CHAPTER III

SOFT Γ- SEMIGROUPS

Definition : 3.1

Let S and Γ be two non-empty sets. Then S is called a **Γ-semigroup** if there exist a mapping $S \times \Gamma \times S \to S$, written as $(x, \gamma, y) \mapsto x\gamma y$, such that $(x\gamma y)\beta z = x\gamma(y\beta z)$ for all $x, y, z \in S$ and $\gamma, \beta \in \Gamma$. For nonempty subsets A and B of S, let

$$A\Gamma B = \{a\gamma b / a \in A, b \in B, \gamma \in \Gamma\}$$

For $x \in S$, let $A\Gamma x = A\Gamma\{x\}$ and $x\Gamma A = \{x\}\Gamma A$.

Definition : 3.2

A nonempty subset T of S is called a **Γ-subsemigroup** of S if for all $x, y \in T$ and $\gamma \in \Gamma$, $x\gamma y \in T$.

S is said to be **commutative** if for all $x, y \in S$ and $\gamma \in \Gamma$, $x\gamma y = y\gamma x$.

Definition : 3.3

Let S be a Γ-semigroup. A nonempty subset A of S is called a **left** (**right**) **ideal** of S if

$$S\Gamma A \subseteq A (A\Gamma S \subseteq A)$$

If A is both a left and right ideal of S, then A is called an ideal of S. An ideal A of S is said to be **idempotent** if $A\Gamma A = A$.

Hereafter, let S be a Γ-semigroup.

Definition : 3.4

The **Γ-restricted product** of soft sets (F, A) and (G, B) over S, denoted by $(F, A)\bar{\Gamma}(G, B)$, is defined as a soft set

$$(K, D) = (F, A)\bar{\Gamma}(G, B)$$

Where $D = A \cap B \neq \emptyset$ and $K: D \rightarrow P(S)$ such that $K(d) = F(d)\Gamma G(d)$ for all d $\in$ D.

Definition : 3.5

A soft set (F, A) over S is called a **soft Γ- semigroup** over S if

$$(F, A)\bar{\Gamma}(F, A) \subseteq (F, A)$$

Example : 3.6

Let $M = \{-i, 0, i\}$ and $\Gamma = A = \{-i, i\}$. We have M is a Γ-semigroup. Define $F: A \rightarrow$ P(M) by $F(i) = F(-i) = A$. Then (F, A) is a soft Γ-semigroup over M.

Theorem : 3.7

A soft set (F, A) over S is a soft Γ-semigroup over S if and only if for all $a \in A$ such that $F(a) \neq \emptyset$, $F(a)$ is a Γ-subsemigroup of S.

Proof :

Assume that a soft set (F, A) over S is a soft Γ-semigroup over S. Let $a \in A$ be such that $F(a) \neq \emptyset$, we have

$$(F, A)\bar{\Gamma}(F, A) = (K, A \cap A) \text{ and } K(a) = F(a)\Gamma F(a) \text{ for all } a \in A.$$

since $K \subseteq F, K(a) \subseteq F(a)$. So, $F(a)\Gamma F(a) \subseteq F(a)$.

Thus $F(a)$ is a Γ-subsemigroup of S.

Conversely, assume that $F(a)$ is a Γ-subsemigroup of S for all $a \in A$ such that $F(a) \neq \emptyset$, we have

$$(F, A)\bar{\Gamma}(F, A) = (K, A \cap A) \text{ and } K(a) = F(a)\Gamma F(a) \text{ for all } a \in A.$$

by assumption, $K(a) \subseteq F(a)$. Thus $(K, A) \subseteq (F, A)$.

Therefore a soft set (F, A) over S is a soft Γ-semigroup over S.

Definition : 3.8

A soft set (F, A) over S is called a **soft l-idealistic** (**r- idealistic**) over S if

$$(S, E)\bar{\Gamma}(F, A) \subseteq (F, A)\big((F, A)\bar{\Gamma}(S, E) \subseteq (F, A)\big).$$

Example : 3.9

We consider $\mathbb{Z}_8 = \{[0], [1], [2], [3], [4], [5], [6], [7]\}$.

Let $\Gamma = \{[1], [4]\}$, then $\mathbb{Z}_8$ is a Γ-semigroup. Take $C = \{[0], [4]\}$. Define $H : C \to P(\mathbb{Z}_8)$ by $H(c) = C$ for all $c \in C$. Then (H, C) is a soft l-idealistic over $\mathbb{Z}_8$.

Theorem : 3.10

A soft set (F, A) over S is a soft l-idealistic (r- idealistic) over S if and only if for all $a \in A$ such that $F(a) \neq \emptyset$, $F(a)$ is a left (right) ideal of S.

Proof :

Assume that a soft set (F, A) over S is a soft l-idealistic over S. Let $a \in A$ be such that $F(a) \neq \emptyset$, we have

$$(S, E)\bar{\Gamma}(F, A) = (K, E \cap A) \text{ and } K(a) = S(a)\Gamma F(a) \text{ for all } a \in A.$$

Since $K = F, K(a) \subseteq F(a)$ and $S\Gamma F(a) = S(a)\Gamma F(a) \subseteq F(a)$, where $F(a)$ is a left ideal of S.

Conversely, assume that $F(a)$ is a left ideal of S for all $a \in A$ such that $F(a) \neq \emptyset$, we have

$$(S, E)\bar{\Gamma}(F, A) = (K, E \cap A) \text{ and } K(a) = S(a)\Gamma F(a) \text{ for all } a \in A.$$

By assumption $K(a) = S(a)\Gamma F(a) = S\Gamma F(a) \subseteq F(a)$.

Thus $(K, A) \subseteq (F, A)$. Therefore, a soft set (F, A)over S is a soft l-idealistic over S.

Hence a soft set (F, A) over S is a soft l-idealistic over S if and only if for all $a \in A$ such that $F(a) \neq \emptyset$, $F(a)$ is a right ideal of S can be proved similarly.

Theorem : 3.11

Let (F, A) and (G, B) be soft Γ-semigroup over S. If $A \cap B \neq \emptyset$, then $(F, A) \cap_R (G, B)$ is a soft Γ-semigroup over S.

Proof :

Assume that $A \cap B \neq \emptyset$. Let $(H, C) = (F, A) \cap_R (G, B)$ where $C \neq \emptyset$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$. To show that (H, C) is soft Γ-semigroup over S, we have to show that $(H, C)\bar{\Gamma}(H, C) \subseteq (H, C)$. Let $(K, D) = (H, C)\bar{\Gamma}(H, C)$ where $D = C \cap C$ and $K(d) = H(d)\Gamma H(d)$ for all $d \in D$. For $d \in D$, we have

$$K(d) = H(d)\Gamma H(d) = (F(d) \cap G(d))\Gamma\big(F(d) \cap G(d)\big) \subseteq F(d) \cap G(d) = H(d)$$

Then $K \subseteq H$. Therefore $(F, A) \cap_R (G, B)$ is a soft Γ-semigroup over S.

Theorem : 3.12

Let (F, A) and (G, B) be soft Γ-semigroup over S. If $A \cap B \neq \emptyset$, then $(F, A) \cup_E (G, B)$ is a soft Γ-semigroup over S.

Proof :

Assume $A \cap B \neq \emptyset$. Let $(H, C) = (F, A) \cup_E (G, B)$ where

$$H(c) = \begin{cases} F(c) & \text{if } c \in A - B \\ G(c) & \text{if } c \in B - A \\ F(c) \cap G(c) & \text{if } c \in A \cap B\,. \end{cases}$$

To show that (H, C) is a soft Γ-semigroup over S, we have to show that

$$(H, C)\bar{\Gamma}(H, C) \subseteq (H, C).$$

Let $(K, D) = (H, C)\bar{\Gamma}(H, C)$ where $D = C \cap C$ and $K(d) = H(d)\Gamma H(d)$ for all $d \in D$.

For $d \in D$, we have

$$K(d) = H(d)\Gamma H(d) \subseteq \begin{cases} F(d) & \text{if } d \in A - B \\ G(d) & \text{if } d \in B - A \\ F(d) \cap G(d) & \text{if } d \in A \cap B\,. \end{cases}$$

Then $K(d) \subseteq H(d)$. Therefore, $(F, A) \cup_E (G, B)$ is a soft Γ-semigroup over S.

Theorem : 3.13

Let (F, A) and (G, B) be soft Γ-semigroup over S. Then $(F, A) \wedge (G, B)$ is a soft Γ-semigroup over S.

Proof :

Let $(F, A) \wedge (G, B) = (H, A \times B)$ where $H(a, b) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$. To show that $(H, A \times B)$ is a soft Γ-semigroup over S.

Let $(K, D) = (H, A \times B)\bar{\Gamma}(H, A \times B)$ where $D = (A \times B) \cap (A \times B)$

and $K(a, b) = H(a, b)\Gamma H(a, b)$ for all $(a, b) \in D$.

For $(a, b) \in D$, we have

$$K(a, b) = H(a, b)\Gamma H(a, b) = (F(a) \cap G(b))\Gamma\big(F(a) \cap G(b)\big) \subseteq Fa \cap Gb = H(a, b)$$

Then $K \subseteq H$. Therefore, $(F, A) \wedge (G, B)$ is a soft Γ-semigroup over S.

Definition : 3.14

Let (F, A) and (G, B) be soft Γ-semigroup over S. Define $(F, A)\boldsymbol{\Gamma}^*(G, B)$ is a soft set $(K, A \times B)$ where $K(a, b) = F(a)\Gamma G(b)$.

Theorem : 3.15

Let (F, A) and (G, B) be soft Γ-semigroups over S. If S is commutative, then $(F, A)\Gamma^*(G, B)$ is a soft Γ-semigroup over S.

Proof :

Let $(F, A)\ \Gamma^{*}\ (G, B) = (H, A \times B)$ where $H(a, b) = F(a)\Gamma G(b)$ for all $(a, b) \in A \times B$. To show that $(H, A \times B)$ is a soft Γ-semigroup over S.

Let $(K, D) = (H, A \times B)\bar{\Gamma}(H, A \times B)$ where $D = (A \times B) \cap (A \times B)$ and $K(a, b) = H(a, b)\ \Gamma\ H(a, b)$ for all $(a, b) \in D$.

For $(a, b) \in D$, since S is commutative, we have

$$K(a, b) = H(a, b)\Gamma H(a, b) = (F(a) \cap G(b))\Gamma\big(F(a)\Gamma G(b)\big) \subseteq F(a)\Gamma G(b) = H(a, b)$$

Then $K \subseteq H$. Therefore, $(F, A)\Gamma^{*}(G, B)$ is a soft Γ-semigroup over S.

Theorem : 3.16

If (F, A) and (G, B) are soft l-idealistics (r- idealistics) over S, then $(F, A) \cap_R (G, B)$ is a soft l-idealistics (r- idealistics) over S contained in both (F, A) and (G, B).

Proof :

Assume that (F, A) and (G, B) are soft l-idealistics over S. Let $(H, C) = (F, A) \cap_R (G, B)$ where $C = A \cap B \neq \emptyset$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$. To show that (H, C) is soft l-idealistics over S, we have to show that $(S, E)\ \bar{\Gamma}\ (H, C) \subseteq (H, C)$. Let $(K, D) = (S, E)\ \bar{\Gamma}\ (H, C)$ where $D = S \cap C$ and $K(d) = S(d)\ \Gamma\ H(d)$ for all $d \in D$. For $d \in D$, we have

$$K(d) = S(d)\Gamma H(d) = S(d)\Gamma\big(F(d) \cap G(d)\big) \subseteq F(d) \cap G(d) = H(d)$$

Then $K \subseteq H$. Therefore $(F, A) \cap_R (G, B)$ is a soft l-idealistics over S. Similarly, if (F, A) and (G, B) are soft l-idealistics over S, then $(F, A) \cap_R (G, B)$ is a soft r-idealistics over S. It is clear by definition that $(F, A) \cap_R (G, B)$ is contained in both (F, A) and (G, B).

Theorem : 3.17

i) If (F, A) and (G, B) are soft l-idealistics (r- idealistics) over S, then $(F, A) \cup_E (G, B)$ is a soft l-idealistics (r- idealistics) over S contained in both (F, A) and (G, B).

ii) If (F, A) and (G, B) are soft l-idealistics (r- idealistics) over S, then $(F, A) \vee (G, B)$ is a soft l-idealistics (r- idealistics) over S contained in both (F, A) and (G, B).

iii) If (F, A) and (G, B) are soft l-idealistics (r- idealistics) over S, then $(F, A) \wedge (G, B)$ is a soft l-idealistics(r- idealistics) over S contained in both (F, A) and (G, B).

Definition : 3.18

Let (F, A) and (G, B) be soft sets over S such that $(G, B) \subseteq (F, A)$. Then (G, B) is called a **soft Γ-subsemigroup** of (F, A) if $G(b)$ is a Γ-subsemigroup of $F(b)$ for all $b \in B$.

Theorem : 3.19

Let (F, A) be a soft Γ-semigroup over S. Let $\{(H_i, B_i) \mid i \in I\}$ be a nonempty family of soft Γ-subsemigroup of (F, A).

i) $\cap_R (H_i, B_i)$ is a soft Γ-subsemigroup of (F, A).

ii) $\wedge_{i \in I} (H_i, B_i)$ is a soft Γ-subsemigroup of $\wedge_{i \in I} (F, A)$.

iii) $\cup_E (H_i, B_i)$ is a soft Γ-subsemigroups of (F, A) if $\{B_i, i \in I\}$, are pairwise disjoint.

Theorem : 3.20

Let (F, A) be a soft Γ-semigroup over S. Let $\{(H_i, B_i) \mid i \in I\}$ be a nonempty family of soft ideals of (F, A).

i) $\cap_R(H_i, B_i)$ is a soft ideal of (F, A).

ii) $\wedge_{i\in I}(H_i, B_i)$ is a soft ideal of $\wedge_{i\in I}(F, A)$.

iii) $\cup_E(H_i, B_i)$ is a soft ideal of (F, A).

iv) $\vee_{i\in I}(H_i, B_i)$ is a soft ideal of $\vee_{i\in I}(F, A)$.