A STUDY OF DERIVATIONS OF BCI-ALGEBRAS

By

SUBHALAKSHMI, S

(11 PM 16)

A DISSERTATION SUBMITTED TO THE
AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND HIGHER
EDUCATION FOR WOMEN COIMBATORE- 641 043

IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

MASTER OF SCIENCE IN MATHEMATICS

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CERTIFIED AS A BONAFIDE RESEARCH WORK

P. Jeyalakshmi

Signature of the
Head of the Department

Signature of the Guide

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CONTENTS

INTRODUCTION

BCK-algebras and BCI-algebras are two classes of non-classical logic algebras which were introduced by Imai and Iseki in 1966 [14, 15]. They are algebraic formulation of BCK-system and BCI-system in combinatory logic. The notion of a BCI-algebra generalizes the notion of a BCK-algebra in the sense that every BCK-algebra is a BCI-algebra but not vice versa.

The notion of derivation in ring theory plays a significant role in analysis, algebraic geometry and algebra. In the year 2004 [18], Jun and Xin have applied the notion of derivation in BCI-algebras which is defined in a way similar to the notion of derivation in rings and near-rings theory which was introduced by Posner in 1957 [30].

This Thesis is devoted to study the derivation of BCI-algebras in different aspects:

The following articles are chosen for our discussion:

(i) *"On Derivations of BCI-algebras"* by Young Bae Jun and Xiao Long Xin [2004].

(ii) *"Some Results On Derivations of BCI-algebras"* by Hamza A. S. Abujabal and Nora O. Al-Shehri [2006].

(iii) *"On f −Derivations of BCI-algebras"* by Jianming Zhan and Yong Lin Liu [2005].

(iv) *"On Left Derivations of BCI-algebras"* by Hamza A.S.Abujabal and Nora O. Al-Shehri [2007].

(v) *"Generalized Derivations of BCI-algebras"* by Mehmet Ali Ozturk, Yilmaz Ceven and Young Bae Jun [2009].

(vi) *"On Symmetric Bi-Derivations of BCI-algebras"* by Sabahattin Ilbira, Alev Firat and Young Bae Jun [2011].

(vii) *"On t −Derivations of BCI-algebras"* by G. Muhiuddin and Abdullah M. Al-roqi [2012].

(viii) *"On (α,β) −Derivations in BCI-algebras"* by G. Muhiuddin and Abdullah M. Al-roqi [2012].

This thesis is divided into four chapters.

In the **first chapter**, preliminary definitions and results on BCI/BCK-algebras are collected. In the second and third section of this chapter, the notion of left-right (resp. right-left) derivation of BCI-algebras and some related properties are studied. Also, characterizations of a p-semisimple BCI-algebra are studied by using the idea of regular derivation.

In this chapter, the following interesting results are discussed.

(i) A BCI-algebra X is commutative if and only if it is Branchwise Commutative.

(ii) A BCI-algebra X is p-semisimple if and only if $d^{-1}(0) = \{0\}$ for every regular derivation d of X.

(iii) Let d be a derivation of a BCI-algebra X. Then d is regular if and only if every ideal of X is $d - invariant$.

(iv) Every derivation of a BCK-algebra is $regular$.

Chapter 2 deals with the study of f −derivations and left derivations of BCI-algebras. In the first section of this chapter the notion of left-right (resp. right-left) f −derivation of a BCI-algebra and some related properties are studied. In the second section of this chapter, the notion of left derivation of a BCI-algebra and some related properties are studied. A condition for left derivation to be regular is also discussed.

In this chapter, the following interesting results are discussed.

(i) Let d_f be a $(l,r) - f - derivation$ of a BCI-algebra X. Then
    (i) $d_f(0) \in L_p(X)$, ie., $d_f(0) = 0 * (0 * d_f(0))$
    (ii) $d_f(a) = d_f(0) * \big(0 * f(a)\big) = d_f(0) + f(a)$ for all $a \in L_p(X)$
    (iii) $d_f(a) \in L_p(X)$ for all $a \in L_p(X)$
    (iv) $d_f(a + b) = d_f(a) + d_f(b) - d_f(0)$ for all $a, b \in L_p(X)$

(ii) Let f be a monic of a commutative BCI-algebra X. Then X is p-semisimple if and only if $\ker d_f = \{0\}$ for every regular f −derivation d_f of X.

(iii) In a p-semisimple BCI-algebra X, a self-map D of X is a left derivation if and only if it is a derivation.

Chapter 3 deals with the study of Generalized Derivations and Symmetric Bi-derivations of BCI-algebras. The concept of Generalized Derivations of BCI-algebra and some related properties are discussed. Also, the concept of generalized derivations of torsion free BCI-algebra are studied. The notion of left-right (resp. right-left) Symmetric Bi-derivation of BCI-algebras and some related properties are discussed in the second section of this chapter.

In this chapter, the following interesting results are discussed.

1. Let D be a self-map of a BCI-algebra X. Then
   a. If D is a generalized (l,r) −derivation of X, then $D(x) = D(x) \wedge x$ for all $x \in X$.
   b. If D is a generalized (r,l) −derivation of X, then $D(0) = 0$ if and only if $D(x) = x \wedge d(x)$ for all $x \in X$ and for some (r,l) −derivation d of X.
2. Let X be a torsion free BCI-algebra and D a generalized derivation. If $D^2 = 0$ on $L_p(X)$, then $D = 0$ on $L_p(X)$.
3. Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $symmetric\ mapping$. Then
   a. If D is a $(l,r) - symmetric\ bi - derivation$, then $D(x,z) = D(x,z) \wedge x$ for all $x, z \in X$
   b. If D is a $(r,l) - symmetric\ bi - derivation$, then $D(x,z) = x \wedge D(x,z)$ for all $x, z \in X$ if and only if $D(0,z) = 0$ for all $z \in X$
4. Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(l,r) - symmetric\ bi - derivation$. If there exist $a \in X$ such that $D(x,z) * a = 0$, for all $x, z \in X$, then D is componentwise regular $(l,r) - symmetric\ bi - derivation.$

In **chapter 4,** the concepts of t −derivations and (α,β) −derivations of BCI-algebra are studied. In the first section of this chapter, the notion of t −derivation of a BCI-algebra and related properties are discussed. Moreover, some results on t −derivations in a p-semisimple BCI-algebra are discussed. In the second section of this chapter, the notion of (regular) (α,β) −derivations of a BCI-algebra X, and some related properties are discussed. The concepts of a $d_{(\alpha,\beta)}$−invariant (α,β) −derivation and α −ideal are studied and their relations are discussed. Finally, some results on regular (α,β) −derivations are obtained.

In this chapter, the following interesting results are discussed.

1. Let d_t be a $(l,r)-t$ −derivation of a p-semisimple BCI-algebra X. Then the following hold:
    - (i) $d_t(0) = d_t(x) * x$ for all $x \in X$
    - (ii) d_t is one-one
    - (iii) If d_t is $t-regular$, then it is an identity map
    - (iv) If there is an element all $x \in X$ such that $d_t(x) = x$, then d_t is identity map
    - (v) If $x \leq y$, then $d_t(x) \leq d_t(y)$ for all $x, y \in X$

2. Let X be a p-semisimple BCI-algebra X and let d_t and $d_t^{'}$ be $t-derivations$ of X. Then $d_t * d_t^{'} = d_t^{'} * d_t$ **.**

3. Let X be a commutative BCI-algebra. Then every $(\alpha,\beta)-derivation$ $d_{(\alpha,\beta)}$ of X satisfies the following assertion:

$$x \leq y \Longrightarrow d_{(\alpha,\beta)}(x) \leq d_{(\alpha,\beta)}(y)),\ \forall x, y \in X$$

   That is, every $(\alpha,\beta)-derivation$ of X is $isotone$.

4. Let X be a torsion free BCI-algebra and $d_{(\alpha,\beta)}, d^{'}_{(\alpha,\beta)}$ be two regular $(\alpha,\beta)-derivation$ on X such that $\alpha \circ d^{'}_{(\alpha,\beta)} = d^{'}_{(\alpha,\beta)}$. If $d_{(\alpha,\beta)} \circ d^{'}_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $d^{'}_{(\alpha,\beta)} = 0$ on $L_p(X)$.

REVIEW OF LITERATURE

The notion of BCK-algebra was proposed by Imai and Iseki in 1966 [14]. In the same year, Iseki introduced the notion of a BCI-algebra [15], which is a generalization of a BCK-algebra. A series of interesting notions concerning BCI-algebras were introduced and studied, several papers have been written on various aspects of these algebras [5, 11, 16]. Recently, in the year 2004 [18], Jun and Xin have applied the notion of derivation in BCI-algebras which is defined in a way similar to the notion of derivation in rings and near-rings theory which was introduced by Posner in 1957 [30]. In non-commutative rings, the notion of derivations is extended to $\alpha-$derivations, left derivations and central derivations. The properties of $\alpha-$derivations and central derivations were discussed in several papers with respect to the ring structures. For left derivations, M. Bresar and J. Vukman [6] used them to give some results in prime and semi-prime rings. For skew polynomial rings, all left derivations are obtained in a similar way to polynomial rings [26].

After the work of Jun and Xin (2004) [18], many research articles have appeared on the derivations of BCI-algebras in different aspects as follows:

In 2005 [37], Zhan and Liu have given the notion of $f-$derivation of BCI-algebras and studied p-semisimple BCI-algebras by using the idea of regular $f-$derivation in BCI-algebras. In 2006 [1], Abujabal and Al-Shehri have extended the results of BCI-algebras. Further, in the next year 2007 [2], they defined and studied the notion of left-derivation of BCI-algebras and investigated some properties of left derivation in p-semisimple BCI-algebras. In 2009 [28], Ozturk and Ceven has defined the notion of derivations and generalized derivation determined by a derivation for a complicated subtraction algebra and discussed some related properties. Also, in 2009 [29], Ozturk et al. have introduced the notion of generalized derivation in BCI-algebras and established some results. Further, they have given the idea of torsion free BCI-algebra and explored some properties. In 2010 [3], Al-Shehri has applied the notion of left-right (resp., right-left) derivation in BCI-algebra to B-algebra and obtained some of its properties. In 2011 [13], Ilbira et al. have studied the notion of left-right (resp., right-left) symmetric bi-derivation in BCI-algebras. Motivated by a lot of work done on derivations of BCI-algebras and on derivations of other related abstract algebraic structures.

Several other authors have also contributed to the study of the concepts mentioned above we give here a brief survey of some of the articles published on various algebras.

(1) Generalized Jordan Derivations on Prime Rings and Standard Operator Algebras

Wu Jing and Shijie Lu (2003) [33]

In this article, the authors initiated the study of generalized Jordan Derivations and generalized Jordan triple derivations on prime rings and standard operator algebras.

(2) On Trace Of Symmetric Bi-Derivations in Near-Rings

Mehmet Ali Ozturk, Young Bae Jun (2004) [22]

Let N be a 3-prime left near-ring with multiplicative center Z. For $x \in N$, let $C(x)$ be the centralizer of x in N. In this article, the authors studied the trace of symmetric Bi-derivations on N. Main results are the following theorems: (1) Let D be a non-zero symmetric bi-derivation of N and d the trace of D. If N is 2-torsion free and $d(N) \subseteq Z$, then N is a commutative ring. (2) Let D be a symmetric bi-derivation of N and d the trace of D. If N is 2-torsion free and $d(y), d(y) + d(y) \in C(D(x, y))$ for all $x, y, z \in N$, then N is a commutative ring.

(3) On Generalized Derivations of Prime Near-Rings

Oznur Golbasi (2006) [27]

In this article, the authors extended the study of some well-known results concerning derivations of prime rings to generalized derivations of prime near-rings.

(4) On Derivations of BCC-algebras

Chanwit Prabpayak and Utsanee Leerawat (2009) [8]

In this article, the notions of left-right (resp. right-left) derivations of BCC-algebras are studied and some properties on derivations of BCC-algebras are investigated. This article also considers regular derivations and the $d-invariant$ on ideals of BCC-algebras.

(5) Jordan Higher Left Derivations And Commutativity In Prime Rings

Kyoo-Hong Park (2010) [20]

In this article, the author showed that the existence of a non-zero Jordan higher left derivation on $R,$ implies R is commutative, where R is a 2-torsionfree prime ring. This result was used to prove a non-commutative extension of the classical Singer-Wermer theorem in the sense of higher derivations.

(6) Derivations of MV-algebras

N. O. Alshehri (2010) [4]

In this article, the author introduced the notion of derivation for an MV-algebra and discussed some related properties. Using the notion of an isotone derivation, the authors gave some characterizations of a derivation of an MV-algebra. Moreover, the authors defined an additive derivation of an MV-algebra and investigated some of its properties. Also, the authors proved that an additive derivation of a linearly ordered MV-algebra is an isotone.

(7) Derivations of *B*-algebras

Nora O. Al-Shehri (2010) [3]

In this article, the notion of left-right (resp. right-left) derivation of *B*-algebra is introduced and some related properties are investigated. Also, the notion of derivation of 0-commutative *B*-algebra is studied and some of its properties are investigated.

(8) $\boldsymbol{f}$ −Derivations of Weak BCC-Algebras

Janus Thomys (2011) [19]

In this article, the author described f −Derivations of weak BCC-algebras in which the condition $(x * y) * z = (x * z) * y$ is satisfied in the case when elements x, y belong to the same branch.

(9) A Note on Generalized Left $(\boldsymbol{\theta}, \boldsymbol{\varphi})$ −Derivations in Prime Rings

Xiao-Wei Xu and Hong-Ying Zhang (2011) [34]

In this article, the authors described the generalized left (θ, φ) −derivations in prime rings, and proved that an additive mapping in a ring R acting as a homomorphism or anti-homomorphism on an additive subgroup S of R must be either a mapping acting as a homorphism on S or a mapping acting as an anti-homomorphism on S, through which some related results are improved.

(10) On Generalized Derivations of Prime and Semiprime Rings

Shuliang Huang (2012) [32]

Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a nonzero derivation d such that $\left(F(x \circ y)\right)^{n} = x \circ y$ for all $x, y \in I$, then R is commutative. The author also examined the case where R is a semiprime ring.

(11) Generalized Skew Left Derivations Characterized by Acting on Zero Products

Chuijia Wang, Xiaowei Xu and Xiaofei Yi (2012) [35]

This short note gives the generalized skew left derivation version of the theorem by T. K. Lee through proving that in a prime ring with a nontrivial idempotent every generalized skew left derivation characterized by acting on zero products must be a generalized skew left derivation.

(12) A Note on f −Derivations of BCC-Algebras

Sang Moon Lee, Kyung Ho Kim (2012) [31]

In this article, the authors considered the properties of f −Derivations of BCC-Algebras. Also, they characterized $\ker d = \{x \in X \ / \ d(x) = 0\}$ by f −Derivations.

CHAPTER 1

ON DERIVATIONS OF BCI-ALGEBRAS

SECTION 1.1:

PRELIMINARIES ON BCI/BCK-ALGEBRAS:

Definition 1.1.1:

Let X be a non-empty set with a binary operation $*$ and a constant 0. Then $(X,*,0)$ is called a $BCI-algebra$, if it satisfies the following axioms for all $x,y,z \in X$:

(BCI-1) $\big((x*y)*(x*z)\big)*(z*y)=0;$

(BCI-2) $\big(x*(x*y)\big)*y=0;$

(BCI-3) $x*x=0;$

(BCI-4) $x*y=0$ and $y*x=0$ imply $x=y$.

Definition 1.1.2:

A subset S of a BCI-algebra X is called $a\ subalgebra\ of\ X$ if $x*y\in S$, whenever $x*y\in S$.

Definition 1.1.3:

A BCI-algebra X satisfying the following condition,

(BCK-1) $0\leq x$ ie., $0*x=0$, is called a $BCK-algebra$.

Note:

In any BCI/BCK-algebra X, we define a partial order "$\leq$" on X by putting $x \leq y$ if and only if $x * y = 0$. Then $(X, \leq)$ is a $partially\ ordered\ set$.

Definition 1.1.4:

A BCI-algebra X is said to be $associative$ if

$$(x * y) * z = x * (y * z), \qquad \forall\, x, y, z \in X.$$

Definition 1.1.5:

A non-empty subset I of a BCI-algebra X is called $an\ ideal\ of\ X$ if it satisfies

(i) $0 \in I$

(ii) $x * y \in I$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.

Note:

Any ideal I has the property $y \in I$ and $x \leq y$ imply $x \in I$.

Proposition 1.1.6:

A BCI-algebra X has the following properties:

For all $x, y, z \in X$,

(i) $x * 0 = x$

(ii) $(x * y) * z = (x * z) * y$

(iii) $0 * (x * y) = (0 * x) * (0 * y)$

(iv) $x * (x * (x * y)) = x * y$

(v) $((x * z) * (y * z)) * (x * y) = 0$

(vi) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$

(vii) $x * 0 = 0$ implies $x = 0$.

Definition 1.1.7:

For a BCI-algebra X,

(i) The BCK-part of X is denoted by $X_+ = \{x \in X \,/\, 0 \leq x\}$

(ii) The BCI-G part of X is denoted by $G(X) = \{x \in X \,/\, 0 * x = x\}$

Note: $G(X) \cap X_+ = \{0\}$ [17]

Definition 1.1.8:

In a BCI-algebra X, if $X_+ = \{0\}$ then X is called a $p - semisimple\ BCI - algebra$.

Proposition 1.1.9: [10, 12]

In a p-semisimple BCI-algebra X, the following properties hold:

(i) $(x * z) * (y * z) = x * y$

(ii) $0 * (0 * x) = x, \forall x \in X$

(iii) $x * (0 * y) = y * (0 * x)$

(iv) $x * y = 0$ implies $x = y$

(v) $x * y = x * z$ implies $y = z$

(vi) $y * x = z * x$ implies $y = z$

(vii) $y * (y * x) = x$

(viii) $x * a = x * b$ implies $a = b$

(ix) $a * x = b * x$ implies $a = b$

Note:

Let X be a p-semisimple BCI-algebra. We define addition $'+'$ as $x + y = x * (0 * y)$ for all $x, y \in X$. Then $(X, +)$ is an abelian group with identity 0 and $x - y = x * y$.

Conversely, let $(X, +)$ be an abelian group with identity 0 and let $x * y = x - y$. Then X is a p-semisimple BCI-algebra and $x + y = x * (0 * y)$ for all $x, y \in X$. [23]

Notations 1.1.10:

(i) For a BCI-algebra X, we denote $x \wedge y = y * (y * x)$. In particular $0 * (0 * x) = a_x$.

(ii) $L_p(X) = \{a \in X / x * a = 0 \Rightarrow x = a, \forall\, x \in X\}$. The elements of $L_p(X)$ are called the p-atoms of X.

(iii) For any $a \in X$, $V(a) = \{x \in X / a * x = 0\}$ is called the branch of X with respect to a.

Note: [21, 24, 36, 38]

(i) Also $x * y \in V(a * b)$, whenever $x \in V(a)$ and $y \in V(b)$ for all $x, y \in X$ and $a, b \in L_p(X)$.

(ii) $L_p(X) = \{x \in X \;/\; a_x = x\}$, which is the p-semisimple part of X. Also X is a p-semisimple BCI-algebra if and only if $L_p(X) = X$.

(iii) $a_x \in L_p(X)$, ie., $0 * (0 * a_x) = a_x$, which implies that $a_x * L_p(X)$ for all $y \in X$.

(iv) $G(X) \subset L_p(X)$and $x * (x * a) = a$ and $a * x \in L_p(X)$ for all $a \in L_p(X)$ and $x \in X$.

Definition 1.1.11: [25]

A BCI-algebra X is said to be $commutative$ if $x = x \wedge y$ whenever $x \leq y$ for all $x, y \in X$.

Note:

Every p-semisimple BCI-algebra is commutative.

Definition 1.1.12: [9]

A BCI-algebra X is said to be $branchwise\ commutative$ if $x \wedge y = y \wedge x$ for all $x, y \in V(a)$ and all $a \in L_p(X)$.

Lemma 1.1.13: [25]

Let X be a commutative BCI-algebra and let $a \in L_p(X)$. Then $x \wedge y = y \wedge x$ for all $x, y \in V(a)$, ie., X is $branchwise\ commutative$.

Proof: Obvious.

Theorem 1.1.14: [7]

A BCI-algebra X is commutative if and only if it is branchwise commutative.

Proof:

Let X be a branchwie commutative BCI-algebra and assume that $x \leq y$ for $x, y \in X$. Also, $0 * (0 * x) \in L_{\mathrm{p}}(X)$ and $0 * (0 * x) \leq x \leq y$.

Hence, $x, y \in V\big(0 * (0 * x)\big)$ which implies that $x \wedge y = y \wedge x = x * (x * y) = x * 0 = x$.

Therefore, X is a commutative BCI-algebra.

Converse part follows from the above lemma.

Definition 1.1.15:

A mapping f of a BCI-algebra X *into* itself is called an *Endomorphism* of X if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Especially, f is *monic* if for any $x, y \in X, f(x) = f(y)$ implies that $x = y$.

Note: $f(0) = 0$.

SECTION 1.2:

DERIVATIONS OF BCI-ALGEBRAS:

Definition 1.2.1:

Let X be a BCI-algebra

(i) By a $left-right\ derivation\ (briefly\ (l,r)-derivation)$ of X, we mean a self-map d of X satisfying the identity $d(x*y)=(d(x)*y)\wedge\big(x*d(y)\big)$ for all $x,y\in X$.

(ii) If d satisfies the identity $d(x*y)=(x*d(y))\wedge(d(x)*y))$ for all $x,y\in X$, then we say that d is a $right-left\ derivation$ $(briefly\ (r,l)-derivation)$ of X.

Moreover, if d is both a (r,l) and a (l,r) derivation, we say that d is a derivation.

Example 1.2.2:

Let $X=\{0,1,2,3,4,5\}$ be a BCI-algebra with Cayley table as follows:

$*$	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	0	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

Define a map $d:X\to X$ by, $d(x)=\begin{cases}0 & if\ x=0,1\\ 2 & if\ x=2,4\\ 3 & if\ x=3,5\end{cases}$

Then, d is both a (r,l) and a (l,r) derivation of X.

Proposition 1.2.3:

Let d be a self-map of a BCI-algebra X defined by $d(x) = a_x$ for all $x \in X$. Then d is a $(l,r) - derivation$ of X. Moreover, if X is commutative, the d is a $(r,l) - derivation$ of X.

Proof:

Let $x, y \in X$. Then,

$$\begin{aligned} d(x * y) = a_{x*y} &= 0 * \big(0 * (x * y)\big) \\ &= \big(0 * (0 * x)\big) * (0 * (0 * y)) = a_x * a_y \\ &= (0 * (0 * a_x)) * (0 * (0 * y)) \\ &= 0 * (0 * (a_x * y)) = a_x * y \\ &= (x * a_y) * ((x * a_y) * (a_x * y)) \\ &= (a_x * y) \wedge (x * a_y) \\ &= (d(x) * y) \wedge \big(x * d(y)\big) \end{aligned}$$

and so d is a $(l,r) - derivation$ of X.

Now, assume that X is commutative. Using theorem 1.1.14, it is sufficient to show that $d(x) * y$ and $x * d(y)$ belong to the same branch for all $x, y \in X$.

Also, we note that $a_x, a_y \in L_p(X)$ for all $x, y \in X$, we have

$$\begin{aligned} d(x) * y = a_x * y &= 0 * (0 * (a_x * y)) \\ &= (0 * (0 * a_x)) * (0 * (0 * y)) \\ &= a_x * a_y \in V(a_x * a_y) \end{aligned}$$

and

$$a_x * a_y = \big(0 * (0 * x)\big) * (0 * (0 * a_y))$$

$$= 0 * (0 * (x * a_y))$$

$$= 0 * \Big(0 * \big(x * d(y)\big)\Big)$$

$$\leq x * d(y),$$

which implies that $x * d(y) \in V(a_x * a_y)$

Hence, $d(x) * y$ and $x * d(y)$ belong to the same branch, and so

$$d(x * y) = (d(x) * y) \wedge \big(x * d(y)\big)$$

$$= (x * d(y)) \wedge (d(x) * y).$$

This completes the proof.

Proposition 1.2.4:

Let d be a self-map of a BCI-algebra X. Then

(i) If d is a $(l, r) - derivation$ of X, then $d(x) = d(x) \wedge x$ for all $x \in X$.

(ii) If d is a $(r, l) - derivation$ of X, then $d(x) = x \wedge d(x)$ for all $x \in X$ if and only if $d(0) = 0$.

Proof:

(i) Let d be a $(l, r) - derivation$ of X. Then

$$d(x) = d(x * 0) = (d(x) * 0) \wedge \big(x * d(o)\big)$$

$$= d(x) \wedge \big(x * d(0)\big)$$
$$= \big(x * d(0)\big) * (\big(x * d(0)\big) * d(x))$$
$$= \big(x * d(0)\big) * (\big(x * d(x)\big) * d(0))$$

$$\leq x * (x * d(x))$$

$$= d(x) \wedge x$$

But $d(x) \wedge x \leq d(x)$ is trivial and so (i) holds.

(ii) Let d be a $(r,l)-$ $derivation$ of X. If $d(x) = x \wedge d(x)$ then

$$d(0) = 0 \wedge d(o)$$
$$= d(0) * (d(0) * 0)$$
$$= d(0) * d(0) = 0\ .$$

Conversely, if $d(0) = 0$, then

$$d(x) = d(x * 0) = x * d(0) \wedge d(x) * 0$$

$= x \wedge d(x)$. Hence the proof.

Proposition 1.2.5:

Let d be a $(l,r) - derivation$ of BCI-algebra X. Then,

(i) $d(0) \in\ L_{\mathrm{p}}(X)$ie., $d(0) = 0 * (0 * d(0))$

(ii) $d(a) = d(0) * (0 * a) = d(0) + a$ for all $a \in L_{\mathrm{p}}(X)$

(iii) $d(a) \in L_{\mathrm{p}}(X)$ for all $a \in L_{\mathrm{p}}(X)$

(iv) $d(a + b) = d(a) + d(b) - d(0)$ for all $a, b \in L_{\mathrm{p}}(X)$

(v) d is the identity on $L_{\mathrm{p}}(X)$ if and only if $d(0) = 0$

Proof:

(i) Follows by proposition 1.2.4(i).

(ii) Let $a \in L_{\mathrm{p}}(X)$.Then $a = 0 * (0 * a)$ and so

$$d(a) = d\big(0 * (0 * a)\big)$$
$$= \big(d(0) * (0 * a)\big) \wedge \big(0 * d(0 * a)\big)$$
$$= \big(0 * d(0 * a)\big) * (\big(0 * d(0 * a)\big) * \big(d(0) * (0 * a)\big))$$

$$= \big(0 * d(0 * a)\big) * \Big(\big(0 * \big(d(0) * (0 * a)\big)\big) * d(0 * a)\Big)$$

$$= 0 * (0 * \big(d(0) * (0 * a)\big))$$

$$= d(0) * (0 * a) = d(0) + a \in L_{\mathrm{p}}(X)$$

(iii) Follows directly from (ii)

(iv) Let $a, b \in L_{\mathrm{p}}(X)$.

Then $a + b \in L_{\mathrm{p}}(X)$.From (ii) we have

$$\begin{aligned} d(a+b) &= d(0) + (a+b) \\ &= d(0) + a + d(0) + b - d(0) \\ &= d(a) + d(b) - d(0) \end{aligned}$$

(v) If d is the identity on $L_{\mathrm{p}}(X)$,then clearly $d(0) = 0$.

Conversely, if $d(0) = 0,$ then

$d(a) = d(0) + a = 0 + a = a$ for all $a \in L_{\mathrm{p}}(X)$.

ie.,d is the identity on $L_{\mathrm{p}}(X)$. This completes the proof.

Proposition 1.2.6:

Let d be a $(r, l) - derivation$ of BCI-algebra X. Then,

(i) $d(a) \in G(X)$ for all $a \in G(X)$

(ii) $d(a) \in L_{\mathrm{p}}(X)$ for all $a \in L_{\mathrm{p}}(X)$

(iii) $d(a) = a * d(0) = a + d(0)$for all $a \in L_{\mathrm{p}}(X)$

(iv) $d(a+b) = d(a) + d(b) - d(0)$ for all $a, b \in L_{\mathrm{p}}(X)$

(v) d is the identity on $L_{\mathrm{p}}(X)$ if an only if $d(0) = 0$

Proof:

(i) For any $a \in G(X)$ we have

$$d(a) = d(0 * a) = \big(0 * d(a)\big) \wedge (d(0) * a)$$
$$= (d(0) * a) * ((d(0) * a) * \big(0 * d(a)\big))$$
$$= 0 * d(a) \text{and so } d(a) \in G(X)$$

(ii) and (iii)

For any $a \in L_{\text{p}}(X)$we get

$$d(a) = d(0 * (0 * a))$$
$$= (0 * d(0 * a)) \wedge (d(0) * (0 * a))$$
$$= \big(d(0) * (0 * a)\big) * (\big(d(0) * (0 * a)\big) * \big(0 * d(0 * a)\big))$$
$$= 0 * d(0 * a) \in L_{\text{p}}(X)$$

and

$$d(a) = d(a * 0) = (a * d(0)) \wedge (d(a) * 0)$$
$$= d(a) * d(a) * (a * d(0)))$$
$$= a * d(0) = a * \big(0 * d(0)\big)$$
$$= a + d(0)$$

(iv)and (v) follows from (iii). This completes the proof.

Note:

The following example shows that there is $a\ (l, r) - derivation$ which is not $a\ (r, l) - derivation.$

Example 1.2.7:

Let $\mathbb{Z}$ be the set of all integers and $" - "$ the minus operation on $\mathbb{Z}$. Then $(\mathbb{Z}, -, 0)$ is a BCI-algebra. Let $d: X \to X$ be defined by $d(x) = x - 1$ for all $x \in \mathbb{Z}$. Then,

$$(d(x) - y) \wedge \big(x - d(y)\big) = ((x - 1) - y) \wedge (x - (y - 1))$$

$$= (x - y - 1) \wedge (x - y + 1)$$

$$= (x - y + 1) - 2 = x - y - 1$$

$$= d(x - y), \text{ for all } x, y \in \mathbb{Z}.$$

and so d is a $(l, r) - derivation.$

But $d(0) = 0 - 1 = -1 \neq 1 = 0 - d(0)$

and thus d is not a $(r, l) - derivation$ by proposition 1.2.6 (i).

Definition 1.2.8:

(i) A self-map d of a BCI-algebra X is said to be $regular$ if $d(0) = 0$.

(ii) The identity map on a BCI-algebra X is a $regular\ derivation$ of X, and it is called as the $trivial\ derivation$.

Proposition 1.2.9:

Let X be a commutative BCI-algebra and let d be a regular derivation of X. Then

(i) Both x and $d(x)$ belong to the same branch for all $x \in X$.

(ii) d is also a $(l, r) - derivation$.

Proof:

(i) Let $x \in X$. Then

$$\begin{aligned} 0 = d(0) &= d(a_x * x) \\ &= (a_x * d(x)) \wedge (d(a_x) * x) \\ &= (d(a_x) * x) * ((d(a_x) * x) * (a_x * d(x))) \\ &= a_x * d(x), \end{aligned}$$

and so $a_x \leq d(x) \Rightarrow d(x) \in V(a_x)$. Clearly, $x \in V(a_x)$.

(ii) By (i), we have $x * d(y) \in V(a_x * a_y)$

and $d(x) * y \in V(a_x * a_y)$

Thus $d(x * y) = \big(x * d(y)\big) \wedge (d(x) * y) = (d(x) * y) \wedge \big(x * d(y)\big)$,

which implies that d is a $(l, r) - derivation$.

This completes the proof.

Note:

The derivation d in example 1.2.2 is a regular derivation but the $(l, r) - derivation\ d$ in example 1.2.7 is not regular.

Proposition 1.2.10:

Let d be a regular derivation of a BCI-algebra X. Then the following hold:

(i) $d(x) \leq x$ for all $x \in X$

(ii) $d(x) * y \leq x * d(y)$ for all $x, y \in X$

(iii) $d(x * y) = d(x) * y \leq d(x) * d(y)$ for all $x, y \in X$

(iv) $d^{-1}(0) = \{x \in X / d(x) = 0\}$ is a subalgebra of X and $d^{-1}(0) \subset X_+$

Proof:

(i) Follows by proposition 1.2.4(ii) and

(ii) Follows from (i)

(iii) For any $x, y \in X$ we have

$$d(x * y) = (x * d(y)) \wedge (d(x) * y)$$

$$= (d(x) * y) * ((d(x) * y) * \big(x * d(y)\big))$$

$= (d(x) * y) * 0$

$= d(x) * y$

$\leq d(x) * d(y)$,which proves (iii)

(iv) Let $x, y \in d^{-1}(0)$

Then $d(x) = 0 = d(y)$, and so

$d(x * y) \leq d(x) * d(y) = 0 * 0 = 0$ by (iii),

and thus $d(x * y) = 0$ or equivalently $x * y \in d^{-1}(0)$.

Hence $d^{-1}(0)$ is a subalgebra of X.

Moreover if $x \in d^{-1}(0)$, then $0 = d(x) \leq x$ by (i) and so $x \in X_{+}$, showing that $d^{-1}(0) \subset X_{+}$.

Proposition 1.2.11:

A BCI-algebra X is p-semisimple if and only if $d^{-1}(0) = \{0\}$ for every regular derivation d of X.

Proof:

Assume that X is a p-semisimple BCI-algebra and let d be a regular derivation of X. Then $X_{+} = \{0\}$ and so $d^{-1}(0) = \{0\}$ by using proposition 1.2.10(iv).

Conversely let $d^{-1}(0) = \{0\}$ for every regular derivation d of X.

Define a self-map d_0 of X by $d_0(x) = a_x$ for all $x \in X$. Using proposition 1.2.3, we know that d is both a (l, r) and $(r, l) - derivation$ of X, since every p-semisimple BCI-algebra is commutative.

Clearly $d_0(0) = 0$, and so d_0 is a regular derivation of X. It follows from the hypothesis that $d^{-1}(0) = \{0\}$.

On the other hand,

$d_0(0) = a_x = 0 * (0 * x) = 0$ for all $x \in X_+$, and thus $x \in d_0^{-1}(0)$ which shows $X_+ \subset d_0^{-1}(0)$.

Hence by proposition 1.2.10 (iv), $X_+ = d_0^{-1}(0) = \{0\}$.

Therefore, X is a p-semisimple BCI-algebra. Hence the proof.

Proposition 1.2.12:

If a p-semisimple BCI-algebra has a regular derivation, then it is trivial.

Proof:

Since, a BCI-algebra X is p-semisimple if and only if $X = L_p(X)$, proposition 1.2.5 (v) implies that every regular derivation of a p-semisimple BCI-algebra X is the identity on X.

Note:

Combining the above propositions we get the characterizations of a p-semisimple BCI-algebra as follows.

Theorem 1.2.13:

Let X be a BCI-algebra. Then the following are equivalent:

(i) X is p-semisimple

(ii) If X has a regular derivation, then it is trivial.

(iii) $d^{-1}(0) = \{0\}$ for every regular derivation d of X.

Proof: Obvious.

Proposition 1.2.14:

Let d be a regular derivation of a BCI-algebra X. Then $d^{-1}(0) = X_+ \Leftrightarrow d(x) = a_x, \forall x \in X$.

Proof:

Assume that $d^{-1}(0) = X_+$. Also $a_x \in L_p(X)$ for all $x \in X$. Since $x * a_x \in X_+$, it follows that $d(x * a_x) = 0$.

By the above proposition, we have

$$0 = d(x * a_x) = d(x) * a_x \text{ and so } d(x) \leq a_x$$

It follows from $a_x \in L_p(X)$ that $d(x) = a_x$ for all $x \in X$.

Conversely, let $d(x) = a_x$ for all $x \in X$. For any $x \in X_+$, $d(x) = a_x = 0$ and hence $x \in d^{-1}(0)$, which proves that $X_+ \subset d^{-1}(0)$.

Using proposition 1.2.10 (iv), we conclude that $d^{-1}(0) = X_+$.

Definition 1.2.15:

Let d be a self-map of a BCI-algebra X. An ideal A of X is said to be $d - invariant$ if $d(A) \subset A$.

Proposition 1.2.16:

Let d be a regular $(r, l) - derivation$ of a BCI-algebra X. Then every ideal A of X is $d - invariant$.

Proof:

By proposition 1.2.4 (ii), we have $d(x) = x \wedge d(x) \leq x$ for all $x \in X$.

Let $y \in d(A)$. Then $y = d(x)$ for some $x \in A$. It follows that $y * x = d(x) * x = 0 \in A \Rightarrow y \in A$.

Hence $d(a) \subset A$ and A is $d - invariant$.

Example 1.2.17:

Let $X = \{0,1,2,3,4,5\}$ be a BCI-algebra with Cayley table as follows:

$*$	0	1	2	3	4	5
0	0	0	2	2	2	2
1	1	0	2	2	2	2
2	2	2	0	0	0	0
3	3	2	1	0	0	0
4	4	2	1	1	0	1
5	5	2	1	1	1	0

Define a map $d: X \rightarrow X$ by $d(x) = \begin{cases} 2 & if\ x = 0,1 \\ 0 & otherwise \end{cases}$

Then, d is a derivation of X, which is not regular. Also, note that $A = \{0,1\}$ is an ideal of X. But $d(A) = \{2\} \not\subset A$ which shows that A is not $d - invariant$.

A characterization of the regularity of a derivation is given in the following theorem.

Theorem 1.2.18:

Let d be a derivation of a BCI-algebra X. Then d is regular if and only if every ideal of X is $d - invariant$.

Proof:

Let d be a derivation of a BCI-algebra X and assume that every ideal of X is $d - invariant$.

Then, since the zero ideal $\{0\}$ is $d - invariant,$ we have $d(\{0\}) \subset \{0\}$ which implies that $d(0) = 0$. Thus d is regular.

Converse follows by the theorem 1.2.16.

SECTION 1.3

SOME MORE RESULTS ON DERIVATIONS OF BCI/BCK-ALGEBRAS

Proposition 1.3.1:

Every $(r,l)-derivations(or\ a\ (l,r)-derivation)$ of a BCK-algebra is regular.

Proof:

Let X be a BCK-algebra and d a $(r,l)-derivation$ of X. Then for all $x \in X$, we have:

$$d(0) = d(0 * x) = (0 * d(x)) \wedge (d(0) * x)$$

$$= 0 \wedge (d(0) * x) = 0$$

Now let d be a $(l,r)-derivation$ of X. Then for all $x \in X$ we have:

$$d(0) = d(0 * x) = (d(0) * x) \wedge (0 * d(x))$$

$$= (d(0) * x) \wedge 0 = 0.$$

Corollary 1.3.2:

A derivation of a BCK-algebra is regular.

Proposition 1.3.3:

Let d be a derivation of a BCI-algebra X and $a \in X$ such that $d(x) * a = 0$, for all $x \in X$. Then d is a regular derivation of X. Moreover, X is a BCK-algebra.

Proof:

Let d be a derivation of a BCI-algebra X and let $a \in X$ such that $d(x) * a = 0$, for all $x \in X$. Since d is $(l, r) - derivation$ we get:

$$0 = d(x * a) * a = ((d(x) * a) \wedge \left(x * d(a)\right) * a$$

$$= \left(x \wedge \left(x * d(a)\right)\right) * a = 0 * a$$

Thus $0 \leq a$ and so $x \in X_+$.

This shows that $d(0) = d(0 * a) = (d(0) * a) \wedge \left(0 * d(a)\right)$

$$= 0 \wedge \left(0 * d(a)\right) = 0$$

Hence d is a regular derivation of X. So by proposition 1.2.10 (i), we have $d(x) \leq x$, for all $x \in X$, and so,

$$0 * x \leq 0 * d(x) = (d(x) * a) * d(x) = 0 * a = 0$$

Thus $0 * x \leq 0$ for all $x \in X$ and so $0 = (0 * x) * 0 = 0 * x$.

Then we have $0 \leq x$, for all $x \in X \Rightarrow X$ is a BCK-algebra.

Proposition 1.3.4:

Let d be a derivation of a BCI-algebra X and $a \in X$ such that $a * d(x) = 0$, for all $x \in X$. Then d is a regular derivation of X. Moreover, X is a BCK-algebra.

Proof: Obvious.

Definition 1.3.5:

Let X be a BCI-algebra and d_1, d_2 two self maps of X we define $d_1 \circ d_2 : X \to X$ as $d_1 \circ d_2(x) = d_1(d_2(x))$, for all $x \in X$.

Proposition 1.3.6:

Let X be a p-semisimple BCI-algebra and d_1, d_2 be the $(l,r)-derivations$ of X. Then $d_1 \circ d_2$ is also a $(l,r)-derivation$ of X.

Proof:

Let X be a p-semisimple BCI-algebra and d_1, d_2 are $(l,r)-derivations$ of X. Then by proposition 1.2.10 (ii) and by proposition 1.1.9 (vii), we get for all $x, y \in X$:

$$(d_1 \circ d_2)(x * y) = d_1(d_2(x * y))$$

$$= d_1((d_2(x) * y) \wedge (x * d_2(y)))$$

$$= d_1(d_2(x) * y)$$

$$= (d_1(d_2(x) * y) \wedge (d_2(x) * d_1(y))$$

$$= d_1(d_2(x)) * y$$

$$= (x * d_1(d_2(y))) * (x * d_1(d_2(y))) * (d_1(d_2(x)) * y)$$

$$= (x * d_1 \circ d_2(y))) * (x * d_1 \circ d_2(y))) * ((d_1 \circ d_2(x)) * y)$$

$$= ((d_1 \circ d_2)(x) * y) \wedge (x * (d_1 \circ d_2(y)))$$

$\Rightarrow$ $d_1 \circ d_2$ is a $(l,r)-derivation$ of X.

Proposition 1.3.7:

Let X be p-semisimple BCI-algebra and d_1, d_2 are $(r,l)-derivations$ of X. Then $d_1 . d_2$ is also a $(l,r)-derivation$ of X.

Proof: Obvious.

Theorem 1.3.8:

Let X be a p-semisimple BCI-algebra and d_1, d_2 derivations of X. Then $d_1 \circ d_2$ is also a derivation of X.

Proof:

Follows by combining the above two propositions.

Proposition 1.3.9:

Let X be a p-semisimple BCI-algebra and d_1, d_2 derivations of X. Then $d_1 \circ d_2 = d_2 \circ d_1$.

Proof:

Let X be a p-semisimple BCI-algebra and d_1, d_2 derivations of X. Since d_2 is a $(l,r) - derivation$ of X, then for all $x, y \in X$:

$$\begin{aligned} d_1 \circ d_2(x * y) &= d_1(d_2(x * y)) \\ &= d_1((d_2(x) * y) \wedge (x * d_2(y))) \\ &= d_1(d_2(x) * y)) \end{aligned}$$

But d_1 is a $(r,l) - derivation$ of X, so

$$\begin{aligned} (d_1 \circ d_2)(x * y) &= d_1(d_2(x) * y) \\ &= (d_2(x) * d_1(y)) \wedge (d_1(d_2(x)) * y) \\ &= d_2(x) * d_1(y), \end{aligned}$$

Thus we have for all $x, y \in X$:

$$(d_1 \circ d_2)(x * y) = d_2(x) * d_1(y) \qquad (1)$$

Also, since d_1 is a $(r,l) - derivation$ of X, then for all $x, y \in X$:

$$(d_1 \circ d_2)(x * y) = d_2((x * d_1(y)) \wedge (d_1(x) * y))$$

$$= d_2(x * d_1(y))$$

But d_2 is a $(l, r) - derivation$ of X, so

$$(d_2 \circ d_1)(x * y) = d_2(x * d_1(y))$$

$$= (d_2(x) * d_1(y)) \wedge (x * d_2(d_1(y)))$$

$$= d_2(x) * d_1(y)$$

Thus we have for all $x, y \in X$:

$$(d_2 \circ d_1)(x * y) = d_2(x) * d_1(y) \qquad (2)$$

From (1) and (2) we get for all $x, y \in X$:

$$(d_1 \circ d_2)(x * y) = (d_2 \circ d_1)(x * y)$$

By putting $y = 0$ we get for all $x \in X$:

$$(d_1 \circ d_2)(x) = (d_2 \circ d_1)(x)$$

$$\Rightarrow d_1 \circ d_2 = d_2 \circ d_1$$

Definition 1.3.10:

Let X be a BCI-algebra and d_1, d_2 be two self maps of X. We define $d_1 * d_2 : X \to X$ as $(d_1 * d_2)(x) = d_1(x) * d_2(x)$, for all $x \in X$.

Proposition 1.3.11:

Let X be a p-semisimple BCI-algebra and d_1, d_2 the derivations of X. Then $d_1 * d_2 = d_2 * d_1$

Proof:

Let X be a p-smisimple BCI-algebra and d_1, d_2 derivations of X. Since d_2 is a $(l,r)-derivation$ of X, then for all $x, y \in X$:

$$(d_1 \circ d_2)(x * y) = d_1((d_2(x) * y) \wedge (x * d_2(y)))$$

$$= d_1(d_2(x) * y)$$

But d_1 is a $(r,l)-derivation$ of X, so

$$d_1(d_2(x) * y) = (d_2(x) * d_1(y)) \wedge (d_1(d_2(x)) * y)$$

$$= d_2(x) * d_{1(}(y)$$

Hence, $(d_1 \circ d_2)(x * y) = d_2(x) * d_1(y)$, for all $x, y \in X$ (3)

Also, we have that d_2 is a $(r,l)-derivataion$ of X, then for all $x, y \in X$:

$$(d_1 \circ d_2)(x * y) = d_1(x * d_2(y) \wedge d_2(x) * y)$$

$$= d_1(x * d_2(y))$$

But d_1 is a $(r,l)-derivation$ of X, so

$$d_1(x * d_2(y)) = (d_1(x) * d_2(y)) \wedge (x * d_1(d_2(y)))$$

$$= d_1(x) * d_2(y)$$

Thus, $(d_1 \circ d_2)(x * y) = d_1(x) * d_2(y)$ for all $x, y \in X$ (4)

From (3) and (4) we get:

$d_2(x) * d_1(y) = d_1(x) * d_2(y)$ for all $x, y \in X$

By putting $x = y$ we get for all $x \in X$:

$$d_2(x) * d_1(x) = d_1(x) * d_2(x)$$

$$(d_2 * d_1)(x) = (d_1 * d_2)(x)$$

$$\Rightarrow d_2 * d_1 = d_1 * d_2$$

CHAPTER-2

ON f-DERIVATIONS AND LEFT DERIVATIONS OF BCI-ALGEBRAS

SECTION 2.1:

ON f-DERIVATIONS OF BCI-ALGEBRAS:

Definition 2.1.1:

Let X be a BCI-algebra. By a $left-right-f-derivation\ (ie., (l,r)-f-derivation)$ of X, a self-map d_f of X satisfying the identity $d_f(x * y) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$ for all $x, y \in X$ is meant, where f is an $endomorphism$ of X.

If d_f satisfies the identity $d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y))$ for all $x, y \in X$, then it is said that d_f is a $right-left\ derivation\ (ie., (r,l)-f-derivation)$ of X.

Moreover, if d_f is both an $(r,l)-f-$ and $(l,r)-f-derivation$, it is said that d_f is an $f-derivation$.

Example 2.1.2:

Let $X = \{0,1,2,3,4,5\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	2	2	2	2
1	1	0	2	2	2	2
2	2	2	0	0	0	0
3	3	2	1	0	0	0
4	4	2	1	1	0	1
5	5	2	1	1	1	0

Define a map $d_f : X \to X$ by $d_f(x) = \begin{cases} 2 & if\ x = 0,1 \\ 0 & otherwise \end{cases}$

and define an endomorphism f of X by $f(x) = \begin{cases} 0 & if\ x = 0,1 \\ 2 & otherwise \end{cases}$

Then d_f is both derivation and f –derivation of X.

Example 2.1.3:

Let X be a BCI-algebra with the Cayley table given in the above example. Define a map $d_f : X \to X$ $d_f(x) = \begin{cases} 2 & if\ x = 0,1 \\ 0 & otherwise \end{cases}$ then d_f is a derivation of X.

Now, define an endomorphism f of X by $f(x) = 0, \forall\, x \in X$

Then d_f is not an $f - derivation$ of X since $d_f(2 * 3) = d_f(0) = 2$

But, $(d_f(2) * f(3)) \wedge (f(2) * d_f(3)) = (0 * 0) \wedge (0 * 0) = 0 \wedge 0 = 0$

and thus $d_f(2 * 3) \neq (d_f(2) * f(3)) \wedge (f(2) * d_f(3))$.

Note:

We see from the above example that there is a derivation of X which is not an $f - derivation$ of X.

Example 2.1.4:

Let $X = \{0,1,2,3,4,5\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	0	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

Define a map $d_f : X \to X$ by $d_f(x) = \begin{cases} 0 & if\ x = 0,1 \\ 2 & if\ x = 2,4 \\ 3 & if\ x = 3,5 \end{cases}$

And define an endomorphism f of X by $f(x) = \begin{cases} 0 & if\ x = 0,1 \\ 2 & if\ x = 2,4 \\ 3 & if\ x = 3,5 \end{cases}$

Then, d_f is both derivation and f −derivation of X.

Example 2.1.5:

Let X be a BCI-algebra with the Cayley table as in example 2.1.4. Define a map $d_f : X \to X$ by $d_f(x) = \begin{cases} 0 & if\ x = 0,1 \\ 2 & if\ x = 2,4 \\ 3 & if\ x = 3,5 \end{cases}$

Then d_f is derivation of X.

Define an endomorphism f of X by $f(0) = 0, f(1) = 1, f(2) = 3, f(3) = 2, f(4) = 5, f(5) = 4$. Then d_f is not a f −derivation of X.

Since $d_f(2 * 3) = d_f(3) = 3$

But $(d_f(2) * f(3)) \wedge (f(2) * d_f(3)) = (2 * 2) \wedge (3 * 3) = 0 \wedge 0 = 0$

and thus $d_f(2 * 3) \neq (d_f(2) * f(3)) \wedge (f(2) * d_f(3))$

Example 2.1.6:

Let X be a BCI-algebra with the Cayley table as in example 2.1.4. Define a map $d_f : X \to X$ $d_f(0) = 0,\ d_f(1) = 1,\ d_f(2) = 3,\ d_f(3) = 2,\ d_f(4) = 5,\ d_f(5) = 4$

Then, d_f is not a derivation of X since $d_f(2 * 3) = d_f(3) = 2$ but

$(d_f(2) * 3) \wedge (2 * d_f(3)) = (3 * 3) \wedge (2 * 2) = 0 \wedge 0 = 0$

and thus $d_f(2 * 3) \neq (d_f(2) * 3) \wedge (2 * d_f(3))$

Define an endomorphism f of X by $f(0) = 0, f(1) = 1, f(2) = 3, f(3) = 2, f(4) = 5, f(5) = 4$. Then d_f is a f $-$derivation of X.

Note:

From the above example, we see that there is a f-derivation of X which is not a derivation of X.

Notation:

Denote $f_x = 0 * (0 * f(x))$ for all $x \in X$. Then $f_x \in L_p(X)$.

Theorem 2.1.7:

Let d_f be a self-map of a BCI-algebra X defined by $d_f(x) = f_x$ for all $x \in X$. Then d_f is an $(l, r) - f - derivation$ of X. Moreover, if X is commutative, then d_f is an $(r, l) - derivation$ of X.

Proof:

Let $x, y \in X$.

Since $0 * (0 * (f_x * f(y))) = 0 * (0 * ((0 * (0 * f(x))) * f(y)))$

$$= 0 * (0 * ((0 * f(y)) * (0 * f(x))))$$

$$= 0 * (0 * (0 * f(y * x)))$$

$$= 0 * f(y * x)$$

$$= 0 * \big(f(y) * f(x)\big) = \big(0 * f(y)\big) * (0 * f(x))$$

$$= \Big(0 * \big(0 * f(x)\big)\Big) * f(y)$$

$$= f_x * f(y)$$

We have $f_x * f(y) \in L_p(X)$ and thus

$$f_x * f(y) = (f(x) * f_y) * ((f(x) * f_y) * (f_x * f(y)))$$

It follows that $d_f(x * y) = f_{x*y} = 0 * (0 * f(x * y)) = 0 * (0 * (f(x) * f(y)))$

$$= \Big(0 * \big(0 * f(x)\big)\Big) * \Big(0 * \big(0 * f(y)\big)\Big)$$

$$= f_x * f_y = 0 * (0 * (f_x * f(y)))$$

$$= f_x * f(y)$$

$$= (f(x) * f_y) * ((f(x) * f_y) * (f_x * f(y)))$$

$$= (f_x * f(y)) \wedge (f(x) * f_y)$$

$$= (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$$

and so d_f is a $(l, r) - f - derivation$ of X.

Now assume that X is commutative.

Using theorem 1.1.14 it is sufficient to show that $d_f(x) * f(y)$ and $f(x) * d_f(y)$ belong to the same branch for all $x, y \in X$, we have

$$d_f(x) * f(y) = f_x * f(y)$$

$$= 0 * (0 * (f_x * f(y)))$$

$$= (0 * (0 * f_x)) * (0 * \big(0 * f(y)\big))$$

$$= f_x * f_y \in V(f_x * f_y)$$

$$\text{and so } f_x * f_y = \big(0 * (0 * f(x))\big) * (0 * (0 * f_y))$$

$$= 0 * (0 * f(x) * f_y))$$

$$= 0 * (0 * (f(x) * d_f(y)))$$

$$\leq f(x) * d_f(y)$$

which implies that $f(x) * d_f(y) \in V(f_x * f_y)$

Hence, $d_f(x) * f(y)$ and $f(x) * d_f(y)$ belong to the same branch, and so

$$d_f(x * y) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$$

$$= (f(x) * d_f(y)) \wedge (d_f(x) * f(y))$$

This completes the proof.

Proposition 2.1.8:

Let d_f be a self-map of a BCI-algebra X. Then the following hold:

(i) If d_f is a $(l, r) - f - derivation$ of X, then $d_f(x) = d_f(x) \wedge f(x)$ for all $x \in X$.

(ii) If d_f is a $(r, l) - f - derivation$ of X, then $d_f(x) = f(x) \wedge d_f(x)$ for all $x \in X$ if and only if $d_f(0) = 0$

Proof:

(i) Let d_f be an $(l, r) - f - derivation$ of X. Then,

$$\begin{aligned} d_f(x) = d_f(x * 0) &= (d_f(x) * f(0)) \wedge (f(x) * d_f(0)) \\ &= (d_f(x) * 0) \wedge (f(x) * d_f(0)) \\ &= d_f(x) \wedge (f(x) * d_f(0)) \\ &= (f(x) * d_f(0)) * ((f(x) * d_f(0)) * d_f(x)) \\ &= (f(x) * d_f(0)) * ((f(x) * d_f(x)) * d_f(0)) \\ &\leq f(x) * (f(x) * d_f(x)) \\ &= d_f(x) \wedge f(x) \end{aligned}$$

ie.,$d_{\mathrm{f}}(x) \leq d_{\mathrm{f}}(x) \wedge f(x)$

but $d_{\mathrm{f}}(x) \wedge f(x) \leq d_{\mathrm{f}}(x)$ is trivial and so (i) holds.

(ii) Let d_{f} be a $(r,l) - f - derivation$ of X.

If $d_{\mathrm{f}}(x) = f(x) \wedge d_{\mathrm{f}}(x)$ for all $x \in X$, then for all $x = 0$,

$$d_{\mathrm{f}}(0) = f(0) \wedge d_{\mathrm{f}}(0) = 0 \wedge d_{\mathrm{f}}(0)$$

$$= d_{\mathrm{f}}(0) * (d_{\mathrm{f}}(0) * 0) = 0$$

Conversely, if $d_{\mathrm{f}}(0) = 0$,then $d_{\mathrm{f}}(x) = d_{\mathrm{f}}(x * 0)$

$$d_{\mathrm{f}}(x) = (f(x) * d_{\mathrm{f}}(0)) \wedge (d_{\mathrm{f}}(x) * f(0))$$

$$= (f(x) * 0) \wedge (d_{\mathrm{f}}(x) * 0)$$

$$= f(x) \wedge d_{\mathrm{f}}(x)$$

Hence the proof.

Proposition 2.1.9:

Let d_{f} be a $(l,r) - f - derivation$ of a BCI-algebra X. Then

(i) Let $d_{\mathrm{f}}(0) \in L_{\mathrm{p}}(X)$, ie.,$d_{\mathrm{f}}(0) = 0 * (0 * d_{\mathrm{f}}(0))$

(ii) $d_{\mathrm{f}}(a) = d_{\mathrm{f}}(0) * \big(0 * f(a)\big) = d_{\mathrm{f}}(0) + f(a)$ for all $a \in L_{\mathrm{p}}(X)$

(iii) $d_{\mathrm{f}}(a) \in L_{\mathrm{p}}(X)$ for all $a \in L_{\mathrm{p}}(X)$

(iv) $d_{\mathrm{f}}(a + b) = d_{\mathrm{f}}(a) + d_{\mathrm{f}}(b) - d_{\mathrm{f}}(0)$ for all $a, b \in L_{\mathrm{p}}(X)$

Proof:

(i) Obviously $d_{\mathrm{f}}(0) \in L_{\mathrm{p}}(X)$

(ii) Let $a \in L_{\mathrm{p}}(X)$, then $a = 0 * (0 * a)$ and so $f(a) = 0 * \big(0 * f(a)\big)$. That is $f(a) \in L_{\mathrm{p}}(X)$. Hence

$$d_{\mathrm{f}}(a) = d_{\mathrm{f}}(0 * (0 * a))$$

$$= (d_{\mathrm{f}}(0) * f(0 * a)) \wedge (f(0) * d_{\mathrm{f}}(0 * a))$$

$$= (d_{\mathrm{f}}(0) * f(0 * a)) \wedge (0 * d_{\mathrm{f}}(0 * a))$$

$$= (0 * d_{\mathrm{f}}(0 * a)) * ((0 * d_{\mathrm{f}}(0 * a)) * (d_{\mathrm{f}}(0) * f(0 * a)))$$

$$= (0 * d_{\mathrm{f}}(0 * a)) * ((0 * (d_{\mathrm{f}}(0) * f(0 * a))) * (d_{\mathrm{f}}(0 * a))$$

$$= 0 * (0 * (d_{\mathrm{f}}(0) * (f(0) * f(a))))$$

$$= 0 * (0 * (d_{\mathrm{f}}(0) * (0 * f(a))))$$

$$= d_{\mathrm{f}}(0) * \big(0 * f(a)\big)$$

$$= d_{\mathrm{f}}(0) + f(a)$$

(iii) Proof follows directly from (ii)

(iv) Let $a, b \in L_{\mathrm{p}}(X)$.

Also, $a + b \in L_{\mathrm{p}}(X)$ so from (ii), we have

$$d_{\mathrm{f}}(a + b) = d_{\mathrm{f}}(0) + f(a + b)$$

$$= d_{\mathrm{f}}(0) + f(a) + d_{\mathrm{f}}(0) + f(b) - d_{\mathrm{f}}(0)$$

$$= d_{\mathrm{f}}(a) + d_{\mathrm{f}}(b) - d_{\mathrm{f}}(0)$$

Proposition 2.1.10:

Let d_{f} be a $(r, l) - f - derivation$ of a BCI-algebra X. Then,

(i) $d_{\mathrm{f}}(a) \in G(X)$ for all $a \in G(X)$

(ii) $d_{\mathrm{f}}(a) \in L_{\mathrm{p}}(X)$ for all $a \in G(X)$

(iii) $d_{\mathrm{f}}(a) = f(a) * d_{\mathrm{f}}(0) = f(a) + d_{\mathrm{f}}(0)$ for all $a \in L_{\mathrm{p}}(X)$

(iv) $d_{\mathrm{f}}(a + b) = d_{\mathrm{f}}(a) + d_{\mathrm{f}}(b) - d_{\mathrm{f}}(0)$ for all $a, b \in L_{\mathrm{p}}(X)$

Proof:

(i) For any $a \in G(X)$, we have

$$\begin{aligned} d_f(a) &= d_f(0 * a) \\ &= (f(0) * d_f(a)) \wedge (d_f(0) * f(a)) \\ &= (d_f(0) * f(a)) * ((d_f(0) * f(a)) * (0 * d_f(a))) \\ &= 0 * d_f(a) \end{aligned}$$

and so $d_f(a) \in G(X)$

(ii) For any $a \in L_p(X)$, we get

$$\begin{aligned} d_f(a) &= d_f\big(0 * (0 * a)\big) \\ &= (0 * d_f(0 * a)) \wedge (d_f(0) * f(0 * a)) \\ &= (d_f(0) * f(0 * a)) * ((d_f(0) * f(0 * a)) * (0 * d_f(0 * a))) \\ &= 0 * d_f(0 * a) \in L_p(X) \end{aligned}$$

(iii) For any $a \in L_p(X)$, we get

$$\begin{aligned} d_f(a) = d_f(a * 0) &= (f(a) * d_f(0)) \wedge (d_f(a) * f(0)) \\ &= d_f(a) * (d_f(a) * (f(a) * d_f(0))) \\ &= f(a) * d_f(0) \\ &= f(a) * (0 * d_f(0)) \\ &= f(a) + d_f(0) \end{aligned}$$

(iv) Proof is obvious from (iii) and hence the proof.

Using the above proposition, we see that there is an $(l,r) - f - derivation$ which is not an $(r,l) - f - derivation$ as shown in the following example.

Example 2.1.11:

Let $\mathbb{Z}$ be the set of all integers and " $-$ " the minus operation on $\mathbb{Z}$. Then $(\mathbb{Z}, -, 0)$ is a BCI-algebra.

Let $d_f : X \to X$ be defined by $d_f(x) = f(x) - 1$ for all $x \in \mathbb{Z}$. Then,

$$
\begin{aligned}
(d_f(x)-f(y)) \wedge (f(x)-d_f(y)) &= (f(x)-1-f(y)) \wedge (f(x)-(f(y)-1)) \\
&= (f(x-y)-1) \wedge (f(x-y)+1) \\
&= (f(x-y)+1)-2 \\
&= f(x-y)-1 = d_f(x-y)
\end{aligned}
$$

Hence d_f is an $(l,r)-f-derivation$ of X. But $d_f(0)=f(0)-1=-1 \neq 1 = f(0)-d_f(0) = 0-d_f(0)$, that is $d_f(0) \notin G(X)$.

Therefore, d_f is not an $(r,l)-f-derivation$ of X by the above proposition.

Definition 2.1.12:

An f –derivation d_f of a BCI-algebra X is said to be $regular$ if $d_f(0)=0$.

Note:

The f –derivations d_f in example 2.2.4 and 2.2.6 are regular.

Proposition 2.1.13:

Let X be a commutative BCI-algebra and let d_f be a regular $(r,l)-f-derivation$ of X. Then the following hold:

(i) Both $f(x)$ and $d_f(x)$ belong to the same branch for all $x \in X$

(ii) d_f is an $(l,r)-f-derivation$ of X.

Proof:

(i) Let $x \in X$. Then,

$$0 = d_f(0) = d_f(a_x * x)$$

$$= (f(a_x) * d_f(x)) \wedge (d(a_x) * f(x))$$

$$= (d(a_x) * f(x)) * ((d(a_x) * f(x)) * (f(a_x) * d_f(x)))$$

$$= (d(a_x) * f(x)) * ((d(a_x) * f(x)) * (f_x * d_f(x)))$$

$= f_x * d_f(x)$, since $f_x * d_f(x) \in L_p(X)$, and so $f_x \leq d_f(x)$.

This shows that $d_f(x) \in V(f_x)$. Clearly $f(x) \in V(f_x)$

(ii) By (i), we have $f(x) * d_f(y) \in V(f_x * f_y)$ and $d_f(x) * f(y) \in V(f_x * f_y)$.

Thus, $d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y))$

$$= (d_f(x) * f(y)) \wedge (f(x) * d_f(y)),$$

which implies that d_f is an $(l,r) - f - derivation$ of X.

Note:

The f −derivations d_f in examples 2.2.4 and 2.2.6 are regular f −derivations but we know that the $(l,r) - f - derivation\ d_f$ in example 2.2.1 is not regular.

In the following we give some properties of regular f-derivations.

Definition 2.1.14:

Let X be a BCI-algebra. Then define $\ker d_f = \{x \in X \ / \ d_f(x) = 0, for\ all\ f - derivations\ d_f\}$.

Proposition 2.1.15:

Let d_f be a f −derivation of a BCI-algebra X. Then the following hold:

(i) $d_f(x) \le f(x)$ for all $x \in X$

(ii) $d_f(x) * f(y) \le f(x) * d_f(y)$ for all $x, y \in X$

(iii) $d_f(x * y) = d_f(x) * f(y) \le d_f(x) * d_f(y)$ for all $x, y \in X$

(iv) $\ker d_f$ is a subalgebra of X.

Especially, if f is monic, then $\ker d_f \subseteq X_+$.

Proof:

(i) Proof is obvious from the proposition 2.1.8(ii)

(ii) Since $d_f(x) \le f(x)$for all $x \in X$, then

$$\begin{aligned} d_f(x) * f(y) &\le f(x) * f(y) \\ &\le f(x) * d_f(y) \end{aligned}$$

(iii) For any $x, y \in X$, we have

$$\begin{aligned} d_f(x * y) &= (f(x) * d_f(y)) \wedge (d_f(x) * f(y)) \\ &= (d_f(x) * f(y)) * ((d_f(x) * f(y)) * (f(x) * d_f(y)) \\ &= (d_f(x) * f(y)) * 0 \\ &= d_f(x) * f(y) \\ &\le d_f(x) * d_f(y) \end{aligned}$$

Hence (iii)

(iv) Let $x, y \in \ker d_f$, then $d_f(x) = 0 = d_f(y)$ and so $d_f(x * y) \le d_f(x) * d_f(y) = 0 * 0 = 0$ by (iii), and thus $d_f(x * y) = 0$. That is $x * y \in \ker d_f$. Hence, $\ker d_f$ is a subalgebra of X.

Especially, if f is monic, and letting $x \in \ker d_f$ then $0 = d_f(x) \le f(x)$ by (i) and so $f(x) \in X_+$, that is $0 * f(x) = 0$, and thus $f(0 * x) = f(x)$, which implies that $0 * x = x$ and so $x \in X_+$, that is, $\ker d_f \subseteq X_+$.

Theorem 2.1.16:

Let f be monic of a commutative BCI-algebra X. Then X is p-semisimple if and only if $\ker d_f = \{0\}$ for every regular f −derivation d_f of X.

Proof:

Assume that X is p-semisimple BCI-algebra and let d_f be a regular f −derivation of X. Then $X_+ = \{0\}$, and so $\ker d_f = \{0\}$ by using proposition 2.1.15 (iv).

Conversely, let $\ker d_f = \{0\}$ for every regular f −derivation d_f of X. Define a self-map d_f of X by $d_f^*(x) = f_x$ for all $x \in X$.

Using 2.1.7 d_f^* is a f −derivation of X.

Clearly, $d_f^*(0) = f_0 = 0 * (0 * f(0)) = 0$,ans so d_f^* is a regular f −derivation of X. Its follows from the hypothesis that $\ker d_f^* = \{0\}$.

In addition $d_f^*(x) = f_x = 0 * (0 * f(x))$

$= f(0 * (0 * x))$

$= f(0) = 0$ for all $x \in X_+$, and thus $x \in \ker d_f^*$, which shows that $X_+ \subseteq \ker d_f^*$.

Hence by the above proposition, $X_+ = \ker d_f^* = \{0\}$. Therefore, X is p-semisimple.

Definition 2.1.17:

An ideal A of a BCI-algebra X is said to be an $f - ideal$ if $f(A) \subseteq A$.

Definition 2.1.18:

Let d_f be a self-map of a BCI-algebra X. An $f-ideal\ A$ of X is said to be $d_f-invariant$ if $d_f(A) \subseteq A$.

Theorem 2.1.19:

Let d_f be a regular $(r,l)-f-derivation$ of a BCI-algebra X, then every ideal A of X is $d_f-invariant$.

Proof:

By proposition 2.1.8 (ii), we have $d_f(x) = f(x) \wedge d_f(x) \leq f(x)$ for all $x \in X$.

Let $y \in d_f(A)$. Then $y = d_f(x)$ for some $x \in A$. It follows that $y * f(x) = d_f(x) * f(x) = 0 \in A$. Since $x \in A$, then $f(x) \in f(A) \subseteq A$ as A is an $f-ideal$.

It follows that $y \in A$ since A is an ideal of X. Hence $d_f(A) \subseteq A$, and thus A is $d_f-invariant$.

Theorem 2.1.20:

Let d_f be an f –derivation of a BCI-algebra X. Then d_f is regular if and only if every $f-ideal$ of X is $d_f-invariant$.

Proof:

Let d_f be a derivation of a BCI-algebra X and assume that every $f-ideal$ of X is $d_f-invariant$. Then, since the zero ideal $\{0\}$ is $f-ideal$ and $d_f-invariant$, we have $d_f(\{0\}) \subseteq \{0\}$, which implies that $d_f(0) = 0$.

Thus d_f is regular. Then by the above theorem, the proof follows.

SECTION-2.2:

LEFT DERIVATIONS OF BCI-ALGEBRAS:

Definition 2.2.1:

Let X be a BCI-algebra. By a left derivation of X, we mean a self-map D of X satisfying $D(x * y) = \big(x * D(y)\big) \wedge \big(y * D(x)\big)$, for all $x, y \in X$.

Example 2.2.2:

Let $X = \{0,1,2\}$ be a BCI-algebra with Cayley table defined by

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define a map $D: X \to X$ by $D(x) = \begin{cases} 2, & if\ x = 0,1 \\ 0, & if\ x = 2 \end{cases}$

Then D is a left derivation of X.

Proposition 2.2.3:

Let D be a left derivation of a BCI-algebra X. Then for all $x, y \in X$, we have

(i) $x * D(x) = y * D(y)$

(ii) $D(x) = a_{D(x)\,x}$

(iii) $D(x) = D(x) \wedge x$

(iv) $D(x) \in L_p(X)$

Proof:

(i) Let $x, y \in X$.

Then $D(0) = D(x * x)$

$= (x * D(x)) \wedge (x * D(x))$

$= x * D(x)$

Similarly, $D(0) = y * D(y)$

So, $x * D(x) = y * D(y)$

(ii) Let $x \in X$. Then $D(x) = D(x * 0)$

$= (x * D(0)) \wedge (0 * D(x))$

$= \left(0 * D(x)\right) * (\left(0 * D(x)\right) * \left(x * D(0)\right))$

$\leq 0 * (0 * \left(x * D(0)\right))$

$= 0 * (0 * \left(x * \left(x * D(x)\right)\right))$

$= 0 * (0 * (D(x) \wedge x))$

$= a_{D(x) \wedge x}$

Thus $D(x) \leq a_{D(x) \wedge x}$

But $a_{D(x) \wedge x} = 0 * (0 * (D(x) \wedge x))$

$\leq D(x) \wedge x \leq D(x)$

Therefore, $D(x) = a_{D(x) \wedge x}$

(iii) Let $x \in X$. Then using (ii), we have $D(x) = a_{D(x) \wedge x} \leq D(x) \wedge x$, but we know that $D(x) \wedge x \leq D(x)$, and hence (iii) holds.

(iv) Since $a_x \in L_p(X)$, for all $x \in X$, we get $D(x) \in L_p(X)$ by (ii)

Note: Proposition 2.2.3 (iv) implies that $D(X)$ is a subset of $L_{\mathrm{p}}(X)$.

Proposition 2.2.4:

Let D be a left derivation of a BCI-algebra X. Then for all $x, y \in X$, we have

(i) $y * (y * D(x)) = D(x)$

(ii) $D(x) * y \in L_{\mathrm{p}}(X)$

Proof: Obvious

Proposition 2.2.5:

Let D be a left derivation of a BCI-algebra X. Then

(i) $D(0) \in L_{\mathrm{p}}(X)$

(ii) $D(x) = 0 + D(x)$, for all $x \in X$

(iii) $D(x + y) = x + D(y)$, for all $x, y \in L_{\mathrm{p}}(X)$

(iv) $D(x) = x$, for all $x \in X$ if and only if $D(0) = 0$

(v) $D(x) \in G(X)$, for all $x \in G(X)$

Proof:

(i) Follows by proposition 2.2.3 (iv)

(ii) Let $x \in X$. From proposition 2.2.3 (iv), we get $D(x) = a_{\mathrm{D(x)}}$, so we have $D(x) = a_{\mathrm{D(x)}} = 0 * (0 * D(x)) = 0 + D(x)$

(iii) Let $x, y \in L_{\mathrm{p}}(X)$. Then

$$\begin{aligned} D(x+y) &= D(x * (0 * y)) = (x * D(0 * y)) \wedge ((0 * y) * D(x)) \\ &= ((0 * y) * D(x)) * (((0 * y) * D(x)) * (x * D(0 * y))) \end{aligned}$$

$$= x * D(0 * y) = x * ((0 * D(y)) \wedge (y * D(0)))$$
$$= x * (0 * D(y)) = x + D(y)$$

(iv) Let $D(0) = 0$ and $x \in X$. Then

$$D(x) = D(x) \wedge x = x * (x * D(x)) = x * D(0) = x * 0 = x$$

Conversely, let $D(x) = x$, for all $x \in X$. So it is clear that $D(0) = 0$.

(v) Let $x \in G(X)$. Then $0 * x = x$ and so

$$D(x) = D(0 * x) = (0 * D(x)) \wedge (x * D(0))$$

$$= (x * D(0)) * ((x * D(0)) * (0 * D(x))) = 0 * D(x)$$

This gives $D(x) \in G(X)$.

Proposition 2.2.6:

A regular left derivation of a BCI-algebra is trivial.

Proof: Obvious by above proposition (iv)

Note:

Proposition 2.2.5 (v) gives that $D(x) \in G(X) \subseteq L_{\mathrm{p}}(X)$.

Definition 2.2.7:

An ideal A of a BCI-algebra X is said to be $D - invariant$, if $D(A) \subset A$.

Theorem 2.2.8:

Let D be a left derivation of a BCI-algebra X. Then D is regular if and only if every ideal of X is $D - invariant$.

Proof:

Let D be a regular left derivation of a BCI-algebra X. Then proposition 2.2.6 gives that $D(x) = x$, for all $x \in X$. Let $y \in D(A)$, where A is an ideal of X. Then $y = D(x)$, for some $x \in A$. Thus, $y * x = D(x) * x = x * x = 0 \in A$.

Then $y \in A$ and $D(A) \subset A$. Therefore, A is $D - invariant$.

Conversely, let every ideal of X be $D - invariant$. Then $D(\{0\}) \subset \{0\}$, and hence $D(0) = 0$ and D is regular.

Proposition 2.2.9:

Let D be a left derivation of a p-semisimple BCI-algebra. Then the following hold for all $x, y \in X$:

(i) $D(x * y) = x * D(y)$
(ii) $D(x) * x = D(y) * y$
(iii) $D(x) * x = y * D(y)$

Proof:

(i) Let $x, y \in X$. Then $D(x * y) = \left(x * D(y)\right) \wedge \left(y * D(x)\right) = x * D(y)$

(ii) We know that $(x * y) * (x * D(y) \leq D(y) * y$ and $(y * x) * \left(y * D(x)\right) \leq D(x) * x$. This means that

$$\left((x * y) * \left(x * D(y)\right)\right) * (D(y) * y) = 0 \text{ and}$$

$$\left((y * x) * \left(y * D(x)\right)\right) * (D(x) * x) = 0$$

So,

$$\big((x * y) * (x * D(y))\big) * (D(y) * y) = ((y * x) * (y * D(x)))) * (D(x) * x) \qquad (1)$$

Using proposition 2.2.3, we get

$$(x * y) * D(x * y) = (y * x) * D(y * x) \qquad (2)$$

By (1) and (2) we have $(x * y) * (x * D(y)) = (y * x) * (y * D(x))$.

Since X is a p-semisimple BCI-algebra, (1) implies that $D(x) * x = D(y) * y$.

(iii) We have, $D(0) = x * D(x)$

From (ii), we get $D(0) * 0 = D(y) * y$ (or) $D(0) = D(y) * y$

So $D(x) * x = y * D(y)$.

Theorem 2.2.10:

In a p-semisimple BCI-algebra X, a self-map D of X is a left derivation if and only if it is a derivation.

Proof:

Assume that D is a left derivation of a BCI-algebra X.

Claim: D is a $(r, l) - derivation$ of X.

$$\begin{aligned} D(x * y) &= x * D(y) \\ &= (D(x) * y) * ((D(x) * y) * (x * D(y))) \\ &= (x * D(y)) \wedge (D(x) * y) \end{aligned}$$

Claim: D is a $(l, r) - derivation$ of X

$$D(x * y) = x * D(y) = (x * 0) * D(y) = (x * (D(0) * D(0)) * D(y))$$

$$= \Big(x * \big((x * D(x)) * (D(y) * y)\big)\Big) * D(y)$$

$$= \Big(x * \big((x * D(y)) * (D(x) * y)\big)\Big) * D(y)$$

$$= (x * D(y)) * ((x * D(y)) * (D(x) * y))$$

$$= (D(x) * y) \wedge (x * D(y))$$

Therefore D is a derivation of X.

Conversely, let D be a derivation of X. So it is a $(r, l) - derivation$ of X. Then

$$D(x * y) = (x * D(y)) \wedge (D(x) * y)$$

$$= (D(x) * y) * ((D(x) * y) * (x * D(y))) = x * D(y)$$

$$= (y * D(x)) * ((y * D(x)) * (x * D(y)))$$

$$= (x * D(y)) \wedge (y * D(x))$$

Hence, D is a left derivation of X.

CHAPTER -3

ON GENERALIZED DERIVATIONS AND SYMMETRIC BI-DERIVATIONS OF BCI-ALGEBRAS

SECTION 3.1:

GENERALIZED DERIVATIONS OF BCI-ALGEBRAS:

Definition 3.1.1:

Let X be a BCI-algebra. A mapping $D: X \to X$ is called a $generalized\ (l,r) - derivation$ if there exist a $(l,r) - derivation\ d: X \to X$ such that $D(x * y) = (D(x) * y) \wedge (x * d(y))$ for all $x, y \in X$. If there exist a $(r,l) - derivation\ d: X \to X$ such that $D(x * y) = \big(x * D(y)\big) \wedge (d(x) * y)$ for all $x, y \in X$, the mapping $D: X \to X$ is called a $generalized\ (r,l) - derivation$. Moreover if D is both a $generalized\ (l,r)$ and $(r,l) - derivation$, we say that D is a $generalized\ derivation.$

Example 3.1.2:

Consider a BCI-algebra $X = \{0, a, b\}$ with the following Cayley table :

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Define a map $d: X \to X$ by $d(x) = \begin{cases} b & if\ x = 0 \\ 0 & if\ x = a \\ a & if\ x = b \end{cases}$

Then d is a $(l,r)-derivation$ of X. But d is not a $(r,l)-derivation$ of X since $d(a*b) \neq \big(a*d(b)\big) \wedge (d(a)*b)$

Now we define a map $D: X \to X$ by $D(x) = \begin{cases} a \ \ if\ x = 0 \\ b \ \ if\ x = a \\ 0 \ \ if\ x = b \end{cases}$

Then D satisfies $D(x*y) = (D(x)*y) \wedge (x*d(y))$ for all $x, y \in X$. Hence D is a $generalized\ (l,r)-derivation$ of X.

Also, let $D: X \to X$ satisfy $D(x*y) = D(x)*y$ for all $x, y \in X$ and X satisfies $x \wedge y = x$ for all $x, y \in X$. Hence, for every $(l,r)-derivation\ d$ of X we have $D(x*y) = (D(x)*y) \wedge (x*d(y))$ for all $x, y \in X$.

$\therefore D$ is a $generalized\ (l,r)-\ derivation$ of X.

Example 3.1.3:

Consider a BCI-algebra $X = \{0, a, b\}$ with the following Cayley table:

$*$	0	a	b
0	0	0	a
a	a	0	b
b	b	b	0

Define a map $d: X \to X$ by $d(x) = \begin{cases} 0 \ \ if\ x = 0, a \\ b \ \ if\ x = b \end{cases}$

Then d is a derivation of X. Now we define a map $D: X \to X$ by

$$D(x) = \begin{cases} b \ \ if\ x = 0, a \\ 0 \ \ if\ x = b \end{cases}$$

Then D is a $generalized\ derivation$ of X.

Proposition 3.1.4:

Let D be a self map of a BCI-algebra X. Then

(i) If D is a $generalized\ (l,r)-derivation$ of X, then $D(x)=D(x)\wedge x$ for all $x\in X$.

(ii) If D is a $generalized\ (r,l)-derivation$ of X, then $D(0)=0$ if and only if $D(x)=x\wedge d(x)$ for all $x\in X$ and for some $(r,l)-derivation\ d$ of X.

Proof:

(i) If D is a $generalized\ (l,r)-derivation$, then there exist a $(l,r)-derivation$ such that $D(x*y)=(D(x)*y)\wedge(x*d(y))$ for all $x,y\in X$.

Hence we get
$$\begin{aligned} D(x) &= D(x*0)=(D(x)*0)\wedge(x*d(0)) \\ &= D(x)\wedge(x*d(0)) \\ &= \big(x*d(0)\big)*(\big(x*d(0)\big)*D(x)) \\ &= \big(x*d(0)\big)*(\big(x*D(x)\big)*d(0)) \\ &\le x*\big(x*D(x)\big)=D(x)\wedge x \end{aligned}$$

But $D(x)\wedge x\le D(x)$ is trivial and so (i) holds.

(ii) Suppose that D is a $generalized\ (r,l)-derivation$ of X. Then there exist a $(r,l)-derivation\ d$ such that $D(x*y)=\big(x*D(y)\big)\wedge(d(x)*y)$ for all $x,y\in X$.

If $D(0)=0$, then we have $D(0)=D(x*0)=\big(x*D(0)\big)\wedge(d(x)*0)=x\wedge d(x)$

Conversely, if $D(x)=x\wedge d(x)$, then $D(0)=0\wedge d(0)=d(0)*(d(0)*0)=d(0)*d(0)=0$. This completes the proof.

Proposition 3.1.5:

Let D be a $generalized\ (l,r)-derivation$ of a BCI-algebra X. Then

(i) $D(0) \in L_{\text{p}}(X)$

(ii) $\forall a \in L_{\text{p}}(X)\ D(a) = D(0) + a \in L_{\text{p}}(X)$

(iii) $\forall a \in L_{\text{p}}(X)$ and $\forall x \in X\ D(a * x) = D(a) * x$

(iv) $\forall a \in L_{\text{p}}(X)$ and $\forall x \in X\ D(a + x) = D(a) + x$

(v) $\forall a \in L_{\text{p}}(X), D(a + b) = D(a) + b = a + D(b)$

Proof:

(i) Using proposition 3.1.4 (i), we have $D(0) = D(0) \wedge 0 = 0 * (0 * D(0))$ and so $D(0) \in L_{\text{p}}(X)$

(ii) Let $a \in L_{\text{p}}(X)$. Then it is known that $a * x \in L_{\text{p}}(X)$ and $x * (x * a) = a$ for all $x \in X$.

Hence $D(0) * (0 * a) \in L_{\text{p}}(X)$ $(\because D(0) \in L_{\text{p}}(X)\ by\ (i))$

Then,
$$\begin{aligned} D(a) &= D(0 * (0 * a)) \\ &= \big(D(0) * (0 * a)\big) \wedge (0 * d(0 * a)) \\ &= \big(0 * d(0 * a)\big) * (\big(0 * d(0 * a)\big) * \big(D(0) * (0 * a)\big)) \\ &= 0 * \Big(0 * \big(D(0) * (0 * a)\big)\Big) = D(0) * (0 * a) \\ &= D(0) + a\ \in L_{\text{p}}(X) \end{aligned}$$

(iii) Let $a \in L_{\text{p}}(X)$ and $x \in X$.

Then we have
$$\begin{aligned} D(a * x) &= (D(a) * x) \wedge (a * d(x)) \\ &= \big(a * d(x)\big) * (\big(a * d(x)\big) * (D(a) * x)) \\ &= D(a) * x\ \ (\because a * d(x), D(a) * x \in L_{\text{p}}(X)) \end{aligned}$$

(iv) Let $a \in L_{\text{p}}(X)$ and $x \in X$.

Using (iii), we have $D(a + x) = D\big(a * (0 * x)\big) = D(a) * (0 * x) = D(a) + x$

(v) Follows directly from (iv).

Proposition 3.1.6:

Let D be a $generalized\ (r,l)-derivation$ of a BCI-algebra X. Then

(i) $\forall\ a \in G(X),\ \ D(a) \in G(X)$

(ii) $\forall\ a \in L_{\mathrm{p}}(X),\ D(a)\ \in L_{\mathrm{p}}(X)$

(iii) $\forall\ a \in L_{\mathrm{p}}(X),\ D(a) = a * D(0) = a + D(0)$

(iv) $\forall\ a \in L_{\mathrm{p}}(X),\ D(a+b) = D(a) + D(b) - D(0)$

(v) D is identity on $L_{\mathrm{p}}(X)$ if and only if $D(0) = 0$

Proof:

If D is a $generalized\ (r,l)-derivation$ of X, then there exists a $(r,l)-derivation\ d$ of X such that $D(x*y) = \big(x * D(y)\big) \wedge (d(x) * y)$ for all $x, y \in X$.

Now let $a \in G(X)$. Then $D(a) = D(0 * a) = \big(0 * D(a)\big) \wedge (d(0) * a)$

$$= (d(0) * a) * ((d(0) * a) * \big(0 * D(a)\big))$$

$$= 0 * D(a)$$

and so $D(a) \in G(X)$. Hence (i) is valid.

For any $a \in L_{\mathrm{p}}(X)$ we get

$$D(a) = D\big(0 * (0 * a)\big) = \big(0 * D(0 * a)\big) \wedge (d(0) * (0 * a))$$

$$= \big(d(0) * (0 * a)\big) * (\big(d(0) * (0 * a)\big) * \big(0 * D(0 * a)\big))$$

$D(a) = 0 * D(0 * a) \in L_{\mathrm{p}}(X)$, and

$$D(a) = D(0 * a) = \big(a * D(0)\big) \wedge (d(a) * 0)$$

$$= d(a) * \Big(d(a) * \big(a * D(0)\big)\Big) = a * D(0) = a * (0 * D(0)) = a + D(0)$$

Therefore (ii) and (iii) are valid.

Now let $a, b \in L_{\mathrm{p}}(X)$. Then $a + b \in L_{\mathrm{p}}(X)$ and so $D(a + b) = a + b + D(0)$

$$= a + D(0) + b + D(0) - D(0)$$

$$= D(a) + D(b) - D(0)$$

Thus (iv) is true.

(v)Follows directly from (iii)

Definition 3.1.7:

A BCI-algebra X is said to be $torsion\ free$ if $x + x = 0\ implies\ x = 0,$ for all $x \in X$.

Note:

(i) If there exists a non-zero element x of a BCI-algebra X such that $x + x = 0$, then X cannot be torsion free.

(ii) If a BCI-algebra X satisfies; (there exists $w \neq 0 \in X, 0 * w = w$; then X cannot be torsion free.

(iii) Every BCK-algebra is torsion free.

(iv) The BCI-algebra X in example 3.1.2 is torsion free.

Example 3.1.8:

Let $X = \{0,1,2,3,4,5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	0	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

Then X is a torsion free BCI-algebra.

Example 3.1.9:

A BCI-algebra $X = \{0,1,2,3\}$ together with the following Cayley table:

$*$	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

is not torsion free because $3 * (0 * 3) = 3 * 3 = 0$.

Proposition 3.1.10:

Let X be a torsion free BCI-algebra and D a generalized derivation. If $D^2 = 0\ on\ L_{\mathrm{p}}(X)$, then $D = 0$ on $L_{\mathrm{p}}(X)$.

Proof:

Assume that $D^2 = 0$ on $L_p(X)$. Let $x \in L_p(X)$ and so

$$\begin{aligned} 0 &= D^2(x+x) \\ &= D(D(x+x)) \\ &= D(0) + D(x+x) \\ &= D(0) + D(x) + D(x) - D(0) \\ &= D(x) + D(x) \end{aligned}$$

Since X is torsion free, it follows that $D(x) = 0$ for all $x \in L_p(X)$ so that $D = 0$ on $L_p(X)$.

Proposition 3.1.11:

Let X be a torsion free BCI-algebra and let D_1 and D_2 be two generalized derivations. If $D_1D_2 = 0$ on $L_p(X)$, then $D_2 = 0$ on $L_p(X)$.

Proof:

Let $x \in L_p(X)$. Then $x + x \in L_p(X)$ and thus

$$\begin{aligned} 0 &= (D_1D_2)(x+x) \\ &= D_1(D_2(x+x)) \\ &= D_1(0) + D_2(x) + D_2(x) - D_2(0) \\ &= D_1(0) - D_2(0) + D_2(x) + D_2(x) \\ &= D_1(0)D_2(0) + D_2(x) + D_2(x) \\ &= D_1(0)(0\, D_2(0)) + D_2(x) + D_2(x) \\ &= D_1(0) + D_2(0) + D_2(x) + D_2(x) \end{aligned}$$

$$= D_1(D_2(0)) + D_2(x) + D_2(x)$$

$$= (D_1D_2)(0) + D_2(x) + D_2(x)$$

$$= D_2(x) + D_2(x)$$

Since X is torsion free, we have $D_2(x) = 0$ for all $x \in L_p(X)$. Hence $D_2 = 0$ on $L_p(X)$.

Definition 3.1.12:

A $(l,r) - derivation\ (or\ (r,l) - derivation)\ d$ of a BCI-algebra X is said to be $regular$ if $d(0) = 0$.

Note:

The generalized $(l,r) - derivation\ D$ in example 3.1.2 is regular, but the (l,r) −derivation d in example 3.1.2 is not regular.

Definition 3.1.13:

A mapping $D: X \to X$ is called a $regular\ generalized\ (l,r) -$ $(or\ (r,l) -)\ derivation$ if there exists $regular\ (l,r)(or\ (r,l) -)\ derivation,\ d$ of X such that

$$D(x * y) = (D(x) * y) \wedge \big(x * d(y)\big), \forall x, y\ \in X$$

$$(or)\ D(x * y) = (x * D(y)) \wedge (d(x) * y)$$

Example 3.1.14:

Let $X = \{0,1,2,3,4,5\}$ be a BCI-algebra described in Example 3.1.8. We give here the $\wedge -$operation on X as follows:

∧	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	0	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	2	4	2	2	4	4
5	3	5	3	3	5	5

Then a map $d: X \to X$ given by $d(x) = \begin{cases} 0 & if\ x = 0,1 \\ 2 & if\ x = 2,4 \\ 3 & if\ x = 3,5 \end{cases}$

is a $regular\ (l,r) - derivation$ of X.

Define a self map D of X by $D(0) = 0, D(2) = 2, D(3) = 3$ and ordered triple $\big(D(1), D(4), D(5)\big)$ is $(5,1,4), (4,5,1), (3,0,2), (2,3,0), (1,4,5) or\ (0,2,3)$. Then D is a $regular\ generalized\ (l,r) - derivation$ of X.

Proposition 3.1.15:

If D is a $regular\ generalized\ (r,l) - derivation$ of a BCI-algebra X, then $D(0) = 0 * D(0)$.

Proof:

If D is a $regular\ generalized\ (r,l) - derivation$ of a BCI-algebra X then there exists a $regular\ (r,l) - derivation\ d$ of X such that

$$D(0) = D(0 * 0) = \big(0 * D(0)\big) \wedge (d(0) * 0)$$

$$= (0 * D(0)) \wedge 0$$

$$= 0 * (0 * (0 * D(0)))$$

$$= 0 * D(0)$$

SECTION 3.2:

SYMMETRIC BI-DERIVATIONS OF BCI-ALGEBRAS:

Definition 3.2.1:

Let X, Y be BCI-algebras. An operation $*$ on the Cartesian product $X * Y$ of X, Y is defined as follows

(i) $(x_1, y_1) * (x_{2,} y_2) = (x_1 * x_2, y_1 * y_2)$
(ii) $0 = (0,0)$

Then $(X \times Y, *, 0)$ is a BCI-algebra, and it is called the $product$ of X, Y.

Definition 3.2.2:

Let X be a BCI-algebra. A mapping $D(.,.): X \times X \to X$ is $symmetric$ if $D(x, y) = D(y, x)$ holds for all pairs $x, y \in X$.

Definition 3.2.3:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a symmetric mapping. A mapping $d: X \to X$ defined by $d(x) = D(x, x)$ is called $trace\ of\ D$.

Definition 3.2.4:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a symmetric mapping. If D satisfies the identity $D(x * y, z) = (D(x, z) * y) \wedge (x * D(y, z))$ for all $x, y, z \in$

X, then D is called $left - right\ symmetric\ bi - derivation\ \ (briefly\ (l,r) - symmetric\ bi - derivation)$.

If D satisfies the identity $D(x * y, z) = \big(x * D(y,z)\big) \wedge \big(D(x,z) * y\big)$ for all $x, y, z \in X$, then we say that D is a $right - left\ symmetric\ bi - derivation$,

$(briefly\ (r,l) - symmetric\ bi - derivation)$.

Moreover if d is both an $(r,l) -$ and a $(l,r) - symmetric\ bi - derivation$, it is said that D is a $symmetric\ bi - derivation$.

Example 3.2.5:

Let $X = \{0,1,2\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define a mapping $D(.,.) \colon X \times X \to X$ by

$$D(x,z) = \begin{cases} 0, & if\,(x,z) = (0,1)\ and\ (x,z) = (1,0) \\ x * z, & otherwise \end{cases}$$

Then D is both $(l,r) - and\ (r,l) - symmetric\ bi - derivation$.

Example 3.2.6:

Let $X = \{0,1,2,3\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	3	3	0

Define a mapping $D(.,.): X \times X \rightarrow X$ by $D(x,z) = \begin{cases} 2, & if(x,z) = (2,2) \\ 0, & otherwise \end{cases}$

Then, D is both $(l,r) -$ and $(r,l) - symmetric\ bi - derivation.$

Example 3.2.7:

Let X be a p-semisimple BCI-algebra. Define a mapping $D(.,.): X \times X \rightarrow X$ such that $D(x,z) = x + z$, for all $x, z \in X$. For all $x, y, z \in X$,

$$D(x * y, z) = (x * y) + z$$

$$= (x * y) * (0 + z)$$

$$= \big(x * (0 * z)\big) * y = (x + z) * y$$

On the other hand,

$$(D(x,z) * y) \wedge \big(x * D(y,z)\big) = \big(x * D(y,z)\big) * \big(x * D(y,z)\big) * (D(x,z) * y))$$

$$= D(x,z) * y = (x + z) * y$$

So, D is $(l,r) - symmetric\ bi - deriation.$ But for the following p-semisimple BCI-algebra $X = \{0,1,2\}$ given Cayley table as

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

$$D(1 * 2,2) = (1 * 2) + 2 = (1 * 2) * (0 * 2) = 2 * 1$$

But $\big(1 * D(2,2)\big) \wedge (D(1,2) * 2) = 1 * D(2,2) = 1 * 2 * (0 * 2) = 1 * (2 * 1)$

$$= 1 * 1 = 0$$

ie., $D(1 * 2,2) \neq (1 * D(2,2)) \wedge (D(1,2) * 2)$

So it is not an $(r,l) - symmetric\ bi - derivation.$

Proposition 3.2.8:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a symmetric mapping. Then

(i) If D is a $(l,r) - symmetric\ bi - derivation$, then $D(x,z) = D(x,z) \wedge x$ for all $x, z \in X$.

(ii) If D is a $(r,l) - symmetric\ bi - derivation$, then $D(x,z) = x \wedge D(x,z)$ for all $x, z \in X$ if and only if $D(0,z) = 0$ for all $z \in X$.

Proof:

(i) Let D be a $(l,r) - symmetric\ bi - derivation$. Then for all $x, z \in X$.

$$\begin{aligned} D(x,z) &= D(x*0,z) \\ &= (D(x,z)*0) \wedge (x*D(0,z)) \\ &= \big(x*D(0,z)\big) * \big(\big(x*D(0,z)\big)*(D(x,z)*0)\big) \\ &= \big(x*D(0,z)\big) * \big(\big(x*D(x,z)\big)*D(0,z)\big) \\ &\le x*(x*D(x,z)) \\ &= D(x,z) = D(x,z) \wedge x \end{aligned}$$

On the other hand $D(x,z) \wedge x \le D(x,z)$ and so (i) holds.

(iii) Let D be a $(r,l) - symmetric\ bi - derivation$. If $D(x,z) = x \wedge D(x,z)$ for all $x, z \in X$, then $D(0,z) = 0 \wedge D(0,z) = D(0,z) * (D(0,z) * 0) = 0$.

Conversely if $D(0,z) = 0$ for all $z \in X$, then

$$\begin{aligned} D(x,z) &= D(x*0,z) \\ &= \big(x*D(0,z)\big) \wedge (D(x,z)*0) \\ &= (x*0) \wedge (D(x,z)*0) = x \wedge D(x,z) \end{aligned}$$

Proposition 3.2.9:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ $(l,r) - symmetric\ bi - derivation$. Then

(i) $D(0,z) \in L_p(X)$ for all $z \in X$

(ii) $D(a,z) = D(0,z) * (0 * a) = D(0,z) + a$ for all $a \in L_{\mathrm{p}}(X), z \in X$

(iii) $D(a,z) \in L_{\mathrm{p}}(X)$ for all $a \in L_{\mathrm{p}}(X), z \in X$

(iv) $D(a+b,z) = D(a,z) + D(b,z) - D(0,z)$, for all $a, b \in L_{\mathrm{p}}(X), z \in X$

(v) $D(.,.): L_{\mathrm{p}}(X) \times X \to L_{\mathrm{p}}(X), D(a,z) = a,$ for all $a \in L_{\mathrm{p}}(X),\ z \in X$ if and only if $D(0,z) = 0$

Proof:

(i) If we show that $D(0,z) = 0 * \big(0 * D(0,z)\big)$, the proof is completed.

(vi) From proposition 3.2.8 (i) we know that $D(x,z) = D(x,z) \wedge x$ for all $x, z \in X$, so

$$D(0,z) = D(x,z) = D(0,z) \wedge 0 = 0 * (0 * D(0,z))$$

(ii) Let $a \in L_{\mathrm{p}}(X)$. Hence $a = 0 * (0 * a)$. Then,

$$\begin{aligned} D(a,z) = D(0 * (0 * a), z) &= \big(D(0,z) * (0 * a)\big) \wedge (0 * D(0 * a, z)) \\ &= \big(0 * D(0 * a, z)\big) * (\big(0 * D(0 * a, z)\big) * \big(D(0,z) * (0 * a)\big)) \\ &= D(0,z) * (0 * a) = D(0,z) + a \end{aligned}$$

(iii) Let $a \in L_{\mathrm{p}}(X)$. From (ii) $D(a,z) = D(0,z) * (0 * a)$

Because of $D(0,z) \in L_{\mathrm{p}}(X), D(0,z) * (0 * a) \in L_{\mathrm{p}}(X)$ so $D(a,z) \in L_{\mathrm{p}}(X)$.

(iv) Let $a, b \in L_{\mathrm{p}}(X)$. Then $a + b = a * (0 * b) \in L_{\mathrm{p}}(X)$ so from (ii)

$$\begin{aligned} D(a+b,z) &= D(0,z) + (a+b) \\ &= D(0,z) + a + D(0,z) + b - D(0,z) \\ &= D(a,z) + D(b,z) - D(0,z) \end{aligned}$$

(v) If $D(a,z) = a$, for all $a \in L_{\mathrm{p}}(X), z \in X$, clearly for $0 \in L_{\mathrm{p}}(X), D(0,z) = 0$.

Conversely if $D(0,z) = 0$, then for all $a \in L_{\mathrm{p}}(X),\ z \in X$.

$$D(a,z) = D(0,z) + a = 0 + a = a$$

Corollary 3.2.10:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(l,r) - symmetric\ bi - derivation$ and $d: X \to X$ be trace of D. Then

(i) $d(0) \in L_{\mathrm{p}}(X)$

(ii) $d(a) \in L_{\mathrm{p}}(X),$ for all $a \in L_{\mathrm{p}}(X)$

Proof: Obvious from the above proposition.

Proposition 3.2.11:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(r,l) - symmetric\ bi - derivation$. Then

(i) $D(a,z) \in G(X)$ for all $a \in G(X)$

(ii) $D(a,z) \in L_{\mathrm{p}}(X)$ for all $a \in L_{\mathrm{p}}(X), z \in X$

(iii) $D(a,z) = a * D(0,z) = a + D(0,z)$ for all $a \in L_{\mathrm{p}}(X),\ z \in X$

(iv) $D(a+b,z) = D(a,z) + D(b,z) - D(0,z)$, for all $a,b \in L_{\mathrm{p}}(X),\ z \in X$

(v) $D(.,.): L_{\mathrm{p}}(X) \times X \to L_{\mathrm{p}}(X),\ D(a,z) = a,$ for all $a \in L_{\mathrm{p}}(X), z \in X$ if and only if $D(0,z) = 0$

Proof:

(i) Let $a \in G(X)$. Hence $a = 0 * a$

Then $D(a,z) = D(0 * a, z) = \big(0 * D(a,z)\big) \wedge (D(0,z) * a)$

$$= (D(0,z) * a) * ((D(0,z) * a) * \big(0 * D(a,z)\big))$$

$$= 0 * D(a,z)$$

(ii) For any $a \in L_{\mathrm{p}}(X), a = 0 * (0 * a),$ so

$$D(a,z) = D(0 * (0 * a), z) = \big(0 * D(0 * a, z)\big) \wedge \big(D(0,z) * (0 * a)\big)$$

$$= \big(D(0,z) * (0 * a)\big) * (\big(D(0,z) * (0 * a)\big) * \big(0 * D(0 * a, z)\big))$$

$$= 0 * D(0 * a, z) \in L_{\mathrm{p}}(X)$$

(iii) For any $a \in L_{\mathrm{p}}(X)$ and $z \in X$

$$\begin{aligned} D(a,z) &= D(a * 0, z) = \big(a * D(0,z)\big) \wedge (D(a,z) * 0) \\ &= \big(a * D(0,z)\big) \wedge (D(a,z)) \\ &= D(a,z) * (D(a,z) * (a * D(0,z)) \\ &= a * D(0,z) = a * (0 * D(0,z)) \\ &= a + D(0,z) \end{aligned}$$

(iv) For all $a, b \in L_{\mathrm{p}}(X),\ a + b \in L_{\mathrm{p}}(X)$, so

$$\begin{aligned} D(a+b,z) &= (a+b) + D(0,z) \\ &= a + D(0,z) + b + D(0,z) - D(0,z) \\ &= D(a,z) + D(b,z) - D(0,z) \end{aligned}$$

(v) If $D(a,z) = a$, for all $a \in L_{\mathrm{p}}(X), z \in X$. Clearly for $0 \in L_{\mathrm{p}}(X), D(0,z) = 0$

Conversely if $D(0,z) = 0$, then for all $a \in L_{\mathrm{p}}(X),\ z \in X$.

$$D(a,z) = a + D(0,z) = a + 0 = a$$

Corollary 3.2.12:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(r,l) - symmetric\ bi - derivation$ and $d: X \to X$ be trace of D. Then

(i) $d(a) \in G(X)$, for all $a \in G(X)$

(ii) $d(a) \in L_{\mathrm{p}}(X)$, for all $a \in L_{\mathrm{p}}(X)$

Proof: Obvious.

Definition 3.2.13:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a symmetric mapping. If $D(0,z) = 0$, for all $z \in X$, D is called $componentwise\ regular$. In particular if $D(0,0) = d(0) = 0, D$ is called $d - regular$.

Note:

1. The symmetric bi-derivation D in example 3.2.5 is d-regular.

2. The symmetric bi-derivation D in example 3.2.6 is componentwise regular.

In the following we show that a $(l,r)-\ (resp,(r,l)-) symmetric\ bi-derivation$ with certain conditions is componentwise regular $(l,r)-(resp(r,l)-),\ symmetric\ bi-derivation.$

Proposition 3.2.14:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(l,r)-symmetric\ bi-derivation.$ If there exist $a \in X$ such that $D(x,z) * a = 0$, for all $x, z \in X$, then D is $componentwise\ regular\ (l,r)-symmetric\ bi-derivation.$

Proof:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(l,r)-symmetric\ bi-derivation.$

Assume that there exist $a \in X$ such that $D(x,z) * a = 0$, for all $x, z \in X$. Since D is $(l,r)-symmetric\ bi-derivation$ we get

$$0 = D(x * a, z) * a = ((D(x,z) * a) \wedge \big(x * D(a,z)\big) * a$$

$$= \big(0 \wedge x * D(a,z)\big) * a = 0 * a$$

Thus $0 \le a$. This shows that

$$D(0,z) = D(0 * a, z) = (D(0,z) * a) \wedge (0 * D(a,z))$$

$$= 0 \wedge \big(0 * D(a,z)\big) = 0$$

Thus D is componentwise regular.

Corollary 3.2.15:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(l, r) - symmetric\ bi - derivation$. If there exist $a \in X$ such that $D(x, z) * a = 0$, for all $x, z \in X$, then D is $d - regular\ (l, r) - symmetric\ bi - derivation$.

Proof: Obvious.

Proposition 3.2.16:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(r, l) - symmetric\ bi - derivation$. If there exist $a \in X$ such that $a * D(x, z) = 0$, for all $x, z \in X$, then D is $componentwise\ regular\ (r, l) - symmetric\ bi - derivation$.

Proof:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(r, l) - symmetric\ bi - derivation$. Assume that there exist $a \in X$ such that $a * D(x, z) = 0$, for all $x, z \in X$. Since D is $(r, l) - symmetric\ bi - derivation$ we get

$$0 = a * D(x * a, z) = a * \big((a * D(x, z)) \wedge (D(a, z) * x)\big)$$

$$= a * (0 \wedge D(a, z) * x) * a = a * 0$$

This shows that $D(0, z) = D(a * 0, z)$

$$= ((a * D(0, z)) \wedge (D(a, z) * 0))$$

$$= 0 \wedge D(a, z) = 0$$

Thus D is componentwise regular.

Corollary 3.2.17:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(r,l) - symmetric\ bi - derivation$. If there exist $a \in X$ such that $a * D(x,z) = 0$, for all $x, z \in X$, then D is $d - regular\ (r,l) - symmetric\ bi - derivation$.

Proof: Obvious.

Theorem 3.2.18:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $(l,r) - symmetric\ bi - derivation$. Then X be a BCK-algebra if and only if D is $componentwise\ regular$.

Proof:

Let X be a BCK-algebra and $D(.,.): X \times X \to X$ be a $(l,r) - symmetric\ bi - derivation$. Then for all $x, z \in X$, we have

$$D(0,z) = D(0 * x, z) = (D(0,z) * x) \wedge (0 * D(x,z))$$

$$= (D(0,z) * x) \wedge 0 = 0 * \big(0 * (D(0,z) * x)\big)$$

$$= 0 * 0$$

Conversely let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a $componentwise\ regular\ (l,r) - symmetric\ bi - derivation$.

Let for a $x \in X$, $o * x \neq 0$. Since D is componentwise regular $D(0 * x, 0) = 0$. And

$$(D(0,0) * x) \wedge \big(0 * D(x,0)\big) = (0 * x) \wedge (0 * 0)$$

$$= (0 * x) \wedge 0$$

$$= 0 * \big(0 * (0 * x)\big)$$

$$= 0 * x \neq 0$$

But it is not possible since D is $(l,r)-symmetric\ bi-derivation$. Thus for all $x\in X, 0*x=0$, ie., X is a BCK-algebra.

Theorem 3.2.19:

Let X be a BCI-algebra and $D(.,.)\colon X\times X\to X$ be a $(r,l)-symmetric\ bi-derivation$. Then X be a BCK-algebra if and only if D is componentwise regular.

Proof:

Let X be a BCK-algebra and $D(.,.)\colon X\times X\to X$ be a $(r,l)-symmetric\ bi-derivation$. Then for all $x,z\in X$, we have

$$D(0,z)=D(0*x,z)=\big(0*D(x,z)\big)\wedge(D(0,z)*x)=0\wedge(D(0,z)*x)$$

$$=(D(0,z)*x)*\big((D(0,z)*x)*0\big)=(D(0,z)*x)*(D(0,z)*x)=0$$

Conversely, let X be a BCI-algebra and $D(.,.)\colon X\times X\to X$ be a componentwise regular $(r,l)-symmetric\ bi-derivation$. Suppose $a\in L_p(X)$ and $a\neq 0$.

Since D is componentwise regular $D(a*0,0)=D(a,0)=0$

But $(a*D(0,0))\wedge(D(a,0)*0)=(a*0)\wedge(0*0)=a\wedge 0=0*(0*a)=a\neq 0$.

But it is not possible since D is $(r,l)-symmetric\ bi-derivation$. Thus the unique p-atom is 0.

Assume that for a $x\in X, 0*x\neq 0$ then $a_{0*x}=0*\big(0*(0*x)\big)=0$, so $0*x\in L_p(X)$, but this is a contradiction. Thus for all $x\in X, 0*x=0$, ie., X is a BCK-algebra.

Corollary 3.2.20:

Let X be a BCI-algebra and $D(.,.): X \times X \to X$ be a symmetric bi-derivation. Then X is a BCK-algebra if and only if D is componentwise regular.

Proof: Obvious.

CHAPTER 4

ON $\boldsymbol{t}$ −DERIVATIONS AND $(\boldsymbol{\alpha},\boldsymbol{\beta})$ −DERIVATIONS OF BCI-ALGEBRAS

SECTION 4.1:

ON $\boldsymbol{t}$ −DERIVATIONS OF BCI-ALGEBRAS:

Definition 4.1.1:

Let X be a BCI-algebra. Then for any $t \in X$, we define a self map $d_{\mathrm{t}}: X \to X$ by $d_{\mathrm{t}}(x) = x * t$ for all $x \in X$.

Definition 4.1.2:

Let X be a BCI-algebra. Then for any $t \in X$, we define a self map $d_{\mathrm{t}}: X \to X$ is called a $left - right\ t - derivation$ or $(l, r) - t - derivation$ of X if it satisfies the identity $d_{\mathrm{t}}(x * y) = (d_{\mathrm{t}}(x) * y) \wedge (x * d_{\mathrm{t}}(y))$ for all $x, y \in X$.

Definition 4.1.3:

Let X be a BCI-algebra. Then for any $t \in X$, we define a self map $d_{\mathrm{t}}: X \to X$ is called a $right - left\ t - derivation$ or $(r, l) - t - derivation$ of X if it satisfies the identity $d_{\mathrm{t}}(x * y) = (x * d_{\mathrm{t}}(y)) \wedge (d_{\mathrm{t}}(x) * y)$ for all $x, y \in X$.

Moreover, if d_{t} is both a $(l, r) -$ and a $(r, l) - t - derivation$ on X, we say that d_{t} is a $t - derivation$ on X.

Example 4.1.4:

Let $X = \{0,1,2\}$ be a BCI-algebra with the following Cayley table:

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

For any $t \in X$, define a self-map d_t: $X \to X$ by $d_t(x) = x * t$ for all $x \in X$. Then d_t is a $t - derivation$ of X.

Proposition 4.1.5:

Let d_t be a self map of an associative BCI-algebra X. Then d_t is a $(l,r) - t - derivation$ of X.

Proof:

Let X be an associative BCI-algebra, then we have

$$d_t(x * y) = (x * y) * t = \big(x * (y * t)\big) * 0$$

$$= \big(x * (y * t)\big) * ((x * (y * t) * (x * (y * t))$$

$$= \big(x * (y * t)\big) * ((x * (y * t)) * (x * y) * t))$$

$$= \big(x * (y * t)\big) * (\big(x * (y * t)\big) * \big((x * t) * y\big))$$

$$= \big((x * t) * y\big) \wedge (x * (y * t))$$

$$= (d_t(x) * y) \wedge (x * d_t(y))$$

Hence d_t is a $(l,r) - t - derivation$ of X.

Proposition 4.1.6:

Let d_t be a self map of an associative BCI-algebra X. Then d_t is a $(r,l)-t-derivation$ of X.

Proof:

Let X be an associative BCI-algebra, then we have

$$
\begin{aligned}
d_t(x*y) = (x*y)*t &= \big((x*t)*y\big)*0 \\
&= \big((x*t)*y\big)*(\big((x*t)*y\big)*\big((x*t)*y\big)) \\
&= \big((x*t)*y\big)*(\big((x*t)*y\big)*\big((x*y)*t\big)) \\
&= \big((x*t)*y\big)*(\big((x*t)*y\big)*\big(x*(y*t)\big)) \\
&= \big(x*(y*t)\big)\wedge((x*t)*y) \\
&= (x*d_t(y))\wedge(d_t(x)*y)
\end{aligned}
$$

Hence d_t is a $(r,l)-t-derivation$ of X.

Theorem 4.1.7:

Let d_t be a self map of an associative BCI-algebra X. Then d_t is a $t-derivation$ of X.

Proof: Follows by combining the above two propositions.

Definition 4.1.8:

A self map d_t of a BCI-algebra X is said to be $t-regular$ if $d_t(0)=0$.

Example 4.1.9:

Let $X = \{0, a, b\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

(i) For any $t \in X$, define a self map $d_t : X \to X$ by

$$d_t(x) = x * t = \begin{cases} b & if\ x = 0, a \\ 0 & if\ x = b \end{cases}$$

Then d_t is (l, r) and $(r, l) - t - derivation$ of X, which is not $t - regular$.

(ii) For any $t \in X$, define a self map $d_t^{'} : X \to X$ by

$$d_t^{'}(x) = x * t = \begin{cases} 0 & if\ x = 0, a \\ b & if\ x = b \end{cases}$$

Then $d_t^{'}$ is (l, r) and $(r, l) - t - derivation$ of X, which is $t - regular$.

Proposition 4.1.10:

Let d_t be a self map of an associative BCI-algebra X. Then

(i) If d_t is a $(l, r) - t - derivation$ of X, then $d_t(x) = d_t(x) \wedge x$ for all $x \in X$

(ii) If d_t is a $(r, l) - t - derivation$ of X, then $d_t(x) = x \wedge d_t(x)$ for all $x \in X$ if and only if d_t is $t - regular$.

Proof:

(i) Let d_t be a $(l, r) - t - derivation$ of X, then

$$\begin{aligned} d_t(x) &= d_t(x * 0) = (d_t(x) * 0) \wedge (x * d_t(0)) \\ &= d_t(x) \wedge (x * d_t(0)) \end{aligned}$$

$$= (x * d_t(0)) * ((x * d_t(0)) * d_t(x))$$
$$= (x * d_t(0)) * ((x * d_t(x)) * d_t(0))$$
$$\leq x * (x * d_t(x)) = d_t(x) \wedge x$$

But $d_t(x) \wedge x \leq d_t(x)$ is trivial so (i) holds.

(ii) Let d_t be a $(r, l) - t - derivation$ of X.

If $d_t(x) = x \wedge d_t(x)$ then

$$d_t(0) = 0 \wedge d_t(0) = d_t(0) * (d_t(0) * 0)$$
$$= d_t(0) * d_t(0) = 0$$

Hence, d_t is $t - regular$.

Conversely, suppose that d_t is $t - regular$, that is $d_t(0) = 0$, then we have

$$d_t(x) = d_t(x * 0) = (x * (d_t(0)) \wedge (d_t(x) * 0)$$

$$= (x * 0) \wedge d_t(x) = x \wedge d_t(x)$$

This completes the proof.

Theorem 4.1.11:

Let d_t be a $(l, r) - t - derivation$ of a p-semisimple BCI-algebra X. Then the following hold:

(i) $d_t(0) = d_t(x) * x$ for all $x \in X$
(ii) d_t is one-one
(iii) If d_t is $t - regular$, then it is an identity map
(iv) If there is an element all $x \in X$ such that $d_t(x) = x$, then d_t is the identity map
(v) If $x \leq y$, then $d_t(x) \leq d_t(y)$ for all $x, y \in X$

Proof:

(i) Let d_t be a $(l,r)-t-derivation$ of a p-semisimple BCI-algebra X. Then for all $x \in X$, we have $x * x = 0$ and so

$$\begin{aligned} d_t(0) &= d_t(x * x) = (d_t(x) * x) \wedge (x * d_t(x)) \\ &= (x * d_t(x)) * ((x * d_t(x)) * (d_t(x) * x)) \\ &= d_t(x) * x \ (\because x * (x * y) = y) \end{aligned}$$

(ii) Let $d_t(x) = d_t(y) \Rightarrow x * t = y * t$ then by the property that $"a * x = b * x \ implies \ a = b"$, we have $x = y$ and so d_t is one-one.

(iii) Let d_t be $t-regular$ and $x \in X$

Then $0 = d_t\,(0)$ so by the above part (i), we have $0 = d_t\,(x) * x$ and hence by the property that "$x * y = 0 \ implies \ x = y$", we obtain $d_t\,(x) = x$ for all $x \in X$. Therefore, d_t is the identity map.

(iv) It is trivial and follows from the above part (iii)

(v) Let $x \leq y$ implying $x * y = 0$

Now, $d_t(x) * d_t(y) = (x * t) * (y * t) = x * y = 0$

Therefore, $d_t(x) \leq d_t(y)$. This completes the proof.

Definition 4.1.12:

Let d_t be a $t-derivation$ of a BCI-algebra X. Then, d_t is said to be an $isotone\ t-derivation$ if $x \leq y \ implies \ d_t(x) \leq d_t(y)$ for all $x, y \in X$.

Example 4.1.13:

In example 4.1.9 (ii), $d_t^{'}$ is an isotone t-derivation, while in example 4.1.9 (i), d_t is not an isotone t-derivation.

Proposition 4.1.14:

Let X be a BCI-algebra and d_t be a t-derivation on X. Then for all $x, y \in X,$ the following hold:

(i) If $d_t(x \wedge y) = d_t(x) \wedge d_t(y)$, then d_t is an isotone t-derivation

(ii) If $d_t(x * y) = d_t(x) * d_t(y),$ then d_t is an isotone t-derivation

Proof:

(i) Let $d_t(x \wedge y) = d_t(x) \wedge d_t(y)$.

If $x \leq y \ \Rightarrow\ x \wedge y = x$ for all $x, y \in X$. Therefore, we have

$$d_t(x) = d_t(x \wedge y) = d_t(x) \wedge d_t(y) \leq d_t(y)$$

Henceforth $d_t(x) \leq d_t(y)$ which implies that d_t is an isotone t-derivation.

(ii) Let $d_t(x * y) = d_t(x) * d_t(y)$. If $x \leq y \Rightarrow x * y = 0$ for all $x, y \in X$. Therefore, we have

$$\begin{aligned} d_t(x) &= d_t(x * 0) = d_t(x * (x * y)) \\ &= d_t(x) * d_t(x * y) \\ &= d_t(x) * (d_t(x) * d_t(y)) \\ &\leq d_t(y)\ (\because \big(x * (x * y)\big) * y = 0) \end{aligned}$$

Thus, $d_t(x) \leq d_t(y)$. This completes the proof.

Theorem 4.1.15:

Let d_t be a $t - regular\ (r, l) - t - derivation$ of a BCI-algebra X. Then the following hold:

(i) $d_t(x) \leq x$ for all $x, y \in X$

(ii) $d_t(x) * y \leq x * d_t(y)$ for all $x, y \in X$

(iii) $d_t(x * y) = d_t(x) * y \leq d_t(x) * d_t(y)$ for all $x, y \in X$

(iv) $\ker(d_t) = \{\, x \in X / d_t(x) = 0 \,\}$ is a subalgebra of X

Proof:

(i) For any $x \in X$, we have

$$d_t(x) = d_t(x * 0) = (x * d_t(0)) \wedge (d_t(x) * 0)$$
$$= (x * 0) \wedge (d_t(x) * 0) = x \wedge d_t(x) \leq x$$

(ii) Since $d_t(x) \leq x$ for all $x \in X$, then

$$d_t(x) * y \leq x * y \leq x * d_t(y)$$

(iii) For any $x, y \in X$, we have

$$\begin{aligned} d_t(x * y) &= (x * d_t(y)) \wedge (d_t(x) * y) \\ &= (d_t(x) * y) * ((d_t(x) * y) * (x * d_t(y))) \\ &= (d_t(x) * y) * 0 \\ &= d_t(x) * y \\ &\leq d_t(x) * d_t(y) \end{aligned}$$

(iv) Let $x, y \in \ker(d_t) \Rightarrow d_t(x) = 0 = d_t(y)$

From (iii), we have $d_t(x * y) \leq d_t(x) * d_t(y) = 0 * 0 = 0$ implying $d_t(x * y) \leq 0$ and so $d_t(x * y) = 0$.

Therefore, $x * y \in \ker(d_t)$

Consequently $\ker(d_t)$ is a subalgebra of X. This completes the proof.

Definition 4.1.16:

Let X be a BCI-algebra and let d_t, $d_t^{'}$ be two self maps of X. Then we define

$d_t \circ d_t^{'} : X \rightarrow X$ by $(d_t \circ d_t^{'})(x) = d_t(d_t^{'}(x))$ for all $x \in X$.

Example 4.1.17:

Let $X = \{0, a, b\}$ be a BCI-algebra which is given in example 4.1.4. Let d_t and d_t' be two self maps on X as defined in example 4.1.9 (i) and (ii), respectively.

Now, define a self map $d_t \circ d_t' : X \to X$ by $(d_t \circ d_t')(x) = \begin{cases} 0 & if\ x = a, b \\ b & if\ x = 0 \end{cases}$

Then $(d_t \circ d_t')(x) = d_t(d_t'(x))$ for all $x \in X$.

Proposition 4.1.18:

Let X be a p-semisimple BCI-algebra X and let d_t and d_t' be $(l, r) - t - derivations$ of X. Then $d_t \circ d_t'$ is also a $(l, r) - t - derivation$ of X.

Proof:

Let X be a p-semisimple BCI-algebra, d_t and d_t' are $(l, r) - t - derivations$ of X. Then for all $x, y \in X$, we get

$$\begin{aligned}
(d_t \circ d_t')(x * y) &= (d_t(d_t')(x * y)) = d_t((d_t'(x) * y) \wedge (x * d_t'(y))) \\
&= d_t((x * d_t'(y)) * ((x * d_t'(y)) * (d_t'(x) * y))) \\
&= d_t(d_t'(x) * y) \\
&= (x * \ d_t(d_t'(y))) * ((x * d_t(d_t'(y)) * (d_t(d_t'(x) * y))) \\
&= (d_t(d_t'(x) * y)) \wedge (x * \ d_t(d_t'(y))) \\
&= ((d_t \circ d_t')(x) * y) \wedge (x * (d_t \circ d_t')(y))
\end{aligned}$$

$\therefore$ $(d_t \circ d_t')$ is a $(l, r) - t - derivation$ of X.

Proposition 4.1.19:

Let X be a p-semisimple BCI-algebra X and let d_t and $d_t^{'}$ be $(r,l)-t-derivations$ of X. Then $d_t \circ d_t^{'}$ is also a $(r,l)-t-derivation$ of X.

Proof: Obvious.

Theorem 4.1.20:

Let X be a p-semisimple BCI-algebra X and let d_t and $d_t^{'}$ be $t-derivations$ of X. Then $d_t \circ d_t^{'}$ is also a $t-derivation$ of X.

Proof: Obvious by the above two propositions.

Theorem 4.1.21:

Let X be a p-semisimple BCI-algebra X and let d_t and $d_t^{'}$ be $t-derivations$ of X. Then $d_t \circ d_t^{'} = d_t^{'} \circ \ d_t$.

Proof:

Let X be a p-semisimple BCI-algebra X. let d_t and $d_t^{'}$ be $t-derivations$ of X. Suppose $d_t^{'}$ is a $(l,r)-t-derivation,$ then for all $x,y \in X,$ we have

$$(d_t \circ d_t^{'})(x*y) = d_t(d_t^{'}(x*y) = d_t((d_t^{'}(x)*y) \wedge (x*d_t^{'}(y)))$$

$$= d_t((x*d_t^{'}(y))*((x*d_t^{'}(y))*(d_t^{'}(x)*y)))$$

$$= d_t(d_t^{'}(x)*y)$$

But d_t is a $(r,l)-t-derivation.$

$$d_t(d_t^{'}(x)*y) = (d_t^{'}(x)*d_t(y)) \wedge (d_t(d_t^{'}(x)*y) = d_t^{'}(x)*d_t(y)$$

Hence $(d_t \circ d_t^{'})(x*y) = d_t^{'}(x)*d_t(y)$ (1)

Again, if d_t is a $(r,l)-t-derivation$, then we have

$$(d_t^{'} \circ d_t)(x*y) = d_t^{'}(d_t(x*y)$$

$$= d_t^{'}((x*d_t(y)) \wedge (d_t(x)*y)$$

$$= d_t^{'}(x*d_t(y))$$

But $d_t^{'}$ is a $(l,r)-t-derivation$, then

$$d_t^{'}(x*d_t(y)) = (d_t^{'}(x)*d_t(y)) \wedge (x*(d_t^{'}(d_t(y))) = d_t^{'}(x)*d_t(y)$$

Hence $(d_t^{'} \circ d_t)(x*y) = d_t^{'}(x)*d_t(y)$ (2)

By (1) and (2) we have $(d_t \circ d_t^{'})(x*y) = (d_t^{'} \circ d_t)(x*y)$

By putting $y=0$, we get $(d_t \circ d_t^{'})(x) = (d_t^{'} \circ d_t)(x)$, $\forall\, x \in X$.

Hence, $d_t \circ d_t^{'} = d_t^{'} \circ d_t$. This completes the proof.

Definition 4.1.22:

Let X be a BCI-algebra and let d_t, $d_t^{'}$ be two self maps of X. Then we define

$d_t * d_t^{'} : X \to X$ by $(d_t * d_t^{'})(x) = d_t(x) * d_t^{'}(x)$ for all $x \in X$.

Example 4.1.23:

Let $X = \{0, a, b\}$ be a BCI-algebra which is given in example 4.1.4. Let d_t and $d_t^{'}$ be two self maps on X as defined in example 4.1.9 (i) and (ii), respectively.

Now define a self map $d_t * d_t^{'} : X \to X$ by $(d_t * d_t^{'})(x) = \begin{cases} 0 & if\ x = a, b \\ b & if\ x = 0 \end{cases}$

Then $(d_t * d_t^{'})(x) = d_t(x) * d_t^{'}(x)$ for all $x \in X$.

Theorem 4.1.24:

Let X be a p-semisimple BCI-algebra and let d_t, d_t' be $t-$derivations of X. Then $d_t * d_t' = d_t' * d_t$.

Proof:

Let X be a p-semisimple BCI-algebra, d_t and d_t', $t-$derivations of X. Since d_t' is a $(r,l)-t-derivation$ of X, we have

$$(d_t \circ d_t')(x*y) = d_t(d_t'(x*y)) = d_t((x* d_t'(y)) \wedge (d_t'(x)*y))$$

$$= d_t(x*d_t'(y))$$

But d_t is a $(l,r)-t-derivation.$

$$\therefore\ d_t(x*d_t'(y)) = (d_t(x)*d_t'(y)) \wedge (x*d_t(d_t'(y))) = d_t(x)* d_t'(y)$$

Hence $(d_t \circ d_t')(x*y) = d_t(x)* d_t'(y)$ (1)

Again, if d_t' is a $(l,r)-t-derivation$ of X, then for all $x,y \in X$, we have

$$(d_t \circ d_t')(x*y) = d_t(d_t'(x*y)) = d_t((d_t'(x)*y)) \wedge (x*d_t'(y)))$$

$$= d_t((x*d_t'(y))*((x*d_t'(y))*(d_t'(x)*y)))$$

$$= d_t(d_t'(x)*y)$$

But d_t is a $(r,l)-t-derivation$. Therefore

$$d_t(d_t'(x)*y) = (d_t'(x)*d_t(y)) \wedge (\ d_t(d_t'(x))*y) = d_t'(x)*d_t(y)$$

Hence $(d_t * d_t')(x*y) = d_t'(x)*d_t(y)$ (2)

By (1) and (2) $d_t(x)* d_t'(y) = d_t'(x)*d_t(y)$

By putting $y = x$, we get $d_t(x)*d_t'(x) = d_t'(x)*d_t(x)$

$$(d_t * d_t')(x) = (d_t' * d_t)(x) \quad \forall x \in X$$

Hence, $d_t * d_t' = d_t' * d_t$. This completes the proof.

SECTION 4.2:

ON (α, β) −DERIVATIONS IN BCI-ALGEBRAS:

Throughout the section, α and β denote the $Endomorphisms$ of a BCI-algebra X.

Definition 4.2.1:

Let X be a BCI-algebra. Then a self map $d_{(}\alpha,\beta_{)}: X \to X$ is called a $(\alpha, \beta) - derivation$ of X if it satisfies:

$$d_{(}\alpha,\beta_{)}(x * y) = (\, d_{(}\alpha,\beta_{)}(x) * \alpha(y)) \wedge (\, d_{(}\alpha,\beta_{)}(y) * \beta(x)) \; \forall x, y \in X$$

Example 4.2.2:

Consider a BCI-algebra $X = \{0, a, b\}$ with the following Cayley table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

(1) Define a map $d_{(}\alpha,\beta_{)}: X \to X, \;\; x \longmapsto \begin{cases} b & if\ x \in \{0, a\}, \\ 0 & if\ x = b, \end{cases}$ and define two endomorphisms

$$\alpha: X \to X, \quad x \longmapsto \begin{cases} 0 & if\ x \in \{0, a\}, \\ b & if\ x = b, \end{cases}$$

$$\beta: X \to X, \quad x \longmapsto \begin{cases} 0 & if\ x \in \{0, b\}, \\ a & if\ x = a. \end{cases}$$

Then $d_{(}\alpha,\beta_{)}$ is a $(\alpha, \beta) - derivation$ of X.

(2) Define a map $d_{(}\alpha,\beta_{)}: X \to X, \;\; x \longmapsto \begin{cases} 0 & if\ x \in \{0, b\}, \\ a & if\ x = a, \end{cases}$ and define two endomorphisms

$$\alpha: X \to X, \quad x \longmapsto \begin{cases} 0 & if\ x \in \{a, b\}, \\ b & if\ x = 0, \end{cases}$$

$$\beta: X \to X, \quad x \longmapsto \begin{cases} 0 & if\ x \in \{0, a\}, \\ a & if\ x = b. \end{cases}$$

Then $d_{(}\alpha, \beta_{)}$ is a $(\alpha, \beta) - derivation$ of X.

Lemma 4.2.3:

Let X be a BCI-algebra. For any $x, y \in X, if\ x \leq y,$ then x and y are contained in the $same\ branch\ of\ X$.

Proof: Obvious.

Lemma 4.2.4:

Let X be a BCI-algebra. For any $x, y \in X$, if x and y are contained in the $same\ branch\ of\ X$, then $x * y, y * x \in X_+$.

Proof: Obvious.

Proposition 4.2.5:

Let X be a commutative BCI-algebra. Then every $(\alpha, \beta) - derivation$ $d_{(}\alpha, \beta_{)}$ of X satisfies the following assertion:

$$(\forall x, y \in X)\ \ (x \leq y \Longrightarrow\ d_{(}\alpha, \beta_{)}(x) \leq\ d_{(}\alpha, \beta_{)}(y))$$

that is, every $(\alpha, \beta) - derivation$ of X is $isotone$.

Proof:

Let $x, y \in X$ be such that $x \leq y$. Since X is commutative, we have $x = x \wedge y$. Hence $d_{(}\alpha, \beta_{)}(x) = d_{(}\alpha, \beta_{)}(x \wedge y)$

$$= (d_{(\alpha,\beta)}(y) * \alpha(y * x)) \wedge (d_{(\alpha,\beta)}(y * x) * \beta(y)) \qquad (1)$$

$$\leq (d_{(\alpha,\beta)}(y) * \alpha(y * x)).$$

Since every endomorphism of X is $isotone$, we have $\alpha(x) \leq \alpha(y)$. It follows from Lemma 4.2.3 that $0 = \alpha(x) * \alpha(y) \in X_+$ and $\alpha(y) * \alpha(x) \in X_+$, so that there exists $a(\neq 0) \in X_+$, such that $\alpha(y * x) = \alpha(y) * \alpha(x) = a$. Hence (1) implies that $d_{(\alpha,\beta)}(x) \leq d_{(\alpha,\beta)}(y) * a$. Using the properties and conditions of BCI-algebra, we have

$$d_{(\alpha,\beta)}(x) * d_{(\alpha,\beta)}(y) \leq (d_{(\alpha,\beta)}(y) * a) * d_{(\alpha,\beta)}(y)$$

$$= (d_{(\alpha,\beta)}(y) * d_{(\alpha,\beta)}(y)) * a$$

$$= 0 * a = 0,$$

and so $d_{(\alpha,\beta)}(x) * d_{(\alpha,\beta)}(y) = 0$, that is, $d_{(\alpha,\beta)}(x) \leq d_{(\alpha,\beta)}(y)$.

Note:

In example 4.2.2(1), the $(\alpha,\beta) - derivation\ d_{(\alpha,\beta)}$ does not satisfy the inequality $(\forall x, y \in X)\ (x \leq y \Longrightarrow d_{(\alpha,\beta)}(x) \leq d_{(\alpha,\beta)}(y))$.

Proposition 4.2.7:

Every $(\alpha,\beta) - derivation\ d_{(\alpha,\beta)}$ of a BCI-algebra X satisfies the following assertion: $(\forall x \in X)\ (d_{(\alpha,\beta)}(x) = d_{(\alpha,\beta)}(x) \wedge d_{(\alpha,\beta)}(0))$.

Proof:

Let $d_{(\alpha,\beta)}$ be an $(\alpha,\beta) - derivation$ of X. Using the properties of BCI-algebra, we have

$$d_{(\alpha,\beta)}(x) = d_{(\alpha,\beta)}(x * 0) = (d_{(\alpha,\beta)}(x) * \alpha(0)) \wedge (d_{(\alpha,\beta)}(0) * \beta(x))$$

$$= (d_{(\alpha,\beta)}(x) * 0) \wedge (d_{(\alpha,\beta)}(0) * \beta(x))$$

$$= d_{(}\alpha,\beta_{)}(x) \wedge (d_{(}\alpha,\beta_{)}(0) * \beta(x))$$

$$= (d_{(}\alpha,\beta_{)}(0) * \beta(x)) * ((d_{(}\alpha,\beta_{)}(0) * \beta(x)) * d(\alpha,\beta)(x))$$

$$= (d_{(}\alpha,\beta_{)}(0) * \beta(x)) * ((d_{(}\alpha,\beta_{)}(0) * d_{(}\alpha,\beta_{)}(x)) * \beta(x))$$

$$\leq d_{(}\alpha,\beta_{)}(0) * (d_{(}\alpha,\beta_{)}(0) * d_{(}\alpha,\beta_{)}(x))$$

$$= d_{(}\alpha,\beta_{)}(x) \wedge d_{(}\alpha,\beta_{)}(0) \qquad (1)$$

Obviously, $d_{(}\alpha,\beta_{)}(x) \wedge d_{(}\alpha,\beta_{)}(0) \leq d_{(}\alpha,\beta_{)}(x)$ (2)

By (1) and (2) $d_{(}\alpha,\beta_{)}(x) = d_{(}\alpha,\beta_{)}(x) \wedge d_{(}\alpha,\beta_{)}(0)$.

Theorem 4.2.8:

Let $d_{(}\alpha,\beta_{)}$ be a $(\alpha,\beta) - derivation$ on a BCI-algebra X. Then

(i) $d(a * x) = d(a) * \alpha(x), for\ all\ a \in L_{\mathrm{p}}(X), x \in X$

(ii) $d(a + x) = d(a) + \alpha(x), for\ all\ a \in L_{\mathrm{p}}(X), x \in X$

(iii) $d(a + b) = d(a) + \alpha(b), for\ all\ a, b \in L_{\mathrm{p}}(X)$

Proof:

(i) For any $a \in L_{\mathrm{p}}(X)$, we have $a * x \in L_{\mathrm{p}}(X)$ for all $x \in X$. Thus $d(a * x) = (d(a) * \alpha(x)) \wedge (d(x) * \beta(a)) = d(a) * \alpha(x)$.

(ii) For any $a \in L_{\mathrm{p}}(X)$ and $x \in X$, it follows from (i) that

$$d(a + x) = a * \big(a * (0 * x)\big) = d(a) * \alpha(0 * x)$$
$$= d(a) * \big(\alpha(0) * \alpha(x)\big) = d(a) * (0 * \alpha(x))$$
$$= d(a) + \alpha(x)$$

(iii) Proof follows directly from (ii).

Definition 4.2.9:

Let X be a BCI-algebra and $d_{(\alpha,\beta)}$, $d'_{(\alpha,\beta)}$ be two self maps of X, we define $d_{(\alpha,\beta)}\circ d'_{(\alpha,\beta)}: X \to X$ by $(d_{(\alpha,\beta)}\circ d'_{(\alpha,\beta)})(x) = d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x))$ for all $x \in X$.

Theorem 4.2.10:

Let X be a p-semisimple BCI-algebra. If $d_{(\alpha,\beta)}$ and $d'_{(\alpha,\beta)}$ are two $(\alpha,\beta)-derivations$ on X such that $\alpha^2 = \alpha$. Then $d_{(\alpha,\beta)}\circ d'_{(\alpha,\beta)}$ is a $(\alpha,\beta)-derivation$ on X.

Proof:

For any $x, y \in X$,

$$(d_{(\alpha,\beta)}\circ d'_{(\alpha,\beta)})(x * y) = d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x * y))$$

$$= d_{(\alpha,\beta)}((d'_{(\alpha,\beta)}(x) * \alpha(y)) \wedge (d'_{(\alpha,\beta)}(y) * \beta(x)))$$

$$= d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x) * \alpha(y))$$

$$= (d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x)) * \alpha(\alpha(y))) \wedge (d_{(\alpha,\beta)}\big(\alpha(y)\big) * \beta(d'_{(\alpha,\beta)}(x)))$$

$$= d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x)) * \alpha(y)$$

$$= (d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(y) * \beta(x)) * ((d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(y) * \beta(x))$$

$$* (d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x)) * \alpha(y))$$

$$= (d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x)) * \alpha(y)) \wedge (d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(y) * \beta(x)))$$

$$= ((d_{(\alpha,\beta)}\circ d'_{(\alpha,\beta)})(x) * \alpha(y)) \wedge ((d_{(\alpha,\beta)}\circ d'_{(\alpha,\beta)})(y) * \beta(x))$$

This completes the proof.

Theorem 4.2.11:

Let α, β be two endomorphisms and $d_{(}\alpha, \beta_{)}$ be a self map on a p-semisimple BCI-algebra X such that $d_{(}\alpha, \beta_{)}(x) = \alpha(x)$ for all $x \in X$. Then $d_{(}\alpha, \beta_{)}$ is a $(\alpha, \beta) - derivation$ on X.

Proof:

Let us take $d_{(}\alpha, \beta_{)}(x) = \alpha(x)$ for all $x \in X$. Since $x, y \in X \Rightarrow x * y \in X$. Using the properties of BCI-algebra, we have

$$d_{(}\alpha, \beta_{)}(x * y) = \alpha(x * y) = \alpha(x) * \alpha(y) = d_{(}\alpha, \beta_{)}(x) * \alpha(y)$$

$$= (d_{(}\alpha, \beta_{)}(y) * \beta(x)) * ((d_{(}\alpha, \beta_{)}(y) * \beta(x)) * (d_{(}\alpha, \beta_{)}(x) * \alpha(y)))$$

$$= (d_{(}\alpha, \beta_{)}(x) * \alpha(y)) \wedge (d_{(}\alpha, \beta_{)}(y) * \beta(x))$$

This completes the proof.

Definition 4.2.12:

A $(\alpha, \beta) - derivation$ $d_{(}\alpha, \beta_{)}$ of a BCI-algebra X is said to be regular if $d_{(}\alpha, \beta_{)}(0) = 0$.

Example 4.2.13:

(1) The $(\alpha, \beta) - derivation$ $d_{(}\alpha, \beta_{)}$ of X in example 4.2.2(1) is not regular.

(2) The $(\alpha, \beta) - derivation$ $d_{(}\alpha, \beta_{)}$ of X in example 4.2.2(2) is regular.

The next theorem provides a condition for a $(\alpha, \beta) - derivation$ $d_{(}\alpha, \beta_{)}$ to be regular.

Theorem 4.2.14:

Let $d_{(}\alpha,\beta_{)}$ be a $(\alpha,\beta)-derivation$ of a BCI-algebra X. If there exists $a \in X$ such that $d_{(}\alpha,\beta_{)}(x) * \alpha(a) = 0$ for all $x \in X$, then $d_{(}\alpha,\beta_{)}$ is regular.

Proof:

Assume that there exists $a \in X$ such that $d_{(}\alpha,\beta_{)}(x) * \alpha(a) = 0$ for all $x \in X$. Then

$$0 = d_{(}\alpha,\beta_{)}(x * a) * a = ((\ d_{(}\alpha,\beta_{)}(x) * \alpha(a)) \wedge (\ d_{(}\alpha,\beta_{)}(a) * \beta(x))) * a$$

$$= (0 \wedge (\ d_{(}\alpha,\beta_{)}(a) * \beta(x))) * a = 0 * a$$

and so $d_{(}\alpha,\beta_{)}(0) = d_{(}\alpha,\beta_{)}(0 * a) = (\ d_{(}\alpha,\beta_{)}(0) * \alpha(a))\ \wedge (\ d_{(}\alpha,\beta_{)}(a) * \beta(0)) = 0$. Hence $d_{(}\alpha,\beta_{)}$ is regular.

Definition 4.2.15:

For a $(\alpha,\beta)-derivation\ d_{(}\alpha,\beta_{)}$ of a BCI-algebra X, we say that an ideal A of X is a $\alpha - ideal(resp. \beta - ideal)$ if $\alpha(A) \subseteq A(resp. \beta(A) \subseteq A)$.

Definition 4.2.16:

For a $(\alpha,\beta)-derivation\ d_{(}\alpha,\beta_{)}$ of a BCI-algebra X, we say that an ideal A of X is $d_{(}\alpha,\beta_{)}{-}invariant$ if $d_{(}\alpha,\beta_{)}(a) \subseteq A$.

Example 4.2.17:

(i) Let $d_{(}\alpha,\beta_{)}$ be a $(\alpha,\beta)-derivation$ of X which is described in Example 4.2.2(1). We know that $A \coloneqq \{0,a\}$ is both a $\alpha - ideal$ and a $\beta - ideal$ of X. But $A \coloneqq \{0,a\}$ is an ideal of X which is not $d_{(}\alpha,\beta_{)}{-}invariant$.

(ii) Let $d_{(}\alpha,\beta_{)}$ be a $(\alpha,\beta)-derivation$ of X which is described in Example 4.2.2(2). We know that $A \coloneqq \{0,a\}$ is both a $\beta-ideal$ and a $d_{(}\alpha,\beta_{)}-invariant\ ideal$ of X. But $A \coloneqq \{0,a\}$ is not a $\alpha-ideal$ of X.

Next we prove some results on regular $(\alpha,\beta)-derivation$ in a BCI-algebra.

Theorem 4.2.18:

Let $d_{(}\alpha,\beta_{)}$ be a regular $(\alpha,\beta)-derivation$ of a BCI-algebra X. Then

(i) $a \in L_{\mathrm{p}}(X) \Rightarrow\ d_{(}\alpha,\beta_{)}(a) \in L_{\mathrm{p}}(X),\ for\ all\ a \in X$

(ii) $a \in L_{\mathrm{p}}(X) \Rightarrow\ \alpha(a),\beta(a) \in L_{\mathrm{p}}(X),\ for\ all\ a \in X$

(iii) $d_{(}\alpha,\beta_{)}(a) = d_{(}\alpha,\beta_{)}(0) + \alpha(a),\ for\ all\ a \in L_{\mathrm{p}}(X)$

(iv) $d_{(}\alpha,\beta_{)}(a+b) = d_{(}\alpha,\beta_{)}(a) + d_{(}\alpha,\beta_{)}(b) - d_{(}\alpha,\beta_{)}(0),\ for\ all\ a,b \in L_{\mathrm{p}}(X)$

Proof:

(i) Let $d_{(}\alpha,\beta_{)}$ be a regular $(\alpha,\beta)-derivation$, that is, $d_{(}\alpha,\beta_{)}(0) = 0$. Then the proof follows from proposition 4.2.7

(ii) Let $a \in L_{\mathrm{p}}(X)$. Then $a = 0 * (0 * a)$, and so $\alpha(a) = \alpha\big(0 * (0 * a)\big) = 0 * \big(0 * \alpha(a)\big)$. Thus $\alpha(a) \in L_{\mathrm{p}}(X)$. Similarly, $\beta(a) \in L_{\mathrm{p}}(X)$.

(iii) Let $a \in L_{\mathrm{p}}(X)$. Using (ii) and the properties of BCI-algebra, we have

$$\begin{aligned} d_{(}\alpha,\beta_{)}(a) &= d_{(}\alpha,\beta_{)}(0 * (0 * a)) \\ &= (d_{(}\alpha,\beta_{)}(0) * \alpha(0 * a)) \wedge (d_{(}\alpha,\beta_{)}(0 * a) * \beta(0)) \\ &= (d_{(}\alpha,\beta_{)}(0) * \alpha(0 * a)) \wedge (d_{(}\alpha,\beta_{)}(0 * a) * 0) \\ &= (d_{(}\alpha,\beta_{)}(0) * \alpha(0 * a)) \wedge d_{(}\alpha,\beta_{)}(0 * a) \\ &= d_{(}\alpha,\beta_{)}(0 * a) * (d_{(}\alpha,\beta_{)}(0 * a) * (d_{(}\alpha,\beta_{)}(0) * \alpha(0 * a))) \\ &= d_{(}\alpha,\beta_{)}(0) * \alpha(0 * a) \\ &= d_{(}\alpha,\beta_{)}(0) * (0 * \alpha(a)) = d_{(}\alpha,\beta_{)}(0) + \alpha(a) \end{aligned}$$

(iv) Let $a, b \in L_p(X)$. Then $a+b \in L_p(X)$. Using (iii), we have

$$
\begin{aligned}
d_{(\alpha,\beta)}(a+b) &= d_{(\alpha,\beta)}(0) + \alpha(a+b) \\
&= d_{(\alpha,\beta)}(0) + \alpha(a) + d_{(\alpha,\beta)}(0) + \alpha(b) - d_{(\alpha,\beta)}(0) \\
&= d_{(\alpha,\beta)}(a) + d_{(\alpha,\beta)}(b) - d_{(\alpha,\beta)}(0)
\end{aligned}
$$

This completes the proof.

Theorem 4.2.19:

Let X be a torsion free BCI-algebra and $d_{(\alpha,\beta)}$ be a regular $(\alpha,\beta)-derivation$ on X such that $\alpha \circ d_{(\alpha,\beta)} = d_{(\alpha,\beta)}$. If $d^2_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $d_{(\alpha,\beta)} = 0$ on $L_p(X)$.

Proof:

Let us suppose $d^2_{(\alpha,\beta)} = 0$ on $L_p(X)$. If $x \in L_p(X)$, then $x+x \in L_p(X)$ and so by using Theorem 4.2.18(iii) and (iv), we have

$$
\begin{aligned}
0 = d^2_{(\alpha,\beta)}(x+x) &= d_{(\alpha,\beta)}(d_{(\alpha,\beta)}(x+x)) \\
&= d_{(\alpha,\beta)}(0) + \alpha(d_{(\alpha,\beta)}(x+x)) \\
&= d_{(\alpha,\beta)}(0) + d_{(\alpha,\beta)}(x+x) \\
&= d_{(\alpha,\beta)}(0) + d_{(\alpha,\beta)}(x) + d_{(\alpha,\beta)}(x) - d_{(\alpha,\beta)}(0) \\
&= d_{(\alpha,\beta)}(x) + d_{(\alpha,\beta)}(x)
\end{aligned}
$$

Since X is torsion free. Therefore, $d_{(\alpha,\beta)}(x) = 0$ for all $x \in X$ implying thereby $d_{(\alpha,\beta)} = 0$. This completes the proof.

Theorem 4.2.20:

Let X be a torsion free BCI-algebra and $d_{(\alpha,\beta)}, d'_{(\alpha,\beta)}$ be two regular $(\alpha,\beta)-derivation$ on X such that $\alpha \circ d'_{(\alpha,\beta)} = d'_{(\alpha,\beta)}$. If $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $d'_{(\alpha,\beta)} = 0$ on $L_p(X)$.

Proof:

Let us suppose $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $L_p(X)$. If $x \in L_p(X)$, then $x + x \in L_p(X)$ and so by using Theorem 4.2.18(i) and (ii), we have

$$\begin{aligned}
0 &= (d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)})(x + x) = d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x + x)) \\
&= d_{(\alpha,\beta)}(0) + \alpha(d'_{(\alpha,\beta)}(x + x)) \\
&= d_{(\alpha,\beta)}(0) + d'_{(\alpha,\beta)}(x + x) \\
&= d_{(\alpha,\beta)}(0) + (d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) - d'_{(\alpha,\beta)}(0)) \\
&= (d_{(\alpha,\beta)}(0) - d'_{(\alpha,\beta)}(0)) + (d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)) \\
&= ((d_{(\alpha,\beta)}(0) * d'_{(\alpha,\beta)}(0))) + (d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)) \\
&= (d_{(\alpha,\beta)}(0) * (0 * d'_{(\alpha,\beta)}(0))) + (d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)) \\
&= (d_{(\alpha,\beta)}(0) + d'_{(\alpha,\beta)}(0)) + (d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)) \\
&= (d_{(\alpha,\beta)}(0) + \alpha d'_{(\alpha,\beta)}(0)) + (d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)) \\
&= d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(0)) + (d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)) \\
&= (d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)})(0) + (d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)) \\
&= d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x)
\end{aligned}$$

Since X is torsion free. Therefore $d'_{(\alpha,\beta)}(x) = 0$ for all $x \in X$ and so $d'_{(\alpha,\beta)} = 0$. This completes the proof.

Proposition 4.2.21:

Let $d_{(\alpha,\beta)}$ be a regular $(\alpha,\beta) - derivation$ of a BCI-algebra X. If $d^2_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $(\alpha \circ d_{(\alpha,\beta)})(x) = \left(\frac{1}{2}\right)((\alpha \circ d_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$.

Proof:

Assume that $d^2_{(\alpha,\beta)} = 0$ on $L_p(X)$. If $x \in L_p(X)$, then $x + x \in L_p(X)$ and so by using Theorem 4.2.18(iii) and (iv), we have

$$0 = d^2_{(\alpha,\beta)}(x + x) = d_{(\alpha,\beta)}(d_{(\alpha,\beta)}(x + x))$$

$$= d_{(\alpha,\beta)}(0) + \alpha(d_{(\alpha,\beta)}(x + x))$$

$$= d_{(\alpha,\beta)}(0) + \alpha(d_{(\alpha,\beta)}(x) + d_{(\alpha,\beta)}(x) - d_{(\alpha,\beta)}(0))$$

$$= d_{(\alpha,\beta)}(0) + 2\alpha(d_{(\alpha,\beta)}(x)) - \alpha(d_{(\alpha,\beta)}(0))$$

Hence $(\alpha \circ d_{(\alpha,\beta)})(x) = \left(\frac{1}{2}\right)((\alpha \circ d_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$. This completes the proof.

Proposition 4.2.22:

Let $d_{(\alpha,\beta)}$ and $d'_{(\alpha,\beta)}$ be two regular $(\alpha,\beta) - derivations$ of a BCI-algebra X. If $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $(\alpha \circ d'_{(\alpha,\beta)})(x) = \left(\frac{1}{2}\right)((\alpha \circ d'_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$.

Proof:

Let $x \in L_p(X)$. Then $x + x \in L_p(X)$, and so $d'_{(\alpha,\beta)}(x + x) \in L_p(X)$ by Theorem 4.2.18(i). It follows from the theorem 4.2.18(iii) and (iv) that

$$0 = (d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)})(x + x) = d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(x + x))$$

$$= d_{(\alpha,\beta)}(0) + \alpha(d'_{(\alpha,\beta)}(x + x))$$

$$= d_{(\alpha,\beta)}(0) + \alpha(d'_{(\alpha,\beta)}(x) + d'_{(\alpha,\beta)}(x) - d'_{(\alpha,\beta)}(0))$$

$$= d_{(\alpha,\beta)}(0) + 2\alpha(d'_{(\alpha,\beta)}(x)) - \alpha(d'_{(\alpha,\beta)}(0))$$

This implies $(\alpha \circ d'_{(\alpha,\beta)})(x) = \left(\frac{1}{2}\right)((\alpha \circ d'_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0))$ for all $x \in L_p(X)$. Hence the proof.

SUMMARY AND CONCLUSION

The notion of BCK-algebra was proposed by Imai and Iseki in 1966 [14]. In the same year, Iseki introduced the notion of BCI-algebra [15], which is a generalization of a BCK-algebra. The notion of derivation in ring theory plays a significant role in analysis, algebraic geometry and algebra. In the year 2004 [18], Jun and Xin have applied the notion of derivation in BCI-algebra which is defined in a way similar to the notion of derivation in rings and near-rings theory which was introduced by Posner in 1957 [30]. In this thesis some interesting results on derivations of BCI-algebras are discussed in different aspects.

In the first chapter, preliminary definitions and results on BCI/BCK-algebra are collected. The notion of left-right (resp. right-left) derivation of BCI-algebras and some related properties are studied. Also, characterizations of a p-semisimple BCI-algebra are studied by using the idea of regular derivation.

In chapter 2, the notion of left-right (resp. right-left) $f-$derivation of a BCI-algebra and some related properties are studied. The notion of left derivation of a BCI-algebra and some related properties are studied. A condition for left derivation to be regular is also discussed.

In chapter 3, the concepts of generalized derivations, symmetric Bi-derivations of BCI-algebras and their properties are discussed.

In chapter 4, the concepts of $t-$derivations and $(\alpha,\beta)-$derivations of BCI-algebra are studied. The notion of $t-$derivations of a BCI-algebra and related properties are discussed. Moreover, some results on $t-$derivations in a p-semisimple BCI-algebra are discussed. The notion of (regular) $(\alpha,\beta)-$derivations of a BCI-algebra X and some related properties are discussed. The concepts of a $d_{(}\alpha,\beta_{)}-invariant\ (\alpha,\beta)-derivation$ and $\alpha-ideal$ are studied and their relations are discussed. Finally, some results on regular $(\alpha,\beta)-derivations$ are obtained.

We hope that these results can be extended to various algebras and fuzzy BCI-algebras for further research.

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