Some Interesting Results
On Topological Groups

By

Vijayalakshmi N.

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND
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N.VIJAYA LAKSHMI

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K.N. Meenakshi
Signature of the Head
of the department
11.5.98.

G. K Chandrika.
Signature of the Guide

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Introduction

INTRODUCTION

The study of general topological groups was initiated by O. Schreier in 1926[10]. From then onwards much work has been carried out on this topic and a good theory of topological groups has been developed. As a generalisation of topological groups, the notion of theta topological groups is introduced by E.J. Beggs and E. Hatir [1]. In a topological group G we require that the multiplication m : (x, y) → xy and inversion $i : x \to x^{-1}$ are continuous whereas in a theta topological group, we need them to be merely theta continuous.

A different study leads to the introduction of quasi-topological groups wherein we need only the multiplication to be continuous. Many authors have contributed to the study of quasi-topological groups and they were mainly interested in obtaining conditions under which a quasi-topological group is a topological group. J. Marin and S. Romaguera [6] have investigated quasi-topological groups from "a bitopological point".

In this thesis, the following two papers are chosen for discussion:

(1) "***Theta topological groups***" by E.J. Beggs and E. Hatir [1].

(2) "***A bitopological view of quasi-topological groups***" by J. Marin and S. Romaguera [6] .

In chapter I we have collected the fundamental definitions and results on topological groups from "***General Topology***" by N. Bourbaki [2] and "***Topology for Analysis***" by A. Wilansky [14]. Many of the results of N. Bourbaki and A. Wilansky have been extended to theta topological groups and quasi-bitopological groups.

Chapter II deals with the concept of theta topological groups. We discuss here the results of E.J. Beggs and E. Hatir [1]. In section 2.1, we have collected the preliminary definitions and results regarding theta continuity. In section 2.2 we discuss some interesting results on the category of theta topological groups. With every topology τ on a set X, a new coarser topology τ' is associated and the map C: $(X, \tau) \rightarrow (X, \tau')$ gives a functor from the category of theta topological spaces and theta continuous mappings to the category of topological spaces and continuous mappings. It has been proved that if G is a theta topological group then CG is a topological group. Section 2.3 is devoted to the study of quotient theta topological groups. With every topological group G, a normal subgroup $H(G) = \cap \{ \overline{U} / U$ is a neighbourhood of $e\}$, is associated and interesting properties of the quotient group $\frac{G}{H(G)}$ are analysed. They [1] have shown that

(1) $\frac{G}{H(G)}$ is Hausdorff.

(2) The group $\frac{G}{H(G)}$ with the quotient topology is a theta topological group.

In section 2.4, we discuss two interesting examples of theta topological groups which are not topological groups. In the first example, inversion is continous but multiplication is not continuous whereas in the second one, multiplication is continuous but inversion is not continuous.

Chapter III is devoted to the study of quasi-bitopological groups. The fact that the topology τ of a quasi-topological group (G, τ) generates a conjugate topology $\tau^{-1} = \{A \subseteq G / A^{-1} \in \tau\}$ has led to the study of quasi-topological groups from "a bitopological view point". J. Marin and S. Romaguera [6] have contributed to this study and have obtained appropriate extensions of classical theorems on topological groups. In section 3.1, we give the preliminary definitions and results on quasi-bitopological groups and also their connections with absolute quasi-valued functions and quasi pseudo-metrics. Section 3.2. deals with quasi-uniformities on quasi-bitopological groups. Among other results, J. Marin and S. Romaguera [6] have proved the following which are extensions of classical results of A. Wilansky [14].

(1) Every quasi-bitopological group is quasi-uniformizable.

(2) Every first countable quasi-bitopological group admits a compatible left invariant quasi-pseudo-metric.

(3) If (G, τ, τ^{-1}) is a first countable quasi-bitopological group and if d is a left (right) invariant quasi-pseudometric on G such that $\tau = \tau(d)$, then the B-quasi-pseudometric D associated to d $\left(D(x, y) = d(x, y) + d(y^{-1}, x^{-1})\right)$ induces the two-sided quasi-uniformity for (G, τ, τ^{-1}).

A classical result of A. Wilansky [14] states that for each Hausdorff topological group G there exists a Hausdorff topological group which is complete in its two-sided uniformity and has G as a dense topological group. The main result of section 3.4, extends this theorem to quasi-bitopological groups.

Review of Literature

REVIEW OF LITERATURE

O.Schreier [10], in 1926, introduced the notion of topological groups. From then many authors have published interesting results on topological groups. The notion of theta-topological groups was introduced by E .J.Beggs and E .Hatir [1] . Quasi - topological groups (paratopological groups in the terminology of N.Bourbaki [2]) have been investigated by several authors. They were mainly interested in obtaining conditions under which a quasi-topological group is a topological group. Relevant contributions in this directions may be found in the works of K .Numakura [8],A.D.Wallace [14], T.G.Raghavan and Ivan L.Reilly [9],etc.J.Marin and S.Romaguera [6] have contributed to the study of quasi - bitopological groups and obtained interesting extensions of classical theorems of A.Wilansky [14] and N. Bourbaki [2]. In this chapter we give a brief survey of some of the interesting articles on topological groups published by various authors.

On the continuity of group operations

[T.G. Raghavan and Ivan L.Reilly, 1978] [9]

A group (G, .) endowed with a topology τ is called a semitopological (quasi - topological) group if and only if the group operation $(x,y)\rightarrow xy$ is continuous in each variable separately (both variable jointly). It is shown in this paper that a T_1 quasi-topological group $(G,\tau , .)$ is a topological group if it satisfies any one of the following conditions:

(1) τ is compact ;

(2) τ is countably compact and the identity element is a G_δ ;

(3) τ is a Lindelof P-space;

(4) τ is sequential and sequentially compact.

Infact,under condition (2),the quasi topological group becomes a compact metric topological group..

Closed images of topological groups

[Van Mill,J. 1985] [12]

It is well known that every space is an open image of a homogeneous space. This suggests the question;whether every space is a closed image of a homogeneous space. The authors have shown that every seperable metrizable space is closed image of a seperable metrizable topological group. In addition they have also proved that there are homogeneous Borel subsets of R that are not a perfect images of a topological group.

Topological groups acting on sets

[EL-Gendy,M.A. and Mostafa, F.E,1986] [4]

If the topological group G acts on the non-empty set X,then a topology designated by τ on X can be defined through a closure operator by using the action $(g,x) \rightarrow gx$. The authors give a list of properties of τ. Showing among other-things that it is the final topology generated by the family $\{\pi_x: x\in X\}$ where $\pi_x(g) = gx$. Further it is proved that if G is T_0 and first countable, then X x X, endowed with the topology $\tau \times \tau$ has a G_δ diagonal.

On the quasi components of regular topological semi groups

[Ursul, M.I and Yunusov, A.S, 1987] [11]

In a topological semi group S, for every $s \in S$ the component and the quasi component of S are denoted by C_s and Q_s respectively. The authors have shown that if S has a zero element 0 then the quasi component Q_0 is transfinitely r-nilpotent modulo the component C_0. This fact implies the following:

If S is also regular then $Q_0 = C_0$. If S is suchthat forevery $s \in S$ there exists an integer $n>1$ forwhich $S^n = S$ then again $Q_0 = C_0$. In the case that S is homogeneous and regular then

$Q_s = C_s$ forevery $s \in S$. For a topological space X let S(X) denote the semigroup of all continuous selfmaps f: $X \rightarrow X$ and $S_2(X)$ its sub-semigroup consisting of those f for which the cardinality of f(X) is ≤ 2. It is proved that if K is any sub-semigroup of S(X) containing $S_2(X)$ then for any τ_2-topology on K with respect to which K becomes a semi topological semi group, the quasi components in K are one-point sets.

Uniformities and uniform continuity on topological groups

[Itzkowitz and Gerald, L. 1991] [5]

In this paper, the authors have tried to answer the following question.

"On a T_0 topological group, are the right and left uniformities equivalent if and only if every real-valued left uniformly continuous function is right uniformly continuous?."

They have described a method of solution of this problem for the case of a locally compact group. In this the answer is in the affirmative.

Chapter I

Chapter - I

TOPOLOGICAL GROUPS

In this chapter we have collected fundamental definitions and results on topological groups from "General topology", by Bourbaki and "Topology for Analysis", by Wilansky.

Section 1.1

Results of Bourbaki [2]

Let (G, .) be a group, and 'e' denote the identity element of G and for each $x \in G$, x^{-1} denote the inverse of x. If $x, y \in G$ we will write xy instead of x.y and G instead of (G, .).

For $A, B \subseteq G$, we write $AB = \{ ab: a \in A, b \in B\}$ and $A^{-1} = \{ a^{-1} : a \in A\}$

Definition 1.1.1

A ***topological group*** is a set G which carries a group structure and a topology and satifies the following two axioms :

(GT_I) The mapping 'm' : $(x, y) \rightarrow xy$ of G x G into G is continuous.

(GT_{II}) The mapping 'i' : $x \rightarrow x^{-1}$ of G into G is continuous.

The mappings 'm' and 'i' are refered to as multiplication and inversion respectively.

A group structure and a topology on a set G are said to be ***compatible*** if they satisfy (GT_I) and (GT_{II}).

Theorem 1.1.2

Axioms (GT_I) and (GT_{II}) are equivalent to the following:

(GT') The mapping $(x, y) \rightarrow x\hat{y}$ of G x G into G is continuous.

Clearly (GT_I) and (GT_{II}) together imply (GT'). Conversely, (GT') implies (GT_{II}), for $x \rightarrow e\,x^{-1} = x^{-1}$ is then continuous; and (GT') and (GT_{II}) together imply (GT_1), for $(x, y) \rightarrow x(y^{-1})^{-1} = xy$ is then continuous.

Definiton 1.1.3

Let (G, τ) be a quasi topological group and $a \in G$. Consider the map $L_a : G \rightarrow G$ defined by $L_a(x) = ax$. Then the map L_a is called the ***left translation*** by a. The map $R_a : G \rightarrow G$ defined by $R_a(x) = xa$ is called the ***right translation*** by a.

By (GT_I) both L_a and R_a are continuous and hence are homeomorphims of G onto itself. The mappings $x \rightarrow a\,x\,b$, as a and b run through G, hence form a ***group of homeomporphisms*** of G ; the mapping $x \rightarrow axa^{-1}$ (respectively $x \rightarrow ax$, $x \rightarrow xa$), where a runs through G, form a subgroup of this group of homeomorphisms. Also axoim (GT_{II}) shows that $x \rightarrow x^{-1}$ is a homeomorphism of G onto G.

Properties 1.1.4

(i) If A is an open (respectively closed) subset of G, and if x is any point of G, then the sets xA, Ax and A^{-1} are open (respectively closed), because they are the images of A under the homeomorphisms-left translation, right translation and inversion 'i' respectively.

(ii) If A is open and if B is any subset of G, then AB and BA are open, because they are unions of open set (Since $AB = \cup\{ A\,x / x \in B\}$ and $BA = \cup\{ x\,A / x \in B\}$.

(iii) If V is any neighbourhood of 'e' in G, then VA and AV are neighbourhoods of A. For if W is an open neighbourhood of e contained in V, then WA and AW are open and contained in A.

(iv) Let $\mathcal{N}(G)$ be the neighbourhood filter of the identity element 'e' in a topological group G and let 'a' be any point of G. Since L_a and R_a are homeomorphisms, it follows that the neighbourhood filter of 'a' is the family

$a\mathcal{N}(G) = \{aV / V \in \mathcal{N}(G)\}$

$\mathcal{N}(G)a = \{ Va / V \in \mathcal{N}(G)\}$

Thus we know the neighbourhood filter of any point of a topological group as soon as we know the neighbourhood filter of the identity element e of the group.

(v) Every filter $\mathcal{B}$ on G, which satifies (GV_I) and (GV_{II}) must satisfy (GV_a), where

(GV_I) Given any $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $VV \subset U$.

(GV_{II}) Given any $U \in \mathcal{B}$, $U^{-1} \in \mathcal{B}$.

(GV_a) Given $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $VV^{-1} \subset U$.

Conversely if a filter $\mathcal{B}$ on G satifies(GV_a)then it must satisfy (GV_I) and (GV_{II}).

(vi) Since $x \to axa^{-1}$ is a homeomorphism which leaves e fixed, $\mathcal{N}(G)$ has the following property.(GV_{III}). For all $a \in G$ and for all $V \in \mathcal{N}(G)$, $aVa^{-1} \in \mathcal{N}(G)$.

(vii) Let G be a group and $\mathcal{B}$ be afilter on G satisfying the axioms (GV_I), (GV_{II}) and (GV_{III}). Then there is a unique topology on G, compatible with the group structure of G, for which $\mathcal{B}$ is the neighbourhood filter of the identity element e. For this topology the neighbourhood filter of any point $a \in G$ is the same as each of the two filters a $\mathcal{B}$ and $\mathcal{B}$ a.

(viii) A topological group G is Hausdorff ⇔ the set $\{e\}$ is closed.

(ix) A topological group G is Hausdorff ⇔ the intersection of the neighbourhood of e consists of the point e.

Quotient topological group

Let G be a topological group and let H be a normal subgroup of G. Then the quotient topology of $\frac{G}{H}$ is compatible with the group structure of $\frac{G}{H}$.

Properties 1.1.5

(i) Let π be the canonical mapping of a topological group G onto the quotient group $\frac{G}{H}$. If $\mathcal{B}$ is a fundamental system of neighbourhoods of e in G, then $\pi(\mathcal{B})$ is a fundamental system of neighbourhoodsof e in G, then $\pi(\mathcal{B})$ is a fundamental system of neighbourhoods of the identity element $\pi(e)$ in $\frac{G}{H}$.

(ii) The quotient group $\frac{G}{H}$ is Hausdorff ⇔ H is closed in G.

(iii) The quotient group $\frac{G}{H}$ is discrete $\Leftrightarrow$ H is open in G.

Definition 1.1.6

An ***isomorphism*** f of a topological group G onto a topological group G′ is a bijective mapping of G onto G′ which is simultaneously an isomorphism of the group structure G onto G′.

In other words, f is an isomorphism of G onto G′ $\Leftrightarrow$

(i) f is bijective,

(ii) $f(xy) = f(x)\, f(y)$ for all $x, y \in G$

(iii) f is bicontinuous

Uniform Spaces

Definition 1.1.7

A ***filter*** on a set X is a collection $\mathcal{F}$ of subsets of X which has the following properties;

(1) Every subset of X which contains a set of $\mathcal{F}$ belongs to $\mathcal{F}$.

(2) Every finite intersection of sets of $\mathcal{F}$ belongs to $\mathcal{F}$.

(3) The empty set is not in $\mathcal{F}$.

Definition 1.1.8

Let X be a topological space and $\mathcal{F}$ a filter on X. A point $x \in X$ is said to be a ***limit point*** of $\mathcal{F}$, if $\mathcal{F}$ is finer than the neighbourhood filter $\mathcal{B}(x)$; $\mathcal{F}$ is also said to ***converge to x***.

Definition 1.1.9

A ***uniform structure*** on a set X is a structure given by a filter $\mathcal{U}$ of subsets of X x X which satisfies the following axioms:

(U_I) Every set belonging to $\mathcal{U}$ contains the diagonal $\Delta = \{(x, x) / x \in X\}$

(U_{II}) If $V \in \mathcal{U}$ then $V^{-1} \in \mathcal{U}$, where $V^{-1} = \{ (x, y) \in X xX / (y, x) \in V\}$

(U_{III}) For each $V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ such that $WW \subset V$ where

$WW = \{ (x, y) \in X \times X$ / there is $z \in X$ such that $(x, z) \in W$ and $(z, y) \in W\}$.

The members of $\mathcal{U}$ are called ***entourages*** of the uniformity defined on X by $\mathcal{U}$. A set endowed with a uniformity is called a ***uniform space.***

If V is an entourage of a uniformity on X, we say x and x′are ***V-close*** if $(x, x') \in V$.

Definition 1.1.10

A ***fundamental system of entourages*** of a uniformity is any set $\mathcal{B}$ of entourages such that every entourage contains a set belonging to $\mathcal{B}$.

Axiom (U_{III}) shows that, if n is any integer > 0 and V runs through a fudamental system of entourages, then the sets V^n again form a fundamental system of entourages.

Definition 1.1.11

Entourages V such that $V = V^{-1}$ are called ***symmetric entourages.***

Note 1.1.12

If V is any entourage, then $V \cap V^{-1}$ and $V \cup V^{-1}$ are symmetric entourages and from (U_{II}) we see that symmetric entourages form a fundamental system of entourages .

Topology induced by a uniform structure $\mathcal{U}$.

Let X be a set endowed with a uniform structure $\mathcal{U}$.

Let $V[x] = \{ y \in X / (x, y) \in V\}$. For each $x \in X$ let $\mathcal{B}(x)$ be the set of subsets V[x] of X, where V runs through the set of entourages of $\mathcal{U}$. Then there is topology on X such that, for each $x \in X$, $\mathcal{B}(x)$ is the neighbourhood filter of x in this topology. This topology is called the ***topology induced by the uniform structure*** $\mathcal{U}$.

Definition 1.1.13

If X is a uniform space and if V is an entourage of X, a subset A of X is said to be **V-small** if every pair of points of A are V-close (i.e , $A \times A \subset V$)

Definition 1.1.14

A filter $\mathcal{F}$ on a uniform space X is a ***Cauchy filter*** if for each entourageV of X there is a subset of X which is V-small and belongs to $\mathcal{F}$. The minimal elements of the set of cauchy filters on a uniform space X are called ***minimal Cauchy filters*** on X.

Definition 1.1.15

A ***complete space*** is a uniform space in which every cauchy filter converges.

Definition 1.1.16

A mapping f of a uniform space X into a uniform space X′ is said to be ***uniformly continuous*** if, for each entourage V′ of X′, there is an entourage V of X such that the relation $(x, y) \in V$ implies $(f(x), f(y)) \in V'$. In other words, if $g = f \times f$, then the above definition means that whenever V′ is an entourage of X′, $g^{-1}(V')$ is an entourage of X.

Completion of a uniform space

Theorem 1.1.17

Let X be a uniform space. Then there exists a complete Hausdorff uniform space $\hat{X}$ and a uniformly continuous mapping $i : X \to \hat{X}$ having the following property :

Given any uniformly continuous mapping f of X into a complete Hausdorff uniform space Y, there is a unique uniformly continuous mapping $g : \hat{X} \to Y$ such that $f = g \circ i$.

If (i_1, X_1) is another pair consisting of a complete Hausdorff uniform space X_1 and a uniformly continuous mapping $i_1 : X \to X_1$ having the property (1), then there is a unique isomorphism $\pi : \hat{X} \to X_1$ such that $i_1 = \pi \circ i_1$.

Proof

We give here only the construction of $\hat{X}$ and i which satisfy the required conditions, proof is omitted.

Definition of $\hat{X}$ *1.1.18*

Let $\hat{X}$ be the set of minimal cauchy filters on X. Now we define a uniform structure on $\hat{X}$.

If V is any symmetric entourage of X, let $\tilde{V}$ denote the set of all pairs $(\mathcal{F}, \mathcal{G})$ of minimal cauchy filters which have in common a V-small set. The sets $\tilde{V}$ form a fundamental system of entourages of a uniform structure on $\hat{X}$ and $\hat{X}$ is Hausdorff.

Definition of 'i' 1.1.19

For each $x \in X$, since the neighbourhood filter $\mathcal{N}(x)$ of x in X is a minimal cauchy filter, it is quite natural to define $i(x) = \mathcal{N}(x)$. Then

(1) the uniform sturcture of X is the inverse image under i of that of $\hat{X}$.

(2) i(X) is dense in $\hat{X}$.

(3) $\hat{X}$ is complete

Definition 1.1.20

The complete Hausdorff uniform space $\hat{X}$ defined above is called the ***Hausdorff Completion of X.***

Uniform Structures on groups

Definition 1.1.21

The right (respectively left) uniformity on a topological group G is the uniformity for which the fundamental system of entourages is obtained by making correspond to each neighbourhood V of the identity element e, the set V_d(respectively V_s) of pairs (x, y) such that $yx^{-1} \in V$ (respectively $x^{-1}y \in V$).

Let $\mathcal{G}_d = \{ V_d / V \in \mathcal{N}(G)\}$

Then $\mathcal{G}_d$ is a fundamental system of entourages of the right uniformity which is compatible with the topology of G (since $V_d(x) = Vx$)

Notation 1.1.22

We denote by G_d the uniform space obtained by giving G its right uniformity and $\hat{G}_d$ its completion (as described in theorem 1.1.17).

Completion of a topological group

Let G be a Hausdorff topological group. The uniform space G_d may be considered as a dense subspace of its completion $\hat{G}_d$.

$$\text{(i.e)}\ \overline{G_d} = \hat{G}_d$$

The following theorem gives the condition under which a Hausdorff topological group can have its completion.

Theorem 1.1.23

A Hausdorff topological G is isomorphic to a dense subgroup of a complete group $\hat{G}_d$ if and only if the image under the symmetry $x \to x^{-1}$, of a cauchy filter with respect to the right uniformity of G is a cauchy filter with respect to this uniformity.

The complete group $\hat{G}_d$ is then unique.

Section 1.2

Results of Wilansky [14]

Definition 1.2.1

Let G be a group. Then a function $\rho : G \to R$ is called an ***absolute value function*** provided it satisfies following conditions:

(1) $\rho(x) \geq 0$, for all $x \in G$ and $\rho(e) = 0$

(2) $\rho(x^{-1}) = \rho(x)$ for all x.

(3) $\rho(xy) \leq \rho(x) + \rho(y)$ for all x, y.

(4) $\rho(x_n) \to 0 \Rightarrow \rho(a\, x_n\, a^{-1}) \to 0$ for all $a \in G$.

Theorem 1.2.2

Let G be a group and let ρ be an absolute value function on G. Then there is a semi-metric d on G for which $(G, \tau(d))$ is a topological group ($\tau(d)$ has basic open neighbourhoods of $x \in G$ of the form $B_d(x, r) = \{ y \in G / d(x, y) < r\}$).

Proof

Define $d(x,y) = \rho(x^{-1}y)$

(1) since $\rho(x) \geq 0$ for all x, $d(x, y) \geq 0$. Consider $d(x, x) = \rho(x^{-1}x) = \rho(e) = 0$.

(2) $d(x, y) = \rho(x^{-1}y) = \rho((x^{-1}y)^{-1}) = \rho(y^{-1}x) = d(y, x)$

(3) consider

$$\begin{aligned} d(x, z) &= \rho(x^{-1}z) = \rho(x^{-1}y\, y^{-1}z) \\ &\leq \rho(x^{-1}y) + \rho(y^{-1}z) \\ &= d(x, y) + d(y, z) \end{aligned}$$

Hence d is a semi-metric.

To prove

multiplication 'm' : G x G $\rightarrow$ G is continuous. It is enough to prove $x_n \rightarrow x$ and $y_n \rightarrow y \Rightarrow x_n y_n \rightarrow xy$.

Consider $d(x_n y_n, xy) = \rho(y_n^{-1} x_n^{-1} xy)$

$$= \rho(y_n^{-1} y\, y^{-1} x_n^{-1} xy)$$

$$\leq \rho(y_n^{-1} y) + \rho(y^{-1} x_n^{-1} xy)$$

Since $x_n \rightarrow x$, $y_n \rightarrow y$, $d(x_n, x) \rightarrow 0$ and $d(y_n, y) \rightarrow 0$

$$\Rightarrow \rho(x_n^{-1} x) \rightarrow 0$$

$$\Rightarrow \rho(y^{-1} x_n^{-1} xy) \rightarrow 0$$

Similarly $\rho(y_n^{-1} y) \rightarrow 0$

$$\Rightarrow d(x_n y_n, xy) \rightarrow 0$$

$$\Rightarrow x_n y_n \rightarrow xy$$

Next to prove

i : G $\rightarrow$ G is continuous

$x_n \rightarrow x \Rightarrow d(x_n, x) \rightarrow 0$

$$\Rightarrow \rho(xx^{-1} x_n x^{-1}) \rightarrow 0$$

Now consider $d(x_n^{-1}, x^{-1}) = \rho(x_n x^{-1})$

$$= \rho(xx^{-1} x_n x^{-1}) \rightarrow 0$$

(i.e) $x_n^{-1} \rightarrow x^{-1}$

Hence inversion is continuous.

As a converse of the above theorem we get the following result.

Theorem 1.2.3

Let G be a group and d, a left invariant semi-metric on G such that $(G, \tau(d))$ is a topological group. Then the function ρ defned on G by $\rho(x) = d(x, e)$ is an absolute value function.

Proof

(1) $\rho(x) = d(x, e) \geq 0$

$\rho(e) = d(e,e) = 0$

(2) $\rho(x^{-1}) = d(x^{-1}, e)$

$= d(x x^{-1}, xe)$ (since d is left invariant)

$= d(e,x)$

$= d(x, e)$

$= \rho(x)$

(3) $\rho(xy) = d(xy, e)$

$= d(y, x^{-1})$

$\leq d(y, e) + d(e, x^{-1})$

$= \rho(y) + \rho(x^{-1})$

$= \rho(y) + \rho(x)$

(4) $\rho(x_n) \to 0 \Rightarrow d(x_n, e) \to 0$

$\Rightarrow x_n \to e$

$\Rightarrow a x_n a^{-1} \to a e a^{-1}$

$\Rightarrow \rho(a x_n a^{-1}) \to 0$

The following result helps to decide which collection of subsets are eligible for the collection of neighbourhoods of e.

Theorem 1.2.4

Let G be a group and τ a topology for G. Then the map $f_1 : x \rightarrow x^{-1}$ is continuous at e $\Leftrightarrow$ whenever U is a neighbourhood of e, U^{-1} is also a neighbourhood of e. The map $f_2 : (x, y) \rightarrow xy$ is continuous at $(e, e) \in G \times G$ if and only if, for every neighbourhood U of e, there exists a neighbourhood V of e such that $VV \subset U$.

The following theorem gives a set of conditions under which a topology τ on a group G, is compatible with the group structure on G.

Theorem 1.2.5

Let G be a group and τ a topology for G. Then τ is compatible with the group structure on G if and only if

(1) every left translate of an open set is open

(2) for every neighbourhood U of e, U^{-1} is a neighbourhood of e.

(3) for each neighbourhood U of e, there exists a neighbourhood V of e such that $VV \subset U$

(4) for each neighbourhood U of e and $a \in X$, there exists a neighbourhood V of e with $aVa^{-1} \subset U$

Proof

First assume that τ is a topology on G

(1) Since left translation is a homeomorphisms of G onto itself we get that every left translation of an open set is an open set

(2) and (3) are true. (From theorem 1.2.4)

(3) Let U be a neighbourhood of e, and $a \in X$. Hence there exists a neighbourhood V of e such that this map carries V into U.

(i.e) $aVa^{-1} \subset U$.

Conversely assume τ obeys the four conditions. The group operation $(x, y) \rightarrow xy$ is continuous at $(e, e) \in X \times X$. (by theorem 1.1.14)

To prove continuity at all other points, first note that for any net (x_δ)

(i) $x_\delta \rightarrow e \Rightarrow a x_\delta a^{-1} \rightarrow e$, for all a and

(ii)

(1) $x_\delta \rightarrow x$,

(2) $x^{-1} x_\delta \rightarrow e$

(3) $x_\delta x^{-1} \rightarrow e$ are equivalent

$x_\delta \rightarrow x$, $y_\delta \rightarrow y$

$\Rightarrow x_\delta y_\delta (xy)^{-1} = x (x^{-1} x_\delta) (y_\delta y^{-1}) x^{-1} \rightarrow e$

This is from (i) and (ii) and from the fact that the operation is continuous at (e, e). Also by(2), we get $x_\delta y_\delta \rightarrow xy$. Next to show the inverse operation is continuous ,

Let $x_\delta \rightarrow x$, then $x x_\delta^{-1} = (x x_\delta^{-1})^{-1} \rightarrow e$

(By (ii) and by the fact that the the inverse operation is continuous at e, (by theorem 1.1.12)

Once again by (ii) $x_\delta^{-1} \rightarrow x^{-1}$

$\therefore$ The inverse operation is continuous at every point.

Theorem 1.2.6

Let G be a group and $\mathcal{F}$ a collection of subsets of X such that

(1) $\mathcal{F}$ is a filter base

(2) every member of $\mathcal{F}$ is symmetric

(3) for each $U \in \mathcal{F}$ there exists $V \in \mathcal{F}$ with $VV \subset U$

(4) for each $U \in \mathcal{F}$ and $a \in X$, there exists $V \in \mathcal{F}$ with $aVa^{-1} \subset U$

Then there exists a unique topology compatible with the group structure on G, such that $\mathcal{F}$ is a local base for the neighbourhood of e.

Proof

For each $U \in \mathcal{F}$,

let $U_L = \{ (x, y) : x^{-1}y \in U\}$

Since for each $U \in \mathcal{F}$, there exists $V \in \mathcal{F}$ such that $VV \subset U$ (by (3)), we get V is not empty.

Let $a \in V$, then $a^{-1} \in V$.

$\Rightarrow aa^{-1} \in U$

$\Rightarrow e \in U$

Let $\mathcal{L}' = \{ U_L : U \in \mathcal{F}\}$

It can be easily verified that $\mathcal{L}'$ and τ the topology induced by this uniformity. A set $\mathcal{N}$ is a neighbourhod of a point x $\Leftrightarrow$ there exists $U_L \in \mathcal{L}'$ with $U_L(x) \subset N$ (where $U_L(x)=\{y \in X / (x, y) \in U_L\}$)

$\Leftrightarrow$ there exists $U \in \mathcal{F}$ with $xU \subset N$. In particular, each member of $\mathcal{F}$ is a neighbour hood of e,

In other words, $\mathcal{F}$ is a base for the neighbourhood system of e.

Next to show that τ is compatible with the group structure, it is enough to check the four conditions in theorem 1.2.5.

(1) Let U be an open set.

Let $x \in aU$. Then $a^{-1}x \in U$. Hence there exists a neighbourhood V of e with $a^{-1}xV \subset U$.

$\Rightarrow xV \subset aU$ where xV is a neighbourhood of x

$\Rightarrow aU$ is a neighbourhood of x.

$\Rightarrow aU$ is open

(2) Let U be a neighbourhood of e.

$\Rightarrow V \subset U$ for some $V \in \mathcal{F}$

$\Rightarrow V^{-1} \subset U^{-1}$

$\Rightarrow V \subset U^{-1}$ (since $V^{-1} = V$)

$\Rightarrow U^{-1}$ is a neighbourhood of e

(3) Let U be a neighbourhood of e. Then $U_1 \subset U$ for some $U_1 \in \mathcal{F}$.

(i.e) There exists $V \in \mathcal{F}$ with $VV \subset U_1 \subset U$

(4) Following a similar argument as in (3) condition (4) can be found to be satisfied

Since the filter of neighbourhoods of e is uniquely determined by $\mathcal{F}$, we get the topology is unique.

We conclude this chapter by stating the following result, which is extended to quasi bitopological groups in chapter III. (Theorem 3.3.10)

Theorem 1.2.7

Let G be a Hausdorff topological group. Then there is a Hausdorff topological group Y which is complete in its two -sided uniformity and has G as a dense topological subgroup.

Chapter II

Chapter-II

THETA TOPOLOGICAL GROUPS

Beggs and Hatir [1] have introduced the notion of theta topological groups. This idea applies to several well known groups and topologies which do not form groups under the usual definition. In section 1 we have collected the results on the category of theta topological groups and theta continuous functions. In section 3, we study in detail quotient theta topological groups. In the last section we discusss two interesting examples of theta topological groups which are not topological groups.

Section 2.1

Theta continuity

In this section we give the definition of theta continuity and discuss some of the properties of theta - continuous mappings.

Definition 2.1.1

A function $f : X \to Y$ is ***theta - continuous*** if for any $x \in X$, and any open neighbourhood V of f(x), there is an open neighbourhood U of x such that $f(\overline{U}) \subset \overline{V}$.

Remark 2.1.2

Every continuous function is theta continuous. But the converse is not true in general. If the co-domain space is regular, then the converse is also true as seen from the following theorem.

Theorem 2.1.3

If $f: X \to Y$ is theta-continuous and Y is regular, then f is continuous.

Proof

Let $x \in X$ and let V be any neighbourhood of f(x). Since Y is regular, there is a neighbourhood W of f(x) such that $\overline{W} \subset V$. Since f is theta-continuous, there exists a neighbourhood U of x such that $f(\overline{U}) \subset \overline{W}$. Therefore $f(U) \subset f(\overline{U}) \subset \overline{W} \subset V$.

Hence f is continuous at x.

The following theorem is an immediate consequence of the definition of theta-continuity.

Theorem 2.1.4

If $f : X \to Y$ and $g : Y \to Z$ are theta-continuous, then $g \circ f$ is also theta-continuous. Hence we can form the ***category*** τ_θ of topological spaces and theta-continuous functions.

In general, it is not possible to restrict the co-domain of a theta-continuous function. This means that if $f : X \to Y$ is theta-continuous, and if the image f(X) is a subset $Z \subset Y$, then $f : X \to Z$ is not necessarily theta continuous. The following example illustrates this.

Example 2.1.5

Let $X = \{a,b\}$, $\tau = \{\phi, X, \{b\}\}$ and $Y = \{c,d,e\}$ with topology $\tau' = \{\phi, Y, \{d\}.\ \{c,d\}, \{d,e\}\}$. Then the function $f : X \to Y$ defined by $f(a) = c$, $f(b) = e$ is theta-continuous.

Let $Z = \{c,e\}$ with the subspace topology τ_z' (i.e., the discrete topology).

Claim

The function $f : X \rightarrow Z$ is not theta-continuous both at a and b.

Given $f(a) = c$.

Now, $V = \{c\} \in \tau_z'$ and $\overline{V} = \{c\}$.

X is the only open set in τ containing a and $\overline{X} = X = \{c,e\} \not\subseteq \overline{V}$.

Therefore f is not theta-continuous at a.

Next to check f is theta-continuous at b or not. Given $f(b) = e$. Neighbourhoods of b in X are X and $\{b\}$.

$V = \{e\}$ is a neighbourhood of $f(b)$ in τ_z'.

$\overline{V} = \{e\}$

Case (i)

If $U = X$, then $f(\overline{U}) = f(\overline{X}) = \{c,e\}$.

Therefore $f(\overline{U}) \not\subseteq \overline{V}$.

Case (ii)

If $U = \{b\}$, then $\overline{U} = X$.

Therefore $f(\overline{U}) = f(X) = \{c,e\}$

Hence $f(\overline{U}) \not\subseteq \overline{V}$. Therefore f is not theta-continuous at b.

Section 2.2

Theta-topological groups

In this section we assume that G is a group (with multiplication 'm' and inversion 'i') and a topological space. The set $\mathcal{N}(G)$ is the set of open neighbourhoods of the identify 'e' in G.

Definition 2.2.1

G is a ***theta-topological group*** if the following are satisfied :

(1) If U is open in G, and $g \in G$, then both gU and Ug are open.

(2) Multiplication 'm' : $G \times G \to G$ is theta-continuous

(3) Inversion 'i' : $G \to G$ is theta-continuous.

Note 2.2.2

(1) implies that the left and right translations on G by a fixed element g is a homeomorphism.

Consider the following conditions :

(2′) For all $V \in \mathcal{N}(G)$, there is a $U \in \mathcal{N}(G)$ such that $\overline{U}\,\overline{U} \subset \overline{V}$.

(3′) For all $V \in \mathcal{N}(G)$, there is a $U \in \mathcal{N}(G)$ such that $i\overline{U} \subset \overline{V}$.

Theorem 2.2.3

(1) and (2′) $\Rightarrow$ (2)

(1) and (3′) $\Rightarrow$ (3)

Proof

(1) and (2′) $\Rightarrow$ (2)

Take g, h $\in$ G. Let V be an open set containing gh. Then $(gh)^{-1}V \in \mathcal{N}(G)$.

By (2′) there is a U $\in \mathcal{N}(G)$ so that $\overline{U}\,\overline{U} \subset (gh)^{-1}\,\overline{V}$

Then $(gh\overline{U}h^{-1})\,(h\overline{U}) \subset \overline{V}$ ------------------- (1)

The set $ghUh^{-1}$ x hU is an open neighbourhood of (g, h) and its closure is

$$gh\overline{U}h^{-1} \text{ x } h\,\overline{U}$$

Hence (1) implies that 'm' is theta-continuous.

1) and (3′) $\Rightarrow$ (3)

Take g $\in$ G, $i(g) = g^{-1}$.

Let V be a neighbourhood of g^{-1}, then gV is a neighbourhood of e.

By (3′) there is a U $\in \mathcal{N}(G)$ such that

$i(\overline{U}) \subset g\,\overline{V}$

$g^{-1}i(\overline{U}) \subset g^{-1}g\,\overline{V}$

$i(\overline{U}g) \subset \overline{V}$

Ug is an open set containing g such that $i(\overline{U}g) \subset \overline{V}$.

Therefore 'i' is theta-continuous.

Remark 2.2.4

In view of the above theorem to check the conditions for a theta topological group it is enough to check (1), (2′) and (3′).

Definition 2.2.5

Let (X, τ) be a topological space. A new ***courser topology*** τ' is defined in the following manner:

The set $W \subset X$ is in τ' if and only if for every $x \in W$, there is an open neighbourhood U of x in τ such that $\overline{U} \subset W$.

Theorem 2.2.6

The map $(X, \tau) \rightarrow (X, \tau')$ gives a functor 'C' from τ_θ to the category τ of topological spaces and continuous functions.

Theorem 2.2.7

If U and V are open neighbourhoods of the identity in the theta topological group G, then U(i(V)) is a neighbourhood of the identity in CG.

Proof

Take any $v \in V$.

Claim

$\overline{U} v^{-1} \subset UV^{-1}$

Suppose not, there is an element yv^{-1} with $y \in \overline{U}$ such that $yv^{-1} \notin UV^{-1}$.

To prove

$yv^{-1} V \cap U = \phi$.

Suppose there exists an element $z \in yv^{-1}V \cap U$.

Then $z \in U$ and $z \in y v^{-1}V$. Therefore $z = yv^{-1}v'$ for some $v' \in V$.

Therefore $z(v')^{-1} = yv^{-1}$.

But $z(v')^{-1} \in UV^{-1}$ (Since $z \in U$ and $((v')^{-1} \in V^{-1})$

Hence $yv^{-1}V \cap UV^{-1}$ which is a contradiction. Therefore $yv^{-1} V \cap U = \phi$.

$\Rightarrow y \notin \overline{U}$ which is a contradiction.

Hence the claim.

To show

$UV^{-1} \in \tau'$

Take any element $x \in UV^{-1}$

Then $x = uv^{-1}$, $u \in U$ and $v \in V$.

Since $U \in \tau$, $Uv^{-1} \in \tau$ and $x \in Uv^{-1}$.

There fore Uv^{-1} is a neighbourhood of x, in τ.

Now, $\overline{Uv^{-1}} = \overline{U}v^{-1} \subset UV^{-1}$ (by the claim).

Therefore by the definition of τ', $UV^{-1} \in \tau'$.

i.e. $U(i(V)) \in \tau'$.

Hence $U(i(V))$ is a neighbourhood of e in CG.

Theorem 2.2.8

If G is a theta topological group then CG is a topological group. The functor $C: \tau_\theta \rightarrow \tau$ restricts to a functor from the category G_θ of theta topological groups to the category C of topological groups.

Proof

First let us show that the multiplication 'm' : CG x CG → CG is continuous.

Take $O \in \mathcal{N}(CG)$.

Then $e \in O$ and $O \in \tau'$

By the definition of τ' there exists $W \in \tau$ such that $e \in W$ and $\overline{W} \subset O$

(i.e), there exists $W \in \mathcal{N}(G)$ such that $\overline{W} \subset O$.

Since (2) $\Rightarrow$ (2′)

there exists $V \in \mathcal{N}(G)$ such that $\overline{V}\,\overline{V} \in \overline{W}$

For this V there exists $U \in \mathcal{N}(G)$ such that

$$\overline{U}\,\overline{U} \subset \overline{V}$$

$$\text{(ie), } \overline{U}\,\overline{U}\,\overline{U}\,\overline{U} \subset \overline{V}\,\overline{V}$$

$$\subset \overline{W}$$

By (3), there isa $V \in \mathcal{N}(G)$ such that $i\overline{V} \subset \overline{U}$

$$\text{i.e, } iV \subset \overline{U}$$

$$\Rightarrow UiV \subset \overline{U}\,\overline{U}$$

$$\Rightarrow (UiV)\,(UiV) \subset \overline{U}\,\overline{U}\,\overline{U}\,\overline{U}$$

$$\subset \overline{W}$$

$$\subset O$$

$\Rightarrow m(UiV \times UiV) \subset O$ (by theorem 2.2.7)

Moreover $UiV \in \mathcal{N}(CG)$

Thus ‘m’ is continuous.

Now to prove, $i : CG \rightarrow CG$ is continuous.

Take $W \in \mathcal{N}(CG)$

Then $W \in \tau'$ there exists $e \in W$

By the definition of τ' there exists $S \in \tau$ such that $e \in S$ and $\overline{S} \subset W$

By (2′) there exists $V \in \mathcal{N}(G)$ such tha $\overline{V}\,\overline{V} \subset \overline{S}$

Since i: $(G, \tau) \rightarrow (G, \tau)$ is theta continuous there is $U \in \mathcal{N}(G)$ such that $i(\overline{U}) \subset \overline{V}$

Since $U, V \in \mathcal{N}(G)$ by theorem 2.2.7

$U(i(V)) \in \mathcal{N}(CG)$

Consider $i(U(i(V))$

$$= V(i(U))$$

$$\subset \overline{V}\, i(\overline{U})$$

$$\subset \overline{V}\ \overline{V}$$

$$\subset \overline{S}$$

$$\subset W$$

Therefore the inversion $i : CG \rightarrow CG$ is continuous

Section 2.3

Quotient theta topological groups

We define a normal subgroup H(G) with respect to which the quotient theta topological group $\dfrac{G}{H(G)}$ is formed.

Some interesting properties of H(G) and $\dfrac{G}{H(G)}$ are discussed in this section.

Definition 2.3.1

If G is a theta topological group, we define ***H(G)*** to be the intersection of all $\overline{U}$, where $U \in \mathcal{N}(G)$.

(i.e), $H(G) = \cap \{ \overline{U} / U \in \mathcal{N}(G)\}$

Theorem 2.3.2

H(G) is a normal subgroup of G.

Proof

Take any $V \in \mathcal{N}(G)$. By (2′), there is a $U \in \mathcal{N}(G)$ such that $\overline{U}\,\overline{U} \subset \overline{V}$.

By definition, $H \subset \overline{U}$ for every $U \in \mathcal{N}(G)$

Therefore $H H \subset \overline{U}\,\overline{U} \subset \overline{V}$

This is true for every $V \in \mathcal{N}(G)$

Therefore $H H \subset \cap \{ \overline{V} / V \in \mathcal{N}(G)\}$

$= H$

Hence $H H \subset H$ and closure property is satisfied. Since i is theta-continuous, for every $V \in \mathcal{N}(G)$ there is a $U \in \mathcal{N}(G)$ such that $i\, \overline{U} \subset \overline{V}$

$\Rightarrow i H \subset i \overline{U} \subset \overline{V}$

This is true for every $V \in \mathcal{N}(G)$

Therefore $i H \subset H$ and existence of inverse is established.

Hence H is a subgroup of G.

Now, to check the normality condition, consider for a fixed $g \in G$, the function $A_g : G \rightarrow G$ defined by $A_g(x) = g x g^{-1}$ is theta continuous, because, if V is an open set containing $g x g^{-1}$, then $U = g^{-1} V g$ is an open set containing x and $A_g(\overline{U}) = \overline{V}$. So for any $V \in \mathcal{N}(G)$, there is a $U \in \mathcal{N}(G)$, so that $g \overline{U} g^{-1} \subset \overline{\overline{V}}$.

Therfore $gHg^{-1} \subset g\overline{U}\ g^{-1} \subset \overline{V}$, for every V.

Hence $gHg^{-1} \subset H$ and H is normal.

Theorem 2.3.3

A theta topological group G is Hausdorff if and only if $H(G) = \{e\}$

Proof

Assume G is Hausdorff

Take $x \neq e$

There exists disjoint open sets U and V such that $x \in U$ and $e \in V$ with

$U \cap V = \phi$

$\Rightarrow U \cap \overline{V} = \phi$

Therefore $x \notin \overline{V}$

$\Rightarrow x \notin \cap \{ V / V \in \mathcal{N}(G)\}$

$\Rightarrow x \notin H(G)$

Therefore $H(G) = \{e\}$

conversely assume $H(G) = \{e\}$

To Prove

G is Hausdorff.

Take $x \neq e$.

Then $x \notin H(G)$

$\Rightarrow x \notin \cap\{\overline{V} / V \in \mathcal{N}(G)\}$

$\Rightarrow$ there is a $V \in \mathcal{N}(G)$ such that $x \notin \overline{V}$.

$\Rightarrow$ there exists a neighbourhood U of x such that $U \cap V = \phi$

Thus we have found two disjoint neighbourhoods U and V of x and e respectively in G such that $U \cap V = \phi$.

Now take any two elements $x, y \in G$ with $x \neq e$, $y \neq e$ and $x \neq y$. Then $xy^{-1} \neq e$.

By the above argument there exists two open sets U and V containing xy^{-1} and e respectively with $U \cap V = \phi$.

Note that $x \in Uy$ and $y \in Vy$.

Also, $Uy \cap Vy = \phi$.

Hence Hausdorff condition is satisfied for every pair of elements x, y with $x \neq y$

Thus the space is Hausdorff.

Theorem 2.3.4

Suppose that N is a normal subgoup of a theta topoligical group G, and that $\pi : G \to G / N$ is the quotient map. Then π is an open map.

Proof

Take U open in G then $\pi U = U N = \cup\{uN / u \in N\}$. Since each uN is an open set, we get πU is open being arbitrary union of open sets.

Theorem 2.3.5

Let N be a normal subgroup of a theta topological group G, and let $\pi : G \to \frac{G}{N}$ be the quotient map. If W is an open set in the quotient topology on $\frac{G}{N}$, then $\overline{W} = \pi\overline{\pi^{-1}W}$

Proof

If $x \notin \overline{\pi^{-1}W}$, then there is atleast one neighbourhood O of x which does not intersect $\pi^{-1}W$

(i.e), $O \cap \overline{\pi^{-1}W} = \phi$

Claim

$\pi^{-1}(ON) \cap \pi^{-1}(W) = \phi$

Suppose not, there exists $y \in \pi^{-1}(ON)$ and $y \in \pi^{-1}(W)$

Then $\pi(y) \in ON$

(i.e) $y N \in ON$

$\Rightarrow y \in O$

As y also belongs to $\pi^{-1}(W)$, $O \cap \pi^{-1}(W) \neq \phi$

This is a contradiction.

Hence the claim.

So $\pi^{-1}\pi(O) \cap \pi^{-1}(W) = \phi$

This means $\pi O \cap W = \phi$

So $\pi x \notin \overline{W}$.

Hence $\overline{W} \subset \overline{\pi\pi^{-1}W}$

conversely if $\pi x \notin \overline{W}$, then there is a neighbourhood V of πx in $\frac{G}{N}$ which does not intersect W.

(i.e) $V \cap W = \phi$

$\Rightarrow \pi^{-1} W \cap \pi^{-1} V = \phi$

So $x \notin \overline{\pi^{-1}W}$

(i.e) $\pi x \notin \pi(\overline{\pi^{-1}(W)})$

Hence $\pi(\overline{\pi^{-1}(W)}) \subset \overline{W}$

Theorem 2.3.6

In a theta topological group G, given $V \in \mathcal{N}(G)$, there is a $U \in \mathcal{N}(G)$ with

$\overline{U\,H(G)} \subset \overline{V}$

Proof

By (2′), for the given $V \in \mathcal{N}(G)$, there is a $U \in \mathcal{N}(G)$ such that $\overline{U}\,\overline{U} \subset \overline{V}$

To prove

$\overline{U\,H(G)} \subset \overline{V}$

By the definition of H(G), $H(G) \subset \overline{U}$

(i.e), $U\,H(G) \subset U\,\overline{U}$

$\subset \overline{U}\,\overline{U}$

$\subset \overline{V}$

Therefore $\overline{U\,H(G)} \subset \overline{\overline{V}} = \overline{V}$

Hence $\overline{U\,H(G)} \subset \overline{V}$

Theorem 2.3.7

The topological space $\dfrac{G}{H(G)}$ with the quotient topology is Hausdorff

Proof

Take $x \in G \setminus H(G)$

$\Rightarrow x \notin H(G)$

$\Rightarrow x \notin \cap\,\{\overline{U} / U \in \mathcal{N}(G)\}$

So there exists a $W \in \mathcal{N}(G)$ such that $x \notin \overline{W}$

For this W, there is a $U \in \mathcal{N}(G)$ such that $\overline{U}\,\overline{U} \subset \overline{W}$. So $x \notin \overline{U}\,\overline{U}$. Also by (3′) there is a $V \in \mathcal{N}(G)$ with i $\overline{V} \subset \overline{U}$

$\Rightarrow \overline{U}\, i\, \overline{V} \subset \overline{U}\, \overline{U}$

Hence $x \notin \overline{U}\, i\, \overline{V}$

Claim

$x\, \overline{V} \cap \overline{U} = \phi$

Suppose not, there exists a $y \in \overline{U}$ and $y \in x\, \overline{V}$

$\Rightarrow x^{-1}y \in \overline{V}$

$\Rightarrow (x^{-1}y)^{-1} \in i\, \overline{V}$

$\Rightarrow (y^{-1}x) \in i\, \overline{V}$

$\Rightarrow x \in y\, (i\, \overline{V})$

(i.e), $x \in \overline{U}\, i\, \overline{V}$, which is a contradiction.

Hence $x\, \overline{V} \cap \overline{U} = \phi$

By the theorem 2.3.6, corresponding to V, there exists $W \in \mathcal{N}(G)$ such that

$\overline{W\, H(G)} \subset \overline{V}$

Corresponding to U, there exists a $O \in \mathcal{N}(G)$

Such that $\overline{OH(G)} \subset \overline{U}$

There fore $x\, \overline{WH(G)} \cap \overline{OH(G)} = \phi$;

So $x\, W\, H(G) \cap OH(G) = \phi$

Thus $\pi(xW)$ and πO are two disjoint open sets containing πx and πe.

Hence the space $\dfrac{G}{H(G)}$ is Hausdorff.

Theorem2.3.8

The group $\dfrac{G}{H(G)}$ with the quotient topology, is a theta topological group.

Proof

First let us prove the theta continuity of the multiplication map

$$m: \frac{G}{H(G)} \times \frac{G}{H(G)} \to \frac{G}{H(G)}$$

Take $W \in \mathcal{N}\left(\frac{G}{H(G)}\right)$

Then $\pi^{-1}(W) \in \mathcal{N}(G)$

By (2′) there exists $V \in \mathcal{N}(G)$ such that $\overline{V}\,\overline{V} \subset \overline{\pi^{-1} W}$

By theorem 2.3.6, there exits $U \in \mathcal{N}(G)$ such that $\overline{U H(G)} \subset \overline{V}$.

Therefore $\overline{U H(G)} . \overline{U H(G)} \subset \overline{V}\,\overline{V} \subset \overline{\pi^{-1}W}$ -----------(1)

Let $O = \pi U$

Then $\pi^{-1}(O) = \pi^{-1}\ (\pi(U))$

$= \pi^{-1}(U H(G))$

Therefore $\pi\, \overline{\pi^{-1}(O)} = \pi\, \overline{\pi^{-1}(UH(G))}$

$= \overline{U H(G)}$ (by theorem 2.3.5)

Now by considering $\overline{U H(G)}$ as an element of $\frac{G}{H(G)}$, the relation (1) becomes

$\overline{U H(G)} . \overline{U H(G)} \subset \pi\ \overline{(\pi^{-1}(W))}$

$= \overline{W}$ (by theorem 2.3.5)

i.e, $\pi\ (\overline{\pi^{-1}(O)}).\ \pi(\overline{\pi^{-1}(O)}) \subset \overline{W}$

$\Rightarrow \overline{O} . \overline{O} \subset \overline{W}$ .... (2)

Since $O = \pi U$ where $U \in \mathcal{N}(G)$

We get $O \in \mathcal{N}\left(\frac{G}{H(G)}\right)$

Hence from (2) we get that

$$m\ (\overline{O} \times \overline{O}) \subset \overline{W}$$

Therefore m is theta continuous.

Next to show Theta continuity of inversion

Take $W \in \mathcal{N}\left(\frac{G}{H(G)}\right)$

Then $\pi^{-1(}W) \in \mathcal{N}(G)$

By (3′), there exists a $V \in \mathcal{N}(G)$ such that

$$i\,\overline{V} \subset (\overline{\pi^{-1}(W)})$$

By theorem 2.3.6 there is a $U \in \mathcal{N}(G)$ such that $\overline{U\,H(G)} \subset \overline{V}$

Hence $i\overline{U\,H(G)} \subset i\,\overline{V} \subset \overline{\pi^{-1}(W)}$

As elements of $\frac{G}{H(G)}$, $i\overline{U\,H(G)} \subset \overline{\pi(\pi^{-1}(W))}$

$= \overline{W}$ (by theorem 2.3.5)

Let $O = \pi U$, which is open in $\frac{G}{H(G)}$

Then $\overline{U\,H(G)} = \pi(\overline{\pi^{-1}(O)})$

$= \overline{O}$ (by theorem 2.3.5)

Hence $i\,\overline{O} \subset \overline{W}$

Therefore i is theta continuous.

Hence $\frac{G}{H(G)}$ is a theta topological group.

Section 2.4

Examples of theta topological groups which are not topological groups

In this section we discuss two interesting examples of Hausdorff, non regular theta topological groups which are not topological groups.

Any topological group is regular. So a theta topological group is a topological group $\Leftrightarrow$ it is regular. The most degenerate examples of the non - regular groups occur when $H(G) = G$.

Example 2.4.1

Consider the set of integer Z with the finite complement topology. The group law is addition. As $\overline{U} = Z$, for every $U \in \mathcal{N}(Z)$, we get $H(Z) = Z$. In this example, the space is compact.

The following example gives a theta topological group which is not compact.

Example 2.4.2

Take any uncountable group G. A set is open if its complement is countable. Here also $H(G) = G$. But the space here is not compact.

In the above two examples, we note that the spaces are not Hausdorff.

Example of a Hausdorff non-regular theta topological group in which inversion is continuous, but multiplication is not continuous :

Definition 2.4.3

Let A be a subset of the set of integers Z. Then the density of a set B ($\subset$ A) in A is defined to be (if the limit exists)

$$\operatorname*{lt}_{n\to\infty} \frac{\#B \cap [-n, n]}{\#A \cap [-n, n]},$$ where # denotes 'the number of elements in'.

The topology on Z is defined by the following base open sets.

Definition 2.4.4

A set $U \subset Z$ is a ***base open set*** if there is an infinite arithmetic progression $A \subset Z$ containing U, such that U has density one in A.

To see that this actually defines a base, we have to prove that the intersection of two base open sets is another base open set (or) empty. This is done in the following theorem.

Theorem 2.4.5

Suppose that U is of density one in an arithmetic progression $A \subset Z$, and that U' is of density one in an infinite arithmetic progession $A' \subset Z$. Then either $A \cap A' = \phi$, (or) $U \cap U'$ is of density one in the infinite arithmetic progression $A \cap A'$.

To prove this theorem we need the following lemma.

Lemma 2.4.6

If $U \subset A$ is of density one in A and if $B \subset A$ is of density > 0 in A, then $U \cap B$ is of density one in B.

Proof

Consider the inequality,

$$\#U \cap B \cap [-n, n] \geq \#B \cap [-n, n] + \#U \cap [-n, n] - \#A \cap [-n, n]$$

Divide both the sides by $\# B \cap [-n, n]$

$$\frac{\# U \cap B \cap [-n, n]}{\# B \cap [-n, n]} \geq \frac{\# U \cap [-n, n]}{\# B \cap [-n, n]} + 1 - \frac{\# A \cap [-n, n]}{\# B \cap [-n, n]}$$

$$= 1 - \left(1 - \frac{\# U \cap [-n, n]}{\# A \cap [-n, n]}\right)\left(\frac{\# A \cap [-n, n]}{\# B \cap [-n, n]}\right) \quad \text{------} \quad (1)$$

Since B is of strictly positive density in A, there is a number M > 0, and an integer N such that , for all $n \geq N$

$$\frac{\# \; U \cap B \cap [-n, n]}{\# \; B \cap [-n, n]} \geq 1 - \left(1 - \frac{\# \; U \cap [-n, n]}{\# \; A \cap [-n, n]}\right) \cdot \frac{1}{M}$$

Since U is of density one in A,

$$\operatorname*{lt}_{n \to \infty} \frac{\# U \cap [-n, n]}{\# A \cap [-n, n]} = 1$$

Therefore equation (1) becomes,

$$\operatorname*{lt}_{n \to \infty} \frac{\# \; U \cap B \cap [-n, n]}{\# \; B \cap [-n, n]} \geq 1$$

But the limit cannot be > 1.

$$\text{Hence } \operatorname*{lt}_{n \to \infty} \frac{\# \; U \cap B \cap [-n, n]}{\# \; B \cap [-n, n]} = 1$$

i.e., $U \cap B$ is of density one in B.

Proof of the theorem

If $A \cap A' \neq \phi$, then it is of strictly positive density in A. By the above lemma 2.4.6, $U \cap A \cap A'$ is of density one in $A \cap A'$. Similarly $U' \cap A \cap A'$ is of density one in $A \cap A'$. Since the intersection of two density one sets is of density one, we get the result.

Hence the collection defined in definition 2.4.4, definies a topology τ_1 **(Z)** on Z.

Theorem 2.4.7

The topological space(Z, τ_1(Z)) is Hausdorff.

Proof

Take $n \neq m$ in Z and

let $d = | n - m | + 1$. Then n is contained in the infinite arithmetic progression $n + dZ$ and m is contained in the infinite arithmetic progression $m + dZ$. Let $U = n + dZ$, $V = m + dZ$. Suppose $n < m$, then $n + d$ will be $m + 1$,

$n + 2d$ will be $m + d + 1$,

$n + 3d$ will be $m + 2d + 1$ and so on.

Hence $U \cap V = \phi$ and the topology is Hausdorff.

Theorem 2.4.8

The topological space$(Z, \tau_1(Z))$ under addition is a theta topological group. Infact inversion is continuous .

Proof

(1) Since the translate of a base open set is open, property(1) is satisfied.

(2) For this we show if V is an open set containing zero, then $\overline{V}$ contains an infinite arithmetic progression dZ. It is sufficient to do this for a base open set V containing zero. Then, by definition V has density one in some progression dZ with $V \subset dZ$. Suppose $n \in dZ$. Let W be a base open set containing n. Then W has density one in an infinite arithmetic progression A with $W \subset A$. Also $A \cap dZ \neq \phi$ ($\therefore n \in A$ and dZ). So by theorem, 2.4.5 $V \cap W$ is of density one in the infinite arithmetic progression $A \cap dZ$ and so cannot be empty.

Therefore $n \in \overline{V}$ and $dZ \subset \overline{V}$

Let $U = dZ$

Then $\overline{U} = dZ$ and $\overline{U} + \overline{U} = dZ \subset \overline{V}$.

$\Rightarrow \overline{U} + \overline{U} \subset \overline{V}$.

(3) To show inversion is continuous, it is enough to note

" If U is of density one in the infinite arithmetic progression A, then iU is of density one in the infinite arithmetic progression iA".

To prove that the topology $\tau_1(Z)$ does not give a topological group we need the following result.

Theorem 2.4.9

For U, a base open set containing zero, if $md \notin U - U$ for some integer $m > 0$, then U is not a density one subset of dZ.

Theorem 2.4.10

The topological space $(Z, \tau_1(Z))$ under addition is not a topological group.

Proof

We have to find a base open set V containing zero such that there is no U which is an open set containing zero with $U + U \subset V$. Without loss of generality we can assume that U is a base open set and that $U = i\,U$ by taking the intersection of the open set and its inverse.

Choose $V = Z \setminus \{n! : n \in N\}$. Then V is of density one in Z. If $U + U \subset V$, then $d! \notin U - U$ for any $d \in N$ (Since $U - U = U + i\,U = U + U$ and $d! \notin U + U$).

By theorem 2.4.9, U is not a density one subset of dZ. This is a contradiction to our assumption that U is a base open set containing zero. Therefore multiplication is not continuous. Hence (Z, τ_1 (Z)) is not a topological group.

Theorem : 2.4.11

The topological space (Z, τ_1 (Z)) is not regular.

Proof

If (Z, τ_1(Z)) is regular, then the multiplication map m : Z x Z → Z will be continuous, and hence (Z, τ_1(Z)) under addition will be a topological group, which is a contradiction to theorem 2.4.10. Hence (Z, τ_1(Z)) is not regular.

Example of a Hausdorff non-regular theta topological group in which multiplication is continuous but inversion is not continuous.

In this example also we consider the group of integers Z under addition. But the topology on Z is definied as follows.

Definition 2.4.12

A set $U \subset Z$ is a base open set if it is the intersection of an infinite arithmetic progression $A \subset Z$ with an interval of the form $[n, \infty)$, where $n \in Z$.

We must check that the intersection of two such sets is another base open set or empty. This follows from the fact that the intersection of two infinite arithmetic progressions is either empty or another infinite arithmetic progression. Hence we get a topology τ_2(Z) on Z.

Theorem 2.4.13

The closure of the base open set $U = A \cap [n, \infty)$ (where A is an infinite arithmetic progression) is A.

Proof

To prove $\overline{U} = A$

If $m \notin A$, then $(A + m - a) \cap [m, \infty)$ (for some $a \in A$) is a base open set containing m. (Since $m = a + m - a \in A + m - a$). Also $U \cap (A + m - a \cap [m,\infty)) = \phi$ (Since if not, there exists $b \in A$ such that $b = b' + m - a$ where $b' \in A$.

Then $m = b - b' + a \in A$, which is a contradiction).

Therefore $m \notin \overline{U}$ -----------------------(1)

If $r \in A$, and r is in the base open set V where $V = B \cap [s,\infty)$. Hence $r \in B \cap A$ and $B \cap A \neq \phi$. Therefore $B \cap A$ is an infinite arithmetic progression and hence $U \cap V \neq \phi$.

Therefore $r \in \overline{U}$ -----------------------(2)

From (1) and (2), we get $\overline{U} = A$.

Theorem 2.4.14

$(Z, \tau_2(Z))$ under addition is a theta topological group. Infact multiplication is continuous.

Proof

(i) Since the translate of a base open set is open, property (1) is satisfied.

(ii) Suppose V is a base open set containing zero, then $U = V \cap [0, \infty)$ is also a base open set containing zero. Then $U + U \subset V$. Hence multiplication (here addition) is continuous.

(iii) If V is a base open set containing zero, then $\overline{V}$ is an arithmetic progression containing zero.

By taking U = V, we find that $i\,\overline{U} \subset \overline{V}$.

Theorem 2.4.15

$(Z, \tau_2(Z))$ is Hausdorff

Proof

As in theorem 2.4.7, any two different integers can be put in disjoint infinite arithmetic progressions. Since an infinite arithmetic progression is open, the result follows.

Theorem 2.4.16

Inversion is not continuous

Proof

Consider the base open set $U = Z \cap [0,\infty)$

$i^{-1}(U) = \{0, -1, -2, \ldots\}$. This does not contain any non-empty base open set. Hence $i^{-1}(U)$ is not open and therefore inversion is not continuous.

Chapter III

Chapter - III

BITOPOLOGICAL VIEW OF QUASI-TOPOLOGICAL GROUPS

Quasi - topological groups have been investigated by several authors. They were mainly interested in obtaining conditions under which a quasi-topological group, J. Marin and S. Romaguera [6] have investigated quasi topological groups from "a bitopology view point" and have obtainded appropriate extensions of classical theorems on topological groups. In this chapter we discuss these results in detail. They have shown that every quasi -bitopological group is quasi-uniformizable and that every first countable quasi-bitopological group admits a compatible left invariant quasi pseudo metric. Moreover for every 2-Hausdorff quasi bitopological group G there is 2-Hausdorff quasi bitopological group which is bicomplete in its two sided uniformity and has g as a 2-dense quasi-bitopological subgroup. First let us give the preliminary definitions and results needed for our discussions.

Section : 3.1

Quasi - bitopo logical groups

In this section we give the definitions and basic properties of quasi - bitopological groups and quasi - pseudo metrics.

Definition 3.1.1

A ***quasi - topological group*** is a pair (G,τ) where G is a group and τ is a topology on G such that the function 'm' : $(G \times G, \tau \times \tau) \rightarrow (G, \tau)$ defined by $m(x,y) = xy$, is continuous.

Note 3.1.2

If (G, τ) is a quasi - topological group, then so is (G, τ^{-1}) where $\tau^{-1} = \{ A \subseteq G : A^{-1} \in \tau\}$ is called the ***conjugate topology*** of τ. Clearly the map $x \rightarrow x^{-1}$ is a homeomorphism of (G, τ) to (G, τ^{-1}).

Definition 3.1.3

A ***quasi - bitopological group*** is an ordered triple (G, τ, τ^{-1}) such that (G, τ) is a quasi - topological group and τ^{-1} is the conjugate topology of τ.

It easily follows from the definition that for any quasi - bitopological group (G, τ, τ^{-1}) the functions

(i) $\phi_1 : (G \times G, \tau^{-1} \times \tau^{-1}) \rightarrow (G, \tau^{-1})$ defined by $\phi_1(x, y) = xy$.

(ii) $\phi_2 : (G \times G, \tau \times \tau^{-1}) \rightarrow (G, \tau)$ defined by $\phi_2(x, y) = xy^{-1}$ and

(iii) $\phi_3 : (G \times G, \tau \times \tau^{-1}) \rightarrow (G, \tau^{-1})$ defined by $\phi_3(x, y) = x^{-1}y$ are continuous.

Example 3.1.4

Let '+' be usual additive law on R and let $u = \{]-\infty, a[: a \in R\}$ be the so called upper topology on R. Then $(R, +, u, u^{-1})$ is a quasi - bitopological group such that $(R, +, u \vee u^{-1})$ is the usual topological group on R.

Example 3.1.5

Let '+' be the usual additive law on R, and let (R,S) be the Sorgenfrey line (basic open sets of S are of the form $[x, a[$, $x < a$). Then $(R, +, S, S^{-1})$ is a quasi - bitopological group.

Example 3.1.6

Let '.' be the usual multiplicative law on $R^+ = \{ x \in R : x > 0\}$ and S^+ the restriction of the Sorgenfrey line to R^+. Then $(R^+, ., S^+, (S^+)^{-1})$ is a quasi - bitopological group.

Notation 3.1.7

Let G be a group and let τ be a topology on G. Then the τ - neighbourhood system of each $x \in G$ will be denoted by $\mathcal{N}(x)$ and the τ- neighbourhood system by $\mathcal{N}^{-1}(x)$. By τ^* we will denote the coarsest topology finer than τ and τ^{-1} ,

(i.e) $\tau^* = \tau \vee \tau^{-1}$. The τ^* - neighbourhood system of each $x \in G$ will be denoted by $\mathcal{N}^*(x)$. In particular we write $\mathcal{N}$G), $\mathcal{N}^{-1}(G)$ and $\mathcal{N}^*(G)$ instead of $\mathcal{N}(e)$, $\mathcal{N}^{-1}(e)$ and $\mathcal{N}^*(e)$ respectively.

Definition 3.1.8

Let (G, τ, τ^{-1}) be a quasi - bitopological group and $a \in G$. Consider the map $L_a : G \to G$ defined by $L_a(x) = ax$. Then L_a is called the ***left translation*** by a. The map $R_a : G \to G$ defined by $R_a(x) = xa$ is called the ***right translation*** by a.

As noted in chapter-I , we get that, for each $a \in G$, both L_a and R_a are homeomorphisms of the topological space (G, τ) onto itself. Clearly they are also homeomorphisms of (G, τ^{-1}) onto itself.

From these observations, one obtains immediately the following result.

Theorem 3.1.9

Let (G, τ, τ^{-1}) be a quasi - bitopological group, $x \in G$ and $U \subseteq G$.

Then

(i) $U \in \mathcal{N}(x) \Leftrightarrow xV \subseteq U$ for some $V \in \mathcal{N}(G) \Leftrightarrow Wx \subseteq U$ for some $W \in \mathcal{N}(G)$

(ii) $U \in \mathcal{N}^{-1}(x) \Leftrightarrow xV^{-1} \subseteq U$ for some $V \in \mathcal{N}(G)$, $W^{-1}x \subseteq U$ for some $W \in \mathcal{N}(G)$.

Theorem 3.1.10

Let (G, τ, τ^{-1}) be a quasi - bitopological group. Then (G, τ^*) is a topological group.

Proof

Let $x, y \in G$ and let A be a τ^*- neighbourhood of xy . Then there exists a τ - neighbourhood U of xy such that $U \cap U^{-1} \subset A$.

Hence there is a $\tau \times \tau$ neighbourhood (V_1, W_1) of (x, y) and $\tau^{-1} \times \tau^{-1}$ neighbourhood (V_2, W_2) of (x, y) such that $V_1W_1 \subset U$ and $V_2W_2 \subset U^{-1}$.

Take $V = V_1 \cap V_2$ and $W = W_1 \cap W_2$

Now to show $VW \subseteq U \cap U^{-1}$.

$$VW = (V_1 \cap V_2)(W_1 \cap W_2)$$

$$\subset V_1W_1$$

$$\subset U$$

and

$$VW = (V_1 \cap V_2)(W_1 \cap W_2)$$

$$\subset V_2W_2$$

$$\subset U^{-1}$$

Therefore $VW \subset U \cap U^{-1}$

$$\subset A$$

$\therefore$ m(V, W) = VW $\subseteq$ A.

Hence the multiplication map from (G x G, τ^* x τ^*) to (G, τ^*) is continuous.

Now to prove the inversion map 'i' : (G, τ^*) $\rightarrow$ (G, τ^*) is continuous.

Let $x \in G$ and $A \in \mathcal{N}^*(x^{-1})$.

Then there is a $U \in \mathcal{N}$ such that $x^{-1} U \cap x^{-1} U^{-1} \subseteq A$

Let $V = U x \cap U^{-1} x$

$$\text{Then } i(V) = x^{-1} U^{-1} \cap x^{-1} U$$
$$\subseteq A$$

$\therefore$ Inversion is continuous.

Hence (G, τ^*) is a topological group.

Quasi - Pseudometric.

The notion of an absolute value function (Definition 1.2.1) on a group G permits us to construct a semi metric d on G for which (G, $\tau(d)$) is a topological group (Theorem 1.2.2). In a similar way, here we introduce the notation of an absolute quasi - valued function and establish connections between absolute quasi - valued function, quasi - bitopological groups and quasi - pseudo metrics.

Definition 3.1.11

A quasi - pseudo metric on a set X is a non - negative real valued function d on X x X such that for x, y, z $\in$ X.

(i) $d(x, x) = 0$, and

(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If in addition d satisfies

(iii) $d(x, y) = 0 \Leftrightarrow x = y$, then d is called a ***quasi-metric*** on X.

Definition 3.1. 12

The topology ***$\tau(d)$ induced by a quasi - pseudo metric d*** on X has basic open neighbourhoods of $x \in X$ of the form $B_d(x, r) = \{ y \in X : d(x, y) < r\}$

Definition 3.1.13

A topological space (X, τ) is called ***quasi-(pseudo) metrizable*** if there is a quasi-(pseudo) metric d on X compatible with τ (we say d is compatible with τ provided $\tau = \tau(d)$)

Definition 3.1.14

Each quasi-(pseudo) metric d on X induces a ***conjugate quasi-(pseudo) metric*** d^{-1}, given by $d^{-1}(x, y) = d(x, y)$.

Thus the pair of topologies induced by a quasi-(pseudo) metric and its conjugate originate the following notion:

Definition 3.1.15

A bitopological space (X, τ_1, τ_2) is called quasi-(pseudo) metrizable if there is a quasi-pseudo metric d on X compatible with (X, τ_1, τ_2). (We say d is compatible with (X, τ_1, τ_2) provided that $\tau_1 = \tau(d)$ and $\tau_2 = \tau(d)^{-1}$).

Notation 3.1.16

If d is a quasi (pseudo) mertic on X we will denote by d* the (pseudo) metric on X given by $d^* = d \vee d^{-1}$.

Definition 3.1.17

Let G be a group. An absolute quasi valued function for G is a non- negative real valued function for ρ on G such that

(i) $\rho(e) = 0$

(ii) $\rho(xy) \leq \rho(x) + \rho(y)$ for every $x , y \in G$ and

(iii) $\rho(x_n) \rightarrow 0$ implies $\rho(ax_na^{-1}) \rightarrow 0$ for all $a \in G$.

As an immediate consequence of this definition we obtain the following theorem which is an extended result of theorem 1.1.10.

Theorem 3.1.18

Let ρ be an absolute quasi valued function on the group G. then the function d defined on G x G by $d(x, y) = \rho(x^{-1}y)$ or by $d(x, y) = \rho(yx^{-1})$ is a quasi - pseudo metric on G such that $(G, \tau(d), \tau(d^{-1}))$ is a quasi - bitopological group.

Definition 3.1.19

Let G be a group and d a quasi - pseudo metric on G. Then d is called ***left invariant*** if for all $a, x, y \in G$, $d(ax, ay) = d(x, y)$: ***right invariant*** if $d(xa, ya) = d(x, y)$ and ***two sided invariant***, if it is both left and right invariant.

The following result is the converse of theorem 3.1.18, the proof of which is similar to theorem 1.2.3.

Theorem 3.1.20

Let G be a group and d, a left (right) invariant quasi - pseudo metric on G such that $(G, \tau(d), \tau(d^{-1}))$ is a quasi - bitopological group. Then the function $\rho(x) = d(e, x)$ is an absolute quasi - valued function.

Example 3.1.21

Let $I = [0, 1]$ and let $G = \{f : I \to I$, such that f is continuous, bijective and increasing $\}$. Let the group operation be composition, $f \circ g = f(g)$. Then the function ρ defined on G by

$$\rho(f) = \begin{cases} \max\{f(x) - x : x \in I\}, & \text{if for each } x \in I,\ f(x) \geq x \\ I, & \text{if for some } x \in I,\ f(x) < x \end{cases}$$

is an absolute quasi-valued function for G, that generates a quasi-bitopological group by theorem 3.1.20.

Section 3.2

Quasi - uniformities on quasi - bitopological groups and quasi-metrization

This section is devoted to the study of quasi-uniformity on quasi-bitopological groups. Three different quasi-uniformities on a quasi-bitopological group are considered. In this section we also discuss the fact that each first countable quasi bitopological group (G, τ, τ^{-1}) admits a compatible left-invariant quasi pseudo metric d which inturn induces the left quasi-uniformity for (G, τ, τ^{-1}). More over the B-quasi pseudo metric D associated to d induces the two sided quasi uniformity for (G, τ, τ^{-1}).

Definition 3.2.1

A ***quasi- uniformity*** on a set X is a filter $\mathcal{U}$ on X x X such that

(1) for each $U \in \mathcal{U}$, $\Delta = \{(x, x) : x \in X\} \subseteq U$;

(2) for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where $V^2 = \{(x, y):$ there is $z \in X$ such that $(x, z) \in V$ and $(z, y) \in V\}$

Definition 3.2.2

A ***quasi-uniform space*** is a pair $(X, \mathcal{U})$ such that X is a non empty set and $\mathcal{U}$ is a quasi uniformity on X.

Definition 3.2.3

The topology $\tau(\mathcal{U})$ induced by a quasi-uniformity $\mathcal{U}$ on X has basic neighbourhoods of $x \in X$ of the form $U[x] = \{y \in X : (x, y) \in U\}$ where $U \in \mathcal{U}$.

Definition 3.2.4

Each quasi - uniformity $\mathcal{U}$ on X induces a ***conjugate quasi-uniformity*** $\mathcal{U}^{-1}$ given by $\mathcal{U}^{-1} = \{ U \subseteq X \text{ x } X : U^{-1} \in \mathcal{U}\}$ where $U^{-1} = \{(x, y) : (y, x) \in U\}$.

Definition 3.2.5

A bitopological space (X, τ_1, τ_2) is called ***quasi-uniformizable*** if there is a quasi-uniformity $\mathcal{U}$ on X compatible with (X, τ_1, τ_2). We say $\mathcal{U}$ is compatible with (X, τ_1, τ_2) provided that $\tau_1 = \tau(\mathcal{U})$ and $\tau_2 = \tau(\mathcal{U}^{-1})$.

Notation 3.2.6

If $\mathcal{U}$ is a quasi-uniformity on X. We will denote by $\mathcal{U}^*$ the uniformity given on X by $\mathcal{U}^* = \mathcal{U} \vee \mathcal{U}^{-1}$.

Definition 3.2.7

Each quasi-pseudo metric d on X induces a **quasi-uniformity $\mathcal{U}$(d)** on X which has as a base, the family of all sets of the form $\{(x, y) : d(x, y) < 2^{-n}\}$, $n \in N$.

Similar to thoerem 1.2.5. of Wilansky[14], we obtain the following result:

Theorem 3.2.8

Let G be a group and τ a topology on G. Then the map **$m : (G \times G, \tau \times \tau) \to (G, \tau)$** given by $m(x, y) = xy$, is continuous at (e, e) $\Leftrightarrow$ for each $U \in \mathcal{N}(G)$ there is $V \in \mathcal{N}(G)$ such that $V^2 \subseteq U$.

Three different quasi-uniformities on a quasi-bitopological group **(G, τ, τ^{-1}).**

For each $U \in \eta$, define $U_L = \{(x, y) : x^{-1}y \in U\}$

From theorem 3.2.6, it follows that $\{U_L : U \in \eta\}$ is a base for a quasi-uniformity $\mathcal{L}$ on G.

Now for $U \in \eta$, let $U_R = \{(x, y) : yx^{-1} \in U\}$. Then $\{U_R : U \in \eta\}$ is also a base for a quasi-uniformity $\mathcal{R}$ on G.

The quasi-uniformities $\mathcal{L}$ and $\mathcal{R}$ are called the ***left and right quasi-uniformity*** for (G, τ, τ^{-1}). The quasi-uniformity $\mathcal{B} = \mathcal{L} \vee \mathcal{R}$ is called the ***two-sided quasi-uniformity*** for (G, τ, τ^{-1}).

Theorem 3.2.9

Each quasi-bitopological group is quasi-uniformizable.

Proof

Let (G, τ, τ^{-1}) be a quasi-bitopological group.

To prove

$\tau(\mathcal{L}) = \tau$ and $\tau(\mathcal{L}^{-1}) = \tau^{-1}$.

It is enough to prove that $U_L[x] = x\,U$ and $U_L^{-1}[x] = x\,U^{-1}$ for each $x \in G$ and for every $U \in \mathcal{N}(G)$.

Consider $U_L[x] = \{y \in X / (x , y) \in U_L\}$

$= \{y \in X / (x^{-1} y) \in U\}$

$= \{y \in X / \ y \in x\,U\}$

$= x\,U$

= neighbourhood of x with respect to τ.

Therefore $\tau(\mathcal{L}) = \tau$

Now $U_L^{-1}[x] = \{y \in X / (x , y) \in U_L^{-1}\}$

$= \{y \in X / (y, x) \in U_L\}$

$= \{y \in X / \ y^{-1}x \in U\}$

$= \{y \in X / (y^{-1}x)^{-1} \in U^{-1}\}$

$= \{y \in X /\ x^{-1}y \in U^{-1}\}$

$= \{y \in X / y \in xU^{-1}\}$

$= xU^{-1}$

= neighbourhood or x with respect to τ^{-1}.

Therefore $\tau(\mathcal{L}^{-1}) = \tau^{-1}$

Thus $\mathcal{L}$ is compatible with (G, τ, τ^{-1}).

Hence it is unifomizable.

Remark 3.2.10

If we consider the quasi-uniformity $\mathcal{R}$ instead of $\mathcal{L}$, we obtain $\tau(\mathcal{R}) = \tau$ and $\tau(\mathcal{R}^{-1}) = \tau^{-1}$, since in this case we get $U_R[x] = Ux$ and $U_R^{-1}[x] = U^{-1}x$, for each $x \in G$ and for each $U \in \mathcal{N}(G)$. Therefore $\mathcal{R}$ is also compatible with (G, τ, τ^{-1}).

Hence by combining this with theorem 3.2.9 we get that $\mathcal{B}$ is also compatible with (G, τ, τ^{-1}).

Theorem 3.2.11

Let (G, τ, τ^{-1}) be a quasi bitopological group. If $\mathcal{L}^{v}$, $\mathcal{R}^{v}$ and $\mathcal{B}^{v}$ denote the left uniformity, the right uniformity and the two sided uniformity for (G, τ^*) respectively, then $\mathcal{L}^* = \mathcal{L}^{v}$, $\mathcal{R}^* = \mathcal{R}^{v}$ and $\mathcal{B}^* = \mathcal{B}^{v}$.

Proof

Since $\mathcal{L}^* = \mathcal{L} \vee \mathcal{L}^{-1}$

$\mathcal{L}^*$ has as its base $\{U_L \cap U_L^{-1} / U \in \mathcal{N}(G)\}$

Take $U \in \mathcal{N}(G)$.

Consider $(U \cap U^{-1})_L$

$= \{(x, y) / x^{-1}y \in U \cap U^{-1}\}$

$= \{(x, y) / x^{-1}y \in U\} \cap \{(x, y) / x^{-1}y \in U^{-1}\}$

$= U_L \cap U_L^{-1}$

Hence $\mathcal{L}^{\vee} = \mathcal{L}^*$, as the corresponding base elements coincide.

Similarly, for each $U \in \mathcal{N}(G)$, $(U \cap U^{-1})_R = U_R \cap U_R^{-1}$.

Hence $\mathcal{R}^{\vee} = \mathcal{R}^*$ and therefore $\mathcal{B}^{\vee} = \mathcal{B}^*$

Definition 3.2.12

A quasi-bitopological group (G, τ, τ^{-1}) is ***first countable*** if (G, τ) (or equivalently (G, τ^{-1})) is first countable.

Theorem 3.2.13

Let (G, τ, τ^{-1}) be a first countable quasi-bitopological group and let d be a left (right) invariant quasi-pseudo metric on G, such that $\tau(d) = \tau$. Then $\tau(d^{-1}) = \tau^{-1}$.

Proof

Suppose that d is left invariant. Since (G, τ) is first countable, (G, τ^{-1}) is also first countable.

Then we have,

$x_n \to x$ (with respect to τ^{-1})

$\Leftrightarrow x_n^{-1} \to x^{-1}$ (with respect to τ)

$\Leftrightarrow x_n^{-1}x \to e$ (with respect to τ)

$\Leftrightarrow d(e, x_n^{-1}x) \to 0$

$\Leftrightarrow d(x_n, x) \to 0$ ($\because$ d is left invariant)

$\Leftrightarrow d^{-1}(x, x_n) \rightarrow 0$

Hence $\tau(d^{-1}) = \tau^{-1}$.

Wilansky [14] has proved that if the topology of a topological group is first countable at e, then it is given by a left invariant semi-metric.

The following result extends this fact to a first countable quasi-bitopological group.

Theorem 3.2.14

Each first countable quasi - bitopological group admits a compatible left invariant quasi - pseudo metric.

Proof

Let (G, τ, τ^{-1}) be a first countable quasi - bitopological group.

Let $< U_n >$ be a countable base of τ neighbourhoods of e.

By theorem 3.2.8, there is a countable base $< V_n >$ of τ- neighbourhoods of e such that $V_0 = G$ and $V_n^3 \subset V_{n-1}$ for all $n \in N$.

Define a non-negative real valued function P on G by

$$P(x) = \begin{cases} 0, \text{if } x \in V_n, \text{for all } n \in N \cup \{0\} \\ 2^{-n}, \text{ if } x \in V_n \setminus V_{n+1}. \end{cases}$$

Then we have

(1) $P(e) = 0$

(2) $x_n \rightarrow e$ (with respect to τ) $\Leftrightarrow P(x_n) \rightarrow 0$

More over, $P(abc) \leq 2 \max \{P(a), P(b), P(c)\}$ for all $a, b, c \in G$.

$$P(a_1, a_2, \ldots.. a_n) \leq 2 \sum_{i=1}^{n} P(a_i)$$

Define a function $\rho : G \to R$ such that

$$\rho(x) = \inf \left\{ \sum_{i=1}^{k} p(a_i\, a_{i-1}^{-1}) : a_0 = e,\ a_k = x,\ k \text{ finite} \right\}$$

It is easily seen that ρ is an absolute quasi-valued function on G. By theorem 3.1.18 the function d defined on G x G by $d(x, y) = \rho(x^{-1}y)$ is a quasi pseudo-metric on G and is clearly left-invariant.

Next we will show that d is compatible with (G, τ, τ^{-1}).

Here $\rho(x) \leq P(x) \leq 2\rho(x)$ for all $x \in G$.

Therefore $x_n \to x$ (with respect to τ)

$\Leftrightarrow x^{-1} x_n \to e$ (with respect to τ)

$\Leftrightarrow P(x^{-1} x_n) \to 0$

$\Leftrightarrow \rho(x^{-1} x_n) \to 0$

$\Leftrightarrow d(x, x_n) \to 0$

Hence $\tau(d) = \tau$

Therefore by theorem 3.2.13, $\tau(d^{-1}) = T^{-1}$. Hence the proof.

Remark 3.2.15

It follows from theorem 3.2.14 that every first countable quasi-bitopological group (G, τ, τ^{-1}) also admits a compatible right invariant quasi-pseudo metric. In fact let d be the left invariant quasi-pseudo metric constructed in the above theorem. Then the function d′ defined on G x G by $d'(x, y) = d(y^{-1}, x^{-1})$ is a right invariant quasi-pseudo metric compatible with (G, τ, τ^{-1}).

The left invariant quasi-pseudo metric obtained in theorem 3.2.14, induces the left quasi uniformity for (G, τ, τ^{-1}) as is seen from the following theorem.

Theorem 3.2.16

Let (G, τ, τ^{-1}) be a first countable quasi-bitopological group and let d be a left (right) invariant quasi-pseudo metric on G such that $\tau = \tau(d)$. Then d induces the left (right) quasi-uniformity for (G, τ, τ^{-1}).

Proof

We have to show that $\mathcal{U}(d) = \mathcal{L}$

For each $\xi > 0$, define

$$U_L^{\xi} = \{(x, y) : x^{-1}y \in B_d(e, \xi)\}$$

and

$$U_{\xi} = \{(x,y) / d(x, y) < \xi\}$$

U_L^{ξ} is a base element for $\mathcal{L}$ and U_{ξ} is a base element for $\mathcal{U}(d)$.

Consider,

$$U_L^{\xi} = \{(x, y) : d(e,x^{-1}y) < \xi\}$$

$$= \{(x, y) : d(x,y) < \xi)\} \qquad \text{(since d is left invariant)}$$

$$= U_{\xi}$$

since the base elements are the same, we have $\mathcal{L} = \mathcal{U}(d)$. Similar proof can be given if d is a right invariant quasi pseudo metric on G.

Definition 3.2.17

Let G be a group and d a quasi-pseudo metric on G. The quasi-pseudo metric D defined on G by

$D(x,y) = d(x, y) + d(y^{-1}, x^{-1})$ is called the ***B-quasi-pseudo metric*** associated to d.

Theorem 3.2.18

Let (G, τ, τ^{-1}) be a first countable quasi-bitopological group and let d be a left (right) invariant quasi-pseudo metric on G such that $\tau = \tau(d)$. Then the B-quasi pseudo metric D associated to d induces the two sided quasi-uniformity for the quasi-bitopological group (G, τ, τ^{-1}).

Proof

For each $\xi > 0$, the set

$U_\xi = \{(x, y) : D(x, y) < \xi\}$ is a base element for $\mathcal{U}(D)$. The set

$U_L^{\xi/2} = \{(x, y) : x^{-1}y \in B_d(e, \xi / 2)\}$ is a base element for the left quasi-uniformity $\mathcal{L}$

$U_R^{\xi/2} = \{(x, y) : yx^{-1} \in B_d(e, \xi / 2)\}$ is a base element for the right quasi-uniformity $\mathcal{R}$.

$$\begin{aligned}\text{Then } U_L^{\xi/2} \cap U_R^{\xi/2} &= \{(x, y) : d(e, x^{-1}y) < \xi / 2\} \cap \{(x, y) : d(e, yx^{-1}) < \xi / 2\}\\ &= \{(x, y) : d(x, y) < \xi / 2\} \cap \{(x, y) : d(y^{-1}, x^{-1}) < \xi / 2\}\\ &\subseteq \{(x, y) : d(x, y) + d(y^{-1}, x^{-1}) < \xi\}\\ &= \{(x, y) : D(x, y) < \xi\}\\ &= U_\xi\end{aligned}$$

Hence $\mathcal{U}(D) \subseteq \mathcal{L} \vee \mathcal{R} = \mathcal{B}$ (1)

To prove the otherway inclusion,

Consider $U_\xi = \{ (x, y) : D(x, y) < \xi\}$

$$= \{ (x, y) : d(x, y) + d(y^{-1}, x^{-1}) < \xi \}$$

$$\subseteq \{ (x, y) : d(x, y) < \xi\} \cap \{ (x, y) : d(y^{-1}, x^{-1}) < \xi \}$$

Since d is left - invariant,

$$U_\xi \subseteq \{(x, y) : d(e, y x^{-1}) < \xi \} \cap \{(x, y) : d(e, x^{-1} y) < \xi \}$$

$$= \{(x, y) : yx^{-1} \in B_d(e, \xi)\} \cap \{(x, y) : x^{-1}y \in B_d(e, \xi)\}$$

Therefore $U_\xi \subseteq U_L^{\xi} \cap U_R^{\xi}$

and $\mathcal{B} \subseteq \mathcal{U}(D)$ ------------------------------ (2)

From (1) and (2) we get $\mathcal{B} = \mathcal{U}(D)$

A similar argument proves the result when d is right invariant.

The section is concluded by stating the conditions under which

(1) a topology on a group generates a quasi-bitopological group. (Theorem 3.2.19).

(2) a group has a structure of a quasi-bitopological group. (Theorem 3.2.20)

The proofs of these theorems are similar to those of theorems 1.2.5 and 1.2.6.

Theorem 3.2.19

Let G be a group and τ a topology on G. Then (G, τ, τ^{-1}) is a quasi-bitopological group $\Leftrightarrow$ the following conditions are satisfied:

(1) Every left translate of a τ-open set is τ-open.

(2) For every τ-neighbourhood U of e, there is a τ-neighbourhood V of e such that $V^2 \subseteq U$.

(3) For every τ-neighbourhood U of e and every $a \in G$, there is a τ-neighbourhood V of e such that $a V a^{-1} \subseteq U$.

Theorem 3.2.20

Let G be a group and $\mathcal{F}$ a collection of subsets of G such that

(1) $\mathcal{F}$ is filter base

(2) for each $U \in \mathcal{F}$ there is $V \in \mathcal{F}$ such that $V^2 \subseteq U$.

(3) For each $U \in \mathcal{F}$ and each $a \in G$, there is $V \in \mathcal{F}$ such that $a V a^{-1} \subseteq U$.

Then there is a unique topology τ on G, such that $\mathcal{F}$ is a τ-neighbourhood base at e (G, τ, τ^{-1}) is a quasi-bitopological group.

Section 3.3

Bicompletion of quasi-bitopological group

A classical result (Wilansky [14]) states that for each Hausdorff topological group G there exists a Hausdorff topological group which is complete in its two sided uniformity and has G as a dense topological subgroup. The main result (Theorem 3.3.10) of this section extends this theorem to quasi-bitopological groups. For the construction of the required space we need the following definitions.

Definition 3.3.1

A ***bitopological space*** (X, τ_1, τ_2) is ***2-Hausdorff*** if $\tau_1 \vee \tau_2$ is a Hausdorff-topology. Thus a ***quasi-uniform space*** $(X, \mathcal{U})$ is said to be ***2-Hausdorff*** provided that $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$ is a 2-Hausdorff bitopological space.

Definition 3.3.2

Let (X, τ_1, τ_2) be a bitopological space. A subset $A \subseteq X$ is called ***2-dense*** in X if it is $\tau_1 \vee \tau_2$ dense in X.

Definition 3.3.3

A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is called $\mathcal{U}$*- ***Cauchy*** if for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that $F \times F \subset U$.

Definition 3.3.4

The quasi-uniform space $(X, \mathcal{U})$ is called ***bi-complete*** if each $\mathcal{U}^*$ - Cauchy filter converges with respect to $\tau(\mathcal{U}^*)$, i.e. if the uniform space $(X, \mathcal{U}^*)$ is complete.

We mention here two theorems due to Fletcher and Lindgren [3] which are needed for discussion.

Theorem 3.3.5 [3]

Let $(X, \mathcal{U})$ be a quasi-uniform space $(Y, \surd)$ a bicomplete 2-Hausdorff quasi-uniform space, and A a 2-dense subset of $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$ and let $f:(A, \mathcal{U}|_{A \times A}) \rightarrow (Y, \surd)$ be a quasi-uniformly continuous function. Then there exists a unique continuous extension $(X, \tau(\mathcal{U}^*)) \rightarrow (Y, \tau(\surd^*))$ of f, and g is quasi-uniformily continuous.

Theorem 3.3.6 [3]

Let $(X, \mathcal{U})$ be a quasi-uniform space and let A be a 2-dense subset of $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$. If every cauchy filter on the uniform space $(A, \mathcal{U}^*|_{A \times A})$ converges in $(X, \tau(\mathcal{U}^*))$, then $(X, \mathcal{U})$ is bicomplete.

Definition 3.3.7

A ***bicompletion*** of a quasi-uniform space (X, $\mathcal{U}$) is a bicomplete quasi uniform space (Y,√) that has a 2-dense subspace quasi - **unimorphic** to (X, $\mathcal{U}$).

Definition 3.3.8

A $\mathcal{U}$*- cauchy filter on a quasi-uniform space (X, $\mathcal{U}$) is said to be ***minimal*** provided that it contains no $\mathcal{U}$* - cauchy filter other than itself.

Construction of the bicompletion of a 2-Hausdorff quasi-uniform space due to Fletcher & Lindgren [3] is provided in the following theorem.

Theorem 3.3.9 [3]

Let (X, $\mathcal{U}$) be a 2-Hausdorff quasi-uniform space. Let Y = { $\mathcal{F}$: $\mathcal{F}$ is a minimal $\mathcal{U}$* -cauchy filter on X}. For each U ∈ $\mathcal{U}$ let V_U = {($\mathcal{F}$, $\mathcal{G}$) ∈ Y x Y: there exist F ∈ $\mathcal{F}$ and G ∈ $\mathcal{G}$ such that F x G ⊆ U} and let √ = { V_U : U ∈ $\mathcal{U}$ }.

Then

(a) (Y, √) is a 2-Hausdorff bicomplete quasi-uniform space that has a 2-dense subspace quasi-unimorphic to (X, $\mathcal{U}$). i.e (Y, √) is a 2-Hausdorff bicompletion of (X, $\mathcal{U}$).

(b) Any 2-Hausdorff bicomplete quasi-uniform space of (X, $\mathcal{U}$) is quasi-unimorphic to (Y, √).

The 2-Hausdorff bicomplete quasi-uniform space (Y, √) of the preceding theorems is called the bicompletion of (X, $\mathcal{U}$).

The chapter is concluded by proving the crucial theorem of this section.

Theorem 3.3.10

Let (G, τ, τ^{-1}) be a 2-Hausdorff quasi -bitopological group. Then there is a 2-Hausdorff quasi-bitopological group which is bicomplete in its two sided quasi-uniformity and has G as a 2-dense quasi-bitopological subgroup.

Proof

Let $\mathcal{B}$ be the two-sided quasi-uniformity for (G, τ, τ^{-1}) and let $(Y, \surd)$ be the bicompletion as in theorem 3.3.9 of $(G, \mathcal{B})$. Then the uniform space $(Y, \surd^*)$ is the completion of the uniform space $(G, \mathcal{B}^*)$.

By theorem 3.2.11, $\mathcal{B}^*$ is the two-sided uniformity for (G, τ^*). Hence the Hausdorff topological space $(Y, \tau(\surd^*))$ can be endowed of a structure of topological group as follows.

For $y_1, y_2 \in Y$, let

$$\mathcal{F}_i = \{ W \cap G : W \text{ is a } \tau(\surd^*) \text{ - neighbourhood of } y_i \},\ i = 1, 2.$$

Then $\mathcal{F}_i$, I = 1, 2, is a $\mathcal{B}^*$- cauchy filter on G and thus $\mathcal{F}_1\mathcal{F}_2$ is a $\mathcal{B}^*$-cauchy filter base on G. Define y_1y_2 as the $\tau(\surd^*)$-limit point of $\mathcal{F}_1\mathcal{F}_2$. Similarly for $y_1 \in Y$ define y_1^{-1} as the $\tau(\surd^*)$ limit point of the $\mathcal{B}^*$-Cauchy filter $\mathcal{F}_1^{-1}$. By theorem 1.2.7 (Wilansky [14]) $(Y, \tau(\surd^*))$ is a topological group and has G as a subgroup.

Now to prove that the multiplication

$m : (Y \times Y, \tau(\surd) \times \tau(\surd)) \rightarrow (Y, \tau(\surd))$ is continuous.

Let $y_1, y_2 \in Y$ and let $V \in \surd$.

Then $V = V_B$ where $B \in \mathcal{B}$

Therefore there exists two τ-neighbourhoods U and W of e such that

$U_L \cap U_R \subseteq B$ and

$W^4 \subseteq U$

Since y_1 and y_2 are $\mathcal{B}^*$ - cauchy filters on G, there exists $F_1' \in y_1$ and $F_2' \in y_2$ such that $F_1' \times F_1' \subseteq W_L \cap W_R$ and $F_2' \times F_2' \subseteq W_L \cap W_R$.

Choose $a_1 \in F_1'$ and $a_2 \in F_2'$

Then by thoerem 3.2.19 there exists a τ-neighbourhood $H \subseteq W$ of e such that $a_1 H a_1^{-1} \subseteq W$ and $a_2^{-1} H a_2 \subseteq W$.

Put $H_L \cap H_R = B_0$

Hence $B_0 \in \mathcal{B}$

Let $V_0 = V_{B_0}$

To prove

$V_0(y_1)\, V_0(y_2) \subseteq V(y_1 y_2)$

Take $y \in V_0(y_1)\, V_0(y_2)$. Then there exists $\mathcal{H}_1 \in V_0(y_1)$ and $\mathcal{H}_2 \in V_0(y_2)$ such that $y = \mathcal{H}_1 \mathcal{H}_2$

$\mathcal{H}_1 \in V_0(y_1)$

$\Rightarrow (y_1, \mathcal{H}_1) \in V_0$.

$\Rightarrow$ there exists $F_1 \in y_1$ and $H_1 \in \mathcal{H}_1$ such that $F_1 \times H_1 \subseteq B_0$.

Similarly since $\mathcal{H}_2 \in V_0(y_2)$

there exists $F_2 \in y_2$ and $H_2 \in \mathcal{H}_2$ such that $F_2 \times H_2 \subseteq B_0$. (Here we assume $F_1 \subseteq F_1'$ and $F_2 \subseteq F_2'$)

Claim

$F_1F_2 \times H_1H_2 \subseteq U_L \cap U_R$. Take $(f_1f_2, h_1h_2) \in F_1F_2 \times H_1H_2$

consider $(f_1f_2)^{-1} h_1h_2 = f_2^{-1} a_2 a_2^{-1} f_1^{-1} h_1 a_2 a_2^{-1} f_2 f_2^{-1} h_2$

since $(f_2,a_2) \in F_2' \times F_2' \subseteq W_L \cap W_R \in W_L$, $f_2^{-1}a_2 \in W$ -------- (1)

Similarly $a_2^{-1} f_2 \in W$ ------------------------------- (2)

Since $(f_1, h_1) \in F_1 \times H_1 \subseteq B_0 = H_L \cap H_R \subset H_L$, $f_1^{-1}h_1 \in H$.

Similarly $f_2^{-1}h_2 \in H$ and since $H \subset W$ $f_2^{-1}h_2 \in W$ ------------ (3)

Also $a_2^{-1} f_1^{-1}h_1a_2 \in a_2^{-1}Ha_2 \subseteq W$ --------------------- (4)

From (1), (2), (3) and (4) we get $(f_1, f_2)^{-1} h_1h_2 \in W^4 \subset U$

Similarly $h_1h_2(f_1f_2)^{-1} = h_1 f_1^{-1} f_1a_1^{-1}a_1 h_2 f_2^{-1} a_1a_1^{-1} f_1^{-1} \in W^4 \subset U$.

$\therefore$ $(f_1f_2, h_1h_2) \in U_L \cap U_R$.

Hence $F_1F_2 \times H_1H_2 \subset U_L \cap U_R \subset B$

Consequently we get

$$(y_1y_2, \mathcal{H}_1 \mathcal{H}_2) \in V_B = \surd$$

$$\text{(i.e).} \qquad y = \mathcal{H}_1 \mathcal{H}_2 \in V(y_1 y_2)$$

$\therefore$ $V_0(y_1)\, V_0(y_2) \subset V(y_1y_2)$ and multiplication is continuous.

Hence $(Y, \tau(\surd), \tau(\surd^{-1}))$ is a 2-Hausdorff quasi-bitopological subgroup that has G as a 2-dense quasi-bitopological subgroup and such that $(Y, \surd)$ is bicomplete.

Let $\widehat{\mathcal{B}}$ denote the two sided quasi uniformity for the quasi-bitopological group $(Y,\tau(\surd), \tau(\surd^{-1}))$

Next to show:-

$$\surd = \widehat{\mathcal{B}}$$

since $\widehat{\mathcal{B}}|_{G \times G} = \mathcal{B}$

and since every cauchy filter on $(G, \widehat{\mathcal{B}^*}|_{G \times G}) = (G, \mathcal{B}^*)$ converges in $(Y, \tau(\surd^*)) = (Y, \tau(\widehat{\mathcal{B}^*}))$, by theorem 3.3.6 we get $(Y, \widehat{\mathcal{B}})$ is bicomplete. By thoerem 3.3.9

$$\surd = \widehat{\mathcal{B}}$$

Hence the theorem.

Summary and Conclusion

SUMMARY AND CONCLUSION

The study of topological groups has proved to be of great interest since 1926. In this thesis, we have concentrated mainly on the following two topics:

(1) A study of ***theta topological groups***.

(2) A study of ***quasi-bitopological groups***.

In chapter II, we have discussed interesting results on the category of theta topological groups and theta continuous mappings. Regarding quotient theta topological groups, the study has been made by E.J. Beggs and E. Hatir [1] only for the special class of quotient theta topological groups, namely, $\frac{G}{H(G)}$, where $H(G) = \cap\{\overline{U} / U$ is a neighbourhood of $e\}$. It will be worthy if this study can be extended to general quotient theta topological groups $\frac{G}{N}$, where N is any normal subgroup of G.

The notion of an absolute valued function on a topological group G permits one to construct a pseudometric on G for which $(G, \tau(d))$ is a topological group.

J. Marin and S. Romaguera [6] have introduced the notion of an absolute quasi-valued function and established connections between absolute quasi-valued functions, quasi-bitopological groups and quasi pseudo-metrics. They [6] have also obtained extensions of classical theorems on topological groups to theta topological groups. Some of the important extensions are as follows.

(1) Every quasi-bitopological group is quasi uniformizable.

(2) Every first countable quasi-bitopological group admits a compatible left-invariant quasi-pseudometric.

(3) If (G, τ, τ^{-1}) is a 2-Hausdorff quasi-bitopological group, then there is a 2-Hausdorff quasi-bitopological group which is bicomplete in its two-sided quasi uniformity and has G as a 2-dense quasi-bitopological subgroup.

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