Chapter IV

CHAPTER IV

FUZZY PARAMETERIZED INTERVAL-VALUED FUZZY SOFT SETS

Definition: 4.1

Let U be an initial universe, E the set of all parameters and X a fuzzy set over E with membership function: $\mu_X : E \to [0, 1]$ and let η_X be an interval-valued fuzzy set over U for all $x \in E$. Then a **Fuzzy Parameterized Interval-Valued Fuzzy Soft Set (** ***fpivfs*** **– set)** Ψ_X over U is a set defined by a function η_X representing a mapping

$$\eta_X : E \to Int(U) \text{ such that } \eta_X(x) = \phi \text{ if } \mu_X(x) = 0$$

Here, $\eta_X = \left[\eta_X^-, \eta_X^+\right]$ is called an **Interval-Valued Fuzzy Approximate Function of the** ***fpivfs*** **– set** Ψ_X , and the value $\eta_X(x)$ is a set called an x- element of the *fpivfs* – set for all $x \in E$. Thus a *fpivfs* – set Ψ_X over U can be represented by the set of ordered pairs

$$\Psi_X = \{(x / \mu_X(x), \eta_X(x)) : x \in E, \eta_X(x) \in Int(U), \mu_X(x) \in [0, 1]\}$$

The set of all *fpivfs* – sets over U will be denoted by $FPIVFSS(U)$.

Example: 4.2

Let U = $\{u_1, u_2, u_3, u_4\}$ be a set of universe, E = $\{x_1, x_2, x_3, x_4\}$ be a set of parameters and let $\mu : E \to [0, 1]$. Suppose X = $\{x_1/0.3, x_2/0.1, x_3/0.4, x_4/1\}$ and $\eta_X(x)$ is defined as follows:

$\eta_X(x_1) = \{ u_1/[0.3, 0.6], u_2/[0.7, 0.8], u_3/[0.5, 0.8], u_4/[0.2, 0.5] \}$,

$\eta_X(x_2) = \{ u_1/[0.0, 0.7], u_2/[0.2, 0.3], u_3/[0.3, 0.5], u_4/[0.1, 0.3] \}$,

$\eta_X(x_3) = \{ u_1/[0.1, 0.4], u_2/[0.4, 0.6], u_3/[0.3, 0.8], u_4/[0.4, 0.5] \}$,

$\eta_X(x_4) = \{ u_1/[0.5, 0.7], u_2/[0.1, 0.4], u_3/[0.0, 0.4], u_4/[0.8, 0.9] \}$

Then the *fpivfs* – set Ψ_X is given by

$$\begin{aligned}\Psi_X = \{ & (x_1/0.3, \{ u_1/[0.3, 0.6], u_2/[0.7, 0.8], u_3/[0.5, 0.8], u_4/[0.2, 0.5] \}),\\ & (x_2/0.1, \{ u_1/[0.0, 0.7], u_2/[0.2, 0.3], u_3/[0.3, 0.5], u_4/[0.1, 0.3] \}),\\ & (x_3/0.4, \{ u_1/[0.1, 0.4], u_2/[0.4, 0.6], u_3/[0.3, 0.8], u_4/[0.4, 0.5] \}),\\ & (x_4/1.0, \{ u_1/[0.5, 0.7], u_2/[0.1, 0.4], u_3/[0.0, 0.4], u_4/[0.8, 0.9] \}) \}\end{aligned}$$

Definition: 4.3

Let Ψ_X and $\Psi_Y \in FPIVFSS(U)$. Then Ψ_X is said to be a **Fuzzy Parameterized Interval-Valued Fuzzy Soft Subset (*fpivfs* – subset)** of Ψ_X and we write $\Psi_X \subseteq \Psi_Y$ if

1) $\mu_X(x) \leq \mu_Y(x)$, $\forall x \in E$
2) $\eta_X(x) \subseteq \eta_Y(x)$, $\forall x \in E$.

Definition: 4.4

Ψ_X and $\Psi_Y \in FPIVFSS(U)$ are said to be **equal** and we write $\Psi_X = \Psi_Y$ if Ψ_X is a *fpivfs* – subset of Ψ_Y and Ψ_Y is a *fpivfs* – subset of Ψ_X. In other words, $\Psi_X = \Psi_Y$ if the following conditions are satisfied:

1) $\mu_X(x) = \mu_Y(x)$, $\forall x \in E$
2) $\eta_X(x) = \eta_Y(x)$, $\forall x \in E$.

Definition: 4.5

Let $\Psi_X \in FPIVFSS(U)$. If $\eta_X(x) = \phi$, $\forall x \in E$, then Ψ_X is called an X **- empty *fpivfs* – set**, denoted by Ψ_{ϕ_X}. If $X = \phi$, then the X - empty *fpivfs* – set Ψ_{ϕ_X} is called an **empty *fpivfs* – set**, denoted by Ψ_ϕ.

Definition: 4.6

Let $\Psi_X \in FPIVFSS(U)$. If $\eta_X(x) = U$, $\forall x \in E$, then Ψ_X is called an X - **universal** ***fpivfs*** **– set**, denoted by $\Psi_{\tilde{X}}$. If $X = E$, then the X - universal *fpivfs* – set $\Psi_{\tilde{X}}$ is called a **universal** ***fpivfs*** **– set**, denoted by $\Psi_{\tilde{E}}$.

Theorem: 4.7

Let Ψ_X, Ψ_Y and $\Psi_Z \in FPIVFSS(U)$. Then the following results hold:

1) $\Psi_X \subseteq \Psi_{\tilde{E}}$
2) $\Psi_{\phi_X} \subseteq \Psi_X$
3) $\Psi_\phi \subseteq \Psi_X$
4) $\Psi_X \subseteq \Psi_X$
5) $\Psi_X \subseteq \Psi_Y$ and $\Psi_Y \subseteq \Psi_Z \Rightarrow \Psi_X \subseteq \Psi_Z$
6) $(\Psi_X = \Psi_Y$ and $\Psi_Y = \Psi_Z) \Leftrightarrow \Psi_X = \Psi_Z$
7) $(\Psi_X \subseteq \Psi_Y$ and $\Psi_Y \subseteq \Psi_X) \Leftrightarrow \Psi_X = \Psi_Y$

Proof:

The proof is straightforward.

Definition: 4.8

Let $\Psi_X \in FPIVFSS(U)$. Then the **Complement** of Ψ_X, denoted by Ψ_X^C, is defined by $c(\mu_X(x))$ and $\tilde{c}(\eta_X(x))$, $\forall x \in E$, where c is a fuzzy complement and $\tilde{c}$ is an interval-valued fuzzy complement.

Example: 4.9

Consider Example 4.2 by using the basic fuzzy complement for $\mu_X(x)$ and interval-valued fuzzy complement for $\eta_X(x)$ we have

$$\begin{aligned}\Psi_X^C = \{ & (x_1/0.7, \{u_1/[0.4, 0.7], u_2/[0.2, 0.3], u_3/[0.2, 0.5], u_4/[0.5, 0.8]\}), \\ & (x_2/0.9, \{u_1/[0.3, 1.0], u_2/[0.7, 0.8], u_3/[0.5, 0.7], u_4/[0.7, 0.9]\}), \\ & (x_3/0.6, \{u_1/[0.6, 0.9], u_2/[0.4, 0.6], u_3/[0.2, 0.7], u_4/[0.5, 0.6]\}), \\ & (x_4/0.0, \{u_1/[0.3, 0.5], u_2/[0.6, 0.9], u_3/[0.6, 1.0], u_4/[0.1, 0.2]\})\}\end{aligned}$$

Theorem: 4.10

Let $\Psi_X \in FPIVFSS(U)$. Then the following results hold:

1) $(\Psi_X^C)^C = \Psi_X$

2) $\Psi_\phi^C = \Psi_{\tilde{E}}$

Proof:

The proof is straightforward.

Definition: 4.11

Let Ψ_X and $\Psi_Y \in$ *FPIVFSS(U)*. The **Union** of Ψ_X and Ψ_Y, denoted by $\Psi_X \cup \Psi_Y$, defined by $\mu_{X \cup Y}(x) = s(\mu_X(x), \mu_Y(x))$ and $\eta_{X \tilde{\cup} Y}(x) = \eta_X(x) \tilde{\cup} \eta_Y(x)$ where s is an s - norm and $\tilde{\cup}$ is an interval-valued fuzzy union.

Example: 4.12

Consider Ψ_X as in Example 4.2 and let Ψ_Y be another $FPIVFSS(U)$ defined as follows:

$$\begin{aligned}\Psi_Y = \{ & (x_1/0.4, \{u_1/[0.3, 0.6], u_2/[0.5, 0.7], u_3/[0.4, 0.5], u_4/[0.1, 0.3]\}), \\ & (x_2/0.7, \{u_1/[0.3, 0.5], u_2/[0.0, 0.2], u_3/[0.5, 0.6], u_4/[0.4, 0.6]\}), \\ & (x_3/0.5, \{u_1/[0.4, 0.6], u_2/[0.2, 0.4], u_3/[0.1, 0.3], u_4/[0.5, 0.6]\}), \\ & (x_4/0.2, \{u_1/[0.5, 0.6], u_2/[0.3, 0.5], u_3/[0.3, 0.4], u_4/[0.2, 0.3]\})\}\end{aligned}$$

By using the basic fuzzy union (maximum) and the interval-valued fuzzy union we have

$$\Psi_X \cup \Psi_Y = \{ (x_1/0.3, \{ u_1/[0.3, 0.6], u_2/[0.7, 0.8], u_3/[0.5, 0.8], u_4/[0.2, 0.5] \}), (x_2/0.7, \{ u_1/[0.3, 0.7], u_2/[0.2, 0.3], u_3/[0.5, 0.6], u_4/[0.4, 0.6] \}), (x_3/0.5, \{ u_1/[0.4, 0.6], u_2/[0.4, 0.6], u_3/[0.3, 0.8], u_4/[0.5, 0.6] \}), (x_4/1.0, \{ u_1/[0.5, 0.7], u_2/[0.3, 0.5], u_3/[0.5, 0.6], u_4/[0.8, 0.9] \}) \}$$

Theorem: 4.13

Let Ψ_X, Ψ_Y and Ψ_Z be any three $FPIVFSS(U)$. Then the following results hold:

1) $\Psi_X \cup \Psi_X = \Psi_X$
2) $\Psi_{\phi_X} \cup \Psi_X = \Psi_X$
3) $\Psi_\phi \cup \Psi_X = \Psi_X$
4) $\Psi_X \cup \Psi_{\tilde{E}} = \Psi_{\tilde{E}}$
5) $\Psi_X \cup \Psi_X = \Psi_Y \cup \Psi_X$

Proof:

The proof is straightforward.

Definition: 4.14

Let Ψ_X and $\Psi_Y \in FPIVFSS(U)$. The **Intersection** of Ψ_X and Ψ_Y, denoted by $\Psi_X \cap \Psi_Y$, defined by $\mu_{X \cap Y}(x) = t\ (\mu_X(x), \mu_Y(x))$ and $\eta_{X \tilde{\cap} Y}(x) = \eta_X(x) \tilde{\cap} \eta_Y(x)$ where t is an t- norm and $\tilde{\cap}$ is an interval-valued fuzzy intersection.

Example: 4.15

Consider Example 4.12 again. By using the basic fuzzy intersection (minimum) and the interval-valued fuzzy intersection we have

$$\Psi_X \cap \Psi_Y = \{ (x_1/0.3, \{ u_1/[0.3, 0.6], u_2/[0.5, 0.7], u_3/[0.4, 0.5], u_4/[0.1, 0.3] \}), (x_2/0.1, \{ u_1/[0.0, 0.6], u_2/[0.0, 0.2], u_3/[0.3, 0.5], u_4/[0.1, 0.3] \}), (x_3/0.4, \{ u_1/[0.1, 0.4], u_2/[0.2, 0.4], u_3/[0.1, 0.3], u_4/[0.4, 0.5] \}), (x_4/0.2, \{ u_1/[0.5, 0.6], u_2/[0.1, 0.4], u_3/[0.0, 0.4], u_4/[0.2, 0.3] \}) \}$$

Theorem: 4.16

Let Ψ_X, Ψ_Y and Ψ_Z be any three $FPIVFSS(U)$. Then the following results hold:

1) $\Psi_X \cap \Psi_X = \Psi_X$
2) $\Psi_{\phi_X} \cap \Psi_X = \Psi_X$
3) $\Psi_\phi \cap \Psi_X = \Psi_X$
4) $\Psi_X \cap \Psi_{\tilde{E}} = \Psi_{\tilde{E}}$
5) $\Psi_X \cap \Psi_X = \Psi_Y \cap \Psi_X$

Proof:

The proof is straightforward.

Theorem: 4.17

Let Ψ_X, Ψ_Y be any two $FPIVFSS(U)$. Then De-Morgan's law is valid:

1) $(\Psi_X \cup \Psi_Y)^C = \Psi_X^C \cap \Psi_Y^C$
2) $(\Psi_X \cap \Psi_Y)^C = \Psi_X^C \cup \Psi_Y^C$

Proof:

1) For all $x \in E$,

$$\begin{aligned}
\mu_{(X \cup Y)^c}(x) &= c(\mu_{(X \cup Y)}(x)) \\
&= c\,(s\,(\mu_X(x)\,,\,\mu_Y(x))\,) \\
&= t\,(\,c\,(\mu_X(x))\,,\,c\,(\mu_Y(x))\,) \\
&= t\,(\,\mu_{X^c}(x)\,,\,\mu_{Y^c}(x)\,) \\
&= \mu_{X^C \cap Y^c}(x)
\end{aligned}$$

and

$$\eta_{(X \tilde{\cup} Y)^{\tilde{c}}}(x) = \tilde{c}\ (\eta_{(X \tilde{\cup} Y)}(x))$$

$$= \tilde{c}\ (\eta_X(x) \tilde{\cup} \eta_Y(x))$$

$$= \tilde{c}\ (\eta_X(x)) \tilde{\cap} \tilde{c}\ (\eta_Y(x))$$

$$= 1 - (\eta_X(x)) \tilde{\cap} 1 - (\eta_Y(x))$$

$$= \eta_{X^{\tilde{c}}}(x) \tilde{\cap} \eta_{Y^{\tilde{c}}}(x)$$

$$= \eta_{X^{\tilde{c}} \tilde{\cap} Y^{\tilde{c}}}(x)$$

2) Likewise, the proof of (2) can be made similarly.

Theorem: 4.18

Let Ψ_X, Ψ_Y and Ψ_Z be any three $FPIVFSS(U)$. Then the following results hold:

1) $\Psi_X \cup (\Psi_Y \cap \Psi_Z) = (\Psi_X \cup \Psi_Y) \cap (\Psi_X \cup \Psi_Z)$
2) $\Psi_X \cap (\Psi_Y \cup \Psi_Z) = (\Psi_X \cap \Psi_Y) \cup (\Psi_X \cap \Psi_Z)$

Proof:

1) For all $x \in E$,

$$\mu_{X \cup (Y \cap Z)}(x) = s\ (\ \mu_X(x)\ ,\ t\ (\mu_Y(x)\ ,\ \mu_Z(x))\ )$$

$$= t\ (\ s\ (\mu_X(x)\ ,\ \mu_Y(x))\ ,\ s\ (\mu_X(x)\ ,\ \mu_Z(x)\ )\ )$$

$$= \mu_{(X \cup Y) \cap (X \cup Z)}(x)$$

and

$$\eta_{X \tilde{\cup} (Y \tilde{\cap} Z)}(x) = \eta_X(x) \tilde{\cup} (\eta_Y(x) \tilde{\cap} \eta_Z(x))$$

$$= (\eta_X(x) \tilde{\cup} \eta_Y(x)) \tilde{\cap} (\eta_X(x) \tilde{\cup} \eta_Z(x))$$

$$= \eta_{(X \tilde{\cup} Y) \tilde{\cap} (X \tilde{\cup} Z)}(x)$$

2) Likewise, the proof of (2) can be made similarly.

Definition: 4.19

Let $\Psi_X \in FPIVFSS(U)$. Then a ***fpivfs*** **–aggregation operator**, denoted by $FPIVFS_{agg}$, is defined by

$$FPIVFS_{agg} : F(E) \times FPIVFSS(U) \rightarrow Int(U)$$

$$FPIVFS_{agg}(X, \Psi_X) = \Psi_X^* \text{ where } \Psi_X^* = \{u / \mu_{\Psi_X^*}(u) : u \in U\}$$

which is an interval-valued fuzzy set over U. The value Ψ_X^* is called an aggregate interval-valued fuzzy set of Ψ_X. Here, the membership degree $\mu_{\Psi_X^*}(u)$ is defined as follows:

$$\mu_{\Psi_X^*}(u) = \left[c^- = \frac{1}{|E|} \sum_{x \in E} \mu_X(x) \mu_{\eta_X^-(x)}(u) \text{ , } c^+ = \frac{1}{|E|} \sum_{x \in E} \mu_X(x) \mu_{\eta_X^+(x)}(u) \right]$$

where $|E|$ is the cardinality of E.