CHAPTER - I

CHAPTER - I

ON GENERALIZED b-CLOSED SETS IN TOPOLOGICAL SPACES

In this chapter the idea cherished by Al-Omari and Noorani [3] on generalized b-closed sets (briefly, gb-closed sets) was discussed. The properties and characterizations of generalized b-closed sets were studied. A characterization of extremely disconnected spaces in terms of gb-closed sets was discussed. The relationship between approximately b-closed maps and contra b-closed maps were analyzed. Also the notion of pre-closed map was discussed and introduced some weaker and stronger forms of continuity, namely, ap-b-continuity, contra-b-continuity and studied their relationship with other forms of continuity. The concepts of irresolute map, b-irresolute map and contra b-irresolute map were added. The chapter was concluded with characterizations of T_{gs}-spaces and $T_{1/2}$-spaces using the concepts of ap-b-continuous maps and ap-b-closed maps.

Section 1.1

Preliminaries

Definition 1.1.1

Let (X, τ) be a topological space. A subset S of a space X is called

(1) an **α-open** if S ⊂ int(cl(int(S))).

(2) **semi-open** if S ⊂ cl(int(S)).

(3) **preopen** if S ⊂ int(cl(S)).

(4) **β-open or semi-preopen** if S ⊂ cl(int(cl(S))).

(5) **b-open** if A ⊂ cl(int(A)) ∪ int(cl(A)).

(6) **regular open** A = int (cl(A)).

Notation 1.1.2

We denote the classes of these sets in a space (X, τ) by, SO(X), PO(X), βO(X) or SPO(X) , BO(X) and RO(X) respectively.

Definition 1.1.3

The complement of a α-open set, semi-open set, pre-open set, semipre-open set, b-open set and regular open set is **α-closed set, semi-closed set, pre-closed set, semipre-closed set, b-closed set, regular closed set** respectively.

Remark 1.1.4

The collection of all α-open sets denoted by τ_α on a space X is a topology on X.

Remark 1.1.5

In general, SO(X) need not be a topology on X, but the intersection of a semi-open set and an open set is semi-open. The same holds for PO(X) and SPO(X).

Definition 1.1.6

Let (X, τ) be a topological space. For a subset S of a space X, the **semi-closure** of S, denoted by **scl S** is the intersection of all semi-closed subsets of X containing S. The semi-interior of S, denoted by **sint S**, is the union of all semi-open subsets of X contained in S.

Definition 1.1.7

Let (X, τ) be a topological space. For a subset S of a space X, the **pre-closure** of S, denoted by **pcl S** is the intersection of all pre-closed subsets of X containing S. The pre- interior of S is denoted by **pint S**, is the union of all pre-open subsets of X contained in S.

Definition 1.1.8

Let (X, τ) be a topological space. For a subset S of a space X, the **semipre-closure** of S, denoted by **spcl S** is the intersection of all semipre-closed subsets of X containing S. The semipre-interior of S is denoted by **spint S**, is the union of all semipre-open subsets of X contained in S. We denote the closure and the interior operator in (X, τ_α) by **clα** and **intα**.

Definition 1.1.9

Let (X, τ) be a topological space. For a subset S of a space X, the **b-closure** of S, denoted by **bcl S** is the intersection of all b-closed subsets of X containing S. The **b-interior** of S, denoted by **bint S**, is the union of all b-open subsets of X contained in S.

Definition 1.1.10

Let (X, τ) be a topological space. Let S be a subset of a space X. Then

(1) $cl\alpha(S) = S \cup cl(int(cl(S)))$, $int\alpha\, S = S \cap int(cl(int(S)))$

(2) $scl(S) = S \cup int(cl(S))$, $sint\, S = S \cap cl(int(S))$

(3) $pcl(S) = S \cup cl(int(S))$, $pint(S) = S \cap int(cl(S))$

(4) $spcl(S) = S \cup int(cl(int(S)))$, $spint(S) = S \cap cl(int(cl(S)))$

Definition 1.1.11

Let (X, τ) be a topological space. Let S be a subset of a space X. Then

(1) scl(sint(S)) = sint(S) ∪ int(cl(int(S)))

(2) pcl(pint(S)) = pint(S) ∪ cl(int(S))

(3) spcl(spint(S)) = spint(spcl(S))

(4) int(scl(S)) = pint (cl(S)) = pint (scl(S)) = scl(pint(S)) = int(cl(S))

(5) int(pcl(S)) = scl(int(S)) = spcl(int(S)) = int(spcl(S)) = int(cl(int(S)))

Definition 1.1.12

A subset A of a space (X, τ) is called

(1) a **generalized closed set** (briefly,**g-closed**) [33] if cl(A) ⊂ U whenever A ⊂ U and U is open.

(2) a **semi generalized closed set** (briefly,**sg-closed**) [16] if scl (A) ⊂ U Whenever A ⊂ U and U is semi-open.

(3) a **generalized semiclosed set** (briefly,**gs-closed**) [11] if scl(A) ⊂ U Whenever A ⊂ U and U is open.

(4) a **generalized semipre-closed set** (briefly,**gsp-closed**) [20] if spcl(A) ⊂ U whenever A ⊂ U and U is open.

(5) a **generalized pre-closed set**(briefly,**gp-closed**) [35] if pcl (A) ⊂ U Whenever A ⊂ U and U is open.

Complement of g-closed, gp-closed, gsp-closed, gs-closed and sg-closed sets are called **g-open, gp-open, gsp-open, gs-open and sg-open sets** respectively.

Theorem 1.1.13 [22]

Let A be a subset of a topological space (X, τ). If A ∈ SO(X), then pcl(A) = cl(A).

Definition 1.1.14

A space (X, τ) is **extremely disconnected** if the closure of every open subset of X is open.

Theorem 1.1.15

For a space X, the following statements are equivalent

(1) (X, τ) is extremely disconnected

(2) scl(A∪B) = scl(A) ∪ scl(B) for all A, B ⊂ X

(3) The union of two semi-closed subsets of X is semi-closed

(4) The union of two sg-closed subsets of X is sg-closed

(5) Every semi-preclosed subset of X is preclosed

(6) Every sg-closed subset of X is preclosed

(7) Every semi-closed subset of X is preclosed

Definition 1.1.16

A function f : X → Y is said to be **b-continuous** [24] if for each $x \in X$ and each open set V of Y containing f(x), there exists $U \in BO(X, x)$ such that $f(U) \subset V$.

Definition 1.1.17

A function f : X → Y is said to be **contra continuous** [21] if $f^{-1}(V)$ is closed in X for each open set V of Y.

Definition 1.1.18

A function f : X → Y is said to be **contra b-continuous** [38] if $f^{-1}(V)$ is b-closed in X for each open set V of Y.

Definition 1.1.19

A function f : X → Y is said to be **b-irresolute** [24] if for each b-open set V in Y, $f^{-1}(V)$ is b-open in X.

Definition 1.1.20

A function f : X → Y is said to be **b-closed** [24] if for every b-closed subset A of X, f(A) is b-closed in Y .

Definition 1.1.21

A function f : X → Y is said to be **b-open** [24] if for every b-open subset A of X, f(A) is b-open in Y .

Definition 1.1.22

A map f : X → Y is said to be **contra b-closed** if f(U) is b-open in Y for each closed set U of X.

Definition 1.1.23

A map f : X → Y is said to be **contra b-open** if f(U) is b-closed in Y for each open set U of X.

Section 1.2

Generalized b-closed sets

In this section, the class of generalized b-closed sets was discussed and studied some of its fundamental properties and characterizations.

Definition 1.2.1

Let X be a topological space. A subset A of X is called a **generalized b-closed set** (briefly, **gb-closed set**) [28] if bcl(A) $\subset$ U, whenever A $\subset$ U and U is open.

The complement of a generalized b-closed set is called **generalized b-open** (briefly, **gb-open**) set.

Notation 1.2.2

The collection of all gb-closed subsets of X is denoted by gbC(X) .

The collection of all gb-open subsets of X is denoted by gbO(X).

We have the following implications diagram.

Implication diagram

Closed $\Rightarrow$ pre-closed $\Rightarrow$ b-closed

$\Downarrow$ $\Downarrow$ $\Downarrow$

g-closed $\Rightarrow$ gp-closed $\Rightarrow$ gb-closed

Theorem 1.2.3

Every b-closed set is gb-closed.

Proof

Let A be a b-closed set .Then bcl A = A. Hence bcl(A) $\subset$ U whenever A $\subseteq$ U and U is open. Hence A is gb-closed.

Remark 1.2.4

The converse of the above theorem need not true be can be seen from the following example.

Example 1.2.5

Let X = {a, b, c} and let = { ϕ, X, {a} }, then the family of all b-closed sets of X is, bC(X) = { ϕ, X, {b}, {c}, {b, c} }.

But the family of all gb-closed sets of X is,

gbC(X) = { ϕ, X, {b}, {c}, {a, b}, {b, c}, {a, c} }.

It is clear that {a, c} is gb-closed but it is not b-closed in X.

Theorem 1.2.6

Let A be a gb-closed subset of (X, τ). Then $(bcl(A) - A)$ does not contain any non-empty closed sets.

Proof

Let F be a closed set such that $F \subset (bcl(A) - A)$.

Then $(X-F)$ is open and $A \subset (X - F)$. Since A is gb-closed, $bcl(A) \subset (X - F)$ and thus, $F \subset (X - bcl(A))$.

This implies that $F \subset (X - bcl(A)) \cap (bcl(A)-A) = \phi$.

Hence $F = \phi$.

Theorem 1.2.7

Let A be a gb-closed set. Then A is b-closed if and only if $(bcl(A) - A)$ is closed.

Proof

Let A be a gb-closed set. If A is b-closed, then $(bcl(A) - A) = \phi$, which is a closed set.

Conversely, let $(bcl(A) - A)$ be closed.

Then, by Theorem 1.2.6, $(bcl(A) - A)$ does not contain any non-empty closed subset and since$(bcl(A) - A)$ is a closed subset of itself,

$(bcl(A) - A) = \phi$.

Hence $A = bcl(A)$ and so A is a b-closed set.

Definition 1.2.8

Let A be a subset of a space(X, τ). A point $x \in X$ is said to be a **b-limit point** of A if for each b-open set U containing x, $U \cap (A - \{x\}) \neq \phi$.

The set of all b-limit points of A is called the **b-derived set** of A and is denoted by **b-d(A).**

Remark 1.2.9

Since every open set is b-open, $b\text{-}d(A) \subset d(A)$ for any open subset $A \subset X$ Where d(A) is the derived set of A.

Lemma 1.2.10

If $d(A) = b\text{-}d(A)$ then $cl(A) = bcl(A)$.

Proof

$cl(A) = A \cup d(A) = A \cup b\text{-}d(A) = bcl(A)$.

Proposition 1.2.11

If $d(A) \subseteq b\text{-}d(A)$ for every subset A of X . Then for subsets F and B of X, $bcl(F \cup B) = bcl(F) \cup bcl(B)$.

Proof

Given $d(A) \subseteq b\text{-}d(A)$ ∀ for every subset A of X.

Now $bcl(F \cup B) = cl(F \cup B) = cl(F) \cup cl(B) = bcl(F) \cup bcl(B)$

Theorem 1.2.12

If A and B are gb-closed sets such that $d(A) \subseteq b\text{-}d(A)$ and $d(B) \subseteq b\text{-}d(B)$. Then $A \cup B$ is gb-closed.

Proof

Let U be an open set such that $A \cup B \subset U$.

Since A and B are gb-closed sets $bcl(A) \subset U$ and $bcl(B) \subset U$.

Given $d(A) \subset b\text{-}d(A)$, we get $d(B) = b\text{-}d(B)$ and by Lemma 1.2.10,

$cl(A) = bcl(A)$.

Similarly, $cl(B) = bcl(B)$.

Thus $bcl(A \cup B) \subset cl(A \cup B) = cl(A) \cup cl(B)$

$= bcl(A) \cup bcl(B) \subset U \Longrightarrow A \cup B$ is gb-closed.

Definition 1.2.13

Let $B \subset A \subset X$. Then we say that **B is gb-closed relative to A** if $bcl_A(B) \subset U$ whenever $B \subset U$ and U is open in A.

Theorem 1.2.14

Let $B \subset A \subset X$ where A is a gb-closed and an open set. Then B is gb-closed relative to A if and only if B is gb-closed in X.

Proof

Given A is both gb-closed and open. Then $bcl(A) \subset A$.

Since $B \subset A$,

$bcl(B) \subset bcl(A) \subset A$.

Therefore, $A \cap bcl(B) = bcl(B)$.

Now from the fact that $A \cap bcl(B) = bcl_A(B)$

$\Rightarrow bcl(B) = bcl_A(B) \subset A$.

Let B be gb-closed relative to A and U is an open subset of X such that $B \subset U$.

Then, $B = B \cap A \subset U \cap A$ and $U \cap A$ is open in A.

Since B is gb-closed relative to A, $bcl(B) = bcl_A(B) \subset U \cap A \subset U$.

Therefore B is gb-closed in X.

Conversely, if B is gb-closed in X and U is an open subset of A such that $B \subset U$.

Then, $U = V \cap A$ for some open subset V of X.

As $B \subset V$ and B is gb-closed in X, $bcl(B) \subset V$.

Then $bcl_A(B) = bcl(B) \cap A \subset V \cap A = U$.

Therefore B is gb-closed relative to A.

Theorem 1.2.15

Let (X, τ) be a topological space and let A be an open and gb-closed set. Then $A \cap F$ is gb-closed whenever $F \in BC(X)$.

Proof

Since A is gb-closed and open,

$bcl(A) \subset A$ and hence A is b-closed.

Thus $A \cap F$ is b-closed in X, which implies that $A \cap F$ is gb-closed in X.

Theorem 1.2.16

Let (X, τ) be a topological space. If A is a gb-closed set and B is any set such that $A \subset B \subset bcl(A)$, then B is a gb-closed set.

Proof

Let $B \subset U$ where U is open.

Since A is gb-closed and $A \subset B$, $A \subset U$,

$bcl(A) \subset U$ and also $bcl(B) \subset bcl(A)$.

Therefore $bcl(B) \subset U$ and hence B is a gb-closed set.

Theorem 1.2.17

A subset $A \subset X$ is gb-open if and only if $F \subset bint(A)$ whenever F is a closed set and $F \subset A$.

Proof

Let A be a gb-open set and suppose $F \subset A$, where F is closed.

Then $(X – A)$ is a gb-closed set contained in the open set $(X – F)$.

Hence $bcl(X – A) \subset (X – F)$ and $(X - bint(A)) \subset (X – F) \Longrightarrow F \subset bint(A)$.

Conversely, if F is a closed set with $F \subset bint(A)$ and $F \subset A$, then

$(X - bint(A)) \subset (X - F) \Longrightarrow bcl(X - A) \subset (X - F)$.

Hence $(X - A)$ is a gb-closed set and A is a gb-open set.

Theorem 1.2.18

A space X is extremely disconnected if and only if every gb-closed subset of X is gp-closed.

Proof

Suppose that X is extremely disconnected. Let A be gb-closed and let U be an open set containing A.

Then $bcl(A) \subset U$.

Since $bcl(A) = A \cup [\, int(cl(A)) \cap cl(int(A)) \,]$

$\Longrightarrow [\, int(cl(A)) \cap cl(int(A)) \,] \subset U$

Since $int(A) \subset int(cl(A))$, $int(A) = int(cl(A)) \cap int(A)$

$$\begin{aligned} \Longrightarrow cl(int\,(A)) &= cl[\, int(cl(A)) \cap (int(A)) \,] \\ &\subset [\, cl(int(cl(A))) \cap cl(int(A)) \,] \\ &= int(cl(A)) \cap cl(int(A)) \text{ (since } int(clA) \text{ is closed)} \\ &\subset U \end{aligned}$$

Now $pcl(A) = A \cup cl(int(A)) \subset U$

Hence A is gp-closed.

Conversely, let every gb-closed subset of X be gp-closed.

Let $A \subset X$ be regular open.

$$\begin{aligned} \text{Then } bcl(A) &= A \cup [\, int(cl(A)) \cap cl(int(A)) \,] \\ &= A \cup [\, A \cap cl(int(A)) \,] \subset A \end{aligned}$$

Then A is gb-closed and so gp-closed.

Since every regular open set is semi open and A is gp-closed, $cl(A) = pcl\ A$.

Therefore A is closed and X is extremely disconnected.

Section 1.3

ap-b-continuous, ap-b-closed and contra b-continuous maps

In this section the notion of ap-b-continuous maps, ap-b-closed maps and contra b-continuous maps were defined using the notion of gb-closed sets and their relationships with various types of mappings were studied.

Definition 1.3.1

A map $f : X \longrightarrow Y$ is said to be **approximately b-continuous**[3] (briefly, **ap-b-continuous**) if $bcl(F) \subset f^{-1}(U)$ whenever U is an open subset of Y and F is a gb-closed subset of X such that $F \subset f^{-1}(U)$.

Definition 1.3.2

A map $f : X \longrightarrow Y$ is said to be **approximately b-closed** [3](briefly, **ap-b-closed**) if $f(F) \subset bint(V)$ whenever V is a gb-open subset of Y, F is a closed subset of X and $f(F) \subset V$.

Definition 1.3.3

A map $f : X \longrightarrow Y$ is said to be **approximately b-open** [3](briefly, **ap-b-open**) if $bcl(F) \subset f(U)$ whenever U is an open subset of X, F is a gb-closed subset of Y and $F \subset f(U)$.

Theorem 1.3.4

Let $f : X \longrightarrow Y$ be a function. Then we have the following,

(1) If f is contra b-continuous, then f is ap-b-continuous

(2) If f is contra b-closed, then f is ap-b-closed

(3) If f is contra b-open, then f is ap-b-open

Proof

(1) Suppose f is contra b-continuous. Let U be an open set in Y and F be a gb-closed subset of X.

Such that $F \subset f^{-1}(U)$.

Since f is contra b-continuous, $bcl(F) \subset bcl(f^{-1}(U)) = f^{-1}(U)$.

Hence f is ap-b-continuous.

(2) Suppose f is contra b-closed.

Let F be an closed subset of X and V be a gb-open subset of Y.

Such that $f(F) \subset V \implies bint(f(F)) \subset bint\ V$

Since f is contra b-closed and F is closed in Y, f(F) is b-open inY.

Therefore $f(F) = bint(f(F)) \subset bint(V)$.

Hence f is ap-b-closed.

(3) Suppose f is contra b-open.

Let U be an open subset in X and F be a gb-closed subset of Y.

Such that $F \subset f(U) \Longrightarrow$ $bcl(F) \subset bcl(f(U))$

Since f is contra b-open, f(U) is b-closed inY.

Then $bcl(F) \subset bcl(f(U)) = f(U)$. Hence f is ap-b-open.

Remark 1.3.5

An ap-b-continuity need not imply contra b-continuity can be seen from the following example.

Example 1.3.6

Let X = {a, b, c} and τ = {ϕ, X, {b}, {c}, {b, c} } then bC(X) = { ϕ, X, {a}, {b}, {c}, {a, c}, {a, b}} = gbC(X).

Let f : (X, τ) $\longrightarrow$ (X, τ) be the identity map.

Then f is ap-b-continuous. Since every gb-closed set is b-closed.

But it is not contra b-continuous.

Remark 1.3.7

An ap-b-closed map need not be contra b-closed can be seen from the following example.

Example 1.3.8

Let X = {a, b, c} with τ = { ϕ, X, {a} } and Y = {a, b, c} with σ = { ϕ, Y, {a}, {a b} } then bO(X) = { ϕ, X, {a}, {a, b}, {a, c} }

gbC(X) = { ϕ, X, {b}, {c}, {a, b}, {a, c},{b, c} }.

And bO(Y) = { ϕ, Y, {a}, {a, b}, {a, c} },

gbC(Y) = { ϕ, Y, {b}, {c}, {b, c}, {a, c} }.

Let f : (X, τ) $\rightarrow$ (Y, σ) be the identity map.

Since the only gb-open subset of (Y, σ) containing the image of closed set V in X is Y, f is ap-b-closed but it is not contra b-closed.

Also f is ap-b-open but it is not contra b-open.

Definition 1.3.9

A function f : X $\rightarrow$Y is said to be **perfectly continuous** [40] if the inverse image of every open set in Y is clopen in X.

Definition 1.3.10

A function f : X $\rightarrow$Y is said to be **perfectly closed** [40] if the image of every closed set in X is clopen in Y.

Definition 1.3.11

A function f : X →Y is said to be **pre-closed** [40] if for every pre-closed subset A of X, f(A) is pre-closed in Y.

Definition 1.3.12

A function f : X →Y is said to be **pre-open** [40] if for every pre-open subset A of X, f(A) is pre-open in Y.

Proposition 1.3.13

(i) Every continuous map is ap-b-continuous

(ii) pre-closed maps are ap-b-closed

(iii) pre-open maps are ap-b-open.

Remark 1.3.14

The converse of the statements (i), (ii) and (iii) need not be true

Example 1.3.15

An ap-b-continuous map which is not continuous

Let X = {a, b, c} with τ = { ϕ, X, {b}, {c}, {b, c} } and

Y = {a, b, c}, with σ = { ϕ, Y, {a, b} }

Then bO(X) = { ϕ, X, {b}, {c}, {a, b}, {a, c}, {b,c} }.

And bO(Y) = { ϕ, Y, {a}, {b}, {a, b}, {a, c}, {b,c} }.

Let f : (X, τ) → (Y, σ) be the identity map.

Then f is ap-b-continuous (Since f is contra b-continuous) but it is not continuous.

Example 1.3.16

An ap-b-closed map, which is not pre-closed

Let X = {a, b, c} with τ = { ϕ, X, {b}, {c}, {b, c} } and

Y = {a, b, c} with σ = { ϕ, Y, {a} }.

Then bO(X) = { ϕ, X, {b}, {c}, {a, b}, {a, c},{b,c} }.

And BO(Y) = { ϕ, Y, {a}, {a, b}, {a, c} }.

Let f : (X, τ) → (Y, σ) be the identity map.

Since f is contra b-closed, then f is ap-b-closed map but not a pre-closed map.

Clearly, the following two diagrams (i) and (ii) hold and none of its implication are reversible:

perfectly continuous $\Rightarrow$ contra-b-continuous

$\Downarrow$ $\Downarrow$

Continuous $\Rightarrow$ ap-b-continuous

perfectly closed $\Rightarrow$ contra-b-closed

$\Downarrow$ $\Downarrow$

pre-closed $\Rightarrow$ ap-b-closed

Theorem 1.3.17

Let $f : (X,\tau) \rightarrow (Y, \sigma)$ be a map.

(1) If the open and b-closed sets of (X, τ) coincide then f is ap-b-continuous if and only if f is contra b-continuous.

(2) If the open and b-closed sets of (Y, σ) coincide then f is ap-b-closed if and only if f is contra-b-closed.

(3) If the open and b-closed sets of (Y, σ) coincide then f is ap-b-open if and only if f is contra-b-open.

Proof

(1) Assume f is ap-b-continuous. Let A be an arbitrary subset of (X, τ) such that $A \subset U$, where U is open in X.

Then $bcl(A) \subset bcl(U) = U$ (since open and b-closed sets coincide in (X,τ)).

Therefore all subsets of (X, τ) are gb - closed .

For any open set V of (Y, σ). we have $f^{-1}(V)$ is gb-closed in (X,τ).

Since f is ap-b-continuous $bcl(f^{-1}(V)) \subset f^{-1}(V)$.

Therefore $bcl(f^{-1}(V)) = f^{-1}(V)$. Thus $f^{-1}(V)$ is b-closed in X and f is contra b-continuous.

Conversely, assume f is contra b-continuous.

Let U be an open set in Y and F be a gb-closed subset of X.

Such that $F \subset f^{-1}(U)$. Since f is contra b-continuous, $bcl(F) \subset bcl(f^{-1}(U)) = f^{-1}(U)$.

Hence f is ap-b-continuous.

(2) Let f be ap-b-closed map. All the subsets of (Y, σ) are gb-open.

Therefore for any closed subset F of (X, τ), f (F) is gb-open in (Y, σ).

Since f is ap-b-closed, f (F) $\subset$ bint(f(F)). Therefore f (F) = bint(f(F)).

Hence f (F) is b-open inY.

Therefore f is contra b-closed.

Conversely suppose f is contra b-closed.

Let F be an closed subset of X and V be a gb-open subset of Y.

Such that f (F) $\subset$ V $\Longrightarrow$ bint(f(F)) $\subset$ bint V.

Since f is contra b-closed and F is closed in Y, f(F) is b-open inY.

Therefore f (F) = bint(f(F)) $\subset$ bint(V). Hence f is ap-b-closed.

(3) Assume f is an ap-b-open map. All the subsets of (Y, σ) are gb-closed. Let V be open subset of Y, then f(V) is gb-closed in Y. Since f is ap-b-open,

bcl(f(V)) $\subset$ f(V) $\Longrightarrow$ bcl(f(V)) = f(V).

Hence f is contra b-open.

Suppose f is contra b-open. Let U be an open subset in X and F be a gb-closed subset of Y.

Such that F $\subset$ f (U) $\Longrightarrow$ bcl(F) $\subset$ bcl(f(U))

Since f is contra b-open and F is closed in Y, f(U) is b-closed inY.

Then bcl(F) $\subset$ bcl(f(U)) = f(U). Hence f is ap-b-open.

Theorem 1.3.18 [10]

If a map f : X $\rightarrow$Y is surjective, b-irresolute and ap-b-closed, then the inverse image of each gb- closed set in Y is gb-closed in X.

Proof

Let A be a gb-closed subset of Y.

Suppose that f^{-1}(A) $\subset$ U where U is an open subset of X.

Then, (X – U) $\subset$ f^{-1}(Y – A) $\Longrightarrow$ f (X – U) $\subset$ (Y – A).

Since f is ap-b-closed, f(X – U) $\subset$ bint(Y – A)

= Y – bcl(A)

Therefore, (X – U) $\subset$ (X – (f^{-1}(bcl(A))).

Hence f^{-1}(bcl(A) $\subset$ U.

Since f is b-irresolute, f^{-1}(bcl(A)) is b-closed.

Hence bcl(f^{-1}(A)) $\subset$ bcl(f^{-1}(bcl(A)))

$= (f^{-1}(bcl(A)) \subset U.$

This implies that $f^{-1}(A)$ is gb-closed in X.

Theorem 1.3.19

If a map $f : X \rightarrow Y$ is surjective b-irresolute and ap-b-open then the inverse image of each gb-open set in Y is gb-open in X.

Proof

Let A be a gb-open set of Y. To prove $f^{-1}(A)$ is gb-open in X.

Let $F \subset f^{-1}(A)$ where F is a closed in X.

Thus $(X- f^{-1}(A)) \subset X- F \Rightarrow f^{-1}(Y-A) \subset X-F \Rightarrow (Y-A) \subset f(X-F)$

Since f is ap-b-open, $bcl(Y-A) \subset f(X-F)$

$\Rightarrow (Y\text{-}bintA) \subset f(X-F)$

$\Rightarrow f^{-1}(Y\text{-}bintA) \subset f^{-1}(f(X-F)) \Rightarrow (X-F)$

$\Rightarrow (X- f^{-1}(bintA)) \subset X-F$

$\Rightarrow F \subset f^{-1}(bintA)$

Since f is b-irresolute f^{-1} (bintA) is b-open in X.

Thus $bint(f^{-1}(A)) \supset bint(f^{-1}(bintA)) = f^{-1}(bintA)) \supset F$

$F \subset bint(f^{-1}(A)) \Rightarrow f^{-1}(A)$ is gb-open in X.

Theorem 1.3.20

If a map $f : X \longrightarrow Y$ is ap-b-continuous and b-closed, then the image of each gb-closed set in X is gb-closed in Y.

Proof

Let F be a gb-closed subset of X.

Let $f(F) \subset V$, where V is an open set of Y.

Then, $F \subset f^{-1}(V)$. Since f is ap-b-continuous, $bcl(F) \subset f^{-1}(V)$.

Then $f(bcl(F)) \subset V$

Therefore, $bcl(f(F)) \subset bcl(f(bcl(F)))$

$= f(bcl(F))$

$\subset V$

Hence f(F) is gb-closed in Y.

Theorem 1.3.21

If f : X ⟶Y is a continuous and a b-closed function, then f(A) is gb-closed in Y for every gb-closed set A of X.

Proof

Let A be gb-closed in X. Let f(A) $\subseteq$ V, where V be any open set in Y.

Since f is continuous, $f^{-1}(V)$ is open in X and A $\subset f^{-1}(V)$.

Since A is gb-closed in X, bcl(A) $\subseteq f^{-1}(V)$ and f(bcl(A)) $\subseteq$ V. Since f is b-closed, f(bcl(A)) is b-closed in Y.

Hence bcl(f(A) $\subseteq$ bcl(f(bcl(A)

= f(bcl(A)) $\subset$ V.

Hence f(A) is gb-closed in Y.

Definition 1.3.22

A map f : X ⟶ Y is said to be **contra-b-irresolute** [3] if $f^{-1}(U)$ is b-closed in X for each U $\subset$ BO(Y).

Theorem 1.3.23

Let f : X ⟶ Y and g : Y ⟶ Z be two maps such that (g∘f) : X ⟶ Z,

(1) If f is pre-closed and g is ap-b-closed then (g∘f) is ap-b-closed.

(2) If f is ap-b-closed and g is b-open and g^{-1} preserves gb-open sets then (g∘f) is ap-b-closed.

(3) If f is ap-b-continuous and g is continuous then (g∘f) is ap-b-continuous.

Proof

(1) Suppose B is an arbitrary closed subset in X and
A is a gb-open subset of Z for which (g ∘f) (B) $\subset$ A.
Then f(B) is closed in Y because f is pre-closed.
Since g is ap-b-closed, g(f(B)) $\subset$ bint(A).
Hence (g ∘f) is ap-b-closed.

(2) Suppose B is an arbitrary closed subset of X and
A is a gb-open subset of Z for which (g ∘f)(B) $\subset$ A.
Hence f(B) $\subset g^{-1}(A)$.
Since g^{-1} preserves gb-open sets, $g^{-1}(A)$ is gb-open and f is ap-b-closed.
Then f(B)) $\subset$ bint($g^{-1}(A)$).
Hence (g ∘f) (B) = g(f(B)) $\subset$ g(bint($g^{-1}(A)$)) $\subset$ bint(g($g^{-1}(A)$)) = bint(A).

Hence $(g \circ f)$ is ap-b-closed.

(3) Suppose F is an arbitrary gb-closed subset of X and U be an open set in Z for which $F \subset (g \circ f)^{-1}(U)$.

Then $g^{-1}(U)$ is open in Y because g is continuous.

Since f is ap-b-continuous, $bcl(F) \subset f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$.

Hence $(g \circ f)$ is ap-b-continuous.

Theorem 1.3.24

Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two maps such that $(g \circ f) : X \longrightarrow Z$ then,

(1) If g is b-continuous and f is contra b-irresolute then $(g \circ f)$ is contra b-continuous.

(2) If g is b-irresolute and f is contra b-irresolute, then $(g \circ f)$ is contra b-irresolute

Proof

(1) Let U be an open set in Z then, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$

Since g is b-continuous, $g^{-1}(U)$ is b-open in Y and since f is contra b-irresolute $f^{-1}(g^{1}(U))$ is b-closed in X

Therefore $(g \circ f)^{-1}(U)$ is b-closed in X.

Hence $g \circ f$ is contra b-continuous.

(2) Let U be an b-open set in Z then, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$

Since g is b-irresolute, $g^{-1}(U)$ is b-open in Y and since f is contra b-irresolute, $f^{-1}(g^{-1}(U))$ is b-closed in X

Therefore $(g \circ f)^{-1}(U)$ is b-closed in X.

Hence $g \circ f$ is contra b-irresolute.

Section 1.4

Characterization of T_{gs}-spaces

In this section the concept of T_{gs} -spaces and the characterizations of these spaces by using the concept of ap-b-continuous maps and ap-b-closed maps were discussed.

Definition 1.4.1

A space X is a **$T_{1/2}$-space** [33] if every g-closed set is closed or equivalently if every singleton is open or closed.

Definition 1.4.2

A topological space X is said to be a $\mathbf{T_{gs}}$**-space** [28] if every gs-closed subset of (X, τ) is sg-closed.

Theorem 1.4.3 [28]

A topological space X is a T_{gs}-space if and only if every gb-closed set is b-closed.

Theorem 1.4.4

Every $T_{1/2}$-space is a T_{gs}-space.

Proof

Let (X, τ) be $T_{1/2}$ space and suppose $A \subset X$ is not a b-closed set.

Let $x \in (bcl(A)\text{-}A)$,

then $\{x\} \subset bcl(A) - A$.

Since X is $T_{1/2}$-space, $\{x\}$ is a closed set.

Hence A is not a gb-closed set.

Every gb-closed set is b-closed.

Definition 1.4.5

A space (X, τ) is said to be a $\mathbf{b\text{-}T_{1/2}}$ [1] space if every each singleton is either b-open or b-closed set.

Lemma 1.4.6

Let (X, τ) be a topological space.Then the space $BO(X,\tau)$ is a b-$T_{1/2}$ space.

Example 1.4.7

A $\mathbf{T_{gs}}$ - space which is not a $\mathbf{T_{1/2}}$ – space

Let $X = \{a, b, c\}$ with $\tau = \{ \phi, X, \{a, b\} \}$

Then $bC(X) = gbC(X) = \{ \phi, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\} \}$.

Then X is a T_{gs}-space since every gb-closed set is b-closed.

But it is not a $T_{1/2}$-space.

Example 1.4.8

A b-$\mathbf{T_{1/2}}$ space which is not a $\mathbf{T_{gs}}$ – space

Let $X = \{a, b, c, d\}$ with $\tau = \{ \phi, X, \{a, b\}, \{b, c, d\} \}$ then

$bC(X) = \{ \phi, X, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\} \}$.

And $U = \{a, b, c\}$ is gb-closed since the only open set containing U is X.

But U is not b-closed.

Therefore X is not a T_{gs}-space but it is a b-$T_{1/2}$-space.

The following diagram holds and none of its implications is reversible:

$T_{1/2}$-space $\Rightarrow$ T_{gs}-space $\Rightarrow$ b-$T_{1/2}$-space

Theorem 1.4.9

For a topological space X, if every gb-closed subset of X is closed, then X is a $T_{1/2}$-space.

Proof

Let $x \in X$. If $\{x\}$ is not closed, then $A = X - \{x\}$ is not open and hence A is gb-closed, since the only open set containing A is X.

Hence A is closed since every gb-closed set is closed thus $\{x\}$ is open in X and so X is a $T_{1/2}$-space.

Definition 1.4.10

A function $f : X \longrightarrow Y$ is called **gb-continuous** [3] if $f^{-1}(V)$ is gb-closed in X for every closed set V of Y.

Definition 1.4.11

A function $f : X \longrightarrow Y$ is called **gb-irresolute** [3] if $f^{-1}(V)$ is gb-closed in X for every gb-closed set V of Y.

Theorem 1.4.12

Let $f : X \longrightarrow Y$ be a function.

(1) If f is gb-irresolute and X is a T_{gs}-space, then f is b-irresolute.

(2) If f is gb-continuous and X is a T_{gs}-space, then f is b-continuous.

Proof

(1) Let V be b-closed in Y.
Then V is gb-closed in Y and since f is gb-irresolute, $f^{-1}(V)$ is gb-closed in X.
Since X is a T_{gs}-space, $f^{-1}(V)$ is b-closed in X.
Hence f is b-irresolute.

(2) Let V be closed in Y.
Since f is gb-continuous, $f^{-1}(V)$ is gb-closed in X.
And since X is T_{gs}-space, $f^{-1}(V)$ is b-closed .
Hence f is b-continuous.

Theorem 1.4.13

If the bijective function $f : X \longrightarrow Y$ is b-irresolute and open then f is gb-irresolute.

Proof

Let V be gb-closed in Y and let $f^{-1}(V) \subset U$, where U is open in X.

Then $V \subset f(U)$, since f(U) is open and V is gb-closed in Y,

$bcl(V) \subset f(U)$ and hence $f^{-1}(bcl(V)) \subset U$.

Since f is b-irresolute and since bcl(V) is a b-closed set, $f^{-1}(bcl(V))$ is a b-closed set in X.

Hence, $bcl(f^{-1}(V)) \subset bcl(f^{-1}(bcl(V))) = f^{-1}(bcl(V)) \subset U$.

So $f^{-1}(V)$ is gb-closed and f is gb-irresolute.

Example 1.4.14

A map which is b-irresolute but not gb-irresolute

Let X = {a, b, c} with τ = { ϕ, X, {a}, {a, b} } and Y = {a, b, c} with

σ = { ϕ, Y, {a} } then bC(X) = { ϕ, X, {b}, {c}, {b, c} } and

bC(Y) = {ϕ, Y, {b}, {c}, {b, c}}.

Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be the identity map.

Then f is b-irresolute but it is not gb-irresolute,

since V = {a, b} belongs to gb-closed sets of (Y, σ) but not in gb-closed sets of (X, τ).

Example 1.4.15

A map which is gb-irresolute but not b-irresolute

Let X = {a, b, c}, with τ = { ϕ,X, {a} }, then bC(X) = { ϕ, X, {b}, {c},{b, c} }, and

gbC(X)={ ϕ, X, {b}, {c}, {b, c}, {a, c}, {a, b}}.

Let $f : (X, \tau) \longrightarrow (X, \tau)$ be defined by f(a) = f(c) = b, f(b) = c.

Then f is gb-irresolute but it is not b-irresolute, since V = {b} is b-closed in Y and $f^{-1}\{b\}$ = {a, c} is not b-closed of X.

Theorem 1.4.16

If $f : X \longrightarrow Y$ is an open, b-irresolute and b-closed surjection map and if X is a T_{gs}-space, then Y is a T_{gs}-space.

Proof

Let F be a gb-closed set of Y. Let G be open subset of X such that $f^{-1}(F) \subset G$.

Then $F \subset f(G)$ and f(G) is open.

Since F is gb-closed, then $bcl(F) \subset f(G)$ and $f^{-1}(bcl(F)) \subset G$.

Since f is b-irresolute, $f^{-1}(bcl(F))$ is b-closed and

$bcl(f^{-1}(bcl(F))) = f^{-1}(bcl(F)) \subset G$ also $bcl(f^{-1}(F)) \subset bcl(f^{-1}(bcl(F))) \subset G$.

Hence $f^{-1}(F)$ is gb-closed in X.

Since X is a T_{gs}-space, $f^{-1}(F)$ is b-closed in X.

Since f is b-closed $f(f^{-1}(F))$ is b-closed in Y.

Thus F is b-closed in Y.

Hence Y is a T_{gs}-space.

Theorem 1.4.17

Let X be a topological space. Then the following statements are equivalent:

(1) X is a T_{gs}-space.

(2) For every space Y and every map $f : X \longrightarrow Y$, f is ap-b-continuous.

Proof

(1) $\Longrightarrow$ (2)

Let F be a gb-closed subset of X and suppose that $F \subset f^{-1}(U)$, where

U is open. Since (X, τ) is $T_{gs,}$-space, F is b-closed

$\Longrightarrow bcl(F) = F \subset f^{-1}(U)$.

Then f is ap-b-continuous.

(2) $\Longrightarrow$ (1)

Let B be a gb-closed subset of X and let Y be the set X with topology $\sigma = \{ \phi, B, Y\}$.

Let $f : X \longrightarrow Y$ be the identity map.

By assumption, f is ap-b-continuous.

Since B is gb-closed in X and open in Y and $B \subset f^{-1}(B)$,

$bcl(B) \subset f^{-1}(B) = B$.

Hence B is b-closed in X and hence (X, τ) is a T_{gs}-space.

Theorem 1.4.18

Let Y be a topological space. Then the following statements are equivalent:

(1) Y is a T_{gs}-space.

(2) For every space (X, τ) and every map $f : X \longrightarrow Y$, f is ap-b-closed (or ap-b-open).

Proof

Let Y be T_{gs}-space. Every gb-closed set is b-closed in Y.

Let F be an closed subset of X.

Such that $f(F) \subset V$ where V is a gb-open subset of Y.

To prove : $f(F) \subset bint(V)$

Every gb-open set is b-open.

Therefore V is b-open.

$V = bint(V)$

Hence $f(F) \subseteq V$.

Conversely, let B be a gb-closed subset of X and B^c be a gb-open.

V is a gb-open subset of Y, B^c is an closed subset of X.

$f(B^c) \subseteq B^c$.

$B^c = f(B^c) \subseteq bint(B^c) \subseteq B^c$.

$B^c \subseteq bint(B^c)$.

B^c is b-open. Therefore B is b-closed. Hence the proof.