Chapter 5

CHAPTER – V
A BATCH ARRIVAL QUEUE WITH A SECOND OPTIONAL SERVICE CHANNEL UNDER (m, N) POLICY AND SINGLE BERNOULLI VACATION

Introduction:

Presently most of the studies have been devoted to batch arrival vacation models under different vacation policies because of their interdisciplinary characters. In Chapter III, we have analysed the (m, N) policy of batch arrival queueing system with second optional service under single and multiple vacations. The model proposed to study in the present chapter is a bi-level control policy batch arrival queueing model along with Bernoulli schedule server vacation. Keilson and Servi (1986) first introduced the concept of Bernoulli vacation to a single service queueing model.

Various authors (Choudhury and Madan 2004 and Madan and Choudhury 2005) have studied the Bernoulli vacation policy due to its numerous applications in many real life situations. Further, from the practical point of view, this type of vacation model can be utilized as a model building of a production process. For example, consider a production process, where the machine, producing certain items, may require two phases of services of which one is essential and the other is optional to complete the processing of raw material. It may so happen that after completion of each service, the process may either need to be stopped for overhauling and maintenance of the system or may continue the further processing of the new materials if there is no fault in the system. This overhauling can be considered as Bernoulli vacation in the system. There may be many other situations such as, the server may require time to switch over from one stage of service to the other stage (or) the server may like to warm him up after each service.

So, in the process of generalization, we propose to study the bi-level control policy of batch arrival two stage heterogeneous service queue with Bernoulli schedule vacation where the vacation is permissible at the ends of each service completion (i.e.,) either at the end of FES (or) at the end of SOS. This type of vacation models suit for some computers and communication networks where messages are processed in two stages with some restrictions by a single server.

5.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

The steady state behavior of M^X / G_{SOS} / 1 /BV queue in which the server may provide two-phases of heterogeneous service (of which the second phase is optional) to incoming units under bi-level threshold and Bernoulli schedule vacation is analysed.

I Model description:

The customers arrive in batches according to the compound Poisson process with rate λ. The server is deactivated whenever the system becomes empty. When the queue length reaches (or) exceeds **m**, the server starts a setup operation which takes a random length D. At the end of the setup if the queue length is greater than or equal to **N**, then the server begins to serve the customers. Otherwise, the server remains dormant in the system waiting for the queue length to reach atleast N before it starts a service. Thus if the queue length reaches N or more either at the end of the setup period or at the end of the dormant period then the server begins a busy period.

During busy period, the server provides two types of heterogeneous service of which one is optional. The service discipline is assumed to be FCFS. After the completion of FES of a unit, the customer may leave the system with probability (1 – r) or may opt for the SOS in additional channel by the same server with probability r ($0 \le r \le 1$).

If a customer leaves the system soon after the FES, the server may take a vacation (Bernoulli schedule) of random V with probability p_1 ($0 \le p_1 \le 1$) or may continue to serve the next customer ,if any ,with probability ($1 - p_1$). On the other hand, if a customer finishes FES and opts for the SOS then the server takes vacation only after finishing the SOS for the customer with probability p_2. (i.e.) After completing each service and sending the customer out of the system the server takes a vacation with probability p_j (j = 1, 2) or continue to stay in the system, with probability $(1 - p_j)$. The vacation time in either case is a random variable and follows the same general distribution with finite moments. It is also assumed that the vacation is single vacation, in the sense that, whenever the vacation period ends, then the server joins the system irrespective of whether there are customers waiting for the service (or) not. (i.e.) The server must join the system even if the system is empty at the end of the vacation. This type of services continues until the system becomes empty again.

Thus a cycle begins, whenever the system becomes empty .Then the server does a setup operation when the queue length reaches at least **m** and continues with FES and SOS along with Bernoulli schedule vacations. The cycle ends when the system becomes empty again. We denote the model by $\mathbf{M^{X}_{(m,N)} / G_{SOS} / 1 / SBV}$ where SBV denotes Single Bernoulli vacation.

II System Size Distribution at Random Epoch

We define the following notations as in Chapter III to obtain the steady state system size equations.

λ	:	group arrival rate
X	:	group size random variable
g_k	:	Pr (X = k), k = 1, 2, 3, . . .
X(z)	:	Probability generating function of X.

Let $N_s(t)$ be the system size at time t. For further development of the model, introduce the random variable **Y(t)** which takes the values 0, 1, 2, 3, 4, and 5 according as the system is, in vacation state, build up state, setup state, dormant state, busy with FES and busy with SOS respectively.

The notations of Random Variables (RV), Cumulative Distribution Function (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k[th] moments are listed as below:

	RV	CDF	Pdf	LST	k[th] moment
Vacation time	V	V(x)	v(x)	$V^*(\theta)$	$E(V^k)$
Setup time	D	D(x)	d(x)	$D^*(\theta)$	$E(D^k)$
FES	S_1	$S_1(x)$	$s_1(x)$	$S_1^*(\theta)$	$E(S_1^k)$
SOS	S_2	$S_2(x)$	$s_2(x)$	$S_2^*(\theta)$	$E(S_2^k)$

Let $V^0(t)$, $D^0(t)$, $S_1^0(t)$ and $S_2^0(t)$ denote the remaining vacation time, setup time, first essential service time and second optional service time respectively at time t. Then the state space K = {N(t), $\delta(t)$} where $\delta(t) = V^0(t)$, 0, $D^0(t)$, 0, $S_i^0(t)$, i= 1,2 according as Y(t) = 0 to 5 respectively defines a bivariate Markov process.

Also we consider the following probabilities to get the steady state system size equations.

$$Q_n(x, t)\,dt = \Pr\{N_s(t) = n, x \le V^0(t) \le x + dt, Y(t) = 0\} \quad n \ge 0$$

$$R_n(t) = \Pr\{N_s(t) = n, Y(t) = 1\} \quad 0 \le n \le m-1$$

$$D_n(x, t)\,dt = \Pr\{N_s(t) = n, x \le D^0(t) \le x + dt, Y(t) = 2\} \quad n \ge m$$

$$U_n(t) = \Pr\{N_s(t) = n, Y(t) = 3\} \quad m \le n \le N-1$$

$$P_{n,1}(x, t)dt = \Pr\{N_s(t) = n, x \le S_1^0(t) \le x + dt, Y(t) = 4\} \quad n \ge 1$$

$$P_{n,2}(x, t)dt = \Pr\{N_s(t) = n, x \le S_2^0(t) \le x + dt, Y(t) = 5\} \quad n \ge 1$$

Assuming that at steady state the probabilities are independent of time exist, and by following the argument of Cox (1955) the steady state equations are written as below :

$$-\frac{d}{dx}(Q_0(x)) = -\lambda Q_0(x) + p_2 P_{1,2}(0)\, v(x) + p_1(1-r) P_{1,1}(0)\, v(x)$$

$$-\frac{d}{dx}(Q_n(x)) = -\lambda Q_n(x) + p_2 P_{n+1,2}(0)\, v(x) + p_1(1-r) P_{n+1,1}(0)\, v(x)$$

$$+ \lambda \sum_{k=1}^{n} Q_{n-k}(x)\, g_k \qquad n \geq 1$$

$$\lambda R_0 = (1-p_2) P_{1,2}(0) + (1-p_1)(1-r) P_{1,1}(0) + Q_0(0)$$

$$\lambda R_n = \lambda \sum_{k=1}^{n} R_{n-k}\, g_k \qquad 1 \leq n \leq m-1$$

$$-\frac{d}{dx}(D_m(x)) = -\lambda D_m(x) + \lambda \sum_{k=1}^{m} R_{m-k}\, g_k\, d(x)$$

$$-\frac{d}{dx}(D_n(x)) = -\lambda D_n(x) + \lambda \sum_{k=n-m+1}^{n} R_{n-k}\, g_k\, d(x) + \lambda \sum_{k=1}^{n-m} D_{n-k}(x)\, g_k, \quad n \geq m+1$$

$$\lambda U_m = D_m(0)$$

$$\lambda U_n = D_n(0) + \lambda \sum_{k=1}^{n-m} U_{n-k}\, g_k, \qquad m+1 \leq n \leq N-1$$

$$-\frac{d}{dx}(P_{1,1}(x)) = -\lambda P_{1,1}(x) + (1-p_2) P_{2,2}(0)\, s_1(x) + Q_1(0)\, s_1(x)$$

$$+ (1-p_1)(1-r) P_{2,1}(0)\, s_1(x)$$

$$\frac{d}{dx}(P_{n,1}(x)) = -\lambda P_{n,1}(x) + (1-p_2) P_{n+1,2}(0)\, s_1(x) + Q_n(0)\, s_1(x)$$

$$+ \lambda \sum_{k=1}^{n-1} P_{n-k,1}(x)\, g_k + (1-p_1)(1-r) P_{n+1,1}(0)\, s_1(x) \qquad 2 \leq n \leq N-1$$

$$-\frac{d}{dx}(P_{n,1}(x)) = -\lambda P_{n,1}(x) + (1-p_2) P_{n+1,2}(0)\, s_1(x) + Q_n(0)\, s_1(x)$$

$$+ \lambda \sum_{k=1}^{n-1} P_{n-k,1}(x)\, g_k + (1-p_1)(1-r) P_{n+1,1}(0)\, s_1(x)$$

$$+ D_n(0)\, s_1(x) + \lambda \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k\, s_1(x) \qquad n \geq N$$

$$-\frac{d}{dx}(P_{1,2}(x)) = -\lambda P_{1,2}(x) + r\, P_{1,1}(0)\, s_2(x)$$

$$-\frac{d}{dx}(P_{n,2}(x)) = -\lambda P_{n,2}(x) + r\, P_{n,1}(0)\, s_2(x) + \lambda \sum_{k=1}^{n-1} P_{n-k,2}(x)\, g_k \qquad n \geq 2$$

By taking LST for the steady state equations as before, we have,

$$\theta\, Q_0^*(\theta) - Q_0(0) = \lambda\, Q_0^*(\theta) - p_2\, P_{1,2}(0)\, V^*(\theta) - p_1\,(1-r)\, P_{1,1}(0)\, V^*(\theta) \quad (5.1)$$

$$\theta\, Q_n^*(\theta) - Q_n(0) = \lambda\, Q_n^*(\theta) - p_2\, P_{n+1,2}(0)\, V^*(\theta) - p_1\,(1-r)\, P_{n+1,1}(0)\, V^*(\theta) - \lambda \sum_{k=1}^{n} Q_{n-k}^*(\theta)\, g_k, \quad n \geq 1 \quad (5.2)$$

$$\lambda\, R_0 = (1-p_2)\, P_{1,2}(0) + Q_0(0) + (1-p_1)\,(1-r)\, P_{1,1}(0) \quad (5.3)$$

$$\lambda\, R_n = \lambda \sum_{k=1}^{n} R_{n-k}\, g_k \quad 1 \leq n \leq m-1 \quad (5.4)$$

$$\theta\, D_m^*(\theta) - D_m(0) = \lambda\, D_m^*(\theta) - \lambda \sum_{k=1}^{m} R_{m-k}\, g_k\, D^*(\theta) \quad (5.5)$$

$$\theta\, D_n^*(\theta) - D_n(0) = \lambda\, D_n^*(\theta) - \lambda \sum_{k=n-m+1}^{n} R_{n-k}\, g_k\, D^*(\theta) - \lambda \sum_{k=1}^{n-m} D_{n-k}^*(\theta)\, g_k, \quad n \geq m+1 \quad (5.6)$$

$$\lambda\, U_m = D_m(0) \quad (5.7)$$

$$\lambda\, U_n = D_n(0) + \lambda \sum_{k=1}^{n-m} U_{n-k}\, g_k \quad m+1 \leq n \leq N-1 \quad (5.8)$$

$$\theta\, P_{1,1}^*(\theta) - P_{1,1}(0) = \lambda\, P_{1,1}^*(\theta) - (1-p_2)\, P_{2,2}(0)\, S_1^*(\theta) - Q_1(0)\, S_1^*(\theta) - (1-p_1)\,(1-r)\, P_{2,1}(0)\, S_1^*(\theta) \quad (5.9)$$

$$\theta\, P_{n,1}^*(\theta) - P_{n,1}(0) = \lambda\, P_{n,1}^*(\theta) - (1-p_2)\, P_{n+1,2}(0)\, S_1^*(\theta) - Q_n(0)\, S_1^*(\theta) - (1-p_1)\,(1-r)\, P_{n+1,1}(0)\, S_1^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k,1}^*(\theta)\, g_k \quad 2 \leq n \leq N-1 \quad (5.10)$$

$$\theta\, P_{n,1}^*(\theta) - P_{n,1}(0) = \lambda\, P_{n,1}^*(\theta) - (1-p_2)\, P_{n+1,2}(0)\, S_1^*(\theta) - Q_n(0)\, S_1^*(\theta) - (1-p_1)\,(1-r)\, P_{n+1,1}(0)\, S_1^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k,1}^*(\theta)\, g_k - D_n(0)\, S_1^*(\theta) - \lambda \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k\, S_1^*(\theta) \quad n \geq N \quad (5.11)$$

$$\theta\, P^*_{1,2}(\theta) - P_{1,2}(0) = \lambda\, P^*_{1,2}(\theta) - r\, P_{1,1}(0)\, S^*_2(\theta) \quad (5.12)$$

$$\theta\, P^*_{n,2}(\theta) - P_{n,2}(0) = \lambda\, P^*_{n,2}(\theta) - r\, P_{n,1}(0)\, S^*_2(\theta) - \lambda \sum_{k=1}^{n-1} P^*_{n-k,2}(\theta)\, g_k$$

$$n \geq 2 \quad (5.13)$$

III Probability Generating Function of the System Size:

By considering the partial PGFs as,

$$R(z) = \sum_{n=0}^{m-1} R_n z^n \; ; \qquad U(z) = \sum_{n=m}^{N-1} U_n z^n$$

$$Q^*(z,\theta) = \sum_{n=0}^{\infty} Q^*_n(\theta) z^n \; ; \qquad Q(z,0) = \sum_{n=0}^{\infty} Q_n(0) z^n$$

$$D^*(z,\theta) = \sum_{n=m}^{\infty} D^*_n(\theta) z^n \; ; \qquad D(z,0) = \sum_{n=m}^{\infty} D_n(0) z^n$$

$$P^*_i(z,\theta) = \sum_{n=1}^{\infty} P^*_{n,i}(\theta) z^n \;\; i = 1, 2 \; ; \quad P_i(z,0) = \sum_{n=1}^{\infty} P_{n,i}(0) z^n \;\; i = 1, 2$$

the following steady state distributions are obtained.

Equations (5.1) and (5.2) imply

$$Q(z,0) = \frac{V^*(w_X(z))}{z} \left(p_2 P_2(z,0) + p_1 (1-r) P_1(z,0)\right) \quad (5.14)$$

and $$Q^*(z,\theta) = \frac{V^*(w_X(z)) - V^*(\theta)}{z(\theta - w_X(z))} \left(p_2 P_2(z,0) + p_1 (1-r) P_1(z,0)\right) \quad (5.15)$$

The PGF of the system size corresponding to the set up time can be obtained by using equations (5.3) to (5.6), as,

$$\theta\, D^*(z,\theta) - D(z,0) = \lambda\, D^*(z,\theta) - \lambda\, D^*(\theta) \sum_{n=m}^{\infty} z^n \sum_{k=n-m+1}^{n} R_{n-k}\, g_k$$

$$- \lambda \sum_{n=m+1}^{\infty} z^n \sum_{k=1}^{n-m} D^*_{n-k}(\theta)\, g_k \quad (5.16)$$

and equations (5.3) and (5.4) imply

$$\lambda R(z) \quad = \quad \lambda R_0 + \lambda \sum_{n=1}^{m-1} z^n \sum_{k=1}^{n} R_{n-k}\, g_k \tag{5.17}$$

By adding equation (5.16) with equation (5.17) multiplied by $(- D^*(\theta))$ we get

$$(\theta - w_X(z))\, D^*(z,\theta) = D(z,0) - (\lambda R_0 - w_X(z) R(z))\, D^*(\theta)$$

At $\theta = w_X(z)$

$$D(z, 0) \quad = \quad (\lambda R_0 - w_X(z) R(z))\, D^*(w_X(z)) \tag{5.18}$$

and hence

$$D^*(z,\theta) \quad = \quad \frac{(D^*(w_X(z)) - D^*(\theta))}{(\theta - w_X(z))} (\lambda R_0 - w_X(z) R(z)) \tag{5.19}$$

With the definition of $\pi_0 = 1$, $\pi_n = \sum_{k=1}^{n} \pi_{n-k}\, g_k \quad (1 \le n \le m-1)$ equation (5.4) recursively implies

$$\lambda R_n = \lambda \pi_n R_0 \text{ where } 0 \le n \le m-1$$

Thus $R(z) = R_0 \sum_{n=0}^{m-1} \pi_n z^n = (\lambda R_0)\, \pi(z)$ (5.20)

where $\pi(z) = \sum_{n=0}^{m-1} \frac{\pi_n z^n}{\lambda}$

substituting for R(z) in (5.18) and (5.19) we get

$$D(z, 0) \quad = \quad D^*(w_X(z))(\lambda R_0)\, (1 - w_X(z)\, \pi(z)) \tag{5.21}$$

$$D^*(z,\theta) \quad = \quad \frac{D^*(w_X(z)) - D^*(\theta)}{(\theta - w_X(z))} (\lambda R_0)\, (1 - w_X(z)\, \pi(z)) \tag{5.22}$$

The partial generating functions corresponding to busy states can be obtained from equations (5.9) to (5.13).

Equations (5.12) and (5.13) imply

$$P_2(z, 0) \quad = \quad r\, S_2^*(w_X(z))\, P_1(z, 0) \tag{5.23}$$

and $$P_2^*(z,\theta) \quad = \quad r\, \frac{(S_2^*(w_X(z)) - S_2^*(\theta))}{(\theta - w_X(z))}\, P_1(z, 0) \tag{5.24}$$

Thus $p_2 P_2(z, 0) + p_1 (1 - r) P_1(z, 0) = (p_2 r S_2^*(w_X(z)) + p_1 (1 - r)) P_1(z, 0)$.

Hence $Q(z, 0)$ and $Q^*(z,\theta)$ in equations (5.14) and (5.15) yield,

$$Q(z, 0) = \left(\frac{V^*(w_X(z))}{z}\right)(p_2 r S_2^*(w_X(z)) + p_1 (1 - r)) P_1(z, 0)) \qquad (5.25)$$

$$\text{and } Q^*(z,\theta) = \left(\frac{V^*(w_X(z)) - V^*(\theta)}{z}\right)\left(\frac{p_2 r S_2^*(w_X(z)) + p_1(1-r)}{(\theta - w_X(z))}\right) P_1(z, 0) \quad (5.26)$$

Next to calculate the partial generating function corresponding to the first essential busy state, equations (5.9) to (5.11) are used and these equations lead to

$$(\theta - w_X(z))P_1^*(z, \theta) = P_1(z, 0) - \frac{S_1^*(\theta)}{z}((1- p_2)P_2(z, 0) + (1 - p_1)(1- r)P_1(z, 0))$$

$$+S_1^*(\theta)(\lambda R_0 - Q(z,0) - \sum_{n=N}^{\infty} D_n(0) z^n - \lambda \sum_{n=N}^{\infty} z^n \sum_{k=n-N+1}^{n-m} U_{n-k} g_k)$$

(5.27)

Equations (5.8) and (5.9) imply

$$\lambda U(z) = \sum_{n=m}^{N-1} D_n(0) z^n + \lambda \sum_{n=m+1}^{N-1} z^n \sum_{k=1}^{n-m} U_{n-k} g_k \qquad (5.28)$$

Multiplying equation (5.28) by $(-S_1^*(\theta))$ and adding with (5.27) and substituting for $P_2(z, 0)$ from equation (5.23), it is found that

$$(\theta - w_X(z)) P_1^*(z, \theta) = P_1(z, 0) + S_1^*(\theta)(\lambda R_0 - Q(z, 0) - D(z, 0) + w_X(z) U(z))$$

$$- \frac{S_1^*(\theta)}{z}((1- p_2) r S_2^*(w_X(z)) + (1 - p_1)(1 - r)) P_1(z, 0)$$

By substituting for Q (z, 0) and D(z, 0) from equations (5.25) and (5.21) the above equation can be written as

$$(\theta - w_X(z)) P_1^*(z, \theta) = \left(\frac{P_1(z, 0)}{z}\right)\left(z - S_1^*(\theta)\phi(z)\right) - S_1^*(\theta)(D(z,0) - \lambda R_0 - w_X(z)U(z))$$

where $\phi(z) = \left(r S_2^*(w_X(z)) + (1-r)\right) + \left(p_2 r S_2^*(w_X(z)) + p_1(1-r)\right)\left(V^*(w_X(z)) - 1\right)$

By letting $\theta = w_X(z)$ the above equation implies

$$P_1(z, 0) = \frac{z S_1^*(w_X(z))(D(z,0) - \lambda R_0 - w_X(z) U(z))}{z - S_{BV}^*(w_X(z))} \quad (5.29)$$

where $S_{BV}^*(w_X(z)) = S_1^*(w_X(z))\, \phi(z)$

$$S_{BV}^*(w_X(z)) = S_{sos}^*(w_X(z)) + (V^*(w_X(z))-1)[p_2 r S_2^*(w_X(z)) + p_1(1-r) S_1^*(w_X(z))$$

$$S_{sos}^*(w_X(z)) = S_1^*(w_X(z))\,((1-r) + r\, S_2^*(w_X(z)))$$ and

$$P_1^*(z, \theta) = \frac{z(S_1^*(w_X(z)) - S_1^*(\theta))(D(z,0) - \lambda R_0 - w_X(z) U(z))}{(z - S_{BV}^*(w_X(z)))(\theta - w_X(z))} \quad (5.30)$$

For further simplification the equations (5.7) and (5.8) are recursively used,

$$\text{(i.e.)} \quad \lambda U_n = \sum_{k=m}^{n} D_k(0)\, \pi_{n-k} \qquad m \le n \le n-1$$

where $D_k(0)$ = coefficient of z^k of $D(z, 0)$.

From equation (5.21)

$$D_k(0) = (\lambda R_0) \text{ co-eff. of } z^k \text{ of } (D^*(w_X(z)))(1 - w_X(z)\, \pi(z))$$

If h_k denotes the probability that k customers arrive in a set up time

then $$D^*(w_X(z)) = \sum_{k=0}^{\infty} h_k z^k$$

further $$(1 - w_X(z)\, \pi(z)) = (X(z) \sum_{n=0}^{m-1} \pi_n z^n - \sum_{n=1}^{m-1} \pi_n z^n)$$

and $$X(z) \left(\sum_{n=0}^{m-1} \pi_n z^n \right) = \left(\sum_{k=1}^{\infty} g_k z^k \right) \left(\sum_{n=0}^{m-1} \pi_n z^n \right)$$

$$= \sum_{n=1}^{m-1} z^n \sum_{i=0}^{n-1} \pi_i g_{n-i} + \sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \pi_i g_{n-i}$$

$$= \sum_{n=1}^{m-1} z^n \pi_n + \sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \pi_i g_{n-i}$$

Thus $$(1 - w_X(z)\, \pi(z)) = \sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \pi_i g_{n-i}$$

$$\text{Hence} \quad D_k(0) = (\lambda R_0) \text{ co-eff. } z^k \text{ in } \left[\sum_{k=0}^{\infty} h_k z^k\right]\left[\sum_{n=m}^{\infty} z^n \sum_{i=0}^{m-1} \pi_i g_{n-i}\right]$$

$$= (\lambda R_0) \text{ co-eff. } z^k \text{ in } \left(\sum_{k=m}^{\infty} z^k \left(\sum_{i=m}^{k} \left(\sum_{j=0}^{m-1} \pi_j g_{i-j}\right) h_{k-i}\right)\right)$$

$$= \lambda R_0 \sum_{i=m}^{k} S_i h_{k-i}$$

$$\text{where} \quad S_i = \sum_{j=0}^{m-1} \pi_j g_{i-j}$$

$$\text{Hence} \quad U_n = R_0 \sum_{k=m}^{n} \left(\sum_{i=m}^{k} S_i h_{k-i}\right) \pi_{n-k}$$

By rearranging the summation,

$$U_n = R_0 \sum_{k=m}^{n} S_k \left(\sum_{i=0}^{n-k} h_i \pi_{n-k-i}\right) \qquad m \le n \le N-1$$

$$\text{Let} \quad \phi_n^{BV} = \sum_{k=m}^{n} S_k \sum_{i=0}^{n-k} h_i \pi_{n-k-i} \quad \text{then}$$

$$U_n = R_0 \phi_n^{BV} \quad \text{and}$$

$$U(z) = \lambda R_0 \phi_{BV}(z), \tag{5.31}$$

$$\text{where } \phi_{BV}(z) = \left(\sum_{n=m}^{N-1} \frac{\phi_n^{BV} z^n}{\lambda}\right)$$

By substituting D(z, 0) and U(z) from equations (5.21) and (5.31) one can find that $D(z, 0) - \lambda R_0 - U(z) w_X(z) = -\lambda R_0 (w_X(z)) (I_{BV}(z))$

$$\text{where} \quad I_{BV}(z) = \left(\frac{(1 - D^*(w_X(z)))}{w_X(z)} + \pi(z) D^*(w_x(z)) + \phi_{BV}(z)\right) \tag{5.32}$$

Thus the equation (5.30) implies,

$$P_1^*(z, \theta) = \frac{z (S_1^*(w_X(z)) - S_1^*(\theta))(-\lambda R_0)(w_X(z)) I_{BV}(z)}{(z - S_{BV}^*(w_X(z))(\theta - w_X(z))} \tag{5.33}$$

Thus the partial generating functions corresponding to different states at arbitrary epochs are given by (using equations (5.26), (5.22), (5.20), (5.31), (5.24) and (5.33))

$$Q^*(z,0) = \lambda\, R_0 \left(\frac{S_1^*(w_X(z))(V^*(w_X(z))-1)}{z - S_{BV}^*(w_X(z))} \right) (p_2\, r\, S_2^*(w_X(z)) + p_1\,(1-r))\, I_{BV}(z)$$

$$D^*(z,0) = \lambda\, R_0\, \frac{(1 - D^*(w_X(z)))}{w_X(z)}\, (1 - w_X(z)\, \pi(z))$$

$$R(z) = \lambda\, R_0\, \pi(z)\ ;\ U(z) = \lambda\, R_0\, \phi_{BV}(z)$$

$$P_2^*(z,0) = (\lambda\, R_0)\, r\, z\, S_1^*(w_X(z))\, \frac{(S_2^*(w_X(z))-1)}{(z - S_{BV}^*(w_X(z)))}\, I_{BV}(z)$$

$$P_1^*(z,0) = (\lambda\, R_0)\, z\, \frac{(S_1^*(w_X(z))-1)}{(z - S_{BV}^*(w_X(z)))}\, I_{BV}(z)$$

By calculation it is found that

$$D^*(z,0) + R(z) + U(z) = (\lambda\, R_0)\, I_{BV}(z) \text{ and}$$

$$Q^*(z,0) + D^*(z,0) + R(z) + U(z) = \frac{(\lambda\, R_0)\, I_{BV}(z)}{(z - S_{BV}^*(w_X(z)))}\,(z - S_{sos}^*(w_X(z)))$$

where $S_{sos}^*(w_X(z)) = S_1^*(w_X(z))\,(r\, S_2^*(w_X(z)) + (1-r))$

$$S_{BV}^*(w_X(z)) = S_{sos}^*(w_X(z)) + (V^*(w_X(z))-1)[p_2\, r\, S_2^*(w_X(z)) + p_1(1-r)]$$

further $P_1^*(z,0) + P_2^*(z,0) = \dfrac{z\,\lambda\, R_0\, I_{BV}(z)}{(z - S_{BV}^*(w_X(z)))}\,(S_{sos}^*(w_X(z))-1)$

Thus the total PGF is given by

$$\mathbf{P}_{BV(m,N)}(z) = Q^*(z,0) + D^*(z,0) + R(z) + U(z) + P_1^*(z,0) + P_2^*(z,0)$$

$$= \frac{(\lambda\, R_0)(I_{BV}(z))}{(z - S_{BV}^*(w_X(z)))}\,(z-1)\,(S_{sos}^*(w_X(z))) \qquad (5.34)$$

IV Performance Measures:

Let P_v, P_{build}, P_{set}, P_{dor} and P_{busy} denote the probability that the system is in vacation, buildup, setup, dormant and in busy state respectively, then their corresponding system size probabilities are given by

(i) $P_v = \lambda R_0\, p\, \lambda E(X)E(V) \left(\frac{D_{BV}(m,N)}{(1-\rho_{BV})}\right)$, where $p = (p_1(1-r) + p_2 r)$

(ii) $P_{set} = \lambda R_0\, E(D)$

(iii) $P_{build} = \lambda R_0 \sum_{n=0}^{m-1} \frac{\pi_n}{\lambda}$

(iv) $P_{dor} = \lambda R_0 \sum_{n=m}^{N-1} \frac{\phi_n^{BV}}{\lambda}$

(v) $P_{busy} = \lambda R_0 \left(\frac{D_{BV}(m,N)}{(1-\rho_{BV})}\right) \rho_{SOS}$

where $\rho_{SOS} = \lambda E(X)\,(E(S_{sos}))$

with $E(S_{sos}) = E(S_1) + r\,E(S_2)$ (5.35)

$\rho_{BV} = \lambda E(X)\,(E(S_{sos}) + p\,E(V))$ (5.36)

$D_{BV}(m, N) = I_{BV}(1) = E(D) + \sum_{n=0}^{m-1} \frac{\pi_n}{\lambda} + \sum_{n=m}^{N-1} \frac{\phi_n^{BV}}{\lambda}$ (5.37)

The values of λR_0 can be calculated by using the normalizing condition $P_{BV(m,N)}(1) = 1$, which implies,

$$1 = \frac{(\lambda R_0) D_{BV}(m,N)}{(1-\rho_{BV})}$$

which in turn gives $\lambda R_0 = \frac{(1-\rho_{BV})}{D_{BV}(m,N)}$

Thus $P_{busy} = \rho_{SOS}$ and $P_v = p\,\lambda\, E(x)\, E(v)$

By substituting for (λR_0) in equation (5.34) ,

$$\mathbf{P_{BV(m,N)}(z)} = \frac{(1-\rho_{BV})(z-1)\, S^*_{sos}(w_X(z))}{(z - S^*_{BV}(w_X(z)))} \frac{I_{BV}(z)}{I_{BV}(1)} \quad (5.38)$$

V Decomposition Property

The total PGF of the model is given by

$$P_{BV(m,N)}(z) = \frac{(1-\rho_{BV})(z-1)(S^{*}_{sos}(w_X(z)))}{(z-S^{*}_{BV}(w_X(z)))} \frac{I_{BV}(z)}{I_{BV}(1)}$$

This shows that under the condition $\rho_{BV} < 1$,the PGF is decomposed into two random variables one of which is the system size of the second optional service, Bernoulli vacation queueing model without N policy and the other $\frac{I_{BV}(z)}{I_{BV}(1)}$ is the PGF of the conditional system size distribution during the server idle period.

VI Mean System Size

The expected system size when the server is in build up (L_{build}), set up (L_{set}), dormant (L_{dor}), busy state (L_{busy}) and in the vacation state (L_v) are given by

(i) $L_{build} = \lambda R_0 \sum_{n=0}^{m-1} \frac{n\,\pi_n}{\lambda}$

(ii) $L_{dor} = \lambda R_0 \sum_{n=m}^{N-1} \frac{n\,\phi_n^{BV}}{\lambda}$

(iii) $L_{set} = \lambda R_0 (\lambda E(X)) (E(D^2)/2 + E(D) \sum_{n=0}^{m-1} \frac{\pi_n}{\lambda})$

(iv) $L_{busy} = \rho_{sos} + \frac{L_{BV}(m,N)}{D_{BV}(m,N)} \lambda E(X) E(S_{sos})$

$$+ \frac{\lambda E(X(X-1)) E(S_{sos}) + (\lambda E(X)^2)((1-\rho_{BV}) E(S^2_{sos}) + \rho_{sos} E(S^2_{BV}))}{2(1-\rho_{BV})}$$

(v) $L_V = \frac{L_{BV}(m,N)}{D_{BV}(m,N)} \lambda E(X)\, p\, E(V) + \frac{1}{2(1-\rho_{BV})} \{ \lambda E(X(X-1))\, p\, E(V)$

$+ (\lambda E(X))^2 [(1 - \rho_{BV})(p\, E(V^2) + 2E(V) (r\, p_2 E(S_2) + p\, E(S_1))$

$+ \lambda E(X)\ p\, E(V)\ E(S^2_{BV})] \}$

Then the expected system size ($L^{s}_{BV(m,N)}$) for the model is given by

$$L^{s}_{BV(m,N)} = \frac{L_{BV}(m,N)}{D_{BV}(m,N)} + \rho_{sos} + \frac{\lambda E(X(X-1)) E(S_{BV}) + (\lambda E(X)^2) E(S^2_{BV})}{2(1-\rho_{BV})}$$

where $E(S_{BV}) = E(S_{sos}) + p\, E(V)$

$E(S^2_{BV}) = E(S^2_{sos}) + p\,(E(V^2)) + 2\,E(V)\,(p\,E(S_1) + p_2\, r\, E(S_2))$; $p = (p_2 + (1-r)p_1)$

$E(S^2_{sos} = E(S^2_1) + r\,E(S^2_2) + 2r\,E(S_1)E(S_2)$

$$L_{BV}(m,N) = (\lambda E(X)) \left(\frac{E(D^2)}{2} + E(D) \sum_{n=0}^{m-1} \frac{\pi_n}{\lambda} \right) + \sum_{n=0}^{m-1} \frac{n\pi_n}{\lambda} + \sum_{n=0}^{m-1} \frac{n\,\phi^{BV}_n}{\lambda}$$

VII Other System Characteristics

Let E(Cycle) and E(Busy) denote the expected length of cycle and expected busy period then by using equation IV- (ii) and (v) imply ,

(i) $E(\text{Cycle}) = \frac{1}{\lambda R_0} = \frac{(1-\rho_{Bv})}{D_{Bv}(m,N)}$

(ii) $E(\text{Busy}) = P_{busy}\, E(\text{cycle}) = \frac{\rho_{sos}}{(1-\rho_{Bv})} D_{Bv}(m, N)$

5.2 Optimal Management Policy

In this section the main objective is to find the optimal values m^* and $N^*(m)$ which minimize the linear cost function, by considering the cost structure as in previous chapters.

Let C_y(cycle cost), C_h(holding cost per customer), C_{set}(setup cost), C_{dor}(dormant cost) ,C_{build}(buildup cost), C_{busy} (operating cost) and C_v(reward cost) per unit time.

Then the average cost per unit time of the system is given by

$$T_C(m, N) = \frac{C_y}{E(cycle)} + C_h L^{s}_{BV(m,N)} + C_{build} P_{build} + C_{set} P_{set} + C_{dor} P_{dor}$$

$$+ C_{busy} P_{busy} - C_v P_v$$

By substituting the performance measures, the above equation becomes

$$T_C(m,N) = A'_{BV} + \frac{1}{D_{BV}(m,N)} \left[A_{BV} + Z_{BV}(m) + C_h \sum_{n=m}^{N-1} \frac{n \phi_n^{BV}}{\lambda} + C_{dor} \sum_{n=m}^{N-1} \phi_n^{BV} \frac{(1-\rho_{BV})}{\lambda} \right]$$

where $A'_{BV} = C_{busy}\, \rho_{sos} + C_h L_1 - C_v \lambda E(X)\ p\, E(V))$

$$L_1 = \rho_{sos} + \frac{\lambda E(X(X-1)) E(S_{BV}) + (\lambda E(X)^2) E(S^2_{BV})}{2(1-\rho_{BV})}$$

$$A_{BV} = (1 - \rho_{BV}) (C_y + E(D)\, C_{set}) + \frac{\lambda E(X) E(D^2)}{2} C_h$$

$$Z_{BV}(m) = C_h \sum_{n=0}^{m-1} \frac{n \pi_n}{\lambda} + (\lambda E(D) E(X) C_h + C_{build} (1 - \rho_{BV})) \sum_{n=0}^{m-1} \frac{\pi_n}{\lambda}$$

By calculation,

$$T_C(m, k+1) - T_C(m, k) = \frac{\phi_k^{Bv}}{\lambda\ D_{Bv}(m, k+1) D_{Bv}(m, k)} H_{Bv}(m, k)$$

where,

$$H_{BV}(m,k) = C_{dor}(1 - \rho_{BV})\, \ell_m^{BV} + C_h \{k\, \ell_m^{BV} + \sum_{n=m}^{k} \frac{(k-n)}{\lambda} \phi_n^{BV}\} - (A_{BV} + Z_{BV}(m))$$

with $\ell_m^{BV} = E(D) + \sum_{n=0}^{m-1} \frac{\pi_n}{\lambda}$

By proceeding as in Chapter III it is found that the condition under which $T_C(m, k+1) - T_C(m,k) > 0$ for the first value of k is given by

$$\mathbf{N^*(m) = \min \{ k / H_{BV}(m, k) > 0 \}} ,$$

The optimal threshold values $(m^*, N^*(m))$ can be obtained by following the algorithm given in section (2.1).

5.3 Particular cases

(1) The (m, N) policy non-vacational $M^X_{(m,N)}$ /G_{sos}/1 queueing model :

When $p_1 = p_2 = 0$, the generating function of the Bernoulli vacation model coincides with $M^X_{(m,N)}$ /G_{sos}/1 queueing model without Bernoulli vacation and it is given by

$$P(z) = \frac{(1-\rho_{sos})(z-1)S^*_{sos}(w_X(z))}{(z-S^*_{sos}(w_X(z))} \frac{I(z)}{I(1)} \quad \text{where}$$

$$I(z) = \left(\frac{(1-D^*(w_X(z)))}{w_X(z)} + \pi(z)D^*(w_X(z)) + \left(\sum_{n=m}^{N-1} \frac{\phi_n^{BV} z^n}{\lambda} \right) \right)$$

$$\text{with} \quad \phi_n^{BV} = \sum_{k=m}^{n} S_k \sum_{i=0}^{n-k} h_i \, \pi_{n-k-i} \quad ; \quad S_k = \sum_{j=0}^{m-1} \pi_i \, g_{k-i}$$

(2) The (m, N) policy single service non-vacational $M^X_{(m,N)}$ / G / 1 model: .

When r = 0 along with the condition p = 0 , the corresponding generating function is given by

$$P(z) = \frac{(1-\rho)(z-1)S^*(w_X(z))}{(z-S^*(w_X(z))} \frac{I(z)}{I(1)} \quad , \rho = \lambda E(x) E(S)$$

(3) The results of single Poisson arrival queueing model can be obtained by putting X(z) = z , and $\pi_n = 1$ for $0 \le n \le m-1$ in the results of batch arrival models.

(4) If $P^Q_{BV(m,N)}(z)$ denotes the PGF of the steady state **queue size** probabilities for the Benronoulli (m, N) policy model considered in this chapter then $P^Q_{BV(m,N)}(z) = \dfrac{(1-\rho_{BV})(z-1)}{(z-S^*_{BV}(w_X(z))} \dfrac{I_{BV}(z)}{I_{BV}(1)}$

5.4 Numerical Analysis

In order to study the effects of (i) mean vacation time E(V) (ii) Bernoulli vacation probabilities p_i (i=1,2) (iii) the second optional service probability (r) (iv)the mean batch size E(X) and the mean setup time E(D) on the optimum policies , numerical values are presented in tables (5.1) to (5.6). The graphical representations are shown in figures (5.1) to (5.5).

For computation, the following distributions are assumed for service time (S_i) (i=1,2) setup time D ,vacation time V and batch size X.

Random Variable (y)	Distribution F(y)	Mean E(y)	Second order moment
S_1	Two stage hyper exponential	$E(S_1)=\frac{a_1}{\mu_{11}}+\frac{1-a_1}{\mu_{12}}$ $0 \le a_1 \le 1$	$E(S_1^2)=2\left(\frac{a_1}{\mu_{11}^2}+\frac{1-a_1}{\mu_{12}^2}\right)$
S_2	Gamma distribution G(2,μ)	$E(S_2)=\frac{2}{\mu}$	$E(S_2^2)=\frac{6}{\mu^2}$
D	Exponential Distribution	$E(D)=\frac{1}{v}$	$E(D^2)=\frac{2}{v^2}$
V	Erlang 3 type distribution	$E(V)=\frac{1}{\eta}$	$E(V^2)=\frac{4}{3\eta^2}$
X	Geometric distribution	$E(X)=\frac{1}{1-p}$	$E(X(X-1))=\frac{2p}{(1-p)^2}$

The impact of the dormant cost on the optimal policies is shown in Table (5.1). The table shows that optimal (m,N) policy would approach the optimal (N,N) policy as C_{dor} increases. The effects of batch size E(X) and the second optional service probability (r) on (i) mean system size $L_{Bv(m^*,N^*)}$ and (ii) on the optimal cost value Tc (m*,N*) are shown in Table (5.2). It is found that the mean batch size E(X) and the sos probability r significantly affect the optimal cost and the mean system size. Both $L_{Bv(m^*,N^*)}$ and Tc (m*,N*) increase as r or E(X) increases. The parametric values used to calculate numerical values are mentioned in the corresponding tables.

Table (5.1): The optimum threshold values (m*,N*) its minimum expected cost Tc(m*,N*) and optimal system size for different values of C_{dor}

$(C_h, C_{build}, C_{set}, C_{busy}, C_y, C_v) = (15,8, 1000, 100,1000,10000,8)$;

$(E(X)\ \lambda, v, \eta, p_1, p_2, r, E(S_{sos})) = (4,.3, .3, 1, .15, .4,.2, .77)$

C_{dor}	(m*,N*)	Tc(m*,N*)	$L_{Bv(m^*,N^*)}$	(N,N*)	Tc(N,N*)	$L_{Bv(N^*,N^*)}$
10	(1,15)	1117.74	23.32	13	1136.8	24.32
50	(5,15)	1121.87	23.50	13	1136.8	24.32
75	(8,15)	1133.43	23.75	13	1136.8	24.32
100	(9,14)	1134.43	23.52	13	1136.8	24.32
150	(11,14)	1135.71	23.89	13	1136.8	24.32
250	(13,13)	1136.56	24.24	13	1136.8	24.32
500	(13,13)	1136.56	24.24	13	1136.8	24.32
1000	(13,13)	1136.56	24.24	13	1136.8	24.32

Table (5.2) : The Optimal policies for different batch size E(X) and SOS probability r

$(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v) = (50,8, 1000, 100,1000,10000,8)$;

$(\lambda, v, \eta, p_1, p_2, E(S_{sos})) = (.04, .3, .5, .2, .1, .77)$

r	E(X) =4		E(X)=5		E(X)=10	
	Tc(m*,N*)	$L_{Bv(m^*,N^*)}$	Tc(m*,N*)	$L_{Bv(m^*,N^*)}$	Tc(m*,N*)	$L_{Bv(m^*,N^*)}$
.1	433.42	2.81	490.03	3.22	857.71	8.01
.2	467.06	2.98	535.09	3.51	1017.63	9.86
.3	501.20	3.16	581.38	3.83	1206.6	12.53
.4	535.90	3.35	629.09	4.17	1442.81	16.24
.5	576.38	3.4	677.86	4.18	1756.06	21.35
.6	606.44	3.36	729.15	4.59	2212.05	29.49
.7	642.45	3.6	780.73	5.05	2972.57	43.63
.8	679.37	3.85	836.03	5.57	4599.78	75.22
.9	717.29	4.12	894.62	6.15	11194.16	206.13

It is important to study the effects of Bernoulli schedule probabilities p_i (i=1,2) of the server on the mean system size. Table (5.3) gives the numerical values for optimum mean system size $L_{Bv(m^*,N^*)}$ for different values of Bernoulli vacation probabilities p_i (i=1, 2), arrival rate λ and mean vacation time E(V). The table values clearly show that as p_i or λ increases, $L_{BV(m^*,N^*)}$ also increases. One can note from the table that, for higher values of λ, the rate of increase in $L_{BV(m^*,N^*)}$ is faster than that of the lower values of λ. The figures (5.1) and (5.2) also reveal the same. One can also note that the system size increases along with the mean vacation time. The last row values when $p_1 = p_2 = 0$ give the expected system size for the non –vacation (m,N) policy queueing model.

Table 5.3 : The optimal system size for different values of λ , p_i and E(V)

$(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v)$= (15,8, 1000, 100,1000,10000,8)

(E(X), v , r, $E(S_{sos})$) = (4, .3, .2, .77)

λ	.2		.25		.3		.35	
(p1,p2)	E(v)=.67	E(v)=1	E(v)=.67	E(v)=1	E(v)=.67	E(v)=1	E(v)=.67	E(v)=1
(.1,.05)	12.08	12.38	14.8	15.26	17.85	18.73	25.27	28.44
(.1,.1)	12.12	12.55	14.91	15.42	18.1	18.69	25.79	30.40
(.1,.2)	12.19	12.91	15.12	15.28	18.42	19.5	27.53	34.58
(.15,.15)	12.46	13.11	15.11	15.47	18.68	20.54	30.37	44.3
(.15,.2)	12.76	13.17	15.01	15.68	19.00	21.06	31.36	49.80
(.2,.1)	12.83	13.28	15.18	15.79	19.35	21.82	33.89	65.98
(.2, .15)	12.88	13.38	15.30	15.74	19.42	22.70	35.82	81.0
(.2,.2)	12.93	13.47	15.43	15.99	19.80	23.17	38.05	104.76
(.25,.1)	13.00	13.59	15.54	16.14	20.27	24.91	42.62	248.13
(0,0)	12.02	12.02	14.59	14.59	17.14	17.14	21.32	21.32

Mean system size $L_{Bv(m^*,N^*)}$ Vs λ for different values of (p_1 , p_2)

Fig. (5.1) when E(v) = .67

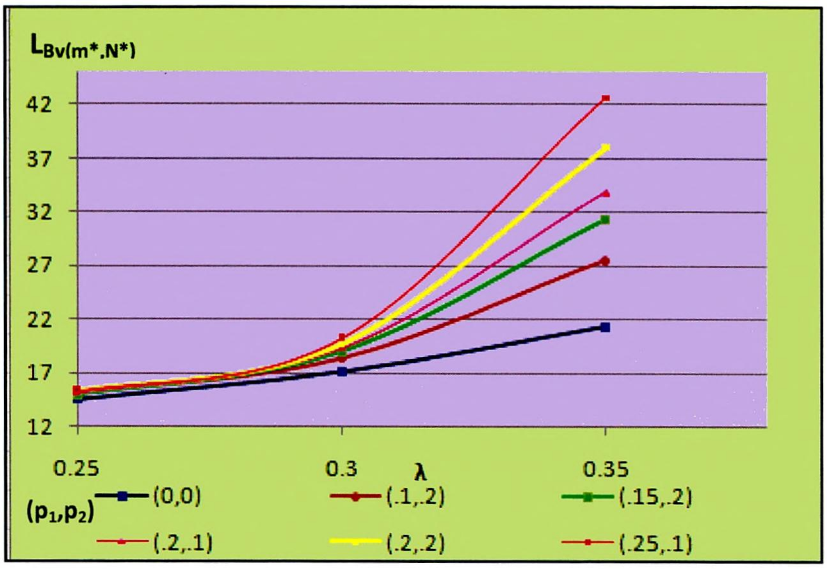
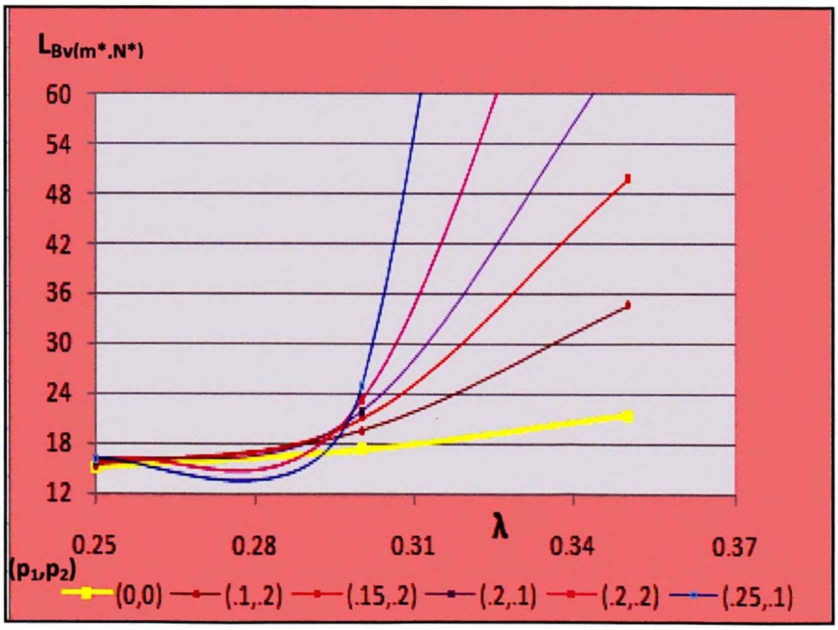


Fig. (5.2): when E(v) = 1



In Table (5.4) the mean system size is compared with Bernoulli vacation probabilities p_i for different values of SOS probability r. The table values show that the optimal system size $L_{BV(m^*,N^*)}$ increases with (r) and p_i (i =1,2). The graphical representations of the table (5.4) are given in figure (5.3).

Table 5.4 : Mean system size with respect to p_i (i =1,2) and SOS r

$(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v) = (15,8, 1000, 100,1000,10000,8)$;

$(E(x), v, \eta, \lambda, E(S_{sos})) = (4,.3, .5,.15, .77)$

r / (p1,p2)	.4	.5	.6	.63	.65	.68	.7	.8
(.1, .05)	11.80	12.34	13.75	14.28	15.08	16.02	16.94	26.62
(.1, .1)	11.88	12.86	14.36	15.46	16.27	18.04	19.24	37.7
(.1, .15)	11.89	12.97	15.48	16.77	18.26	20.51	22.66	71.35
(.1, .2)	11.91	13.65	16.7	18.68	20.66	24.59	28.62	840.3
(.05, .1)	11.72	12.39	14.15	14.88	15.63	16.95	18.29	34.26
(.15, .1)	11.94	12.91	15.15	16.13	17.09	18.82	20.52	42.59
(.2, .1)	11.98	13.52	15.86	16.99	18.13	20.13	21.79	48.77

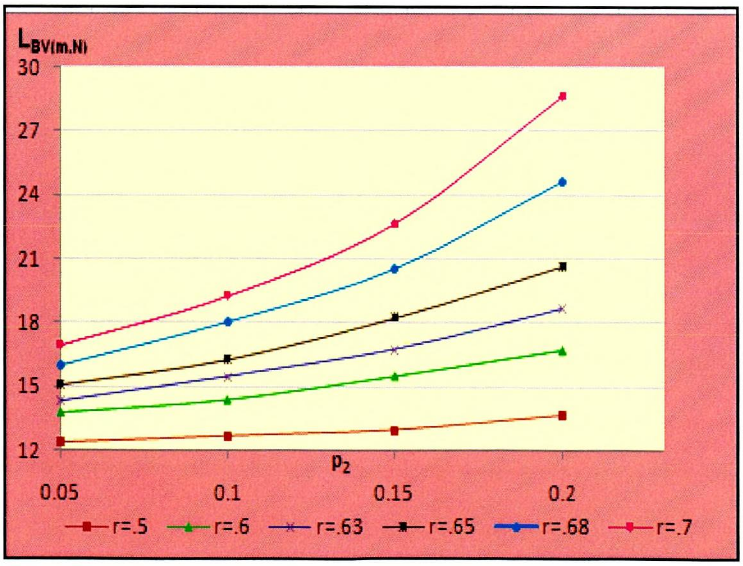


Fig. (5.3) : $L_{Bv(m^*,N^*)}$ with respect to p_2 for different sos probability r

The expected cost values of Tc(m,N) for different values of m and N are plotted in figure (5.4) and the optimal threshold values (m* ,N*) and Tc(m*,N*) are given by m*=9, N* =14 and Tc(m*,N*) =1134.44. This satisfies the solution procedure specified in section (5.2) .Table (5.5) gives the data for the figure (5.4)

Fig. (5.4) The expected cost Tc (m,N) for Bernoulli vacations

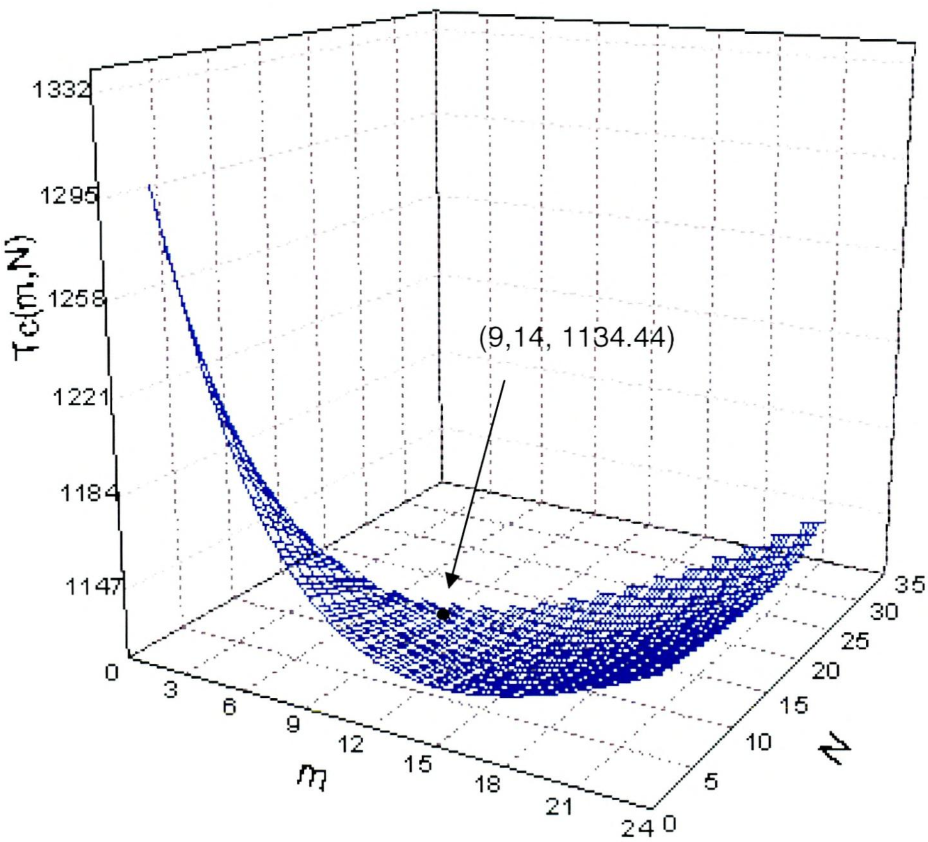


Table 5.5 : The expected cost Tc(m,N) Vs (m,N)

$(C_h, C_{build}, C_{set}, C_{dor}, C_{busy}, C_y, C_v) = (15,8,1000,100,1000,10000,8)$;

$(E(x), \lambda, v, \eta, p_1, p_2, r, E(S_{sos})) = (4,.3, .3, 1, .15, .4, .2, .77)$

m \ N	7	8	9	10	11	12	13	14	15	16	17	18
5	1166.5	1158.23	1151.24	1145.63	1141.37	1138.38	1136.54	1135.73	1135.82	1136.7	1138.27	1140.44
6	1162.5	1155.57	1149.46	1144.41	1140.49	1137.7	1135.97	1135.21	1135.33	1136.22	1137.79	1139.95
7	1158.34	1152.84	1147.68	1143.24	1139.7	1137.13	1135.52	1134.81	1134.95	1135.83	1137.39	1139.54
8		1150.06	1145.93	1142.15	1139.02	1136.69	1135.2	1134.55	1134.69	1135.56	1137.09	1139.21
9			1144.2	1141.15	1138.48	1136.4	1135.04	**1134.44**	1134.57	1135.42	1136.9	1138.98
10				1140.26	1138.08	1136.29	1135.07	1134.51	1134.64	1135.43	1136.87	1138.88
11					1137.86	1136.38	1135.31	1134.8	1134.9	1135.64	1137	1138.93
12						1136.69	1135.8	1135.33	1135.41	1136.07	1137.33	1139.15
13							1136.56	1136.15	1136.19	1136.76	1137.89	1139.59
14								1137.27	1137.27	1137.74	1138.73	1140.27
15									1138.7	1139.06	1139.89	1141.24
16										1140.75	1141.4	1142.55