CHAPTER III

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HAUSDORFF GENERALISED HAUSDORFF AND ∂- HAUSDORFF BICLOSURE SPACES

In this chapter the concepts Hausdorff biclosure spaces, g- Hausdorff biclosure spaces and ∂-Hausdorff biclosure spaces are discussed and the basis concepts and preliminaries are studied.

SECTION 3.1

HAUSDORFF BICLOSURE SPACES

In this section, the concepts of Hausdorff biclosure spaces and the properties of Hausdorff biclosure spaces are discussed.

Definition: 3.1.1

A biclosure space (X, u_1, u_2) is called a **Hausdorff biclosure space** if, whenever x and y are distinct points of X there exists an open subset U of (X, u_1) and an open subset V of (X, u_2) such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Example: 3.1.2

Let $X=\{a, b\}$ and u_1 be a closure operator on X defined by $u_1\phi = \phi$ and $u_1\{a\} = \{a\}$, $u_1\{b\} =\{b\}$ and $u_1X = X$. Let u_2 be a closure operator on X defined by $u_2\phi = \phi$ and $u_2\{a\} = \{a\}$, $u_2\{b\} =\{b\}$ and $u_2X = X$. Then (X, u_1, u_2) is a Hausdorff biclosure space.

Proposition: 3.1.3

Let (X, u_1, u_2) be a biclosure space and (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If (X, u_1, u_2) is a Hausdorff biclosure space, then (Y, v_1, v_2) is a Hausdorff biclosure space.

Proof:

Let y and y' be any two distinct points of Y.

Then y and y' are distinct points of X.

Since (X, u_1, u_2) is a Hausdorff biclosure space, there exist disjoint open subsets U of (X, u_1) and V of (X, u_2) containing y and y' respectively.

Then $y \in U \cap Y$ and $y' \in V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = \phi$.

By Lemma 1.2.26, $U \cap Y$ is an open subset of (Y, v_1) and $V \cap Y$ is an open subset of (Y, v_2).

Hence (Y, v_1, v_2) is a Hausdorff biclosure space.

Proposition: 3.1.4

Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. Then $\prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2)$ is a Hausdorff biclosure space if and only if $(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a Hausdorff biclosure space for each $\alpha \in I$.

Proof:

Let $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ be a Hausdorff biclosure space.

Claim 1: $(X_\beta, u_\beta^1, u_\beta^2)$ is a Hausdorff biclosure space for each $\beta \in I$.

Let $\beta \in I$ and x_β, y_β be any two disjoint points of X_β.

Let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be disjoint points of $\prod_{\alpha\in I} X_\alpha$ such that $x_\beta \neq y_\beta$. Since $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a Hausdorff biclosure space, there exist an open subset U of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1)$ and an open subset V of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^2)$ such that $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ and $U \cap V = \phi$.

Hence $x_\beta \in \pi_\beta(U)$ and $y_\beta \in \pi_\beta(V)$ and $\pi_\beta(U) \cap \pi_\beta(V) = \phi$.

Since π_β is the projection map, $\pi_\beta(U)$ and $\pi_\beta(V)$ are disjoint open sets in X_β containing x_β and y_β respectively and hence $(X_\beta, u_\beta^1, u_\beta^2)$ is a Hausdorff biclosure space.

Conversely, let $(X_\alpha, u_\alpha^1, u_\alpha^2)$ be a Hausdorff biclosure space for each $\alpha \in I$.

Claim 2: $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a Hausdorff biclosure space.

Let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be any two distinct points of $\prod_{\alpha\in I} X_\alpha$. Then there exists atleast one $\beta \in I$, such that $x_\beta \neq y_\beta$.

Then x_β and y_β are two disjoint points of X_β.

Since $(X_\beta, u^1_\beta, u^2_\beta)$ is a Hausdorff biclosure space, there exist disjoint open subset U of (X_β, u^1_β) and an open subset V of (X_β, u^2_β) such that $x_\beta \in U$ and $y_\beta \in V$ and $U \cap V = \phi$.

Therefore, $\pi_\beta^{-1}(U) = U \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and $\pi_\beta^{-1}(V) = V \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$, such that $(x_\alpha)_{\alpha\in I} \in U \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ and $(y_\alpha)_{\alpha\in I} \in V \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ and $(U \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha) \cap (V \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha) = \phi$.

Therefore, $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a Hausdorff biclosure space.

Proposition: 3.1.5

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. Let $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ be injective and continuous. If (Y, v_1, v_2) is a Hausdorff biclosure space, then (X, u_1, u_2) is a Hausdorff biclosure space.

Proof:

Let x and y be any two distinct points of X. Then f(x) and f(y) are distinct points of Y. Since (Y, v_1, v_2) is a Hausdorff biclosure space, there exist disjoint open subsets U of (Y, v_1) and V of (Y, v_2) containing f(x) and f(y) respectively. Since f is continuous and $U \cap V = \phi$, $f^{-1}(U)$ is an open subset of (X, u_1) containing x and $f^{-1}(V)$ is an open subset of

(X, u_2) containing y such that $f^{-1}(U) \cap f^{-1}(V) = \phi$ and $x \in f^{-1}(U)$, $y \in f^{-1}(V)$.

Therefore, (X, u_1, u_2) is a Hausdorff biclosure space.

SECTION 3.2

GENERALISED HAUSDORFF BICLOSURE SPACES

In this section generalized Hausdorff biclosure spaces and its properties are analyzed.

Definition: 3.2.1

A biclosure space (X, u_1, u_2) is called a **generalized Hausdorff biclosure space (g- Hausdorff biclosure space)** if, whenever x and y are distinct points of X there exists a g- open subset U of (X, u_1) and g- open subset V of (X, u_2) such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Remark: 3.2.2

Every Hausdorff biclosure space is a g- Hausdorff biclosure space. The converse need not be true.

Example: 3.2.3

Let $X = \{a, b\}$ and u_1 be a closure operator on X defined by $u_1\phi = \phi$ and $u_1\{a\} = u_1\{b\} = u_1X = X$. Let u_2 be a closure operator on X defined by $u_2\phi = \phi$ and $u_2\{a\} = u_2\{b\} = u_2X = X$. Then (X, u_1, u_2) is a g- Hausdorff biclosure space but it is not a Hausdorff biclosure space.

Proposition: 3.2.4

Let (X, u_1, u_2) be a biclosure space and (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If (X, u_1, u_2) is a g- Hausdorff biclosure space, then (Y, v_1, v_2) is a g- Hausdorff biclosure space.

Proof:

Let y and y' be any two distinct points of Y.

Then y and y' are distinct points of X.

Since (X, u_1, u_2) is a g- Hausdorff biclosure space, there exist disjoint g- open subsets U of (X, u_1) and V of (X, u_2) containing y and y' respectively.

Then $y \in U \cap Y$ and $y' \in V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = \phi$.

By Lemma 1.2.27, $U \cap Y$ is a g-open subset of (Y, v_1) and $V \cap Y$ is a g- open subset of (Y, v_2).

Hence (Y, v_1, v_2) is a g- Hausdorff biclosure space.

Proposition: 3.2.5

Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. Then $\prod_{\alpha \in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a g- Hausdorff biclosure space if and only if $(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a g- Hausdorff biclosure space for each $\alpha \in I$.

Proof:

Let $\prod_{\alpha \in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ be a g- Hausdorff biclosure space.

Claim 1: $(X_\beta, u_\beta^1, u_\beta^2)$ is a g- Hausdorff biclosure space for each $\beta \in I$.

Let $\beta \in I$ and x_β, y_β be any two disjoint points of X_β.

Then $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be disjoint points of $\prod_{\alpha\in I} X_\alpha$.

Since $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a g- Hausdorff biclosure space, there exists a g- open subset U of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and a g- open subset V of $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$ such that $(x_\alpha)_{\alpha\in I} \in U$ and $(y_\alpha)_{\alpha\in I} \in V$ and $U \cap V = \phi$.

Since π_β is the projection map, $\pi_\beta(U)$ and $\pi_\beta(V)$ are disjoint g- open subsets of (X_β, u^1_β) and (X_β, u^2_β) such that $x_\beta \in \pi_\beta(U)$, $y_\beta \in \pi_\beta(V)$ and $\pi_\beta(U) \cap \pi_\beta(V) = \phi$.

Therefore, $(X_\beta, u^1_\beta, u^2_\beta)$ is a g- Hausdorff biclosure space.

Conversely, let $(X_\alpha, u^1_\alpha, u^2_\alpha)$ be a g- Hausdorff biclosure space for each $\alpha \in I$.

Claim 2: $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a g- Hausdorff biclosure space.

Let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be any two distinct points of $\prod_{\alpha\in I} X_\alpha$. Then there exists atleast one $\beta \in I$, such that $x_\beta \neq y_\beta$.

Then x_β and y_β are two disjoint points of X_β.

Since $(X_\beta, u^1_\beta, u^2_\beta)$ is a g- Hausdorff biclosure space, there exist disjoint g- open subset U of (X_β, u^1_β) and a g- open subset V of (X_β, u^2_β) such that $x_\beta \in U$ and $y_\beta \in V$.

Therefore, $U \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a g- open subset of $\prod_{\alpha\in I} (X_\alpha, u_\alpha^1)$ and $V \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a g- open subset of $\prod_{\alpha\in I} (X_\alpha, u_\alpha^2)$, such that $(x_\alpha)_{\alpha\in I} \in U \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ and $(y_\alpha)_{\alpha\in I} \in V \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ and $(U \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha) \cap (V \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha) = \phi$.

Therefore, $\prod_{\alpha\in I} (X_\alpha, u_\alpha^1, u_\alpha^2)$ is a g- Hausdorff biclosure space.

Proposition: 3.2.6

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. Let $f: (X, u_1, u_2) \to (Y, v_1, v_2)$ be injective and g- irresolute. If (Y, v_1, v_2) is a g- Hausdorff biclosure space, then (X, u_1, u_2) is a g- Hausdorff biclosure space.

Proof:

Let x and y be any two distinct points of X. Then f(x) and f(y) are distinct points of Y.

Since (Y, v_1, v_2) is a g- Hausdorff biclosure space, there exist disjoint g- open subsets U of (Y, v_1) and V of (Y, v_2) containing f(x) and f(y) respectively.

Since f is g- irresolute and $U \cap V = \phi$, $f^{-1}(U)$ is a g- open subset of (X, u_1) and $f^{-1}(V)$ is a g- open subset of (X, u_2), then $f^{-1}(U) \cap f^{-1}(V) = \phi$ and $x \in f^{-1}(U)$, $y \in f^{-1}(V)$.

Therefore, (X, u_1, u_2) is a g- Hausdorff biclosure space.

SECTION 3.3

∂- HAUSDORFF BICLOSURE SPACES

In this section, the concepts of ∂-Hausdorff biclosure spaces and the properties of ∂-Hausdorff biclosure spaces are discussed.

Definition: 3.3.1

A biclosure spaces (X, u_1, u_2) is said to be **∂-Hausdorff biclosure spaces** if, whenever x and y are distinct points of X there exist ∂-open subset U of (X, u_1) and ∂-open subset V of (X, u_2) such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Example: 3.3.2

Let X= {a,b} and u_1 be a closure operator on X defined by $u_1\phi=\phi, u_1\{a\}= u_1\{b\}= u_1X=X$. Let u_2 be a closure operator on X defined by $u_2\phi=\phi, u_2\{a\}= u_2\{b\}$ $u_2X=X$. Then (X, u_1, u_2) is a ∂-Hausdorff biclosure space.

Definition: 3.3.3

Let (X, u) and (Y, v) be closure spaces. A map $f: (X, u) \rightarrow (Y, v)$ is called **∂-irresolute**, if $f^{-1}(F)$ is a ∂-closed subset of (X, u) for every ∂-closed subset F of (Y, v).

(or) A map $f: (X, u) \rightarrow (Y, v)$ is called **∂-irresolute**, if and only if $f^{-1}(G)$ is a ∂-open subset of (X, u) for every ∂-open subset G of (Y, v).

Definition: 3.3.4

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1,2\}$.A map $f: (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ is called **i-∂-irresolute** if the map f: $(X, u_i) \rightarrow (Y, v_i)$ is ∂-irresolute. A map f is called ∂-irresolute if f is i-∂-irresolute for each $i \in \{1,2\}$.

Proposition: 3.3.5

Let (X, u_1, u_2) be a biclosure space and let A, B$\subseteq$ X. If A is a closed subset of (X, u_1, u_2) and B is both a ∂-closed subset of (X, u_1) and (X, u_2), then A$\cap$B is both a ∂-closed subset of (Y, v_1) and (Y, v_2).

Proof

Let G be a g-open subset of (X, u_1) such that A$\cap$B$\subseteq$ G. Then B$\subseteq$ (X-A) $\cup$G. Since (X-A) $\cup$G is a g-open subset of (X, u_1), $u_1B \subseteq (X\text{-}A) \cup G$.We have, $A \cap u_1B \subseteq G$.

Since A is a closed subset of (X, u_1),$u_1(A\cap B) \subseteq u_1A\cap u_1B = A\cap u_1B \subseteq G$. Hence, is a ∂-closed subset of (X, u_1).

Similarly A is a closed subset of (X, u_1, u_2) and B is a ∂-closed subset of (X, u_2), then A$\cap$B is a ∂-closed subset of (Y, v_2).

Lemma: 3.3.6

Let (X, u_1, u_2) be a biclosure space and (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If G is both a ∂-open subset of (X, u_1) and (X, u_2), then G$\cap$Y is both a ∂-open subset of (Y, v_1) and (Y, v_2).

Proof

Let G be a ∂-open subset of (X, u_1). Then X-G is a ∂-closed subset of (X, u_1). Since Y is a closed subset of (X, u_1), (X-G) $\cap$Y is a ∂-closed subset of (X, u_1). But (X-G) $\cap$Y=Y-(G$\cap$Y).

Therefore, Y-(G$\cap$Y) is a ∂-closed subset of (X, u_1). Hence, G$\cap$Y is a ∂-open subset of (X, u_1). Similarly, if G is a ∂-open subset of (X, u_2), then G$\cap$Y is a ∂-open subset of (Y, v_2).

Proposition: 3.3.7

Let (X, u_1, u_2) be a biclosure space and (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If (X, u_1, u_2) is a ∂-Hausdorff biclosure space, then (Y, v_1, v_2) is a ∂-Hausdorff biclosure space.

Proof

Let y and y' be any two distinct points of Y. Then y and y' are distinct points of X. Since (X, u_1, u_2) is a ∂-Hausdorff biclosure space, there exist disjoint ∂-open subset U of (X, u_1) and ∂-open subset V of (X, u_2) containing y and y', respectively. Hence, y$\in$ U$\cap$Y, y' $\in$ V$\cap$Y and (U$\cap$Y)$\cap$(V$\cap$Y) =ϕ.

By lemma 3.3.6, U$\cap$Y is a ∂-open subset of (Y, v_1) and V$\cap$Y is ∂-open subset of (Y, v_2).

Hence, (Y, v_1, v_2) is a ∂-Hausdorff biclosure space.

Proposition: 3.3.8

Let {(X_α, u_α^1, u_α^2) : $\alpha \in$ I } be a family of biclosure spaces. Then $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a ∂- Hausdorff biclosure space if and only if $(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a ∂- Hausdorff biclosure space for each $\alpha \in$ I.

Proof

Let $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ be a ∂- Hausdorff biclosure space. Let $\beta \in$ I and x_β, y_β be any two distinct points of X_β. Then $(x_\alpha)_{\alpha\in I}$ and

$(y_\alpha)_{\alpha\in I}$ are distinct points of $\prod_{\alpha\in I} X_\alpha$. Since $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a ∂- Hausdorff biclosure space, there exists ∂-open subset U of (X_β, u_β^1) and ∂- open subset V of (X_β, u_β^2) such that $x_\beta \in$ U, $y_\beta \in$ V and $U \cap V = \phi$. Therefore, $(X_\beta, u_\beta^1, u_\beta^2)$ is a ∂- Hausdorff biclosure space.

Conversely, let $(X_\alpha, u_\alpha^1, u_\alpha^2)$ be a ∂- Hausdorff biclosure space for each $\alpha \in I$. Let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be any two distinct points of $\prod_{\alpha\in I} X_\alpha$. Then $x_\beta \neq y_\beta$ for atleast one $\beta \in I$. Then x_β and y_β are distinct points of X_β. Since $(X_\beta, u_\beta^1, u_\beta^2)$ is a ∂- Hausdorff biclosure space, there exist disjoint ∂- open subset U of (X_β, u_β^1) and ∂- open subset V of (X_β, u_β^2) such that $x_\beta \in$ U and $y_\beta \in$ V. Hence, $U \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a ∂- open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1)$ and $V \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a ∂- open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^2)$ such that

$(x_\alpha)_{\alpha\in I} \in U \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha, (y_\alpha)_{\alpha\in I} \in V \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$

and $\left(U \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha\right) \cap \left(V \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha\right) = \phi$.

Hence, $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a ∂- Hausdorff biclosure space.

Proposition: 3.3.9

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. Let f: $(X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ be an injective and ∂-irresolute map. If (Y, v_1, v_2) is a ∂- Hausdorff biclosure space, then (X, u_1, u_2) is a ∂- Hausdorff biclosure space.

Proof

Let x and y be any two distinct points of X. Then f(x) and f(y) are distinct points of Y. Since (Y, v_1, v_2) is a ∂- Hausdorff biclosure space, there exist disjoint ∂- open subset U of (Y, v_1) and ∂- open subset V of (Y, v_2) containing f(x) and f(y), respectively. Since f is ∂-irresolute and $U \cap V = \phi$, $f^{-1}(U)$ is a ∂- open subset of (X, u_1), $f^{-1}(V)$ is a ∂- open subset of (X, u_2), $f^{-1}(U) \cap f^{-1}(V) = \phi$ and $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Therefore, (X, u_1, u_2) is a ∂- Hausdorff biclosure space.