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SECOND ORDER FUZZY TOPOLOGICAL SPACES - II

Dr.A.KALAICHELVI
Associate Professor,
Department of Mathematics,
Avinashilingam Deemed University for Women,
Coimbatore – 641 043, Tamil Nadu, India.
kalaichelviadu@yahoo.co.in
Mobile : 9486355475

ABSTRACT

In this paper some interesting crisp topologies (second order fuzzy topologies) associated with second order fuzzy topologies (crisp topologies) are introduced and discussed.

INTRODUCTION

A **fuzzy set** on a set X is a map defined on X with values in I, where I is the closed unit interval [0, 1]. Equivalently fuzzy sets which are named as first order fuzzy sets in this paper deal with crisply defined membership functions or degrees of membership. It is doubtful whether, for instance, human beings have or can have a crisp image of membership functions in their minds. Zadeh [7] therefore suggested the notion of a fuzzy set whose membership function itself is a fuzzy set. This leads to the following definition of a second order fuzzy set or a fuzzy set of type 2.

A **second order fuzzy set** on a nonempty set X is a map from X to I^I.

First order fuzzy sets are denoted by f, g, h, . . . and second order fuzzy sets are denoted by $\hat{f}$, $\hat{g}$, $\hat{h}$, ...

In this paper the terms 'fuzzy set' and 'first order fuzzy set' are used synonymously.

Whenever a fuzzy set is considered without mentioning the order, it always refers to a first order fuzzy set.

Similar terminology applies to all concepts related to first order fuzzy sets.

Definition and examples of second order fuzzy topological spaces are given in [4].

Six important and interesting connections $¢_1$, $¢_2$, $¢_3$, $¢_4$, $¢_5$ and $¢_6$ between first order and second order fuzzy topological spaces are also discussed in [4]. In this paper we use the first connection $¢_1$.

Every first order fuzzy topology $\delta = \{f_\lambda \,/\, \lambda \in \Lambda\}$ on a nonempty set X defines a second order fuzzy topology $\hat{\delta} = \{\hat{f}_\lambda \,/\, f_\lambda \in \delta\}$ on X where $\hat{f}_\lambda(x)(\alpha) = f_\lambda(x)$, for every $x \in X$ and for every $\alpha \in I$. The correspondence $\delta \to \hat{\delta}$ is denoted as $¢_1$ [4].

With every second order fuzzy topology $\hat{\delta}$ on a nonempty set X, three crisp topologies $i(\hat{\delta})$, $i_\varepsilon(\hat{\delta})$ and $i^*(\hat{\delta})$ on X are associated.

These associations are introduced in section one of this paper.

With every crisp topology τ on a nonempty set X, three second order fuzzy topologies $\omega_2(\tau)$, $(\omega_2(\tau))_\varepsilon$ and $\omega_2{}^*(\tau)$ on X are associated.

These associations are introduced in section two of this paper. Some interesting results regarding these associations are proved.

SECTION – 1

CRISP TOPOLOGIES ASSOCIATED WITH SECOND ORDER FUZZY TOPOLOGIES

Definition : 1.1

Let X be any nonempty set. Let $\hat{f} \in (I^I)^X$. For $\varepsilon \in (0, 1)$ define

$$(A_{\hat{f}})_\varepsilon = \{x \in X \,/\, (\hat{f}(x))^{-1}(\varepsilon, 1] = I\}.$$

Definition : 1.2

Let X be any nonempty set. Let $\hat{f} \in (I^I)^X$.

Define $A_{\hat{f}} = \{x \in X \,/\, (\hat{f}(x))^{-1}(\varepsilon, 1] = I$ for some $\varepsilon \in (0, 1)\}$.

The following result is a direct consequence of the above definitions.

Remark : 1.3

(1) $(A_{\hat{f}})_\varepsilon \subseteq A_{\hat{f}}$, for every $\varepsilon \in (0, 1)$

(2) $A_{\hat{f}} = \bigcup_{\varepsilon \in (0, 1)} (A_{\hat{f}})_\varepsilon$

Proposition : 1.4

Let $(X, \hat{\delta})$ be a second order fuzzy topological space. Then the collection $\{(A_{\hat{f}})_\varepsilon \,/\, \hat{f} \in \hat{\delta}\}$ is closed with respect to finite intersection.

Proof : Consider

$(A_{\hat{f}})_\varepsilon \cap (A_{\hat{g}})_\varepsilon$

$= \{x \in X \,/\, (\hat{f}(x))^{-1}(\varepsilon, 1] = I$ and $(\hat{g}(x))^{-1}(\varepsilon, 1] = I\}$

= $\{x \in X \,/\, (\hat{f}(x)(\alpha) > \varepsilon$ and $\hat{g}(x)(\alpha) > \varepsilon$, for every $\alpha \in I\}$.

= $\{x \in X \,/\, (\hat{f}(x)(\alpha) \wedge \hat{g}(x)(\alpha)) > \varepsilon$, for every $\alpha \in I\}$.

= $\{x \in X \,/\, (\hat{f} \wedge \hat{g})(x)(\alpha) > \varepsilon$, for every $\alpha \in I\}$.

= $\{x \in X \,/\, ((\hat{f} \wedge \hat{g})(x))^{-1}(\varepsilon, 1] = I\}$

= $(A_{\hat{f} \wedge \hat{g}})_\varepsilon$

$\therefore$ The given collection is closed with respect to finite intersection.

Proposition : 1.5

Let $(X, \hat{\delta})$ be a second order fuzzy topological space. Then $(A_{\hat{f}} \,/\, \hat{f} \in \hat{\delta}\}$ is closed with respect to finite intersection.

Proof : Consider

$A_{\hat{f}} \cap A_{\hat{g}}$

= $\{x \in X \,/\, \hat{f}(x))^{-1}(\varepsilon_1, 1] = I$ and $(\hat{g}(x))^{-1}(\varepsilon_2, 1] = I$, for some $\varepsilon_1, \varepsilon_2 \in (0, 1)\}$

= $\{x \in X \,/\, \hat{f}(x))^{-1}(\varepsilon, 1] = I$ and $(\hat{g}(x))^{-1}(\varepsilon, 1] = I$, where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)\}$

= $\{x \in X \,/\, ((\hat{f} \wedge \hat{g})(x))^{-1}(\varepsilon, 1] = I$, for some $\varepsilon \in (0, 1)\}$

= $A_{\hat{f} \wedge \hat{g}}$

$\therefore$ The given collection is closed with respect to finite intersection.

Definition : 1.6

Let $(X, \hat{\delta})$ be a second order fuzzy topological space. Define

(1) $i_\varepsilon(\hat{\delta})$ to be the topology generated by the collection $\{(A_{\hat{f}})_\varepsilon \,/\, \hat{f} \in \hat{\delta}\}$

(2) $i^*(\hat{\delta})$ to be the topology generated by the collection $\{A_{\hat{f}} \,/\, \hat{f} \in \hat{\delta}\}$.

(3) $i(\hat{\delta})$ to be the topology having the collection $\{(A_{\hat{f}})_\varepsilon \,/\, \hat{f} \in \hat{\delta}, \varepsilon \in (0, 1)\}$ as a subbasis.

Theorem : 1.7

If the second order fuzzy topology $\hat{\delta}$ on X is got from the first order fuzzy topology δ on X through the association $¢_1$, then

(1) $i_\varepsilon(\hat{\delta}) = i_\varepsilon(\delta)$

(2) $i^*(\hat{\delta}) = i_0(\delta)$

(3) $i(\hat{\delta}) = i(\delta)$

Proof : In this case, $(A_{\hat{f}})_\varepsilon = f^{-1}(\varepsilon, 1]$

For,

$x \in (A_{\hat{f}})_\varepsilon$

$\Leftrightarrow$ $(\hat{f}(x))^{-1}(\varepsilon, 1] = I$

$\Leftrightarrow$ $\hat{f}(x)(\alpha) > \varepsilon$, for every $\alpha \in I$

$\Leftrightarrow$ $f(x) > \varepsilon$

$\Leftrightarrow$ $x \in f^{-1}(\varepsilon, 1]$

(1) Since $(A_{\hat{f}})_\varepsilon = f^{-1}(\varepsilon, 1]$, $i_\varepsilon(\hat{\delta}) = i_\varepsilon(\delta)$

(2) $$A_{\hat{f}} = \bigcup_{\varepsilon \in (0, 1)} (A_{\hat{f}})_\varepsilon$$

$$= \bigcup_{\varepsilon \in (0,1)} f^{-1}(\varepsilon, 1]$$

$$= \{x \,/\, f(x) > 0\}$$

$$= f^{-1}(0, 1]$$

$$\therefore \; i^*(\hat{\delta}) = i_0(\delta)$$

(3) The collection $\{(A_{\hat{f}})_\varepsilon \,/\, \hat{f} \in \hat{\delta}, \varepsilon \in (0, 1)\}$ is a subbasis for $i(\hat{\delta})$.

The collection $\{f^{-1}(\varepsilon, 1] \,/\, f \in \delta, \varepsilon \in (0, 1)\}$ is a subbasis for $i(\delta)$.

Since $(A_{\hat{f}})_\varepsilon = f^{-1}(\varepsilon, 1]$, $i(\hat{\delta}) = i(\delta)$.

SECTION – 2

SECOND ORDER FUZZY TOPOLOGIES ASSOCIATED WITH CRISP TOPOLOGIES

Definition : 2.1

Let (X, τ) be a topological space.

(1) For $\varepsilon \in (0, 1)$, define

$$\hat{K}_\varepsilon = \{\hat{f} \in (I^I)^X \,/\, (A_{\hat{f}})_\varepsilon \in \tau\}$$

(2) Define $\hat{K} = \{\hat{f} \in (I^I)^X \,/\, (A_{\hat{f}})_\varepsilon \in \tau,$ for every $\varepsilon \in (0, 1)\}$

(3) Define $\hat{K}_* = \{\hat{f} \in (I^I)^X \,/\, A_{\hat{f}} \in \tau\}$.

Proposition : 2.2

Each of the above three sets $\hat{K}_\varepsilon$, $\hat{K}$ and $\hat{K}_*$ is closed with respect to finite intersection.

The proof is immediate from the following two results.

(1) $(A_{\hat{f}})_\varepsilon \cap (A_{\hat{g}})_\varepsilon = \left(A_{\hat{f} \wedge \hat{g}}\right)_\varepsilon$

(2) $A_{\hat{f}} \cap A_{\hat{g}} = A_{\hat{f} \wedge \hat{g}}$

Definition : 2.3

Let (X, τ) be a topological space.

Then define

(1) $(\omega_2(\tau))_\varepsilon$ to be the second order fuzzy topology generated by $\hat{K}_\varepsilon$.

(2) $\omega_2(\tau)$ to be the second order fuzzy topology generated by $\hat{K}$.

(3) $\omega_2^*(\tau)$ to be the second order fuzzy topology generated by $\hat{K}_*$.

Theorem : 2.4

Let (X, τ) be a topological space. Then

(1) $\widehat{\omega(\tau)} \subseteq \omega_2(\tau)$ if $\widehat{\omega(\tau)}$ is got from $\omega(\tau)$ through the association $¢_1$.

(2) $\omega_2(\tau) \subseteq (\omega_2(\tau))_\varepsilon$, for every $\varepsilon \in (0, 1)$

(3) $\omega_2(\tau) \subseteq \omega_2^*(\tau)$

Proof : (1) Consider $\hat{f} \in \widehat{\omega(\tau)}$

$\therefore$ $f \in \omega(\tau)$

$\Rightarrow$ $f^{-1}(\varepsilon, 1] \in \tau$, for every $\varepsilon \in (0, 1)$.

$\Rightarrow$ $(A_{\hat{f}})_\varepsilon \in \tau$, for every $\varepsilon \in (0, 1)$.

($\because$ $\widehat{\omega(\tau)}$ is got from $\omega(\tau)$ through the association $¢_1$).

$\Rightarrow \quad \hat{f} \in \hat{K}$

$\Rightarrow \quad \hat{f} \in \omega_2(\tau)$

$\therefore \quad \hat{\omega}(\tau) \subseteq \omega_2(\tau)$

(2) $\hat{f} \in \hat{K}$

$\Rightarrow \quad (A_{\hat{f}})_\varepsilon \in \tau$, for every $\varepsilon \in (0, 1)$.

$\Rightarrow \quad \hat{f} \in \hat{K}_\varepsilon$, for every $\varepsilon \in (0, 1)$.

$\therefore \quad \omega_2(\tau) \subseteq (\omega_2(\tau))_\varepsilon$, for every $\varepsilon \in (0, 1)$.

(3) $\hat{f} \in \hat{K}$

$\Rightarrow \quad (A_{\hat{f}})_\varepsilon \in \tau$, for every $\varepsilon \in (0, 1)$.

$\Rightarrow \quad \bigcup_{\varepsilon \in (0,1)} (A_{\hat{f}})_\varepsilon \in \tau$

$\Rightarrow \quad A_{\hat{f}} \in \tau$ ($\because$ From remark 1.3)

$\Rightarrow \quad \hat{f} \in \hat{K}_*$

$\therefore \quad \omega_2(\tau) \subseteq \omega_2^*(\tau)$.

Proposition : 2.5

Let τ_1, τ_2 be two topologies on X such that $\tau_1 \subseteq \tau_2$. Then

(1) $\omega_2(\tau_1) \subseteq \omega_2(\tau_2)$

(2) For $\varepsilon \in (0, 1)$, $(\omega_2(\tau_1)_\varepsilon \subseteq (\omega_2(\tau_2))_\varepsilon$

(3) $\omega_2^*(\tau_1) \subseteq \omega_2^*(\tau_2)$

Theorem : 2.6

Let (X, τ) be a topological space. Then

(1) $\tau \subseteq i^*(\omega_2{}^*(\tau))$

(2) For $\varepsilon \in (0, 1)$, $\tau \subseteq i_\varepsilon((\omega_2(\tau))_\varepsilon)$

(3) $\tau \subseteq i(\omega_2(\tau))$

Proof :

Consider $M \in \tau$.

Consider the second order characteristic function $\hat{\chi}_M$ on X as follows :

$$\hat{\chi}_M(x) = \mathbf{1}, \quad \text{if } x \in M$$

$$= \mathbf{0}, \quad \text{if } x \notin M.$$

where **0** and **1** are the constant fuzzy sets taking the values 0 and 1 respectively.

Consider

$$A_{\hat{\chi}_M} = \{x \in X \,/\, (\hat{\chi}_M(x))^{-1}(\varepsilon, 1] = \mathrm{I} \text{ for some } \varepsilon \in (0, 1)\}.$$

$$= \{x \in X \,/\, (\hat{\chi}_M(x)(\alpha) > \varepsilon, \text{ for some } \varepsilon \in (0, 1) \text{ and for every } \alpha \in \mathrm{I}\}$$

$$= \{x \in X \,/\, (\hat{\chi}_M(x) \neq \mathbf{0}\}$$

$$= \{x \in X \,/\, (\hat{\chi}_M(x) = \mathbf{1}\}$$

$$= M$$

$$M \in \tau \Rightarrow A_{\hat{\chi}_M} \in \tau$$

$$\Rightarrow \hat{\chi}_M \in \hat{K}_*$$

$$\Rightarrow \hat{\chi}_M \text{ is a basis element of } \omega_2{}^*(\tau)$$

$\Rightarrow$ $A_{\hat{\chi}_M}$ is a basis element of $i^*(\omega_2^*(\tau))$

$\Rightarrow$ M is a basis element of $i^*(\omega_2^*(\tau))$

$\therefore$ $\tau \subseteq i^*(\omega_2^*(\tau))$

Proofs of (2) and (3) are similar.

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