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gπ - Compactness and gπ-Connectedness Topological Spaces

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ABSTRACT

This paper deals with gπ-compact spaces and their properties by using nets, filter base and gπ-complete accumulation points. The notion of gπ-connectedness in topological spaces is also introduced and their properties are studied.

Mathematics Subject Classification: *54D05, 54D30*

Keywords: *gπ-open set, gπ-closed sets, gπ-compact spaces, gπ-complete accumulation point, gπ-connectedness.*

1. Introduction:

The notions of compactness and connectedness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Many researchers have investigated the basic properties of compactness and connectedness. The productivity and fruitfulness of these notions of compactness and connectedness motivated mathematicians to generalize these notions. In the course of these attempts many stronger and weaker forms of compactness and connectedness have been introduced and investigated.

In 1970, Levine [6] initiated the study of so called g-closed sets, that is, a subset A of a topological space (X, τ) is said to be g-closed if the closure of A is included in every open superset of A and defined a T_{12} space to be one in which the closed sets and g-closed sets coincide. Recently, Dontchev and Noiri [4] introduced the notion of πg-closed sets and used this notion to obtain a characterizations and some preservation theorems for quasi normal spaces.

More recently, Park [7] has introduced and studied the notion of πgp-closed sets which is implied by that of gp-closed sets. Park and Park [8] continued the study of πgp-closed sets and associated functions and introduced the concepts of πGP-compactness and πGPconnectedness. Also Aslim, Guler and Noiri [3] introduced the concept of πgs-closed sets and studied it's basic properties. Moreover they also introduced the notions of πgs-$T_{1/2}$ spaces and πgs-continuity in topological spaces. The aim of this paper is to introduce the concept of gπ-compactness and gπ-connectedness in topological spaces and is to give some characterizations of gπ-compact spaces in terms of nets and filter bases. The notion of gπ-complete accumulation points is introduce and is used to characterize gπ-compactness. Further it is proved that gπ-connectedness is preserved under gπ-irresolute surjections.

2 Preliminary Notes:

Throughout this paper (X, τ), (Y, σ) are topological spaces with no separation axioms assumed unless otherwise stated. Let A ⊆ X. The closure of A and the interior of A will be denoted by Cl(A) and Int(A) respectively.

Definition: 2.1 A subset A of a topological space (X, τ) is said to be gπ-closed [9] if πcl(A) ⊆ U whenever A ⊆ U and U is open in X.

Definition: 2.2 The gπ-closure [9] of a set A, denoted by gπCl(A), is the intersection of all gπ-closed sets containing A.

Definition: 2.3 The gπ-interior [9] of a set A, denoted by gπInt(A), is the union of all gπ-open sets contained in A.

Remark: 2.4 [9] Every π-closed set is gπ-closed.

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3 gπ - Compactness:

Definition: 3.1 A collection $\{A_i: i \in \Lambda\}$ of gπ-open sets in a topological space X is called a gπ-open cover of a subset B of X if $B \subset \{A_i: i \in \Lambda\}$ holds.

Definition 3.2 A topological space X is gπ-compact if every gπ-open cover of X has a finite sub-cover.

Definition: 3.3 A subset B of a topological space X is said to be gπ-compact relative to X if, for every collection $\{A_i : i \in \Lambda\}$ of gπ-open subsets of X such that $B \subset \{A_i : i \in \Lambda\}$ there exists a finite subset Λ_0 of Λ such that $B \subset \{A_i : i \in \Lambda\}$.

Definition: 3.4 A subset B of a topological space X is said to be gπ-compact if B is gπ-compact as a subspace of X.

Theorem: 3.5 Every gπ-closed subset of a gπ-compact space is gπ-compact relative to X.

Proof: Let A be gπ-closed subset of gπ-compact space X. Then A^C is gπ-open in X.

Let $M = \{G\alpha : \alpha \in \Lambda\}$ be a cover of A by gπ-open sets in X. Then $M_* = M \cup A^c$ is a gπ-open cover of X. Since X is gπ-compact M_* is reducible to a finite subcover of X, say

$X = G_{\alpha 1} \cup G_{\alpha 2} \cup \cdots \cup G_{\alpha m} \cup A^c$, $G_{\alpha k} \in M$. But A and A^C are disjoint hence

$A \subset G_{\alpha 1} \cup \cdots \cup G_{\alpha m}$, $G_{\alpha k} \in M$, which implies that any gπ-open cover M of A contains a finite sub-cover. Therefore A is gπ-compact relative to X. Thus every gπ-closed subset of a gπ-compact space X is gπ-compact.

Definition: 3.6 A function f: X → Y is said to be gπ-continuous [5] if f^{-1} (V) is gπ-closed in X for every closed set V of Y.

Definition: 3.7 A function f: X → Y is said to be gπ-irresolute [5] if f^{-1} (V) is gπ-closed in X for every gπ-closed set V of Y.

Theorem: 3.8 A gπ-continuous image of a gπ-compact space is compact.

Proof: Let f: X → Y be a gπ-continuous map from a gπ-compact space X onto a topological space Y. Let $\{A_i: i \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in \Lambda\}$ is a gπ-open cover of X. Since X is gπ-compact it has a finite sub-cover say $\{f^{-1}(A_1), f^{-1}(A_2), \cdots, f^{-1}(A_n)\}$. Since f is onto $\{A_1, \cdots, A_n\}$ is a cover of Y, which is finite. Therefore Y is compact.

Theorem: 3.9 If a map f: X → Y is gπ-irresolute and a subset B of X is gπ-compact relative to X, then the image f (B) is gπ-compact relative to Y.

Proof: Let $\{A_\alpha : \alpha \in \Lambda\}$ be any collection of gπ-open subsets of Y such that $f(B) \subset \cup \{A_\alpha : \alpha \in \Lambda\}$. Then $B \subset \{f^{-1}(A_\alpha) : \alpha \in \Lambda\}$ holds. Since by hypothesis B is gπ-compact relative to X there exists a finite subset Λ_0 of Λ such that $B \subset \cup \{f^{-1}(A_\alpha) : \alpha \in \Lambda_0\}$.

Therefore we have $f(B) \subset \cup \{A_\alpha : \alpha \in \Lambda_0\}$, which shows that f(B) is gπ-compact relative to Y.

Definition: 3.10 Let Λ be a directed set. A net $\xi = \{x_\alpha : \alpha \in \Lambda\}$ gπ accumulates at a point $x \in X$ if the net is frequently in every $U \in g\pi O(X, x)$, i.e., for each $U \in g\pi O(X, x)$ and for each $\alpha_0 \in \Lambda$, there is some $\alpha \geq \alpha_0$ such that $x_\alpha \in U$. The net ξ gπ-converges to a point x of X if it is eventually in every $U \in g\pi O(X, x)$.

Definition: 3.11 We say that a filter base $\Theta = \{F_\alpha : \alpha \in \Gamma\}$ gπ-accumulates at a point $x \in X$ if $x \in D \cap_{\alpha \in \Gamma} g\pi Cl(F_\alpha)$. A filter base $\Theta = \{F_\alpha : \alpha \in \Gamma\}$ gπ-converges to a point x in X if for each $U \in g\pi O(X, x)$, there exists an F_α in Θ such that $F_\alpha \subset U$.

Definition: 3.12 A point x in a space X is said to be gπ-complete accumulation point of a subset S of X if $Card(S \cap U) = Card(S)$ for each $U \in g\pi O(X, x)$, where Card(S) denotes the cardinality of S.

Definition: 3.13 In a topological space X, a point x is said to be a gπ-adherent point of a filter base Θ on X if it lies in the gπ-closure of all sets of Θ.

Theorem: 3.14 A space X is gπ-compact if and only if each infinite subset of X has a gπ-complete accumulation point.

Proof: Let the space X be gπ-compact and let S be an infinite subset of X. Let K be the set of points x in X which are not gπ-complete accumulation points of S. Now it is obvious that for each x in K, we are able to find U(x) ∈ gπO(X, x) such that Card(S ∩ U(x)) ≠ Card(S). If K is the whole space X, then Θ = {U(x): x ∈ X} is a gπ-cover of X. By the hypothesis X is gπ-compact, so there exists a finite sub-cover ψ = {U (x_i)}, where i = 1, 2, · · ·, n such that S ⊂ {U (x_i) ∩ S : i = 1, 2, · · ·, n}. Then Card(S) = max {Card (U (x_i) ∩ S)}, where i = 1, 2, · ·, n which does not agree with what we assumed. This implies that S has gπ-complete accumulation point.

Conversely, suppose that X is not gπ-compact and that every infinite subset S ⊂ X has a gπ-complete accumulation point in X. It follows that there exists a gπ-cover E with no finite sub-cover. Set δ = min {Card(Φ) : Φ ⊂ E}, where Φ is a gπ-cover of X. Fix ψ ⊂ E for which Card(ψ) = δ and ∪{U : U ∈ ψ} =X. Let N denote the set of natural numbers. Then by hypothesis δ ≥ Card (N). By well ordering of ψ by some minimal well ordering ~ suppose that U is any member of ψ. By minimal well ordering ~ we have Card ({V: V ∈ ψ, V ~ U}) < Card ({V: V ∈ ψ}). Since ψ can not have any sub-cover with cardinality less than δ, then for each U ∈ ψ we have X ≠ ∪{V: V □ ψ, V ~ U}. For each U ∈ ψ choose a point x(U) ∈ X – U {V ∪{x(V)}: V ∈ ψ, V ~ U}. We are always able to do this if not one can choose a cover of smaller cardinality from ψ. If H = {x(U) : U ∈ ψ}, then to finish the proof we will show that H has no gπ-accumulation points in X. Suppose that z is a point of X. Since ψ is a gπ-cover of X, then z is a point of some set W in ψ. By the fact that U ~ W, we have x(U) ∈W. But Card (T) < δ. Therefore, Card (H ∩ W) < δ. But Card (H) = δ ≥ Card (N), since for two distinct points U and W in ψ, we have x(U) ≠ x(W).

This means that H has no gπ-complete accumulation point in X which contradicts our assumptions. Therefore X is gπ-compact.

Theorem: 3.15 For a space X the following are equivalent.

(1) X is gπ-compact.
(2) Every net in X with a well ordered directed set as its domain gπ-accumulates to some point of X.

Proof: (1) ⇒ (2):
Suppose that (X, τ) is gπ-compact and ξ = {x_α : α ∈ Λ} a net with a well ordered directed set Λ as domain. Assume that ξ has no gπ adherent point in X. Then for each point x in X, there exist V (x) ∈ gπO(X, x) and an α(x) ∈Λ such that V (x) ∩ {x_α : α ≥ α(x)} = φ. This implies that {x_α : α ≥ α(x)} is a subset of X – V (x). Then the collection C = {V (x): x ∈ X} is a gπ-cover of X. By hypothesis of theorem, X is gπ-compact and so C has a finite subfamily {V (x_i)}, where i = 1, 2, · · ·, n such that X = ∪{V (x_i)}. Suppose that the corresponding elements of Λ are {α (x_i)}, where i = 1, 2,.., n. Since Λ is well ordered and {α(x_i)}, where i = 1, 2, · ·, n is finite, the largest element of {α(x_i)} exists. Suppose it is {α(x_l)}, then for γ ≥ {α(x_l)}. We have {x_δ: δ ≥ γ} ⊂ D$\cup_{i=1}^{n}(X - V(x_i)) = X - \cup_{i=1}^{n} V(x_i)$ = φ which is impossible.

This shows that ξ has at least one gπ-adherent point in X.

(2) ⇒ (1): Now by the last Theorem 3.14, it is enough to prove that each infinite subset has a gπ-complete accumulation point. Suppose that S ⊂ X is an infinite subset of X. According to Zorn's lemma, the infinite set S can be well ordered. This means that we can assume S to be a net with a domain, which is a well ordered index set. It follows that S has a gπ-adherent point z. Therefore z is a gπ-complete accumulation point of S. This shows that X is gπ-compact.

Theorem: 3.16 A space X is gπ-compact if and only if each family of gπ-closed subsets of X with the finite intersection property has a non-empty intersection.

Proof: Given a collection **A** of subsets of X, let C = {X – A : A ∈ **A**} be the collection of their complements. Then the following statements hold.

(a) **A** is a collection of gπ-open sets if and only if C is a collection of gπ –closed sets.

(b) The collection **A** covers X if and only if the intersection $\cap_{c \in c}$C of all the elements of C is non-empty.

(c) The finite sub collection {A_1 · · ·A_n} of A covers X if and only if the intersection of the corresponding elements C_i = X – Ai of C is empty. The statement (a) is trivial, while the (b) and (c) follow from DeMorgan's law.

$X - (D\bigcup_{\alpha\in J} A_\alpha) = D\bigcap_{\alpha\in J} (X - A_\alpha)$ The proof of the theorem now proceeds in two steps, taking the contrapositive of the theorem and then the complement.

The statement X is gπ-compact is equivalent to: Given any collection $\mathcal{A}$ of gπ-open subsets of X, if $\mathcal{A}$ covers X, then some finite sub collection of $\mathcal{A}$ covers X. This statement is equivalent to its contrapositive, which is the following.

Given any collection $\mathcal{A}$ of gπ-open sets, if no finite sub-collection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X. Letting C be as earlier, the collection $\{X - A : A \in \mathcal{A}\}$ and applying (a) to (c), we see that this statement is in turn equivalent to the following:

Given any collection C of gπ-closed sets, if every finite intersection of elements of C is non-empty, then the intersection of all the elements of C is non-empty. This is just the condition of our theorem.

Theorem: 3.17 A space X is gπ-compact if and only if each filter base in X has at least one gπ-adherent point.

Proof: Suppose that X is gπ-compact and $\theta = \{F_\alpha : \alpha \in \Gamma\}$ a filter base in it. Since all finite intersections of F_α's are non-empty, it follows that all finite intersections of gπCl (F_α)'s are also non-empty. Now it follows from Theorem 3.16 that $D\bigcap_{\alpha\in\Gamma}$ gbCl (F_α) is non-empty. This means that θ has at least one gπ-adherent point.

Conversely, suppose θ is any family of gπ-closed sets. Let each finite intersection be non-empty. The sets (F_α) with their finite intersection establish a filter base θ. Therefore θ gπ-accumulates to some point z in X. It follows that $z \in D\bigcap_{\alpha\in\Gamma} (F_\alpha)$. Now by Theorem 3.16, we have that X is gπ-compact.

Theorem: 3.18 A space X is gπ-compact if and only if each filter base on X with at most one gπ-adherent point is gπ-convergent.

Proof: Suppose that X is gπ-compact, x a point of X and a filter base on X. The gπ-adherence of θ is a subset of {x}. Then the gπ-adherence of θ is equal to {x} by Theorem 3.17. Assume that there exists V ∈ gπO(X, x) such that for all F ∈ θ, F ∩ (X –V) is non-empty. Then

$\psi = \{F - V : F \in \theta\}$ is a filter base on X. It follows that the gπ-adherence of θ is non-empty. However,

$D\bigcap_{F\in\theta}$ gπCl$(F - V) \subset (D\bigcap_{F\in\theta}$ gπCl$(F) \cap (X - V)) = \{x\} \cap \{X - V\} = \varphi$.

But this is a contradiction. Hence for each V ∈ gπO(X, x) there exists an F ∈ θ with F ⊂ V. This shows that θ gπ-converges to x.

To prove the converse, it suffices to show that each filter base in X has at least one gπ-accumulation point. Assume that θ is a filter base on X with no gπ-adherent point. By hypothesis, θ gπ-converges to some point z in X. Suppose F_α is an arbitrary element of θ. Then for each V ∈ gπO(X, z), there exists $F_\beta \in \theta$ such that $F_\beta \subset V$. Since θ is a filter base, there exists a γ such that $F_\gamma \subset F_\alpha \cap F_\beta \subset F_\alpha \cap V$, where F_γ non-empty. This means that $F_\alpha \cap V$ is non-empty for every V ∈ gπO(X, z) and correspondingly for each α, z is a point of gπCl (F_α) it follows that $z \in D\bigcap_\alpha$ gπCl (F_α). Therefore z is a gπ-adherent point of θ which is contradiction.

This shows that X is gπ-compact.

4 gπ - Connectedness:

Definition: 4.1 A topological space X is said to be gπ-connected if X can not be expressed as a disjoint union of two non-empty gπ-open sets. A subset of X is gπ-connected if it is gπ-connected as a subspace.

Example: 4.2 Let X = {a, b} and let τ = {X, φ, {a}}. Then it is gπ-connected.

Remark: 4.3 Every gπ-connected space is connected but the converse need not be true in general, which follows from the following example.

Example: 4.4 Let X = {a, b} and let τ = {X, φ}. Clearly (X, τ) is connected. The gπ-open sets of X are {X, φ, {a}, {b}}. Therefore (X, τ) is not a gπ-connected space, because X = {a} ∪ {b} where {a} and {b} are non-empty gπ-open sets.

Theorem: 4.5 For a topological space X the following are equivalent.

(i) X is gπ-connected.
(ii) X and φ are the only subsets of X which are both gπ-open and gπ-closed.
(iii) Each gπ-continuous map of X into a discrete space Y with at least two points is a constant map.

Proof: (i) ⇒ (ii):
Let O be any gπ-open and gπ-closed subset of X. Then O^C is both gπ-open and gπ-closed. Since X is disjoint union of the gπ-open sets O and O^C implies from the hypothesis of (i) that either O = φ or O = X.

(ii) ⇒ (i):
Suppose that X = A ∪B where A and B are disjoint non-empty gπ-open subsets of X. Then A is both gπ-open and gπ-closed. By assumption A = φ or X.

Therefore X is gπ-connected.

(ii) ⇒ (iii):
Let f: X → Y be a gπ-continuous map. Then X is covered by gπ-open and gπ-closed covering $\{f^{-1}(Y): y \in (Y)\}$. By assumption $f^{-1}(y) = \varphi$ or X for each y ∈ Y. If $f^{-1}(y) = \varphi$ for all y ∈ Y, then f fails to be a map. Then there exists only one point y ∈ Y such that $f^{-1}(y) \neq \varphi$ and hence $f^{-1}(y) = X$. This shows that f is a constant map.

(iii) ⇒ (ii):
Let O be both gπ-open and gπ-closed in X. Suppose O ≠ φ. Let f: X → Y be a gπ-continuous map defined by f (O) = y and f (O^C) = {w} for some distinct points y and w in Y.

By assumption f is constant. Therefore we have O = X.

Theorem: 4.6 If f : X → Y is a gπ-continuous and X is gπ-connected, then Y is connected.

Proof: Suppose that Y is not connected. Let Y = A∪B where A and B are disjoint non-empty open set in Y. Since f is gπ-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty gπ-open sets in X. This contradicts the fact that X is gπ-connected. Hence Y is connected.

Theorem: 4.7 If f: X → Y is a gπ-irresolute surjection and X is gπ-connected, then Y is gπ-connected.

Proof: Suppose that Y is not gπ-connected. Let Y = A ∪ B where A and B are disjoint non-empty gπ-open set in Y. Since f is gπ-irresolute and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty gπ-open sets in X. This contradicts the fact that X is gπ-connected. Hence Y is gπ -connected.

Theorem: 4.8 In a topological space (X, τ) with at least two points, if πO(X, τ) = πCL(X, τ) then X is not gπ-connected.

Proof: By hypothesis we have πO(X, τ) = πCL(X, τ) and by Remark 2.4 we have every π closed set is gπ-closed, there exists some non-empty proper subset of X which is both gπ-open and gπ-closed in X. So by last Theorem 4.5 we have X is not gπ-connected.

Definition: 4.9 A topological space X is said to be $T_{g\pi}$-space if every gπ-closed subset of X is closed subset of X.

Theorem: 4.10 Suppose that X is a $T_{g\pi}$-space then X is connected if and only if it is gπ-connected.

Proof: Suppose that X is connected. Then X can not be expressed as disjoint union of two non-empty proper subsets of X. Suppose X is not a gπ-connected space. Let A and B be any two gπ-open subsets of X such that X = A∪B, where A ∩ B = φ and A ⊂ X, B ⊂ X. Since X is $T_{g\pi}$-space and A, B are gπ-open, A, B are open subsets of X, which contradicts that X is connected. Therefore X is gπ-connected.

Conversely, every open set is gπ-open. Therefore every gπ-connected space is connected.

Theorem: 4.11 If the gπ-open sets C and D form a separation of X and if Y is gπ-connected subspace of X, then Y lies entirely within C or D.

Proof: Since C and D are both gπ-open in X the sets C ∩ Y and D ∩ Y are gπ-open in Y these two sets are disjoint and their union is Y. If they were both non-empty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely in C or in D.

Theorem: 4.12 Let A be a gπ-connected subspace of X. If A ⊂ B ⊂ gπCl(A) then B is also gπ-connected.

Proof: Let A be gπ-connected and let A ⊂ B ⊂ gπCl(A). Suppose that B = C∪D is a separation of B by gπ-open sets. Then by Theorem 4.11 above A must lie entirely in C or in D. Suppose that A ⊂ C, then gπCl(A) ⊆ gπCl(C). Since

gπCl(C) and D are disjoint, B cannot intersect D. This contradicts the fact that D is non-empty subset of B. So D = φ which implies B is gπ-connected.

References:

[1] D. Andrijevic: Semi-preopen sets, Mat. Vesnik. 38(1) (1986), 24–32.

[2] S. P. Arya, T. M. Nour: Characterization of s-normal spaces, Indian. J. Pure. Appl. Math., 21 (1990), 717–719.

[3] G. Aslim, A. Cakshu Guler, T. Noiri: On πgs-closed sets in topological spaces, Acta. Math. Hungar., 112(4) (2006), 275–283.

[4] J. Dontchev, T. Noiri: Quasi normal spaces and πg-closed sets, Acta. Math. Hungar., 89 (2000), 211–219.

[5] E. D. Khalimsky: Applications of connected ordered topological spaces in topology, in: Conferenceof Math, Department of Povolsia (1970).

[6] N. Levine: Generalized closed sets in topology, Rend. Circ. Mat. Palermo. 19(2) (1970), 89–96.

[7] J. H. Park: On πgp-closed sets in topological spaces, Indian J. Pure Appl. Math (to appear).

[8] J. H. Park, J. K. Park: On πgp-continuous functions in topological spaces, Chaos, Solitons and Fractals, 20 (2004), 467–477.

[9] S.Saranya, A. Parvathi , gπ-closed sets in Topological spaces (Submitted)

[10] V.Zaitsav: On certain classes of topological spaces and their bicompacti fications, Dokl. Akad Nauk SSSR, 178 (1978), 778–779.

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ON gπ US SPACES

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Abstract

In this paper we introduce and study the notion of gπ –US spaces, gπ -convergency, sequentially gπ -compactness, sequentially gπ -continuity and sequentially gπ -sub-continuity by utilizing gπ -open sets.

***Keywords and phrases**: topological spaces, gπ – open sets, gπ US – spaces, gπ – convergence, sequentially GπO –compactness.*

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1. Introduction and Preliminaries:

In 1967, A. Wilansky [11] introduced and studied the concept of US -spaces. Levine [8] introduced the concept of generalized closed sets of a topological space. Since the advent of these notions, several research papers with interesting results in different respects came to existence (see, [1], [2], [3], [4], [5], [6], [9]). This paper is devoted to deal with the concepts of g π -US spaces, gπ -convergence, sequentially G π O-compactness, sequentially gπ -continuity and sequentially gπ -sub-continuity.

Throughout the present paper (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by Int(A) and Cl(A), respectively. A subset A is said to be regular open (resp. regular closed) if A = int(cl(A) (resp.A = cl(int(A)). The finite union of regular open sets is said to be π -open. The complement of a π-open set is said to be π-closed. $A \subset X$ is called a generalized closed set (briefly g-closed set) of X [8] if $Cl(A) \subset G$ holds whenever $A \subset G$ and G is open in X. A subset A of X is called a g -open set of X, if its complement A^c is g -closed in X. $A \subset X$ is called a gπ-closed set of X [10] if $\pi Cl(A) \subset G$ holds whenever $A \subset G$ and G is open in X. A subset A of X is called a gπ -open set of X, if its complement A^c is gπ -closed in X.

A space X is G Ocompact if every g-open cover of X has a finite subcover. A subset A of a space X is said to be GO-compact if A is GO-compact as a subspace of X. The product space of two non-empty spaces is GO-compact, if each factor space is GO compact [1]. If A is g-open in X and B is g-open in Y, then A× B is g-open in X ×Y [8]. A function f: X → Y is said to be g-continuous (resp.gπ continuous) [1] ([10]) if the inverse image of every closed set in Y is g-closed (resp.gπ-closed) in X.

2. gπ –US spaces:

Definition: 1 A sequence $\{x_n\}$ in a space X gπ-converges to a point x ∪ X if $\{x_n\}$ is eventually in every gπ-open set containing x.

Definition: 2 A space X is said to be gπ-US if every sequence in X gπ –converges to a point of X.

Definition: 3 A space X is said to be:

(1) gπ-T_1 if for each pair of distinct points x and y in X there exist a gπ-open set U in X such that $x \cup U$ and $y \notin U$ and a gπ-open set V in X such that $y \in V$ and $x \notin V$.

(2) gπ-T_2 if for each pair of distinct points x and y in X there exist gπ-open sets U and V such that $U \cap V = \emptyset$ and $x \in U$, $y \in V$.

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Theorem: 2.1 Every gπ-US space is gπ-T_1.

Proof: Let X be a gπ-US space and x, y be two distinct points of X. Consider the sequence $\{x_n\}$, where $x_n = x$ for any $n \in N$. Clearly, $\{x_n\}$ gπ converges to x. Since $x \neq y$ and X is gπ-US, $\{x_n\}$ does not gπ-converge to y, i.e., there exists a gπ-open set U containing x but not y. Similarly, we obtain a gπ-open set V containing y but not x. Thus, X is gπ $-T_1$.

Theorem: 2.2 Every gπ-T_2 space is gπ- US.

Proof: Let X be a gπ-T_2 space and $\{x_n\}$ a sequence in X. Assume that $\{x_n\}$ gπ-converges to two distinct points x and y. Then $\{x_n\}$ is eventually in every gπ-open set containing x and also in every gπ-open set containing y. Since X is gπ-T_2 then $\{x_n\}$ is eventually in two disjoint gπ-open sets. This is a contradiction. Therefore, X is gπ-US.

Definition: 4 A subset A of a space X is said to be:

(1) Sequentially gπ-closed if every sequence in A gπ-converges to a point in A,

(2) Sequentially GπO-compact if every sequence in A has a subsequence which gπ converges to a point in A.

Theorem: 2.3 A space is gπ-US if and only if the diagonal set Δ is a sequentially gπ-closed subset of the product space $X \times X$.

Proof: Suppose that X is a gπ-US space and $\{(x_n, x_n)\}$ is a sequence in the diagonal Δ. If follows that $\{x_n\}$ is a sequence in X. Since X is gπ-US, the sequence $\{x_n\}$ gπ-converges to a unique point, say $x \in X$. This implies that the sequence $\{(x_n, x_n)\}$ gπ-converges to (x, x) which clearly belongs to Δ. Therefore, Δ is a sequentially gπ-closed subset of $X \times X$.

Conversely, suppose that the diagonal Δ is a sequentially gπ-closed subset of $X \times X$. Assume that a sequence $\{x_n\}$ is gπ-converging to x and y. Then it follows that $\{(x_n, x_n)\}$ gπ-converges to (x, y). By hypothesis, since Δ is sequentially gπ-closed, we have $(x, y) \in \Delta$. Thus, x = y. Therefore, X is gπ-US.

Theorem: 2.4 If a space X is gπ-US and a subset M of X is sequentially GπO compact, then M is sequentially gπ-closed.

Proof: Assume that $\{x_n\}$ is any sequence in M which gπ-converges to a point $x \in X$. Since M is sequentially GπO-compact, there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $\{x_{nk}\}$ gπ-converges to $m \in M$. Since X is gπ-US, we have x = m. This shows that M is sequentially gπ-closed.

Theorem: 2.5 The product space of an arbitrary family of gπ-US topological spaces is a gπ-US topological space.

Proof: Let $\{X_\lambda: \lambda \in \Delta\}$ be a family of gπ-US topological spaces with the index set Δ. The product space of $\{X_\lambda: \lambda \in \Delta\}$ is denoted by ΠX_λ .Let $\{x_n(\lambda)\}$ be a sequence in ΠX_λ. Suppose that $\{x_n(\lambda)\}$ gπ-converges to two distinct points x and y in ΠX_λ. Then there exists a $\lambda_0 \in \Delta$ such that $x(\lambda_0) \neq y(\lambda_0)$. Then $\{x_n(\lambda_0)\}$ is a sequence in $X_{\lambda 0}$. Let $V_{\lambda 0}$ be any gπ-open set in $X_{\lambda 0}$ containing $x(\lambda_0)$. Then $V = V_{\lambda 0} \times \Pi_{\lambda \neq \lambda 0} X_\lambda$ is a gπ-open set of ΠX_λ containing x. Therefore, $\{x_n(\lambda)\}$ is eventually in V. Thus, $\{x_n(\lambda_0)\}$is eventually in $V_{\lambda 0}$ and it gπ-converges to $x(\lambda_0)$. Similarly, the sequence $\{x_n(\lambda_0)\}$ gπ-converges to $y(\lambda_0)$. This is a contradiction as $X_{\lambda 0}$ is a gπ-US space. Therefore, the product space ΠX_λ is gπ-US.

3. Sequentially GπO-compact preserving functions:

Definition: 5 A function f: X →Y is said to be:

(1) Sequentially gπ-continuous at $x \in X$ if the sequence $\{f(x_n)\}$ gπ- converges to f (x) whenever a sequence $\{x_n\}$ gπ-converges to x. If f is sequentially gπ-continuous at each $x \in X$, then it is said to be sequentially gπ-continuous.

(2) Sequentially nearly gπ-continuous, if for each sequence $\{x_n\}$ in X that gπ-converges to $x \in X$, there exists subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that the sequence $\{f(x_{nk})\}$ gπ-converges to $\{(x_n)\}$.

(3) Sequentially sub gπ-continuous if for each point $x \in X$ and each sequence $\{x_n\}$ in X gπ-converging to x, there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ and a point $y \in Y$ such that the sequence $\{f(x_{nk})\}$ gπ-converge to y.

(4) Sequentially GπO-compact preserving if the image f(M) of every sequentially GπO-compact set M of X is a sequentially GπO-compact subset of Y.

Theorem: 3.1 Let f_1: X →Y and f_2: X →Y be two sequentially $g\pi$- continuous functions. If Y is $g\pi$-US, then the set E = {x ∈ X: $f_1(x) = f_2(x)$} is sequentially $g\pi$-closed.

Proof: Suppose that Y is $g\pi$-US and {x_n} is any sequence in E that f_1-converges to x ∈ X. Since f_1 and f_2 are sequentially $g\pi$-continuous functions, the sequence {$f_1(x_n)$} (respectively, {$f_n(x_n)$}) converges to $f_1(x)$ respectively, $f_2(x)$). Since x_n ∈ E for each n ∈ N and Y is $g\pi$-US, $f_1(x) = f_2(x)$ and hence x ∈ E. This shows that E is sequentially $g\pi$-closed.

Lemma: 3.2 Every function f: X →Y is sequentially sub-$g\pi$-continuous if Y is sequentially GπO-compact.

Proof: Let {x_n} be a sequence in X that $g\pi$ converges to x ∈ X. It follows that {f (x_n)} is a sequence in Y. Since Y is sequentially GπO compact, there exists a subsequence {(x_{nk})} of {f(x_n)} that $g\pi$-converges to a point y ∈Y .

Therefore, f: X →Y is sequentially sub-$g\pi$-continuous.

Theorem 3.3 Every sequentially nearly $g\pi$-continuous function is sequentially GπO-compact preserving.

Proof: Let f: X →Y be a sequentially nearly $g\pi$-continuous function and M be any sequentially GπO-compact subset of X. We will show that f (M) is a sequentially GπO-compact subset of Y. So, assume that {y_n} is any sequence in f (M). Then for each n ∈ N, there exists a point x_n ∈ M such that f (x_n) = y_n. Now M is sequentially GπO-compact, so there exist a subsequence {x_{nk}} of {x_n} that $g\pi$-converges to a point x ∈M. Since f is sequentially nearly $g\pi$-continuous, there exists a subsequence {x_{nk} (i)} of { x_{nk} } such that {f (x_{nk} (i))} $g\pi$-converges to f (x) . Therefore, there exists a subsequence {y_{nk} (i)} of {y_n} that $g\pi$-converges to f (x) . This implies that f (M) is a sequentially GπO-compact set of Y.

Theorem: 3.4 Every sequentially GπO-compact preserving function is sequentially sub-$g\pi$-continuous.

Proof: Suppose that f: X → Y is a sequentially GπO-compact preserving function. Let x be any point of X and {x_n} a sequence that $g\pi$ converges to x. We denote the set {x_n : n ∈ N} by A and put M = A ∪{x}. Since {x_n} $g\pi$-converges to x, M is sequentially GπO-compact. By hypothesis, f is sequentially GπO-compact preserving and hence f (M) is a sequentially GπO-compact subset of Y. Now in f (M) there exists a subsequence {f (x_{nk}) } of {f (x_n)} that $g\pi$-converges to a point y ∈ f (M) . This implies that f sequentially sub-$g\pi$-continuous.

Theorem 3.5 A function f: X →Y is sequentially GπO-compact preserving if and only if f $|_M$: M → f (M) is sequentially sub $g\pi$-continuous for each sequentially GπO-compact set M of X.

Proof: Necessity: suppose that f: X →Y is a sequentially GπO-compact preserving function. Then f (M) is sequentially GπO-compact in Y for each sequentially GπO-compact subset M of X. Therefore, by Theorem 3.4 f $|_M$: M → f (M) is sequentially sub-$g\pi$-continuous.

Sufficiency: Let M be any sequentially GπO-compact set of X. We will show that f (M) is sequentially GπO-compact subset of Y. Let {y_n} be any sequence in f (M). Then for each n ∈ N , there exists a point x_n ∈ M such that f (x_n) = y_n

Since {x_n} is a sequence in the sequentially GπO-compact set M there exists a subsequence {x_{nk}} of {x_n} that $g\pi$-converges to a point in M. By hypothesis f $|_M$: M → f (M) is sequentially sub-$g\pi$-continuous hence there exists a subsequence {y_{nk}} of {y_n} that $g\pi$-converges to y ∈ f (M). This implies that f (M) is sequentially GπO-compact in Y.

Corollary: 3.6 If a function f: X →Y is sequentially sub-$g\pi$-continuous and f (M) is sequentially $g\pi$-closed in Y for each sequentially GπO-compact set M of X, then f is sequentially GπO-compact preserving.

Proof: It will suffice to show that is sequentially sub-$g\pi$-continuous for each sequentially GπO-compact set M of X, and by Lemma 3.2 we are done. So, let {x_n} be any sequence in M that $g\pi$-converges to a point x ∈ M. Then, since f is sequentially sub-$g\pi$-continuous there exists a subsequence {x_n} of {x_n} and a point y ∈ Y such that {f (x_{nk})} $g\pi$ converges to y.

Since {f(x_{nk})} is a sequence in the sequentially $g\pi$-closed set f (M) of Y, we obtain y ∈ f (K) . This implies that f $|_M$: M → f (M) is sequentially sub-$g\pi$-continuous.

REFERENCES:

[1] K. Balachandran, P. Sundaram and H. Maki, On generalized continuous maps in topological spaces, Mem. Fac. Sci. Kochi Univ (Math.), 12 (1991), 5-13.

[2] M. Caldas, On g-closed sets and g-continuous mappings, Kyungpook Math. J., 33(1993), 205-209.

[3] M. Caldas, Further results on generalized open mappings in topological spaces, Bull. Cal. Math. Soc. 88(1996), 37-42.

[4] W. Dunham and N. Levine, Further results on generalized closed sets in topology, Kyungpook Math. J., 20(1980), 169-175.

[5] W. Dunham, 1/ 2 T -spaces, Kyungpook Math. J., 17(1977), 161-169.

[6] W. Dunham, A new closure operator for non- 1 T topologies, Kyungpook Math. J., 22(1982), 55-60.

[7] J. Dontchev and T. Noiri. Quasi-normal spaces and gπ-closed sets, Acta Math Hungar., 89 (2000) 211-219.

[8] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo (2), 19(1970), 89-96.

[9] S. R. Malghan, Generalized closed sets maps, J. Karnatak Univ. Sci., 27(1982), 82-88.

[10] S. Saranya, A.Parvathi, Gπ-closed sets in topological spaces (Submitted).

[11] A. Wilanski, Between 1 T and 2 T, Amer. Math. Monthly Vol. (1967), 261-266.

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