$.

$B \in I(A) \Rightarrow B \subseteq A$.

$C \subseteq B \Rightarrow C \subseteq B \subseteq A \Rightarrow C \subseteq A$.

So, $C \in I(A)$.

$B \in I(A)$ and $C \in I(A)$ then to prove that $B \cup C \in I(A)$.

$$\left.\begin{array}{l} B \in I(A) \Rightarrow B \subseteq A \\ C \in I(A) \Rightarrow C \subseteq A \end{array}\right\} \text{-----}(1)$$

Consider $B \cup C$.

$B \cup C \subseteq A$ from (1)

So, $B \cup C \in I(A)$.

Hence $I(A)$ is an ideal on X.

Let $(X, \tau)$ be a space and $p \in X$.

Define $I(X - \{p\}) = \{A \subseteq X ; p \notin A\}$.

$$\begin{aligned} \psi^*(I(X - \{p\})) &= \{A \subseteq X \,/\, cl^*(X - A) = X - A\} \cup \{\phi\} \\ &= \{A \subseteq X \,/\, X - A \in I(X - \{p\})\} \cup \{\phi\}. \end{aligned}$$

Since $cl^*(A) = \begin{cases} X & \text{if } A \notin I(X - \{p\} . \\ A & \text{if } A \in I(X - \{p\} . \end{cases}$

$\psi^*(I\ (X - \{p\} ) = \{A \subseteq X \ / \ A \notin I\ (X - \{p\}\ )\} \cup \{\phi\}.$

$= \{A \subseteq X : p \in A\} \cup \{\phi\}.$

Hence $I(X - \{p\}\ ) = \{A \subseteq X : p \notin A\}$ generates a simple topology $\psi^* = \{A \subseteq X : p \in A\} \cup \{\phi\}$ known as the particular point topology.

## SECTION 1.4 COMPATIBILITY OF $\tau$ WITH I.

**Introduction.**

This section deals with the concept of compatibility of $\tau$ with **I.**

**Definition 1.4.1.**

Let $(X, \tau)$ be a space and **I** an ideal on X. We say the topology $\tau$ is compatible with the ideal **I,** denoted by $\tau \sim I$, if the following holds for every $A \subseteq X$ : if for every $x \in A$ there exists a $U \in N(x)$ such that $U \cap A \in I$ then $A \in I$.

**Definition 1.4.2.**

An ideal **I** defined on X is called a $\sigma$ - ideal iff it is closed with respect to countable union (i.e) $A_i \in I$, $i = 1,2,3\ldots\ldots$ then $\bigcup_{i=1}^{\infty} A_i \in I$.

**Theorem 1.4.3.**

Let $(X, \tau)$ be a hereditarily Lindelof space and **I** a $\sigma$ - ideal on X, then $\tau \sim$**I.**

**Proof : -**

Let $A \subseteq X$ and assume that for every $x \in A$, there exists a $U_x \in N(x)$ such that $U_x \cap A \in$ **I.** Now $\{U_x \cap A : x \in A\}$ is an open cover of A and hence has a countable subcover $\{V_n \cap A : n \in N\}$. Since **I** is a $\sigma$ - ideal, $A = \bigcup\{V_n \cap A : n \in N\}$ is in **I.** Hence $\tau \sim$**I.**

**Theorem 1.4.4.**

Let $(X, \tau)$ be a space with **I** an ideal on X. Then the following are equivalent

a. $\tau \sim$ **I** .

b. If A has a cover of open sets each of whose intersection with A is in **I**, then A is in **I.**

c. For every $A \subseteq X$, $A \cap A^* = \phi \Rightarrow A \in$ **I.**

d. For every $A \subseteq X$, $A - A^* \in$ **I** .

e. For every $\tau^*$ - closed subset A, $A - A^* \in$ **I.**

f. For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B^*$, then $A \in$ **I.**

**Proof : -**

Assume (a) (i.e)$\tau \sim$**I** (i.e) if for every $x \in A$ there exists a $U \in N(x)$ such that $U \cap A \in$ **I** then $A \in$ **I.**

To prove (b) (i.e) if A has a cover of open sets each of whose intersection with A is in **I** then A is in **I** .

Assume A has a cover of open sets each of whose intersection with A is in **I** then to prove that A is in **I**.

By our assumption $A = \bigcup_{U \in \tau} (U \cap A)$ where U is an open set and $U \cap A \in \mathbf{I} \Rightarrow \bigcup_{U \in \tau} (U \cap A) \in \mathbf{I} \Rightarrow A \in \mathbf{I}$. Hence the proof of (b).

Assume (b) (i.e) if A has a cover of open sets each of whose intersection with A is in **I** then A is in **I**.

To prove (c) (i.e) for every $A \subseteq X$, $A \cap A^* = \phi \Rightarrow A \in \mathbf{I}$ .

Assume $A \cap A^* = \phi$ . Then $x \in A \Rightarrow x \notin A^*$.

$x \notin A^* \Rightarrow$ there exists $U \in N(x)$ such that $U \cap A \in \mathbf{I} \Rightarrow A \in \mathbf{I}$.

Hence the proof of (c).

Assume (c) (i.e) for every $A \subseteq X$ $A \cap A^* = \phi \Rightarrow A \in \mathbf{I}$.

We prove (d) (i.e) for every $A \subseteq X$, $A - A^* \in \mathbf{I}$. (i.e) to prove that $(A-A^*) \cap (A-A^*)^* = \phi$.

If not, suppose $x \in A - A^*$ and $x \in (A - A^*)^*$.

$x \in A - A^* \Rightarrow x \in A$ and $x \notin A^*$.

$x \in (A-A^*)^* \Rightarrow U \cap (A - A^*) \notin \mathbf{I}$ for every $U \in N(x)$.

$\Rightarrow U \cap A \notin \mathbf{I}$ for every $U \in N(x)$.

which is a contradiction to $x \notin A^*$.

$\therefore (A-A^*) \cap (A-A^*)^* = \phi$ . Hence $A - A^* \in \mathbf{I}$.

Hence the proof of (d).

Assume (e) (i.e) for every $\tau^*$ - closed subset A, $A - A^* \in \mathbf{I}$. To prove (a) (i.e) $\tau \sim \mathbf{I}$.

Let $A \subseteq X$ and assume that for every $x \in A$ there exists a $U \in N(x)$ such that $U \cap A \in \mathbf{I}$. Then $A \cap A^* = \phi$ and since $A \cup A^*$ is $\tau^*$ - closed, we have $(A \cup A^*) - (A \cup A^*)^* \in \mathbf{I}$.

$$\therefore\ (A \cup A^*) - (A \cup A^*) = (A \cup A^*) - (A^* \cup A^{**})$$
$$= (A \cup A^*) - A^* = A \in \mathbf{I}.$$

Hence the proof of (a).

Assume (d) (i.e) for every $A \subseteq X$, $A - A^* \in \mathbf{I}$. To prove (f) (i.e) for every $A \subseteq X$, if $A$ contains no nonempty subset $B$ with $B \subseteq B^*$, then $A \in \mathbf{I}$.

Let $A \subseteq X$ and assume that $A$ has no nonempty subset $B$ with $B \subseteq B^*$, since $A - A^* \in \mathbf{I}$, $A \cap A^* \subseteq (A \cap A^*)^*$ and hence $A \cap A^* = \phi$. Therefore $A = A - A^*$ and $A \in \mathbf{I}$. Hence the proof of (f).

Assume (f) (i.e) for every $A \subseteq X$, if $A$ contains no nonempty subset $B$ with $B \subseteq B^*$ then $A \in \mathbf{I}$.

To prove (d) (i.e) for every $A \subseteq X$, $A - A^* \in \mathbf{I}$.

Let $A \subseteq X$, since $(A - A^*) \cap (A - A^*)^* = \phi$, $A - A^*$ contains no nonempty subset $B$ with $B \subseteq B^*$. Hence $A - A^* \in \mathbf{I}$.

Hence the proof of (d).

Hence the result.

**Theorem 1.4.5.**

Let $(X, \tau)$ be a space with $\mathbf{I}$ an ideal on $X$. Then the following are equivalent

a. For every $A \subseteq X$, $A \cap A^* = \phi \Rightarrow A^* = \phi$.

b. For every $A \subseteq X$, $(A - A^*)^* = \phi$ .

c. For every $A \subseteq X$, $(A \cap A^*)^* = A^*$.

**Proof : -**

Assume for every $A \subseteq X$, $A \cap A^* = \phi \Rightarrow A^* = \phi$. To prove that $(A - A^*)^* = \phi$ (i.e) to prove that $(A - A^*) \cap (A - A^*)^* = \phi$.

If not, let $x \in (A - A^*)$ and $x \in (A - A^*)^*$.

$x \in (A - A^*)^* \Rightarrow U \cap (A - A^*) \notin I$ for every $U \in N(x)$. - - - - (1)

$x \in A - A^* \Rightarrow x \in A$ and $x \in A^*$.

$x \in A^* \Rightarrow$ there exists $U_1 \in N(x)$ such that $U_1 \cap A \in I$.

$\Rightarrow U_1 \cap (A - A^*) \in I$ for some $U_1 \in N(x)$ which is a contradiction to (1).

$\therefore (A - A^*) \cap (A - A^*)^* = \phi$.

$(A - A^*)^* = \phi$.

Assume for every $A \subseteq X$, $(A - A^*)^* = \phi$.

To prove $(A \cap A^*)^* = A^*$.

We know that $(A - A^*)^* - A = A^* - A$.

$\therefore (A - A^*)^* = A^*$ .

But $(A - A^*)^* = \phi$.

$\Rightarrow A^* = \phi$.

To prove that $(A \cap A^*)^* = \phi$ .

$A^* = \phi \Rightarrow$ for every $x$ there exists $U \in N(x)$ such that $U \cap A \in I$.

So, $U \cap (A \cap A^*) \in I$ for some $U \in N(x)$.

Hence $x \notin (A \cap A^*)^*$ for every $x$. $\therefore (A \cap A^*)^* = \phi$.

Hence $A^* = (A \cap A^*)^*$.

Assume for every $A \subseteq X, (A \cap A^*)^* = A^*$.

To prove that for every $A \subseteq X$, $A \cap A^* = \phi$.

$\Rightarrow A^* = \phi$.

$A^* = (A \cap A^*)^* = \phi^* = \phi$.

Hence $A^* = \phi$.

Hence the result.

**Theorem 1.4.6.**

Let $(X, \tau)$ be a space and $\mathbf{I}$ an ideal on X compatible with $\tau$. A set is closed with respect to $\tau^*$ iff it is the union of a set which is closed with respect to $\tau$ and a set in $\mathbf{I}$.

**Proof :-**

Let A be $\tau^*$ - closed then $A^* \subseteq A$.

$\Rightarrow A = (A - A^*) \cup A^*$.

Given that $\tau \sim \mathbf{I}$ then by theorem 1.4.4. (d) for every $A \subseteq X$, $A - A^* \in \mathbf{I}$. Also $cl(A^*) = A^*$.

So, $A^*$ is $\tau$ - closed. Hence A is the union of a set which is closed with respect to $\tau$ and a set in $\mathbf{I}$.

Conversely, assume that if $A = B \cup I$ where B is $\tau$ - closed and $I \in \mathbf{I}$ then to prove that A is closed with respect to $\tau^*$.

$A = B \cup I$.

$\Rightarrow A^* = B^* \cup I^* = B^* \subseteq cl(B) = B \subseteq A$.

Since $I^* = \{x \,/\, U \cap I \in \mathbf{I}$, for every $U \in N(x)\} = \phi$.

Thus $A^* \subseteq A$. Hence A is $\tau^*$ - closed.

Hence the result.

**Definition 1.4.7.**

We say $\tau$ is weakly compatible with I if an ideal satisfies $A^* = \phi \Rightarrow$

$A \in I$ for every $A \subseteq X$. It is denoted by $\tau \overset{W}{\sim} I$.

**Theorem 1.4.8.**

$\tau \overset{W}{\sim} I_f$ iff $(X, \tau)$ is countably compact.

**Proof : -**

Assume $\tau \overset{W}{\sim} I_f$ then $A^* = \phi \Rightarrow A \in I_f$.

To prove that $(X, \tau)$ is countably compact.

Let $\{G_i / i \in I^+\}$ be a countable covering. To prove that it contains a finite sub covering.

Assume that it has no finite sub covering.

Select $x_i$ from the above covering in such a way that $x_1 \in G_1$, $x_2 \in G_2 - G_1$, $x_3 \in G_3 - G_1 \cup G_2$, . . . .

Let $A = \{x_1, x_2, \ldots \}$. Then A is countable. Also $G_i \cap A$ is finite. We know that $\tau \overset{W}{\sim} I_f$. So, A is finite, which is a contradiction. Hence $(X, \tau)$ is countably compact.

The converse part of the theorem can be discussed in a similarly way.

**Theorem 1.4.9.**

A space $(X, \tau)$ is hereditarily Lindelof iff $\tau \sim I_c$.

**Proof :-**

Assume the space is hereditarily Lindelof space. Also we know that $I_c$ is a $\sigma$- ideal on X. Using theorem 1.4.3., we get $\tau \sim I_c$.

Assume $\tau \sim I_c$. Now to prove that it is hereditarily Lindelof space.

Assume the space is not hereditarily Lindelof space. Let $Y \subset X$ be not Lindelof.

Take $Y = \bigcup_{\lambda \in \Lambda} G_\lambda$

Pick the elements $x_1 \in G_1$, $x_2 \in G_2 - G_1$,

$\ldots \; x_\lambda \in G_\lambda - (G_1 \cup G_2 \cup \ldots \cup G_{\lambda-1})$

$x_\mu \in G_\mu - (G_1 \cup G_2 \cup \ldots \cup G_{\lambda-1})$

Let $A = \{x_\lambda \,/\, \lambda \in \Lambda\}$. Then A is uncountable . Also Also $G_\mu \cap A$ which is countable then A is countable because $\tau \sim I_c$, which is a contradiction. So, the space $(X, \tau)$ is hereditarily Lindelof.

**Definition 1.4.10.**

A space is $x_1$ - compact iff every uncountable set in the space has atleast one limit point.

**Theorem 1.4.11.**

$\tau \overset{W}{\sim} I_c$ iff $(X, \tau^*)$ is $x_1$ - compact.

**Proof : -**

Assume $\tau \overset{W}{\sim} I_c$ (i.e) whenever $A^* = \phi$ then $A \in I_c$.

(i.e) If for every $x \in X$, there exists $U \in N(x)$ such that $U \cap A$ is countable then A is countable.

Let Y be any uncountable set in X. Assume it has no limit point. Then for every x there exists $U \in N(x)$ such that $U \cap (Y - \{x\}) = \phi \in I_c$.

$\Rightarrow Y - \{x\} \in I_c$

$\Rightarrow Y \in I_c$.

$\Rightarrow$ Y is countable which is a contradiction. Hence the space is $x_1$ - compact.

The converse part of the theorem can be proved in a similar way.

**Theorem 1.4.12.**

Let $(X, \tau)$ be a space and I an ideal on X. Then $\tau \sim I$ iff $\tau^* \sim I$.

**Proof :-**

Assume $\tau^* \sim I$ (i.e) for every x there exists $U \in N^*_\tau(x)$ such that $U \cap A \in I$ then $A \in I$.

To prove that $\tau \sim I$ (i.e) If for every $x \in X$ there exists $U_1 \in N_\tau(x)$ such that $U_1 \cap A \in I$ then $A \in I$.

Since every $\tau$ neighbourhood is a $\tau^*$ neighbourhood .

So, $U_1 \cap A \in I$ for some $U_1 \in N_\tau(x)$ .

$\Rightarrow U \cap A \in I$ for some $U \in N_{\tau^*}(x)$

$\Rightarrow A \in I$

Hence $\tau \sim I$.

Assume $\tau \sim I$.

To prove that $\tau^* \sim I$.

(i.e) to prove that if $(U - E) \cap A \quad I$, for some $U \in N(x)$ and $E \in I$ then $A \in I$.

Consider $U \cap A = [(U - E) \cup E] \cap A$

$$= [((U - E) \cap A) \cup (E \cap A)] \in I$$

$\therefore U \cap A \in I \Rightarrow A \in I$ since $\tau \sim I$.

$\therefore \tau^* \sim I$.

Hence the result.

## SECTION 1.5 IDEALS CONTAINING $I_f$.

**Introduction.**

The case when the ideal under consideration contains $I_f$ is very interesting. This section is devoted to the study of this case.

**Theorem 1.5.1.**

Let $(X, \tau)$ be a space and $I$ an ideal on X. Then $\{x\} \in I$ for every $x \in X$ iff $A^* = A^{d^*}$ for every $A \subseteq X$, where $A^{d^*}$ denotes the derived set of A respect to $\tau^*$.

**Proof :-**

By theorem 1.1.3.(g) we get $A^* = (A - \{x\})^*$.

Now to prove that $A^{d^*} = (A - \{x\})^*$.

$x \in A^{d^*}$ iff $x \in cl^*(A - \{x\})$ iff $x \in (A - \{x\}) \cup (A - \{x\})^*$

iff $x \in (A - \{x\})^*$.

Hence $A^{d^*} = (A - \{x\})^*$

Hence $A^* = (A - \{x\})^* = A^{d^*}$.

$\therefore A^* = A^{d^*}$.

Hence the result.

**Definition 1.5.2.**

A subset A of a space $(X, \tau)$ is dense - in - itself if $A \subseteq A^d$ that is, A is without isolated points.

**Definition 1.5.3.**

A set which is dense - in - itself and closed is said to be perfect.

**Theorem 1.5.4.**

Let $(X, \tau)$ be a space with an ideal **I** on X such that $\tau \sim$ **I** and $\{x\} \in$ **I** for each $x \in X$. If a set $A \subseteq X$ is closed with respect to $\tau^*$, then A is the union of a set which is perfect with respect to $\tau$ and a set in **I**.

**Proof : -**

Let $A \subseteq X$ be closed with respect to $\tau^*$. Hence using theorem 1.4.6. we get $A = A^* \cup I$ where $A^*$ is closed with respect to $\tau$ and $I \in$ **I**. Given $\tau \sim$ **I** using theorem 1.4.4. (d) we get $A - A^* \in$ **I** for every $A \subseteq X$. Let $A - A^* = I$. Then $A^* \cap I = \phi$. Also using theorem 1.1.3. (e) we get $A^* - A^{**} \subseteq (A - A^*)^*$.

Using theorem 1.4.5. (b) we get $(A - A^*)^* = \phi$. Hence $A^* - A^{**} = \phi \Rightarrow A^* = A^{**}$. Now using 1.5.1.

We get

$$\begin{aligned} A^* &= A^{**} \\ &= (A^*)^* \\ &= (A^*)^{d*}. \end{aligned}$$

So, $A^* = (A^*)^{d*}$ and consequently $A^* \subseteq (A^*)^d$. So, $A^*$ is dense - in - itself. Hence $A^*$ is perfect with respect to $\tau$.

**Theorem 1.5.5.**

**Cantor - Bendixson theorem.**

A second countable space can be represented as the union of two sets, one of which is perfect and the other countable.

**Proof : -**

In the previous theorem (1.5.4.) take $A = X$ and $I = I_c$. Then it satisfies all the conditions of this theorem. Hence the above theorem 1.5.5. is an immediate consequence of 1.5.4.

**Definition 1.5.6.**

A set $A \subseteq X$ is said to be scattered if A contains no nonempty dense - in - itself subset.

**Definition 1.5.7.**

A set A is said to discrete if $A \cap A^d = \phi$.

**Remark 1.5.8.**

Discrete sets are scattered.

**Proof :** -

Let A be any discrete set then $A \cap A^d = \phi$. To prove that A is scattered.

Let $B \subset A$ be any subset of A. Then to prove that $B \nsubseteq B^d$.

$B \subset A \Rightarrow B^d \subset A^d$.

$$A \cap A^d = \phi \Rightarrow B \cap A^d = \phi \quad \therefore B \subset A$$

$$\Rightarrow B \cap B^d = \phi \quad \because B^d \subset A^d$$

$$\Rightarrow B \nsubseteq B^d.$$

$\therefore$ A is scattered.

**Theorem 1.5.9.**

Let $(X, \tau)$ be a space and **I** an ideal on X. The following are equivalent.

a. $\tau \sim \mathbf{I}$ and $\{x\} \in \mathbf{I}$ for each $x \in X$.

b. Scattered sets in $(X, \tau^*)$ are in **I**.

c. Discrete sets in $(X, \tau^*)$ are in **I**.

**Proof :-**

Assume $\tau \sim \mathbf{I}$ and $\{x\} \in \mathbf{I}$ for each $x \in X$. By using theorem 1.5.1. we get $B^* = B^{d^*}$ for every $B \subseteq X$.

Let A be a scattered set in $(X, \tau^*)$. Then A contains no nonempty set B with $B \subseteq B^{d^*} = B^*$. Then using 1.4.4. (f) we get $A \in \mathbf{I}$. Hence the scattered sets in $(X, \tau^*)$ are in **I**.

Assume (b) (i.e) scattered sets in $(X, \tau^*)$ are in **I** then by using Remark 1.5.8. we get discrete sets in $(X, \tau^*)$ are in **I**. Hence the proof of (c).

Assume discrete sets in $(X, \tau^*)$ are in **I**. Now to prove that $\tau \sim I$ and $\{x\} \in I$ for every $x \in X$.

Obviously $\{x\} \in I$ for every $x \in X$. Let $A \subseteq X$ then to prove that $A - A^*$ is discrete in $(X, \tau^*)$ (i.e) to prove that $(A - A^*) \cap (A - A^*)^{d^*} = \phi$. Since $(A - A^*)^{d^*} \subseteq A^{d^*} = A^*$, since $\{x\} \in I$. $\therefore$ $(A - A^*) \cap (A - A^*)^{d^*} \subseteq (A - A^*) \cap A^* = \phi$

$$\Rightarrow (A - A^*) \cap (A - A^*)^{d^*} = \phi.$$

So, $A - A^*$ is discrete in $(X, \tau^*)$.

Hence $A - A^* \in I$ for every $A \subseteq X$. Using theorem 1.4.4. (d) we get $\tau \sim I$.

**Theorem 1.5.10.**

Let $(X, \tau)$ be a space and $I_c$ be the ideal of countable subsets of X. Then the following are equivalent.

a. $(X, \tau)$ is hereditarily Lindelof

b. $(X, \tau^*(I_c))$ is hereditarily Lindelof.

c. If I is scattered in $(X, \tau^*)$, then $I \in I_c$.

d. If I is discrete in $(X, \tau^*)$, then $I \in I_c$.

**Proof :-**

By theorem 1.4.9. we get $(X, \tau)$ is hereditarily Lindelof iff $\tau \sim I_c$. Also we know that $\tau \sim I_c$ iff $\tau^* \sim I_c$ by theorem 1.4.12.

Hence combining we get, $(X, \tau)$ is hereditarily Lindelof iff $\tau \sim I_c$ iff $\tau^* \sim I_c$ iff $(X, \tau^*(I_c))$ is hereditarily Lindelof. Hence (a)$\Leftrightarrow$(b).

Using theorem 1.5.9. we get, if $I$ is scattered in $(X, \tau^*)$ then $I \in I_c$ iff $I$ is discrete in $(X, \tau^*)$ then $I \in I_c$. Hence (c)$\Leftrightarrow$(d).

Assume $(X, \tau)$ is hereditarily Lindelof then by theorem 1.4.9. we get $\tau \sim I_c$ and also we know $\{x\} \in I_c$ for each $x \in X$. So, by theorem 1.5.9. we get scattered sets in $(X, \tau^*)$ are in $I_c$. Hence (a)$\Rightarrow$(c).

Now assume (c) (i.e) if $I$ is scattered in $(X, \tau^*)$ then $I \in I_c$.

Using theorem 1.5.9. we get $\tau \sim I_c$ and $\{x\} \in I_c$ for each $x \in X$. Hence using theorem 1.4.9. once again we get $(X, \tau)$ is hereditarily Lindelof. Hence the proof of (a). Hence (c) $\Rightarrow$(a). Hence the result.

## SECTION 1.6. SOME APPLICATIONS

In this section we shall discuss some interesting applcations of the results proved in the previous sections.

### Definition 1.6.1.

An open subset U of a space $(X, \tau)$ is said to be regular open if $U = \mathrm{Int}(\mathrm{cl}(U))$.

**Definition 1.6.2.**

The regular open sets in $(X, \tau)$ form a basis for a new topology on X, known as the semiregularization of $\tau$, denoted by $\tau_s$. The topology $\tau_s$ is coarser than $\tau$, and is said to be semiregular if $\tau = \tau_s$.

The following theorem gives a characterisation of spaces for which $X = X^*$.

**Theorem 1.6.3.**

Let $(X, \tau)$ be a space with **I** an ideal on X. The following are equivalent

a. $X = X^*$.

b $\tau \cap I = \{\phi\}$.

c. If $I \in \mathbf{I}$, then $\mathrm{Int}(I) = \phi$, and

d. For every $U \in \tau$, $U \subseteq U^*$.

As an interesting consequence we have the following theorem.

**Theorem 1.6.4.**

Let $(X, \tau)$ be a space with an ideal **I** on X such that $X = X^*$. Then $\tau_s = (\tau^*)_s$.

**Definition 1.6.5.**

Spaces in which compact sets are finite have been called cf - spaces or pseudofinite or anticompact.

**Definition 1.6.6.**

The ideal of relatively compact sets are denoted by $I_k$, defined as $I_k = \{A \subseteq X \; ; \; cl(A) \text{ is compact in } (X, \tau)\}$.

The following theorem gives a condition under which $(X, \tau^*)$ is anticompact.

**Theorem 1.6.7.**

Let $(X, \tau)$ be a Hausdorff space. Then $(X, \tau^* (I_c \cap I_s \cap I_k))$ is anticompact.

Sufficient condition for $(X, \tau^*)$ to be anticompact is given by the following theorem.

**Theorem 1.6.8.**

Let $(X, \tau)$ be a Hausdorff space with an ideal $I$ on $X$. If $\tau \sim I$ and $\{x\} \in I$ for each $x \in X$, then $(X, \tau^*(I))$ is anticompact.

As an immediate consequence we have the following corollaries.

**Corollary 1.6.9.**

Let $(X, \tau)$ be a dense-in-itself Hausdorff space. Then $(X, \tau^*(I_c \cap I_n \cap I_k))$ is anticompact .

**Corollary 1.6.10.**

Let $(X, \tau)$ be a dense-in-itself Hausdorff space. Then $(X, \tau^*(I_n))$ is anticompact.

## SECTION 1.7. MORE ON IDEAL TOPOLOGIES.

In this section we shall collect some important results proved in the following two papers

1. A topology formed from a given topology and ideal by Samuels.P.

2. Idealy equivalent topologies and semitopological properties by David A.Rose and T.R.Hamlett.

**Theorem 1.7.1.**

$\tau^*(I) = \tau$ iff every member of I is $\tau$ - closed.

**Theorem 1.7.2.**

If $(X, \tau)$ is Hausdorff and I is any ideal on X not contained in the family of $\tau$ - closed sets, $(X, \tau^*(I))$ is not compact.

**Theorem 1.7.3.**

Let $(X, \tau)$ be Hausdorff and I any ideal on X. Let $E \subset X$ be an infinite $\tau$ -compact set which is not in I. Then E is $\tau^*(I)$ compact if and only if $E \cap z$ is $\tau$ - compact and $\tau$ -closed discrete for all $z \in I$.

**Theorem 1.7.4.**

$(X, \tau)$ is Hausdorff in $(X, \tau^*(I))$ is Hausdorff.

**Theorem 1.7.5.**

If $(X, \tau^*(I))$ is regular then $\tau$ and $\tau^*(I)$ coincide.

**Theorem 1.7.6.**

If $\tau$ is completely regular and $\tau^*(I)$ is regular, then $\tau^*(I)$ is completely regular.

**Theorem 1.7.7.**

Let $(X,d)$ be a metric space which is compact, and let **I** be an ideal which contains all singletons. Then a set $E \subset X$ is $\tau^*(I)$ - perfect iff for every $\epsilon > 0$ it is a finite union of $\tau$ - closed pieces of diameter $< \epsilon$ not belonging to **I**.

**Definition 1.7.8.**

Given a topological space $(X, \tau)$ and an ideal **I** of subsets of X the topology $\tau$ on X is said to be **I** - equivalent to $\tau$ if $\sigma^*(I) = \tau^*(I)$.

$[\tau]^*$ denotes the set of all topologies **I** - equivalent to $\tau$ .

**Theorem 1.7.9.**

The topology $\tau^*(I)$ is the maximum of $[\tau]^*$.

In the paper "Ideally Equivalent Topologies And Semitopological Properties" by "David A. Rose and T.R.Hamlett", the authors study $\tau^*$ in details. Using this, author is able to characterise the semiregular properties and semitopological properties.

# Chapter - II

# CHAPTER - II

## PRELIMINARY DEFINITIONS.

**Definition 1.**

A topological space with no disjoint open sets will be called **hyperconnected.**

**Definition 2.**

If each point in a topological space is contained in an open set whose closure is compact, we call the space **strongly locally compact.**

**Definition 3.**

A topological space is said to be of **second category** in X iff it cannot be expressed as the union of a countable collection of nowhere dense subsets of X.

**Definition 4.**

A space with no disjoint closed sets will be called **ultraconnected.**

**Definition 5.**

A topological space is called **locally compact** if each point is contained in a compact neighbourhood.

**Definition 6.**

A space is called **scattered** if it contains no nonempty dense-in-itself subsets.

**Definition 7.**

A space is **almost compact** iff each open covering has a finite subcollection whose closures form a covering.

**Definition 8.**

A space is called **locally connected** if it has a basis consisting of connected sets.

## SECTION 2.1. PRIME IDEAL TOPOLOGY.

An ideal P in A is prime if $ab \in P$ implies $a \in P$ or $b \in P$ (i.e) the complement of P is closed with respect multiplication.

**Notation.**

$P_p$ denotes the prime ideal P generated by the prime number p.

Let X be the set of all prime ideals of integers. Define $V_x = \{P \in X / x \notin P\}$. Then the family $\{V_x / x \in Z^+\}$ forms a basis for a topology known as prime ideal topology on X.

We shall prove that this topological space is $T_0$, compact, hyperconnected, second countable and locally connected. Further this space does not satisfy the $T_1$, $T_4$ axioms.

$$V_x = \{P \in X \,/\, x \in P\}$$

$$\therefore\ P_p \in V_x \Rightarrow x \notin P_p \Rightarrow p \nmid x$$

Hence $V_x = \{P_p \in X \,/\, p \nmid x\}$

If $x \neq 0$, $0 \in V_x$.

$V_0 = \{P_p \in X \,/\, 0 \notin P_p\} = \phi$

$$\therefore\ V_0 = \phi.$$

For a prime $p' \in Z^+$, $V_{p'} = \{P_{p'} \in X \,/\, p \nmid p'\}$

$$= X - \{P_{p'}\} \quad .$$

Therefore every non zero member is closed in X.

Every $x \in Z^+$ has a finite number of prime factors. Let $x \in Z^+$ and let $p_1, p_2 \ldots p_n$ be its prime factors.

Then $V_x = \{P_p \in X \,/\, p \nmid x\}$

$$= \{P_p \in X \,/\, p \neq p_i, \text{ for } i = 1,2, \ldots n\}$$

$$= x - \{P_{p_1}, P_{p_2}, \ldots P_{p_n}\}$$

Therefore the complement of $V_x$ is finite. Hence a subset of X is open iff it contains 0 and its complement is finite.

Let U be a subset of X. We shall prove that U is open iff there exists an ideal $I \subset Z$ such that $U = \{P \in X \,/\, I \not\subset P\}$ .

$$U \text{ is open} \Rightarrow U = \bigcup_{x \in M} V_x$$

Let I be the ideal generated by M.

Then $P_p \in U \Leftrightarrow P_p \in V_x$ for some $x \in M$.

$\Leftrightarrow$ $p \nmid x$ for some $x \in M$.

$\Leftrightarrow$ x is not a multiple of p .

$\Leftrightarrow$ $x \notin P_p$ for some $x \in M$.

(i.e) $P \in U$ iff there exists $x \in I$ such that $x \notin P$.

(i.e) $P \in U$ iff $I \not\subset P$ .

Hence $U = \{ P \in X \ / \ I \not\subset P \}$ .

(i.e) U is open $\Rightarrow U = \{ P \in X \ / \ I \not\subset P \}$ for an ideal $I \subset Z$.

Conversely, assume $U = \{ P \in X \ / \ I \subset P \}$ for an ideal $I \subset Z$.

To prove that U is open.

$P \in U \Rightarrow I \subset P \Rightarrow$ there exists $x \in I$ such that $x \notin P$.

$\Rightarrow P \in V_x$ for some $x \in I$.

Therefore $U = \bigcup_{x \in I} V_x$.

Therefore U is open in X.

So, a set C is closed iff there is an ideal I in Z such that $C = \{ P \in X \ / \ I \subset P \}$ .

$P_1 \neq P_2$ in $X \Rightarrow P_1 \in X - \{P_2\}$ and $P_2 \in X - \{P_1\}$ where $X - \{P_1\}$ and $X - \{P_2\}$ are open in X. Therefore X is $T_0$.

Since '0' is in every open set. Hence X is not $T_1$ and hence X is not $T_2$ or $T_3$.

Since the singleton sets of prime ideals are closed and disjoint. X has disjoint closed sets. But X does not have disjoint open sets, since '0' is in every open set. Hence X is not $T_4$ or $T_5$ or metrizable.

A topological sapce is hyperconnected iff it has no disjoint open sets. Hence X is hyperconnected. Hyperconnectedness of X implies connectedness and hence X is locally connected.

The fact that every open set of X has a finite complement implies the compactness of X. This can be proved as follows:

Let $\mathbf{R} = \{U_\alpha \,/\, \alpha \in \Lambda\}$ be an open covering for X.

Let $U_{\alpha_0} \in \mathbf{R}$ for some $\alpha_0 \in \Lambda$.

Then $X - U_{\alpha_0}$ is finite say $\{P_1, P_2 \ldots P_n\}$.

Consider $U_{\alpha_1}$, $U_{\alpha_2}$ ... $U_{\alpha_n} \in \mathbf{R}$ such that $P_i \in U_{\alpha_i}$ for every $i = 1, 2, \ldots n$.

Then the family $\{U_{\alpha_0}, U_{\alpha_1} \ldots U_{\alpha_n}\}$ forms a finite subcovering of $\mathbf{R}$.

Hence every open covering of X has a finite subcovering.

Since the base $\{V_x \,/\, x \in Z^+\}$ is countable, X is second countable.

## SECTION 2.2. DIVISOR TOPOLOGY.

Let $X = \{x \in Z^+ \,/\, x \geqslant 2\}$, together with the topology generated by the family $\{U_n \,/\, n \geqslant 2\}$ where $U_n = \{x \in Z^+ \,/\, x$ divides $n\}$.

We shall prove that this topological space is $T_0$, ultraconnected, locally connected, locally compact, scattered, separable and it is second countable. This space does not satisfy the $T_1$ - axiom. Also this space is not countable compact and not strongly locally compact.

$U_p = \{p\}$ if p is a prime. Therefore each prime is open with respect to this topology.

If $x < y$ then $y \notin U_x$. $\therefore$ X is $T_0$.

Consider n and 2n. Since n is a divisor of 2n, every neighbourhood of 2n contains n. Therefore X is not $T_1$ and hence not $T_2$ or $T_3$ .

If x is a point of X then to prove that closure of x consists of all multiples of x. Take $y \in \overline{\{x\}} \Leftrightarrow U_y \cap \{x\} \neq \phi \Leftrightarrow x \in U_y \Leftrightarrow x/y \Leftrightarrow$ y is a multiple of x.

$\therefore$ closure of x consists of all mutiples of x.

To show that X is $T_4$ it is enough to observe that X can not have non-empty disjoint closed sets. Given x and y we find $\{x\} \cap \{y\}$ should contain the product element xy. Hence X is ultraconnected and therefore X is pathconnected and connected.

Suppose $x \nmid y$. Then y is not a multiple of x (i.e) $y \notin \{x\}$ . Hence $\{y\} \cap \overline{\{x\}} = \phi$. So, $\{x\}$ and $\{y\}$ are separated sets.

Let $U_1$ and $U_2$ be open sets containing x and y respectively. Then both $U_1$ and $U_2$ will contain the greatest common divisor of x and y.Consider the case when g.c.d.(x,y) $\neq$ 1. Hence there are no open sets $U_1$ and $U_2$ such that $x \in U_1$ and $y \in U_2$ with $U_1 \cap U_2 = \phi$. Therefore X is not $T_5$.

Let P be the set of all primes. Then P is dense in X, since every open set $U_n$ contains all prime divisors of n.

Since each singleton set $\{p\}$ where p is a prime is open, $\{p\}$ is not nowhere dense and hence X is of second category.

Since each point in X has a finite neighbourhood, X is locally compact.

X is not countably compact, because the open covering $\{ S_n \ / \ n \geqslant 2 \}$ , where $S_n = \bigcup\limits_{x \leq n} U_x = \{ x \ / \ x \leq n \}$ , has no finite subcover.

As each closed set is infinite, X is not strongly locally compact and therefore it is not compact.

Since X is countable, it is separable and Lindelof and since all basis elements are finite, it is second countable.

For each $n \in X$ the set $U_n$ is the smallest open set containing n. So, X is locally connected and hence connected.

There is no nonempty subset of X which is dense in itself, because for every non empty subset A of X the first element x of A is an isolated point of A. Therefore X is scattered, though it is connected.

## SECTION 2.3. MAXIMAL COMPACT TOPOLOGY.

In this example we shall construct a space $(X, \tau)$ which has the following properties: (1) $(X, \tau)$ is compact.
(2) If $\tau^* \supset \tau$ is compact topology for X then $\tau^* = \tau$ (i.e) $\tau$ is a maximal compact topology on X.

Description of the set X.

Let X be the set of all lattice points (i.j) of positive integers together with two ideal points x and y.

Description of the topology $\tau$.

Let **A** be the collection of all sets A, where A is any set of lattice points with atmost finitely many points from each row and **B** be the collection of all sets B, where B is any set of lattice points selected from atmost finitely many rows.

Define $\mathcal{T}$ to be the topology on X under which each lattice point (i,j) in X is open and the basic open neighbourhoods of x are of the form $X - (A \cup \{y\})$, $A \in \mathbf{A}$ and the basic open neighbourhoods of y are of the form $X - (B \cup \{x\})$, $B \in \mathbf{B}$.

Each neighbourhood of x intersects each neighbourhood of y and conversely. Therefore the pair x and y does not have disjoint neighbourhoods. Hence the space X is not Hausdorff.

Each neighbourhood of x does not contain the point y and conversely. So, the space X is $T_1$.

If X - A and X - B are open neighbourhoods of x and y respectively. Then the complement of the open set $(X - A) \cup (X - B)$ in X is $A \cap B$ which is finite. Hence the space is compact.

It can be proved that every compact subset of X is closed. Suppose that there exists a compact subset E of X which is not closed (i.e) $\bar{E} \neq E$. Then there is a limit point of E which is not in E. Such a point must be x or y, since each lattice point in X is isolated. Suppose that the ideal point y is in $\bar{E}$ but not in E. Then E must contain points from infinitely many rows. Let $A = \{(i_n, j_n)\}$ be the collection of points in E selected exactly one from each of such rows. Then $X - (A \cup \{y\})$ is an open neighbourhood of x and this set together with discrete points $(i_n, j_n)$ forms an open covering of E having no finite subcovering. Suppose that the ideal point x is in $\bar{E}$ but not in E. (i.e) $x \in \bar{E} - E$. Then E must contain infinitely many points from some row. Let B be the set of points from that row belonging to E. Then X - B $\cup \{x\}$) is an open neighbourhood of y and hence the set X - (B $\cup \{x\}$) together with the open points in E forms an open covering for E which has no finite subcovering. Hence E is not compact whenever E is not closed.

We shall prove that the topology $\tau$ defined is the maximal compact topology on X.

Suppose $\tau^*$ is a topology on X strictly finner than $\tau$ and compact. Then there is an open set N in $\tau^*$ which is not in $\tau$. Hence the complement X - N of N is closed under $\tau^*$ but not under $\tau$, which implies that the set X - N is compact in $\tau^*$ but not under $\tau$. This is not possible, since $\tau^* \supset \tau$. Hence $\tau$ is a maximal compact non-Hausdorff topology.

## SECTION 2.4. MINIMAL HAUSDORFF TOPOLOGY.

In this example we shall construct a topological space (X, $\tau$ ) which is Hausdorff such that if $\tau' \supset \tau$ then $\tau'$ is not Hausdorff. In this sense (X, $\tau$ ) is a minimal Hausdorff topological space.

Description of the set X.

Let A be the linearly ordered set $\{1, 2, 3, \ldots w, \ldots, -3, -2, -1\}$ with the interval topology, and if $Z^+$ is the set of positive integers with the discrete topology. We define X to be A x $Z^+$ together with two ideal points a and -a.

Description of the topology $\tau$.

Let $\tau$ be the topology on X determined by the product topology on $A \times Z^+$ together with the basic neighbourhoods.

$M_n^+(a) = \{a\} \cup \{(i,j) \,/\, i < w,\ j > n\}$ and

$M_n^-(-a) = \{-a\} \cup \{(i,j) \,/\, i > w,\ j > n\}$.

Note that $M_n^+(a) \cap M_n^-(-a)$ is empty for any pair of integers n and m and since $(n, n+2) = \{n-1\}$ for every n in A, every point $(n,j) \in A \times Z^+$ is open. Hence for each pair of points in X we can find disjoint neighbourhoods and therefore X is Hausdorff.

The points $(w,j) \in A \times Z^+$ are contained only in their own neighbourhoods. Hence the open covering formed by the basis neighbourhoods of points of X has no finite subcovering. So, X is not compact.

A space is almost compact iff each open covering has a finite subcollection whose closures form a covering.

Closures of basic neighbourhoods of a and -a contain all but finitely many points (w,j). So, each open covering of X has a finite subcollection whose closures form a covering. Hence X is almost compact.

Since the complement of $M_m^-(-a) = \{-a\} \cup \{(i,j) \,/\, j < m\}$ is finite, every open covering of $X - M_n^+(a)$ has a finite subcollection whose closure form a covering. Hence $X - M_n^+(a)$ is almost compact. Similarly we can prove that the complement of any basic neighbourhood of -a is also almost compact.

Basic neighbourhoods of the point $(n,j) \in A \times Z^+$ is of the form $\{(t,j) \ / \ y < t < z\}$ where $y,z$ is a pair of elements in A with $y < n < z$ . Similar argument we will show that the complements of such neighbourhoods are also almost compact.

Let $\tau^*$ be a topology on X finer than $\tau$. Let N be a basic open set in $\tau$. Then by the above argument, X - N is almost compact. Hence every $\tau$ - open covering of X - N has a finite subcollection whose $\tau$ - closures form a covering. Let $\{O_\alpha \ / \ \alpha \in \Lambda\}$ be a $\tau^*$ - open covering of X - N, it is then a $\tau$ -open covering of X - N, which has a finite subcollection whose $\tau$ -closures form a covering say $\{O_1, O_2, \ldots O_n\}$ is the sub family, the union of whose $\tau$ - closures covers X - N. It is well known that the closure of a set with respect to the finer topology contains its closure with respect to the weaker topology. Hence the union of $\tau^*$ - closures of $O_1, O_2 \ldots O_n$ covers X - N. Hence X - N is almost compact with respect to $\tau^*$ also whenever N is a basic neighbourhood of $\tau$.

We shall prove that the topology $\tau$ defined is the minimal Hausdorff topology on X. Suppose $\tau^*$ is a topology on X strictly weaker than $\tau$ and Hausdroff. Then there exists an open set N in $(X, \tau)$ which is not open in $(X, \tau^*)$ so that X - N closed in $(X, \tau)$ but not in $(X, \tau^*)$. X - N is not closed in $(X, \tau^*)$ implies that there is a point $x \notin X - N$ such that $x \in \overline{(X - N)}^*$ where $\overline{(X - N)}^*$ denotes the $\tau^*$ - closure of X - N.

Let $\{C_\alpha \,/\, \alpha \in \Lambda\}$ be the family of $\tau^*$ - closed neighbourhoods of x. Then we shall prove that the family $\{X - C_\alpha \,/\, \alpha \in \Lambda\}$ of $\tau^*$ -open sets does not cover X - N. Suppose the family $\{X - C_\alpha \,/\, \alpha \in \Lambda\}$ covers X - N. Then there exists a finite subfamily $\{C_1, C_2, \dots C_n\}$ such that the union of $\{\overline{X - C_i} \,/\, i = 1, 2, \dots n\}$ covers X - N because of the almost compactness of X - N with respect to $\tau^*$.

That is $X - N \subset (\overline{X - C_i})^*$

Since $\bigcup_{i=1}^{n} (\overline{X - C_i})^*$ is closed and $x \in (\overline{X - N})^*$, $x \in \bigcup_{i=1}^{n} (\overline{X - C_i})^*$. Since $C_i$ are neighbourhoods of x, this is impossible, so $\{X - C_\alpha \,/\, \alpha \in \Lambda\}$ could not have covered X - N. Thus $\bigcap C_\alpha$ contains more than just the point x, So $\tau^*$ cannot be Hausdorff. This means that $\tau$ is a minimal Hausdorff topology on X.

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