Chapter V

CHAPTER-5

s*g-LOCALLY CLOSED SETS IN TOPOLOGICAL AND BITOPOLOGICAL SPACES

In this chapter, s*g locally closed sets and s*g sub maximal spaces and its properties in topological and in bitopological spaces are discussed. The notions of s*glc continuity, the composition of two s*glc contiouous functions and the restriction maps of s*glc-continuity in topological spaces are analysied.

SECTION: 5.1

PRELIMINARIES

Definition: 5.1.1

A set A of a topological space (X, τ) is called **semi star generalized closed** (s*g-closed) if cl (A) $\subseteq$ U whenever A$\subseteq$U and U is semi open in X.

Definition: 5.1.2

A set A of a topological space (X, τ) is called **semi star generalized open (s*g-open)** if X – A is s*g-closed in X.

Definition: 5.1.3

A subset A of a topological space (X, τ) is said to be a **locally semi closed set** if A = G $\cap$F where G is open and F is semi closed in X.

Definition: 5.1.4

A subset A of a topological space (X, τ) is said to be a **semi locally closed set** if A = G $\cap$F where G is semi open and F is semi closed in X.

Definition: 5.1.5

A subset A of a topological space (X, τ) is said to be a **g-locally closed set** if $A = G \cap F$ where G is g-open and F is g-closed in X

***Definition*: 5.1.6**

A subset A of a topological space (X, τ) is said to be a **sg-locally closed set** if $A = G \cap F$ where G is sg-open and F is sg-closed in X

Definition: 5.1.7

A subset A of a topological space (X, τ) is said to be a **sg-locally closed*** set if $A = G \cap F$ where G is sg-open and F is closed in X

Definition: 5.1.8

A subset A of a topological space (X, τ) is said to be a **sg-locally closed**** set if $A = G \cap F$ where G is open and F is sg- closed in X

Definition: 5.1.9

A subset A of a topological space (X, τ) is said to be a **gs - locally closed** set if $A = G \cap F$ where G is gs-open and F is gs - closed in X

Definition: 5.1.10

A map f: $(X, \tau) \rightarrow (Y, \sigma)$ is **LC-continuous** if $f^{-1}(U)$ is locally closed for each open set U in Y .

Definition: 5.1.11

A subset A of a bitopological space (X, τ_1, τ_2) is called **$\tau_1\tau_2$– Semi open** if there exists a τ_1-open set U such that $U \subseteq A \subseteq \tau_2\text{– cl}(U)$.

Definition: 5.1.12

A subset A of a bitopological space (X, τ_1, τ_2) is called **$\tau_1\tau_2$ – Semi closed** if $X - A$ is $\tau_1\tau_2$ – Semi open

Equivalently, a set A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$– Semi closed if there exists a τ_1 – closed set F such that τ_2– $\text{int}(F) \subseteq A \subseteq F$.

Definition: 5.1.13

A subset A of a bitopological space (X, τ_1, τ_2) is called **$\tau_1\tau_2$ – Generalized closed** ($\tau_1\tau_2$– g closed) if τ_2– $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1– open in X .

Definition: 5.1.14

A subset A of a bitopological space (X, τ_1, τ_2) is called **$\tau_1\tau_2$ – Generalized open** ($\tau_1\tau_2$ – g open) if $X - A$ is $\tau_1\tau_2$ – g closed.

Definition: 5.1.15

A subset A of a bitopological space (X, τ_1, τ_2) is called **$\tau_1\tau_2$ - semi star generalized closed** ($\tau_1\tau_2$ – s*g closed) if τ_2 – $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1 - semi open in X.

Definition: 5.1.16

A subset A of a bitopological space (X, τ_1, τ_2) is called **$\tau_1\tau_2$ - semi star generalized open ($\tau_1\tau_2$ –s*g open)** if $X - A$ is $\tau_1\tau_2$ –s*g closed in X.

SECTION: 5.2

SEMI STAR GENERALIZED LOCALLY CLOSED SETS AND CONTINUITY IN TOPOLOGICAL SPACES

Definition: 5.2.1

A subset A of a topological space (X, τ) is said to be **s*g-locally closed** set if $A = G \cap F$ where G is s*g -open set and F is s*g-closed set in X.

Definition: 5.2.2

A subset A of a topological space (X, τ) is said to be **s*g-locally closed*** if $A = G \cap F$ where G is s*g-open set and F is closed in X.

Definition: 5.2.3

A subset A of a topological space (X, τ) is said to be **s*g-locally closed**** if $A = G \cap F$ where G is open and F is s*g-closed in X.

Notation: 5.2.4

The class of all s*g-locally closed sets, s*g-locally closed* sets, s*g-locally closed** sets in (X, τ) are denoted by S*GLC (X, τ), S*GLC* (X, τ) and S*GLC** (X, τ) respectively.

Remark: 5.2.5

Every s*g - locally closed set is the intersection of a s*g-open set and s*g-closed set.

Remark: 5.2.6

Every open set is s*g - open and every closed set is s*g – closed.

Theorem: 5.2.7

(a) Every open set is s*g-locally closed and every closed set is s*g -locally closed.

(b) Every locally closed set is s*g-locally closed, s*g-locally closed* and s*g-locally closed**.

Remark: 5.2.8

But the converses of the assertions of above theorem 5.2.7 are not true.

Example: 5.2.9

a)Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {a, b}}. Then {c, d} is a s*g -locally closed set, but not open in (X, τ) and {b} is a s*g - locally closed set, but not closed in (X, τ).

(b) Let X = {a, b, c, d}, τ = {ϕ, X, {a, b}, {a, b, c},{a, b, d}}.Then {a} is a s*g-locally closed set, but not locally closed, s*g-locally closed* and s*g-locally closed** in (X, τ).

Remark: 5.2.10

Since every s*g-closed set is g-closed, sg-closed and gs-closed, we conclude the following.

Theorem: 5.2.11

(a) Every s*g-locally closed is g-locally closed.

(b) Every s*g-locally closed is sg-locally closed.

(c) Every s*g-locally closed is gs-locally closed.

Remark: 5.2.12

But the converses of the assertions of above theorem are not true

Example: 5.2.13

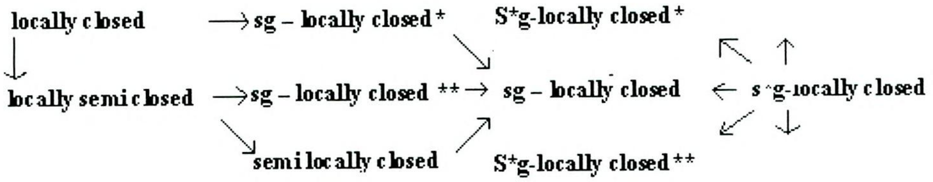
In Example: 5.2.9(a),

(a) {a, c} is a g-locally closed set, but not s*g-locally closed in (X, τ).

(b) {a, c} is a sg-locally closed set, but not s*g-locally closed in(X,τ).

(c) {c} is a gs-locally closed set, but not s*g-locally closed in (X, τ).

From the above results we conclude the following



Theorem: 5.2.14

In any topological space (X, τ), intersection of two s*g-locally closed**sets is s*g-locally closed**.

Theorem: 5.2.15

If $A \in S^*GLC(X, \tau)$ and B is closed in X, then $A \cap B \in S^*GLC(X, \tau)$.

Proof:

It is obvious since every closed set is s*g-closed and the intersection of two s*g-closed sets is s*g-closed.

Theorem: 5.2.16

If $A \in S^*GLC(X, \tau)$ and B is s*g - closed in X, then $A \cap B \in S^*GLC(X, \tau)$.

Definition: 5.2.17

The spaces in which every singleton is locally closed are called **T_D-spaces.**

Remark: 5.2.18

Since every locally closed set is s*g-locally closed, every singleton is s*g-locally closed in T_D-spaces. Also a dense subset is open if and only if it is locally closed. Consequently a dense subset is open if and only if it is s*g-locally closed.

Remark: 5.2.19

The complement of s*g-locally closed set in (X, τ) is not s*g-locally closed in general and hence the finite union of s*g-locally closed sets need not be s*g-locally closed in (X, τ). The next examples show the claim.

Example: 5.2.20

In Example: 5.2.9(a), $\{b\}$ is a s*g-locally closed set, but its complement $\{a, c, d\}$ is not s*g-locally closed in (X, τ).

Example: 5.2.21

In Example: 5.2.9(a), $A = \{a\}$, $B = \{c, d\}$ are s*g-locally closed sets, but $A \cup B = \{a, c, d\}$ is not s*g-locally closed in (X, τ).

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Theorem: 5.2.22

If A, B are any two separated s*g-locally closed* subsets of (X, τ), then $A \cup B \in S^*GLC^*(X, \tau)$.

Proof:

Let A, B be two s*g-locally closed* subsets of (X, τ). Then there exist s*g- open sets G, F such that $A = G \cap cl(A)$ and $B = F \cap cl(B)$. Let $U = G \cap [X - cl(B)]$ and $V = F \cap [X - cl(A)]$. Then U, V are s*g-open sets and Hence $U \cap V$ is s*g-open in X. Clearly $A = U \cap cl(A)$, $B = V \cap cl(B)$, $U \cap cl(B) = \phi$ and $V \cap cl(A) = \phi$. Consequently,

$A \cup B = (U \cup V) \cap cl(A \cap B)$.

Therefore, $A \cup B \in S^*GLC^*(X, \tau)$.

Remark: 5.2.23

It is essential that A and B are separated in the theorem 5.2.22.

Example: 5.2.24

In Example: 5.2.9(a), $A = \{a\}$, $B = \{c, d\}$ are s*g - locally closed sets, but $A \cup B = \{a, c, d\}$ is not s*g-locally closed in (X, τ)

Theorem: 5.2.25

In a topological space (X, τ), the following are equivalent.

(a) A is s*g-locally closed if and only if A^C is s*g-locally closed.

(b) s*g-locally closed sets are closed under finite union.

Proof:

(a)$\Rightarrow$(b)

Let A be s*g-locally closed if and only if A^C is s*g-locally closed. Let A, B be s*g-locally closed. Then by our assumption A^C, B^C are s*g-locally closed. Therefore, $(A \cup B)^C = A^C \cap B^C$ is s*g -locally closed. Therefore, $A \cup B$ is s*g-locally closed.

(b) $\Rightarrow$(a)

Let s*g-locally closed sets be closed under finite union. Let A be s*g-locally closed. Then $A = G \cap F$ where G is s*g-open and F is s*g- closed in X. Since G^C is s*g-closed and F^C is s*g-open in X and every s*g- open and s*g-closed set are s*g-locally closed, Hence A^C is s*g-locally closed by our assumption.

SECTION: 5.3

s*glc-CONTINUITY

Definition: 5.3.1

A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is s*glc -continuous if $f^{-1}(U)$ is s*g - locally closed for each open set U in Y .

Definition: 5.3.2

A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is s*glc*-continuous if $f^{-1}(U)$ is s*g - locally closed* for each open set U in Y.

Definition: 5.3.3

A function f: $(X, \tau) \rightarrow (Y, \sigma)$ is s*glc** -continuous if $f^{-1}(U)$ is s*g - locally closed** for each open set U in Y .

Example: 5.3.4

Let X = Y = {a, b, c, d}, $\tau = \{\phi, X, \{a, b\}, \{a, b, d\}\}$.

$\sigma = \{\phi, Y, \{a\}, \{c, b\}\}$. Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be a function defined by f(a) = b , f(b) = c , f(c) = a , f(d) = d . Then f is both s*glc-continuous and s*glc**continuous.

Example: 5.3.5

Let X = Y = {a, b, c, d}, τ = {ϕ, X, {a, b}, {a, b, c}, {a, b, d}}. σ = {ϕ, Y, {a}, {a, b}}. Let f : (X, τ) → (Y, σ) be a function defined by f(a) = b , f(b) = a , f(c) = b , f(d) = a . Then f is s*glc*-continuous.

Definition: 5.3.6

A function f: (X, τ) → (Y, σ) is **s*glc - irresolute** if f^{-1}(U) is s*g-locally closed for each s*g -locally closed U in Y.

Definition: 5.3.7

A function f: (X, τ) → (Y, σ) is s***glc*-irresolute** if f^{-1}(U) is s*g- locally closed* for each s*g - locally closed U in Y

Definition: 5.3.8

A function f: (X, τ) → (Y, σ) is **s*glc**-irresolute** if f^{-1}(U) is s*g-locally closed** for each s*g - locally closed U in Y

Example: 5.3.9

In Example: 5.3.4, the function f is both s*glc-irresolute and s*glc**-irresolute and in Example: 5.3.5, the function f is s*glc*-irresolute. Obviously the notions of s*glc-irresolute (resp. s*lgc*-irresolute, s*glc**- irresolute) functions are articular cases of m-continuous functions.

Theorem: 5.3.10

(a) Every lc-continuous function is s*glc-continuous.

(b) Every s*glc*-continuous function is s*glc-continuous

(c) Every s*glc**-continuous function is s*glc-continuous.

Proof:

(a)Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be locally continuous. Then f^{-1} (U) is locally closed for each open set U in Y. Then by Theorem 5.2.7(b), f^{-1} (U) is s*g locally closed in X. Consequently f: $(X, \tau) \rightarrow (Y, \sigma)$ is s*glc - continuous.

The proofs of (b) and (c) are similar.

Remark: 5.3.11

a) Every s*glc-continuous function need not be lc-continuous.

b) Every s*glc-continuous function need not be s*glc*-continuous.

c) Every s*glc-continuous function need not be s*glc**-continuous

Example: 5.3.12

In Example: 5.3.4,

(a) f is s*glc-continuous, but not lc-continuous.

(b) f is s*glc-continuous, but not s*glc*-continuous.

Since every s*g closed set is g-closed, sg-closed and gs-closed , every s*glc-continuous function is glc-continuous, sglc-continuous and gslc -continuous.

Example: 5.3.13

Let X = Y = {a, b, c, d}, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\phi, Y, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by f(a) = b, f(b) = c , f(c) = a , f(d) = d . Then f is glc-continuous, sglc-continuous and gs-continuous but not s*glc-continuous.

Proposition: 5.3.14

Concerning composition of functions, the composition of two s*glc-continuous functions is s*glc-continuous.

Proposition: 5.3.15

The composition of two s*glc- irresolute functions is s*glc-irresolute.

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Theorem: 5.3.16

(a) Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be s*glc**-continuous and Z is a subset of X. Then the restriction map f|Z: $(Z, \tau_Z) \rightarrow (Y, \sigma)$ is s*glc** - continuous.

(b) Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be s*glc* - continuous and Z is a subset of X. Then the restriction map f|Z: $(Z, \tau_Z) \rightarrow (Y, \sigma)$ is s*glc*-continuous.

Proof:

Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be s*glc**-continuous. Then $f^{-1}(U)$ is s*g -locally closed** for each open set U in Y. Consequently $f^{-1}(U)$ is the intersection of an open set G and s*g - closed set F in (X, τ). Now, $(f|Z)^{-1}(U) = (G \cap Z) \cap (F \cap Z)$. Since G is open in X, $G \cap Z$ is open in Z and since F is s*g -closed in X, $F \cap Z$ is s*g- closed in Z. Therefore, f|Z: $(Z, \tau_Z) \rightarrow (Y, \sigma)$ is s*glc** -continuous. The proof of (b) is similar.

Definition: 5.4.2

A subset A of a bitopological space (X, τ_1, τ_2) is said to be **$\tau_1\tau_2$ − s*g locally closed*** if A = G$\cap$ F where G is τ_1 − s*g open set and F is τ_2 −closed in X.

Definition: 5.4.3

A subset A of a bitopological space (X, τ_1, τ_2) is said to be **$\tau_1\tau_2$ − s*g locally closed**** if A = G $\cap$ F where G is τ_1 − open and F is τ_2 − s*g closed in X.

Remark: 5.4.4

(a) The class of all $\tau_1\tau_2$ − s*g locally closed sets in (X, τ_1, τ_2) is denoted by $\tau_1\tau_2$− S*GLC(X, τ_1, τ_2).

(b) The class of all $\tau_1\tau_2$ − s*g locally closed$\ast$ sets in (X, τ_1, τ_2) is denoted by $\tau_1\tau_2$ − S*GLC*(X, τ_1, τ_2).

(c) The class of all $\tau_1\tau_2$ − s*g locally closed** sets in (X, τ_1, τ_2) is denoted by $\tau_1\tau_2$ − S*GLC**(X, τ_1, τ_2).

Theorem: 5.3.17

A topological space (X, τ) is s*g -submaximal if and only if every function having (X, τ) as its domain is s*glc* - continuous.

Proof:

Necessity:

Let f: $(X, \tau) \to (Y, \sigma)$ be a function and $V \in \sigma$. Then $f^{-1}(V) \in P(X)$ = S*GLC*(X, τ) since (X, τ) is s*g-submaximal. Therefore every function having (X, τ) as its domain is s*glc*-continuous.

Suffciency:

Let Y = {0, 1} be the Sierpinski space with topology $\sigma = \{\phi, Y, \{0\}\}$. Let V be a subset of (X, τ). Define f: $(X, \tau) \to (Y, \sigma)$ such that f(x) = 0 for every $x \in V$ and f(x) = 1 for every $x \in X - V$. Then f (V) = {0} and hence $V = f^{-1}(\{0\})$. Since {0} is open in (Y, σ), f is s*glc* -continuous,V is s*g -locally closed*. Therefore, P(X) $\subseteq$ S*GLC*(X, τ). Obviously , S*GLC*(X, τ) $\subseteq$ P(X). Hence P(X) = S*GLC*(X, τ). Therefore, (X, τ) is s*g -submaximal.

SECTION: 5.4

SEMI STAR GENERALIZED LOCALLY CLOSED SETS IN BITOPOLOGICAL SPACES

Definition: 5.4.1

A subset A of a bitopological space (X, τ_1, τ_2) is said to be **$\tau_1\tau_2$ – s*g locally closed** set if $A = G \cap F$ where G is τ_1 – s*g open set and F is τ_2 – s*g closed set in X.

Example: 5.4.5

Let X = {a, b, c}, τ_1 = {ϕ,X, {b, c}}, τ_2 = {ϕ,X, {a}}. Then τ_1–s*g open sets in (X, τ_1, τ_2) are ϕ,X, {b}, {c}, {b, c} and τ_2–s*g closed sets in (X, τ_1, τ_2) are X, ϕ, {b, c}. Then

(a) $\tau_1\tau_2$ – s*g locally closed sets in (X, τ_1, τ_2) are ϕ,X, {b}, {c}, {b, c}.

(b) $\tau_1\tau_2$ – s*g locally closed* sets in (X, τ_1, τ_2) are ϕ,X, {b}, {c}, {b, c}.

(c) $\tau_1\tau_2$ – s∗g locally closed** sets in (X, τ_1, τ_2) are ϕ,X, {b, c}.

Remark: 5.4.6

Every $\tau_1\tau_2$ – s*g locally closed set in (X, τ_1, τ_2) need not be τ_2 - closed in general as can be seen from the following example.

Example: 5.4.7

In Example 5.4.5 (b) is $\tau_1\tau_2$ – s*g locally closed set in (X, τ_1, τ_2) but {b} is not τ_2 - closed in (X, τ_1, τ_2).

Remark: 5.4.8

Every $\tau_1\tau_2$ – s*g locally closed set in (X, τ_1, τ_2) need not be τ_1 – open in general as can be seen from the following example.

Example: 5.4.9

In Example 5.4.5(c) is $\tau_1\tau_2$ – s*g locally closed set in (X, τ_1, τ_2), but {c} is not τ_1 - open in (X, τ_1, τ_2).

Theorem: 5.4.10

In any bitopological space(X, τ_1, τ_2),

(i) A $\in \tau_1\tau_2$ – S*GLC*(X, τ_1, τ_2) $\Rightarrow$ A $\in \tau_1\tau_2$ – S*GLC(X, τ_1, τ_2).

(ii) A $\in \tau_1\tau_2$ – S*GLC**(X, τ_1, τ_2) $\Rightarrow$ A $\in \tau_1\tau_2$ – S*GLC(X, τ_1, τ_2).

(iii) A $\in \tau_2$ – S*GC(X, τ_1, τ_2) $\Rightarrow$ A $\in \tau_1\tau_2$ – S*GLC(X, τ_1, τ_2).

(iv) A $\in \tau_1$ – S*GO(X, τ_1, τ_2) $\Rightarrow$ A $\in \tau_1\tau_2$ – S*GLC(X, τ_1, τ_2).

Proof:

i) Since A is $\tau_1\tau_2$ –s*g locally closed* subset in (X, τ_1, τ_2), we have A= G$\cap$F where G is τ_1–s*g open set and F is τ_2 – closed in X. Since every τ_2 –closed set is τ_2–s*g closed in (X, τ_1, τ_2), A = G$\cap$F where G is τ_1–s*g open and F is τ_2 – s*g closed in (X, τ_1, τ_2). Therefore A $\in \tau_1\tau_2$ – S*GLC(X, τ_1, τ_2).

(ii) Since A is $\tau_1\tau_2$ – s*g locally closed** subset in (X, τ_1, τ_2), A =G$\cap$F where G is τ_1 – open and F is τ_2 – s*g closed in (X, τ_1, τ_2). Since every τ_1- open sets are τ_1 – s*g open in(X, τ_1, τ_2), A = G$\cap$F where G is τ_1 – s*g open and F is τ_2 – s*g closed in (X, τ_1, τ_2). Therefore A $\in \tau_1\tau_2$ – S*GLC(X, τ_1, τ_2).

(iii) Since $A = A \cap X$ and A is $\tau_2 -$ s*g closed and X is τ_1-s*g open in (X, τ_1, τ_2). $A \in \tau_1\tau_2-$ S*GLC(X, τ_1, τ_2).

(iv) Since $A = A \cap X$ and A is $\tau_1 -$ s*g open and X is $\tau_2 -$ s*g closed in (X, τ_1, τ_2), we have $A \in \tau_1\tau_2-$ S*GLC(X, τ_1, τ_2).

Remark: 5.4.11

The converses of (i), (ii), (iii) and (iv) of the above theorem are not true in general as can be seen from the following examples.

Example: 5.4.12

Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{c, d\}\}$ Then $\{a, c\}$ is $\tau_1\tau_2 -$s*g locally closed in(X, τ_1, τ_2), but not $\tau_1\tau_2 -$ s*g locally closed* in (X, τ_1, τ_2).

Example: 5.4.13

In Example 5.4.5(b) is $\tau_1\tau_2 -$ s*g locally closed in (X, τ_1, τ_2), but not $\tau_1\tau_2 -$ s*g locally closed** in (X, τ_1, τ_2).

Example: 5.4.14

In Example 5.4.12, $\{a, c\}$ is $\tau_1\tau_2-$ s*g locally closed in (X, τ_1, τ_2), but not $\tau_1 -$ s*g open in (X, τ_1, τ_2) and $\{a\}$ is $\tau_1\tau_2 -$ s*g locally closed in (X, τ_1, τ_2), but not $\tau_2 -$ s*g closed in (X, τ_1, τ_2).

Theorem: 5.4.15

If (X, τ_1, τ_2) is pairwise door space, then every subset of X is both $\tau_1\tau_2 -$ s*g locally closed and $\tau_2\tau_1 -$ s*g locally closed.

Proof:

Since (X, τ_1, τ_2) is pairwise door space, every subset of (X, τ_1, τ_2) is either τ_1 - open or τ_2 - closed and τ_2 - open or τ_1 - closed. Since every τ_1 – open (resp. τ_2 - closed) subset of (X, τ_1, τ_2) is τ_1 – s*g open (resp. τ_2 – s*g closed), we have every subset of (X, τ_1, τ_2) is either τ_1 – s*g open or τ_2 – s*g closed. Since every τ_1 – s*g open and $\tau_1\tau_2$ – s*g closed subset of (X, τ_1, τ_2) is $\tau_1\tau_2$ – s*g locally closed, every subset of X is $\tau_1\tau_2$– s*g locally closed. Similarly we can prove that every subset of X is $\tau_2\tau_1$ – s*g locally closed.

Theorem: 5.4.16

For a subset A of a bitopological space (X, τ_1, τ_2) the following are equivalent.

(a) $A \in \tau_1\tau_2 - S^*GLC^*(X, \tau_1, \tau_2)$.

(b) $A = G \cap [\tau_2 \text{ - cl } (A)]$ for some τ_1 – s*g open set G.

(c) $A \cup \{X - [\tau_2 - \text{cl } (A)]\}$ is τ_1 – s*g open.

(d) $[\tau_2 \text{ - cl } (A)] - A$ is τ_1 – s*g closed.

Proof:

(a) $\Rightarrow$ (b)

Since A is $\tau_1\tau_2$ – s*g locally closed* set in (X, τ_1, τ_2), $A = G \cap F$ where G is τ_1–s*g open set and F is τ_2 – closed in X. Since $A \subseteq \tau_2\text{- cl } (A)$ and $A \subseteq G$, $A \subseteq G \cap [\tau_2 \text{ - cl } (A)]$ Since $A \subseteq F$ and F is τ_2 – closed in X, $\tau_2 - \text{cl } (A) \subseteq F$. Therefore $G \cap [\tau_2 \text{ - cl } (A)] \subseteq G \cap F = A$. Hence $G \cap [\tau_2 - \text{cl } (A)] \subseteq A$. Therefore $A = G \cap [\tau_2 \text{ - cl } (A)]$ for some τ_1–s*g open set G in (X, τ_1, τ_2).

(b) $\Rightarrow$ (a)

Suppose that $A = G \cap [\tau_2 \text{ - cl } (A)]$ for some $\tau_1 - s^*g$ open set G in (X, τ_1, τ_2). Since $\tau_2 - \text{cl } (A)$ is τ_2 - closed in (X, τ_1, τ_2) and G is $\tau_1 - s^*g$ open in (X, τ_1, τ_2), $A \in \tau_1\tau_2 - S^*GLC^*(X, \tau_1, \tau_2)$.

(b) ⇒ (c)

Since $A = G \cap [\tau_2 \text{ - cl } (A)]$ for some $\tau_1 - s^*g$ open set G in (X, τ_1, τ_2), $A \cup \{X - [\tau_2 - \text{cl } (A)]\} = \{G \cap [\tau_2 \text{ - cl } (A)]\} \cup \{X - [\tau_2 \text{ - cl } (A)]\} = G$. Therefore $A \cup \{X - [\tau_2 \text{ - cl } (A)]\}$ is $\tau_1 - s^*g$ open.

(c) ⇒ (b)

Suppose that $A \cup \{X - [\tau_2 - \text{cl } (A)]\}$ is $\tau_1 - s^*g$ open in (X, τ_1, τ_2). Let $G = A \cup \{X - [\tau_2 \text{ - cl } (A)]\}$. Then G is $\tau_1 - s^*g$ open in (X, τ_1, τ_2). Now, $G \cap [\tau_2 - \text{cl}(A)] = [A \cup \{X - [\tau_2 - \text{cl}(A)]\}] \cap [\tau_2 - \text{cl}(A)]$

$= \{[A \cup [\tau_2 - \text{cl}(A)]^C\} \cap [\tau_2 - \text{cl}(A)]$

$= \{A \cap [\tau_2 - \text{cl}(A)]\} \cup \{[\tau_2 - \text{cl}(A)]^C \cap [\tau_2 - \text{cl}(A)]\}$

$= A \cup \phi$

$= A.$

Therefore $A = G \cap [\tau_2 - \text{cl } (A)]$ for some $\tau_1 - s^*g$ open set G in (X, τ_1, τ_2).

(c) ⇒ (d)

Suppose that $A \cup \{X - [\tau_2 - \text{cl } (A)]\}$ is $\tau_1 - s^*g$ open in (X, τ_1, τ_2). Let $G = A \cup \{X - [\tau_2 - \text{cl } (A)]\}$. Since G is $\tau_1 - s^*g$ open in (X, τ_1, τ_2), $X - G$ is $\tau_1 - s^*g$ closed in (X, τ_1, τ_2).

Now,

$X - G = X - [A \cup \{X - [\tau_2 - \text{cl}(A)]\}]$

$= (X - A) \cap \{X - [\tau_2 - \text{cl}(A)]\}$

$= (X - A) \cap [\tau_2 - \text{cl}(A)]$

$= \tau_2 - \text{cl}(A) - A.$

Therefore, $\tau_2 - \text{cl } (A) - A$ is $\tau_1 - s^*g$ closed in (X, τ_1, τ_2).

(d) $\Rightarrow$ (c)

Suppose that τ_2 – cl (A) – A is τ_1 – s*g closed in (X, τ_1, τ_2). Let F = τ_2 -cl(A) –A. Then F is τ_1 – s*g closed in (X, τ_1, τ_2) implies that X –F is τ_1 – s*g open in (X, τ_1, τ_2).

Now,

$$\begin{aligned} X - F &= X - \{[\tau_2 - cl(A)] - A\} \\ &= X \cap \{[\tau_2 - cl(A)] - A\}^C \\ &= X \cap \{[\tau_2 - cl(A)] \cap A^C\}^C \\ &= X \cap \{[\tau_2 - cl(A)]^C \cup (A^C)^C\} \\ &= X \cap \{[\tau_2 - cl(A)]^C \cup A\} \\ &= \{X \cap [\tau_2 - cl(A)]^C\} \cup \{X \cap A\} \\ &= [\tau_2 - cl(A)]^C \cup A \\ &= \{X - [\tau_2 - cl(A)]\} \cup A. \end{aligned}$$

Hence A $\cup$ {X – [τ_2 – cl (A)]} is τ_1 – s*g open in (X, τ_1, τ_2).

Theorem: 5.4.17

In a bitopological space (X, τ_1, τ_2), the following are equivalent.

(a) A-- [τ_1 - int (A)] is τ_2 – s*gopen in (X, τ_1, τ_2).

(b) [τ_1– int (A)] $\cup$ [X – A] is τ_2 – s*g closed in (X, τ_1, τ_2).

(c) G $\cup$ [τ_1 - int (A)] = A for someτ_2– s*g open set G in (X, τ_1, τ_2).

Proof:

(a) $\Rightarrow$ (b)

Now,

$$\begin{aligned} X - \{A - [\tau_1 - int(A)]\} &= X \cap \{A - [\tau_1 - int(A)]\}^C \\ &= X \cap [A \cap \{\tau_1 - int(A)\}^C]^C \\ &= X \cap \{A^C \cup [\{\tau_1 - int(A)\}^C]^C\} \end{aligned}$$

$$= X \cap \{A^C \cup [\tau_1 - \mathrm{int}(A)]\}$$
$$= \{A^C \cup [\tau_1 - \mathrm{int}(A)]\}$$
$$= [\tau_1 - \mathrm{int}(A)] \cup [X - A].$$

Since $A - [\tau_1 - \mathrm{int}\,(A)]$ is $\tau_2 - s^*g$ open, we have $X - \{A - [\tau_1 - \mathrm{int}\,(A)]\}$ $= [\tau_1 - \mathrm{int}(A)] \cup [X - A]$ is $\tau_2 - s^*g$ closed in (X, τ_1, τ_2).

(b) ⇒ (a)

Suppose that $[\tau_1 - \mathrm{int}\,(A)] \cup [X - A]$ is $\tau_2 - s^*g$ closed in (X, τ_1, τ_2). Since $[\tau_1 - \mathrm{int}\,(A)] \cup [X - A]$ is $\tau_2 - s^*g$ closed, $X - \{[\tau_1 - \mathrm{int}\,(A)] \cup [X - A]\}$ is $\tau_2 - s^*g$ open.

Now,

$$X - \{[\tau_1 - \mathrm{int}(A)] \cup [X - A]\} = X \cap \{[\tau_1 - \mathrm{int}(A)] \cup [X - A]\}^C$$
$$= X \cap \{[\tau_1 - \mathrm{int}(A)] \cup A^C\}^C$$
$$= X \cap \{[\tau_2 - \mathrm{int}(A)]^C \cap (A^C)^C\}$$
$$= X \cap \{[\tau_1 - \mathrm{int}(A)]^C \cap A\}$$
$$= A \cap [\tau_1 - \mathrm{int}(A)]^C$$
$$= A - [\tau_1 - \mathrm{int}(A)].$$

Therefore $A - [\tau_1 - \mathrm{int}\,(A)]$ is $\tau_2 - s^*g$ open in (X, τ_1, τ_2).

(b) ⇒ (c)

Suppose that $[\tau_1 - \mathrm{int}\,(A)] \cup [X - A]$ is $\tau_2 - s^*g$ closed. Let $U = [\tau_1 - \mathrm{int}(A)] \cup [X - A]$. Then U is $\tau_2 - s^*g$ closed. Then U^C is $\tau_2 - s^*g$ open.

Now,

$$U^C \cup [\tau_1 - \mathrm{int}(A)] = \{[\tau_1 - \mathrm{int}(A)] \cup [X - A]\}^C \cup [\tau_1 - \mathrm{int}(A)]$$
$$= \{[\tau_1 - \mathrm{int}(A)]^C \cap (A^C)^C\} \cup [\tau_1 - \mathrm{int}(A)]$$
$$= \{[\tau_1 - \mathrm{int}(A)]^C \cap A\} \cup [\tau_1 - \mathrm{int}(A)]$$
$$= \{[\tau_1 - \mathrm{int}(A)]^C \cup [\tau_1 - \mathrm{int}(A)]\} \cap \{A \cup [\tau_1 - \mathrm{int}(A)]\}$$

$= X \cap A$

$= A.$

Take $G = U^C$. Then $A = G \cup [\tau_1 - \text{int}(A)]$ for some $\tau_2 - s^*g$ open set G in (X, τ_1, τ_2).

(c) $\Rightarrow$ (b)

Suppose that $A = G \cup [\tau_1 - \text{int}(A)]$ for some $\tau_2 - s^*g$ open set G in (X, τ_1, τ_2).

Now,

$[\tau_1 - \text{int}(A)] \cup [X - A] = \tau_1 - \text{int}(A) \cup A^C$

$= [\tau_1 - \text{int}(A)] \cup \{G \cup [\tau_1 - \text{int}(A)]\}^C$

$= [\tau_1 - \text{int}(A)] \cup \{G^C \cap [\tau_1 - \text{int}(A)]^C\}$

$= \{[\tau_1 - \text{int}(A)] \cup G^C\} \cap \{[\tau_1 - \text{int}(A)] \cup [\tau_1 - \text{int}(A)]^C\}$

$= \{[\tau_1 - \text{int}(A)] \cup G^C\} \cap X\}$

$= \{[\tau_1 - \text{int}(A)] \cup G^C\}$

$= X - G.$

Since G is $\tau_2 - s^*g$ open in (X, τ_1, τ_2), $X - G$ is $\tau_2 - s^*g$ closed in (X, τ_1, τ_2). Therefore $[\tau_1 - \text{int}(A)] \cup [X - A]$ is $\tau_2 - s^*g$ closed in (X, τ_1, τ_2).

Remark: 5.4.18

The union of two $\tau_1\tau_2 -$ s*g locally closed sets in (X, τ_1, τ_2).need not $\tau_1\tau_2 -$ s*g locally closed in general as can be seen from the following example.

Example: 5.4.19

Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$,$\tau_2 = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. Then $A = \{a, d\}$, $B = \{b, d\}$ are

$\tau_1\tau_2$ – s*g locally closed sets in (X, τ_1, τ_2), but A $\cup$ B = {a, b, d} is not $\tau_1\tau_2$ – s*g locally closed set in (X, τ_1, τ_2).

Remark: 5.4.20

Eventhough A and B are not $\tau_1\tau_2$– s*g locally closed sets in (X, τ_1, τ_2), A $\cup$B is $\tau_1\tau_2$ – s*g locally closed in general as can be seen from the following example.

Example: 5.4.21

Let X = {a, b, c, d}, τ_1 = {ϕ, X, {a}, {a, b}}, τ_2 = {ϕ, X, {c, d}, {b, c, d}}. Then A = {b}, B = {a, d} are not $\tau_1\tau_2$ – s*g locally closed sets in (X, τ_1, τ_2) but A $\cup$ B = {a, b, d} is $\tau_1\tau_2$ – s*g locally closed set in (X, τ_1, τ_2).

SECTION: 5.5

s*g - SUBMAXIMAL SPACES IN BITOPOLOGICAL SPACES

Definition: 5.5.1

A bitopological space (X, τ_1, τ_2) is **$\tau_1\tau_2$ – submaximal space** if every τ_1 - dense subset of X is τ_2 - open in X.

Definition: 5.5.2

A bitopological space (X, τ_1, τ_2) is **$\tau_2\tau_1$ – submaximal space** if every τ_2 - dense subset of X is τ_1 - open in X.

Definition: 5.5.3

A bitopological space (X, τ_1, τ_2) is **$\tau_1\tau_2$ − s*g submaximal space** if every τ_1 - dense subset of X is τ_2 − s*g open in X.

Definition: 5.5.4

A bitopological space (X, τ_1, τ_2) is **$\tau_2\tau_1$ − s*g submaximal space** if every τ_2 – dense subset of X is τ_1 – s*g open in X.

Example: 5.5.5

In Example 5.4.19,

(i) τ_1 - dense subsets of (X, τ_1, τ_2) are X, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}.

(ii) τ_2 − s*g open sets of (X, τ_1, τ_2) are ϕ,X, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}.

(iii) τ_2 - open sets of (X, τ_1, τ_2) are ϕ,X, {a}, {a, b}{a, c}{a, d}, {a, b, c}, {a, b, d}, {a, c, d}. Therefore (X, τ_1, τ_2) is both $\tau_1\tau_2$ – s*g submaximal space and $\tau_1\tau_2$ – submaximal space.

Theorem: 5.5.6

If (X, τ_1, τ_2) is a $\tau_1\tau_2$ – submaximal space, then X is $\tau_1\tau_2$ – s*g submaximal space.

Proof:

Since (X, τ_1, τ_2) is $\tau_1\tau_2$ – submaximal space, every τ_1 – dense subset of X is τ_2 - open in X. Since every τ_2 - open set in X is τ_2 − s*g open in X, every τ_1 - dense subset of X is τ_2 – s*g - open in X. Therefore (X, τ_1, τ_2) is $\tau_1\tau_2$ – s*g submaximal space.

Remark: 5.5.7

The converse of the above theorem need not true in general as can be seen from the following example.

Example: 5.5.8

Let X = {a, b, c, d}, τ_1 = {ϕ, X, {a}, {a, b}}, τ_2 = {ϕ,X, {a}, {b, c, d}}. Then τ_1 – dense subsets of (X, τ_1, τ_2) are X, {a}, {a, b},{a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}. Therefore (X, τ_1, τ_2) is $\tau_1\tau_2$–s*g submaximal space but not $\tau_1\tau_2$ – submaximal space.

Theorem: 5.5.9

A bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$–s*g submaximal space if and only if $\tau_2\tau_1$ – S*GLC*(X, τ_1, τ_2) = P(X).

Proof:

Suppose that (X, τ_1, τ_2) is $\tau_1\tau_2$ – s*g submaximal space. Obviously $\tau_2\tau_1$ – S*GLC*(X, τ_1, τ_2)$\subseteq$ P(X). Let A $\in$ P(X) and U = A $\cup$ {X – [τ_1–cl(A)]}. Since τ_1 – cl (U) = X, we have U is τ_1 - dense subset of X. Since (X, τ_1, τ_2) is $\tau_1\tau_2$ – s*g submaximal space, we have U is τ_2 – s*g open in X. Since every τ_2 – s*g open set in X is $\tau_2\tau_1$ – s*g locally closed* set in (X, τ_1, τ_2), U $\in$ $\tau_2\tau_1$ – S*GLC*(X, τ_1, τ_2).

Therefore P(X) $\subseteq$ $\tau_2\tau_1$ – S*GLC*(X, τ_1, τ_2).

Hence $\tau_2\tau_1$ – S*GLC*(X, τ_1, τ_2) = P(X).

Conversely, suppose that $\tau_2\tau_1 - S^*GLC^*(X, \tau_1, \tau_2). = P(X)$. Let A be the τ_1 dense subset of (X, τ_1, τ_2). Then $A \cup \{X - [\tau_1 - cl\ (A)]\} = A \cup\ [\tau_1 - cl\ (A)]^C = A$. Therefore $A \in \tau_2\tau_1 - S^*GLC^*(X, \tau_1, \tau_2)$ implies that A is $\tau_2 - s^*g$ open in X {By Theorem 5.4.16}. Hence X is a $\tau_1\tau_2 - s^*g$ submaximal space.