CHAPTER - 6

SECOND ORDER BIPOLAR FUZZY MATRIX

Section 6.1

Definition:6.1.1

Let X be a nonempty set. A **second order bipolar fuzzy matrix** (SOBPFM) $\left((\widehat{A}_{bp})_{ij}\right)_{p\times q}$ is defined as $\left((\widehat{A}_{bp})_{ij}\right)_{p\times q} = \left(\left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)_{ij}\right)_{p\times q}$ where $\left(\widehat{A}_{bp}{}^{+}\right)_{ij}(x)(\alpha) \in [0,1]$ and $\left(\widehat{A}_{bp}{}^{-}\right)_{ij}(x)(\alpha) \in [-1,0]$, $\forall i, j$, where $\alpha \in I$ & $x \in X$.

Definition:6.1.2 (Operations on second order bipolar fuzzy matrix)

Let $\left((\widehat{A}_{bp})_{ij}\right)_{p\times q} = \left(\left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)_{ij}\right)_{p\times q}$ $\left((\widehat{B}_{bp})_{ij}\right)_{p\times q} = \left(\left(\widehat{B}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-}\right)_{ij}\right)_{p\times q}$ be two second order bipolar fuzzy matrices. Define

(i) $\left((\widehat{A}_{bp})_{ij} + (\widehat{B}_{bp})_{ij}\right)_{p\times q} = \left((\widehat{C}_{bp})_{ij}\right)_{p\times q}$ where $\widehat{C}_{bp} = \left(\widehat{C}_{bp}{}^{+}, \widehat{C}_{bp}{}^{-}\right)$ and is defined as

$\widehat{C}_{bp}{}^{+}(x)(\alpha) = \max\left\{\widehat{A}_{bp}{}^{+}(x)(\alpha), \widehat{B}_{bp}{}^{+}(x)(\alpha)\right\}$ &

$\widehat{C}_{bp}{}^{-}(x)(\alpha) = \min\left\{\widehat{A}_{bp}{}^{-}(x)(\alpha), \widehat{B}_{bp}{}^{-}(x)(\alpha)\right\}$, $\forall i, j$,

for every $x \in X$ for every $\alpha \in I$.

(ii) $\left((\widehat{A}_{bp})_{ij} * (\widehat{B}_{bp})_{ij}\right)_{p\times q} = \left((\widehat{D}_{bp})_{ij}\right)_{p\times q}$ where $\widehat{D}_{bp} = \left(\widehat{D}_{bp}{}^{+}, \widehat{D}_{bp}{}^{-}\right)$ and is defined as

$\widehat{D}_{bp}{}^{+}(x)(\alpha) = \min\left\{\widehat{A}_{bp}{}^{+}(x)(\alpha), \widehat{B}_{bp}{}^{+}(x)(\alpha)\right\}$ &

$\widehat{D}_{bp}{}^{-}(x)(\alpha) = \max\left\{\widehat{A}_{bp}{}^{-}(x)(\alpha), \widehat{B}_{bp}{}^{-}(x)(\alpha)\right\}$, $\forall i, j$,

for every $x \in X$ for every $\alpha \in I$.

(iii) $\left(\left((\widehat{A}_{bp})_{ij}\right)^{c}\right)_{p\times q} = \left(\left(\left(\widehat{A}_{bp}{}^{+}\right)^{c}, \left(\widehat{A}_{bp}{}^{-}\right)^{c}\right)_{ij}\right)_{p\times q}$ is defined as

$\left(\widehat{A}_{bp}{}^{+}\right)^{c}(x)(\alpha) = 1 - \widehat{A}_{bp}{}^{+}(x)(\alpha)$, $\forall i, j$, for every $x \in X$ for every $\alpha \in I$.

$\left(\widehat{A}_{bp}{}^{-}\right)^{c}(x)(\alpha) = -1 - \widehat{A}_{bp}{}^{-}(x)(\alpha)$, $\forall i, j$, for every $x \in X$ for every $\alpha \in I$.

Example:6.1.3

Let $(\widehat{A}_{bp})_{ij}$, $(\widehat{B}_{bp})_{ij}$ be two 3×3 second order bipolar fuzzy matrices.

$$(\widehat{A}_{bp})_{ij} = \begin{bmatrix} (0.7,-0.3) & (0.5,-0.1) & (0.6,-0.4) \\ (0.2,-0.8) & (0.1,-0.5) & (0.9,-0.3) \\ (0.4,-0.7) & (0.3,-0.2) & (0.8,-0.6) \end{bmatrix}_{3\times 3}$$

$$(\widehat{B}_{bp})_{ij} = \begin{bmatrix} (0.9,-0.2) & (0.4,-0.5) & (0.7,-0.3) \\ (0.5,-0.1) & (0.6,-0.4) & (0.8,-0.6) \\ (0.1,-0.8) & (0.2,-0.7) & (0.5,-0.9) \end{bmatrix}_{3\times 3}.$$

Then compute (i) $(\widehat{A}_{bp})_{ij} * (\widehat{B}_{bp})_{ij}$

(ii) $(\widehat{A}_{bp})_{ij} + (\widehat{B}_{bp})_{ij}$

(iii) $\left((\widehat{A}_{bp})_{ij}\right)^c$.

Solution:

(i) $$(\widehat{A}_{bp})_{ij} * (\widehat{B}_{bp})_{ij} = \begin{bmatrix} (0.7,-0.2) & (0.4,-0.1) & (0.6,-0.3) \\ (0.2,-0.1) & (0.1,-0.4) & (0.8,-0.3) \\ (0.1,-0.7) & (0.2,-0.2) & (0.5,-0.6) \end{bmatrix}_{3\times 3}$$

(ii) $$(\widehat{A}_{bp})_{ij} + (\widehat{B}_{bp})_{ij} = \begin{bmatrix} (0.9,-0.3) & (0.5,-0.5) & (0.7,-0.4) \\ (0.5,-0.8) & (0.6,-0.5) & (0.9,-0.6) \\ (0.4,-0.8) & (0.3,-0.7) & (0.8,-0.9) \end{bmatrix}_{3\times 3}$$

(iii) $$\left((\widehat{A}_{bp})_{ij}\right)^c = \begin{bmatrix} (0.3,-0.7) & (0.5,-0.9) & (0.4,-0.6) \\ (0.8,-0.2) & (0.9,-0.5) & (0.1,-0.7) \\ (0.6,-0.3) & (0.7,-0.8) & (0.2,-0.4) \end{bmatrix}_{3\times 3}.$$

Definition:6.1.4

The **transpose of a second order bipolar fuzzy $\mathbf{m \times n}$ matrix** $\left((\widehat{A}_{bp})_{ij}\right) = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)_{ij}$ is defined as the $n \times m$ second order bipolar fuzzy matrix $\left((\widehat{B}_{bp})_{ij}\right) = \left(\widehat{B}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-}\right)_{ij}$ with $\left((\widehat{B}_{bp})_{ij}\right) = \left((\widehat{A}_{bp})_{ji}\right)$ for all $1 \le i \le m$ and $1 \le j \le n$. The **transpose** of $\left((\widehat{A}_{bp})_{ij}\right)$ is denoted as $\left((\widehat{A}_{bp})_{ij}\right)^T$.

Example:6.1.5

$$\left(\left(\widehat{A}_{bp}\right)_{ij}\right) = \begin{bmatrix} (0.8,-0.7) \\ (0.1,-0.6) \\ (0.5,-0.5) \end{bmatrix}$$

Then the transpose is given by

$$\left(\left(\widehat{A}_{bp}\right)_{ij}\right)^{T} = [(0.8,-0.7) \quad (0.1,-0.6) \quad (0.5,-0.5)]$$

Definition:6.1.6

A square matrix $\left(\left(\widehat{A}_{bp}\right)_{ij}\right) = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)_{ij}$ with $\left(\left(\widehat{A}_{bp}\right)_{ij}\right) = \begin{cases} (1,-1) & \text{if } i = j \\ (0,0) & \text{if } i \neq j \end{cases}$ is called as **second order bipolar fuzzy identity matrix**, denoted as $\left(\hat{I}_{bp}\right)_{n}$.

Example:6.1.7

$$\left(\hat{I}_{bp}\right)_{n} = \begin{bmatrix} (1,-1) & (0,0) & (0,0) \\ (0,0) & (1,-1) & (0,0) \\ (0,0) & (0,0) & (1,-1) \end{bmatrix}$$

Remark:6.1.8

Let $\left(\left(\widehat{A}_{bp}\right)_{ij}\right) = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)_{ij}$ be a square matrix of order n and $\hat{I}_{bp}$ be a **second order bipolar fuzzy identity matrix**. Then

$$\left(\widehat{A}_{bp}\right)_{ij} * \hat{I}_{bp} = \hat{I}_{bp} * \left(\widehat{A}_{bp}\right)_{ij} = \left(\widehat{A}_{bp}\right)_{ij}$$

Definition:6.1.9

Let $\left(\left(\widehat{A}_{bp}\right)_{ij}\right) = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)_{ij}$ be a square matrix (order n). Then the **trace of a second order bipolar fuzzy matrix** $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)$ is denoted by $tr\left(\widehat{A}_{bp}\right)_{ij}$ and is defined by

$$tr\left(\widehat{A}_{bp}\right)_{ij} = \left(\max\left(\widehat{A}_{bp}{}^{+}\right)_{ij}, \min\left(\widehat{A}_{bp}{}^{-}\right)_{ij}\right), \text{ where } i = j$$

Example:6.1.10

$\left(\widehat{A}_{bp}\right)_{ij} = \begin{bmatrix} (0.1,-0.2) & (0.7,-0.9) \\ (0.3,-0.5) & (0.5,-0.1) \end{bmatrix}$ Then the trace of SOBPFM $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)$ is given by

$$\text{tr}(\hat{A}_{bp})_{ij} = \left(\max\left({\hat{A}_{bp}}^{+}\right)_{ij}, \min\left({\hat{A}_{bp}}^{-}\right)_{ij}\right)$$

$$= (\max\{0.1, 0.5\}, \min\{-0.2, -0.1\})$$

$$= (0.5, -0.2)$$

Properties of second order bipolar fuzzy matrix:6.1.11

Let $(\hat{A}_{bp})_{ij}$, $(\hat{B}_{bp})_{ij}$ and $(\hat{C}_{bp})_{ij}$ be three second order bipolar fuzzy matrices of order $m \times n, n \times p$ and $p \times q$ respectively, then

(i) $(\hat{A}_{bp})_{ij} * \left((\hat{B}_{bp})_{ij} * (\hat{C}_{bp})_{ij}\right) = \left((\hat{A}_{bp})_{ij} * (\hat{B}_{bp})_{ij}\right) * (\hat{C}_{bp})_{ij}$ (Associativity)

(ii) $(\hat{A}_{bp})_{ij} * \left((\hat{B}_{bp})_{ij} + (\hat{C}_{bp})_{ij}\right) = (\hat{A}_{bp})_{ij} * (\hat{B}_{bp})_{ij} + (\hat{A}_{bp})_{ij} * (\hat{C}_{bp})_{ij}$
(Distributive law)

The same results hold for second order bipolar fuzzy complement matrices

(i) $({\hat{A}_{bp}}^{c})_{ij} * \left(({\hat{B}_{bp}}^{c})_{ij} * ({\hat{C}_{bp}}^{c})_{ij}\right) = \left(({\hat{A}_{bp}}^{c})_{ij} * ({\hat{B}_{bp}}^{c})_{ij}\right) * ({\hat{C}_{bp}}^{c})_{ij}$ (Associativity)

(ii) $({\hat{A}_{bp}}^{c})_{ij} * \left(({\hat{B}_{bp}}^{c})_{ij} + ({\hat{C}_{bp}}^{c})_{ij}\right) = (\hat{A}_{bp})^{c}_{ij} * ({\hat{B}_{bp}}^{c})_{ij} + ({\hat{A}_{bp}}^{c})_{ij} * ({\hat{C}_{bp}}^{c})_{ij}$
(Distributive law)

Example:6.1.12

(i) $(\hat{A}_{bp})_{ij} = \begin{bmatrix} (0.7, -0.1) & (0.8, -0.5) \\ (0.1, -0.8) & (0.3, -0.2) \end{bmatrix}$ $(\hat{B}_{bp})_{ij} = \begin{bmatrix} (0.8, -0.5) & (0.2, -0.7) \\ (0.6, -0.4) & (0.3, -0.9) \end{bmatrix}$

$$(\hat{C}_{bp})_{ij} = \begin{bmatrix} (0.1, -0.3) & (0.3, -0.8) \\ (0.4, -0.5) & (0.9, -0.7) \end{bmatrix}$$

$$(\hat{B}_{bp})_{ij} * (\hat{C}_{bp})_{ij} = \begin{bmatrix} (0.1, -0.3) & (0.2, -0.7) \\ (0.4, -0.4) & (0.3, -0.7) \end{bmatrix}$$

$$(\hat{A}_{bp})_{ij} * \left((\hat{B}_{bp})_{ij} * (\hat{C}_{bp})_{ij}\right) = \begin{bmatrix} (0.1, -0.1) & (0.2, -0.5) \\ (0.1, -0.4) & (0.3, -0.2) \end{bmatrix}$$

$$(\hat{A}_{bp})_{ij} * (\hat{B}_{bp})_{ij} = \begin{bmatrix} (0.7, -0.1) & (0.2, -0.5) \\ (0.1, -0.4) & (0.3, -0.2) \end{bmatrix}$$

$$(\hat{A}_{bp})_{ij} * \left((\hat{B}_{bp})_{ij} * (\hat{C}_{bp})_{ij}\right) = \begin{bmatrix} (0.1,-0.1) & (0.2,-0.5) \\ (0.1,-0.4) & (0.3,-0.2) \end{bmatrix}$$

Property (i) holds

(ii) $(\hat{A}_{bp})_{ij} = \begin{bmatrix} (0.7,-0.1) & (0.4,-0.2) \\ (0.3,-0.5) & (0.6,-0.9) \end{bmatrix}$ $(\hat{B}_{bp})_{ij} = \begin{bmatrix} (0.1,-0.3) & (0.5,-0.4) \\ (0.7,-0.1) & (0.8,-0.5) \end{bmatrix}$

$$(\hat{C}_{bp})_{ij} = \begin{bmatrix} (0.8,-0.7) & (0.1,-0.5) \\ (0.2,-0.9) & (0.7,-0.6) \end{bmatrix}$$

$$(\hat{B}_{bp})_{ij} + (\hat{C}_{bp})_{ij} = \begin{bmatrix} (0.8,-0.7) & (0.5,-0.5) \\ (0.7,-0.9) & (0.8,-0.6) \end{bmatrix}$$

$$(\hat{A}_{bp})_{ij} * \left((\hat{B}_{bp})_{ij} + (\hat{C}_{bp})_{ij}\right) = \begin{bmatrix} (0.7,-0.1) & (0.4,-0.2) \\ (0.3,-0.5) & (0.6,-0.6) \end{bmatrix}$$

$$(\hat{A}_{bp})_{ij} * (\hat{B}_{bp})_{ij} = \begin{bmatrix} (0.1,-0.1) & (0.4,-0.2) \\ (0.3,-0.1) & (0.6,-0.5) \end{bmatrix}$$

$$(\hat{A}_{bp})_{ij} * (\hat{C}_{bp})_{ij} = \begin{bmatrix} (0.7,-0.1) & (0.1,-0.2) \\ (0.2,-0.5) & (0.6,-0.6) \end{bmatrix}$$

$$(\hat{A}_{bp})_{ij} * (\hat{B}_{bp})_{ij} + (\hat{A}_{bp})_{ij} * (\hat{C}_{bp})_{ij} = \begin{bmatrix} (0.7,-0.1) & (0.4,-0.2) \\ (0.3,-0.5) & (0.6,-0.6) \end{bmatrix}$$

Property (ii) holds

Similarly we can prove for the complement of SOBPFM.

Theorem:6.1.13

Let $(\hat{A}_{bp})_{ij}$ and $(\hat{B}_{bp})_{ij}$ be two SOBPFMs (order n) and λ - scalar such that $0 \leq \lambda \leq 1$. Then

(i) $tr(\hat{A}_{bp} + \hat{B}_{bp})_{ij} = tr(\hat{A}_{bp})_{ij} + tr(\hat{B}_{bp})_{ij}$

(ii) $tr\left(\lambda(\hat{A}_{bp})_{ij}\right) = \lambda tr(\hat{A}_{bp})_{ij}$

(iii)$tr(\hat{A}_{bp})_{ij} = tr\left((\hat{A}_{bp})_{ij}\right)^T$

Proof :

(i) Let $\left(\widehat{A}_{bp}\right)_{ij}$ and $\left(\widehat{B}_{bp}\right)_{ij}$ be two SOBPFMs (order n)

$$tr\left(\widehat{A}_{bp}\right)_{ij} = \left(\max\left(\widehat{A}_{bp}{}^{+}\right)_{ij}, \min\left(\widehat{A}_{bp}{}^{-}\right)_{ij}\right)$$

$$tr\left(\widehat{B}_{bp}\right)_{ij} = \left(\max\left(\widehat{B}_{bp}{}^{+}\right)_{ij}, \min\left(\widehat{B}_{bp}{}^{-}\right)_{ij}\right)$$

Then $\left(\widehat{A}_{bp} + \widehat{B}_{bp}\right)_{ij} = \left(\widehat{C}_{bp}\right)_{ij}$, where $\widehat{C}_{bp} = \left(\widehat{C}_{bp}{}^{+}, \widehat{C}_{bp}{}^{-}\right)$. By the definition of trace of SOBPFM, we have

$$tr\left(\widehat{C}_{bp}\right)_{ij} = \left(\max\left\{\max\left\{\left(\widehat{A}_{bp}{}^{+}\right)_{ij}, \left(\widehat{B}_{bp}{}^{+}\right)_{ij}\right\}\right\}, \min\left\{\min\left\{\left(\widehat{A}_{bp}{}^{-}\right)_{ij}, \left(\widehat{B}_{bp}{}^{-}\right)_{ij}\right\}\right\}\right)$$

$$= \left(\max\left\{\max\left(\widehat{A}_{bp}{}^{+}\right)_{ij}, \max\left(\widehat{B}_{bp}{}^{+}\right)_{ij}\right\}, \min\left\{\min\left(\widehat{A}_{bp}{}^{-}\right)_{ij}, \min\left(\widehat{B}_{bp}{}^{-}\right)_{ij}\right\}\right)$$

$$= tr\left(\widehat{A}_{bp}\right)_{ij} + tr\left(\widehat{B}_{bp}\right)_{ij}$$

(ii) $$tr\left(\lambda\left(\widehat{A}_{bp}\right)_{ij}\right) = \left(\max\left(\lambda\left(\widehat{A}_{bp}{}^{+}\right)_{ij}\right), \min\left(\lambda\left(\widehat{A}_{bp}{}^{-}\right)_{ij}\right)\right)$$

$$= \lambda\left(\max\left(\widehat{A}_{bp}{}^{+}\right)_{ij}, \min\left(\widehat{A}_{bp}{}^{-}\right)_{ij}\right)$$

$$= \lambda tr\left(\widehat{A}_{bp}\right)_{ij}$$

(iii) Proof is obvious.

SECTION-6.2

DECISION MAKING WITH SECOND ORDER BIPOLAR FUZZY TOPSIS METHOD

Second Order Bipolar Fuzzy TOPSIS Method:6.2.1

To solve some problems in the real world, the second order bipolar fuzzy set on the score function to the TOPSIS method (SBPFS-TOPSIS) is applied.

1) $D = \{D_1, D_2, \dots, D_k\}$ is a set of evaluators, where $k = 1,2, \dots, K$.
2) $P = \{P_1, P_2, \dots, P_i\}$ is a set of assessing alternatives, where $i = 1,2, \dots, m$.
3) $Q = \{Q_1, Q_2, \dots, Q_j\}$ is a set of criteria, where $j = 1,2, \dots, n$.

The working procedure of second order bipolar fuzzy-TOPSIS method are as follows:

i. Let $W = [w_1, w_2, \dots w_n]^T$ be the weight vector $0 \leq w_j \leq 1$ and $\sum_{i=1}^{n} W_i = 1$.

ii. The value of each alternative with respect to each criterion is given in the form of BPFS and it is expressed in the decision matrix as

$$\kappa = [\kappa_{ij}]_{m\times n} = \begin{bmatrix} \kappa_{11} & \cdots & \kappa_{1n} \\ \vdots & \ddots & \vdots \\ \kappa_{m1} & \cdots & \kappa_{mn} \end{bmatrix}$$

Each entry $\kappa_{ij} = \left(\left(\widehat{A}_{bp}{}^{+}\right)_{ij}, \left(\widehat{A}_{bp}{}^{-}\right)_{ij}\right)$ where, $\left(\widehat{A}_{bp}{}^{+}(x)(\alpha)\right)_{ij} \in [0,1]$ represents the positive membership function and $\left(\widehat{A}_{bp}{}^{-}(x)(\alpha)\right)_{ij} \in [-1,0]$ represents the negative membership function respectively for $i = 1,2, \dots m; j = 1,2, \dots n$.

iii. In order to obtain better accuracy for selecting the best project proposal, second order bipolar fuzzy membership values are used to associate the expectation level e_{ij} with the score by multiplying the score with the evaluators expectation level e_{ij} for the respective criteria.

$$e_i\kappa = [\kappa_{ij}]_{m\times n} = \begin{bmatrix} e_i(\kappa_{11}) & \cdots & e_i(\kappa_{1n}) \\ \vdots & \ddots & \vdots \\ e_i(\kappa_{m1}) & \cdots & e_i(\kappa_{mn}) \end{bmatrix}$$

Each entry $e_i(\kappa_{ij}) = \left(e_i\left(\widehat{A}_{bp}{}^{+}\right)_{ij}, e_i\left(\widehat{A}_{bp}{}^{-}\right)_{ij}\right)$

iv. Weighted second order bipolar fuzzy decision matrix is computed by multiplying weights to the membership values in the aggregated second order bipolar fuzzy decision matrix as follows:

$$V = w_j\kappa = [\kappa_{ij}]_{m\times n} = \begin{bmatrix} w_1(\kappa_{11}) & \cdots & w_n(\kappa_{1n}) \\ \vdots & \ddots & \vdots \\ w_1(\kappa_{m1}) & \cdots & w_n(\kappa_{mn}) \end{bmatrix}$$

where $v_{ij} = w_i(\kappa_{ij}) = \left(w_i\left(\widehat{A}_{bp}{}^{+}\right)_{ij}, w_i\left(\widehat{A}_{bp}{}^{-}\right)_{ij}\right)$

v. The second order bipolar fuzzy relative positive ideal solution (SBPFPIS), $\dot{\widehat{A}}_{bp}{}^{p}$ and second order bipolar fuzzy relative negative ideal solution (SBPFNIS), $\dot{\widehat{A}}_{bp}{}^{n}$ for the benefit type attributes and cost type attributes are defined as follows:

SOBPFPIS for the benefit type criteria, $j = 1,2,\dots,n$

$\dot{\widehat{A}}_{bp}{}^{p} = \max\left(w_j\left(\widehat{A}_{bp}{}^{+}\right)_{ij}\right)$ and $\ddot{\widehat{A}}_{bp}{}^{p} = \min\left(w_j\left(\widehat{A}_{bp}{}^{-}\right)_{ij}\right)$

SOBPFNIS for the cost type criteria, $j = 1,2,\dots,n$

$\dot{\widehat{A}}_{bp}{}^{n} = \min\left(w_j\left(\widehat{A}_{bp}{}^{+}\right)_{ij}\right)$ and $\ddot{\widehat{A}}_{bp}{}^{n} = \max\left(w_j\left(\widehat{A}_{bp}{}^{-}\right)_{ij}\right)$

vi. The normalized Euclidean distance of each alternative $\left(w_j\left(\widehat{A}_{bp}{}^{+}\right)_j, w_j\left(\widehat{A}_{bp}{}^{-}\right)_j\right)$ from the SOBPFPIS $\left(\dot{\widehat{A}}_{bp}{}^{p}, \ddot{\widehat{A}}_{bp}{}^{p}\right)$ can be calculated as

$$d_{bp}{}^{p} = \left(\dot{d}_{bp}{}^{p}, \ddot{d}_{bp}{}^{p}\right) = \left(\sqrt{\sum_{j=1}^{n}\left(d(v_{ij}{}^{P}, \dot{v}_j{}^{P})\right)^2}, \sqrt{\sum_{j=1}^{n}\left(d(v_{ij}{}^{P}, \ddot{v}_j{}^{P})\right)^2}\right), i = 1,2,\dots m$$

and the normalized Euclidean distance of each alternative $\left(w_j\left(\widehat{A}_{bp}{}^{+}\right)_j, w_j\left(\widehat{A}_{bp}{}^{-}\right)_j\right)$ from the SOBPFNIS $\left(\dot{\widehat{A}}_{bp}{}^{n}, \ddot{\widehat{A}}_{bp}{}^{n}\right)$ can be calculated as

$$d_{bp}{}^{n} = \left(\dot{d}_{bp}{}^{n}, \ddot{d}_{bp}{}^{n}\right) = \left(\sqrt{\sum_{j=1}^{n}\left(d(v_{ij}{}^{n}, \dot{v}_j{}^{n})\right)^2}, \sqrt{\sum_{j=1}^{n}\left(d(v_{ij}{}^{n}, \ddot{v}_j{}^{n})\right)^2}\right), i = 1,2,\dots m$$

vii. A closedness coefficient Second order bipolar fuzzy set $(\mathcal{C}_{bp})_i$ of each alternative P_i, as $(\mathcal{C}_{bp})_i = \left((\mathcal{C}_{bp})_i{}^{p}, (\mathcal{C}_{bp})_i{}^{n}\right) = \left(\frac{\ddot{d}_{bp}{}^{p}}{\dot{d}_{bp}{}^{p}+\ddot{d}_{bp}{}^{p}}, -\frac{\ddot{d}_{bp}{}^{n}}{\dot{d}_{bp}{}^{n}+\ddot{d}_{bp}{}^{n}}\right)$

viii. Calculate the score function as

$$\zeta(\mathcal{C}_{bp})_i = (\mathcal{C}_{bp})_i^{\,p} + (\mathcal{C}_{bp})_i^{\,n}$$

Improve the score function as

$$\mathcal{I}(\mathcal{C}_{bp})_i = \left((\mathcal{C}_{bp})_i^{\,p}\right)^2 \zeta(\mathcal{C}_{bp})_i + \left((\mathcal{C}_{bp})_i^{\,n}\right)^2 \zeta(\mathcal{C}_{bp})_i - \left((\mathcal{C}_{bp})_i^{\,p}(\mathcal{C}_{bp})_i^{\,n}\right)\zeta(\mathcal{C}_{bp})_i$$

And the double improve score function as

$$\mathfrak{D}(\mathcal{C}_{bp})_i = \zeta(\mathcal{C}_{bp})_i + \mathcal{I}(\mathcal{C}_{bp})_i$$

After obtaining these score function values, $\boldsymbol{\mathfrak{D}(\mathcal{C}_{bp})_i}$ select the best alternative which is given by,

$$\mathfrak{B}(P_i) = \max\left\{\mathfrak{D}(\mathcal{C}_{bp})_i \ / \ i = 1,2,\dots,m\right\}$$

Application of Second Order Bipolar Fuzzy TOPSIS Method: 6.2.2

Let us consider a process of selecting best project proposal for project funding. Suppose there are six projects $\{P_i \mid i = 1,2,3,4,5,6\}$ submitted for consideration. Let $\{D_i \mid i = 1,2,3\}$ be a set of three evaluators for selecting the best project proposal. The best project proposal is chosen for funding based on appropriate criteria established by the evaluators. These criteria encompass quality Q_1, feasibility Q_2, modernity Q_3, utilization Q_4, and cost Q_5. The proposed method is applied to solve this problem and the computational procedure is summarized as follows: (This problem has been taken from the journal "Journal of Mathematical and Computational Science" (51))

Step 1: Expectation Levels e_{ij} of the evaluators with respect to each criteria are given as follows:

Table 1. Expectation Levels of the evaluators

e_{ij}	Q_1	Q_2	Q_3	Q_4	Q_5
D_1	0.8	0.6	0.4	0.9	0.9
D_2	0.7	0.6	0.7	0.8	0.8
D_3	0.8	0.7	0.5	0.8	0.7

Step 2: Score of the project proposal evaluation by each evaluators using bipolar fuzzy membership values are given in Table 2

Table 2. Bipolar fuzzy decision matrix

D_1		Q_1	Q_2	Q_3	Q_4	Q_5
	P_1	(0.8, -0.2)	(0.5, -0.5)	(0.0, -1.0)	(0.6, -0.5)	(0.4, -0.6)
	P_2	(0.5, -0.5)	(0.6, -0.5)	(0.6, -0.4)	(0.8, -0.2)	(1.0, 0.0)
	P_3	(1.0, 0.0)	(0.4, 0.6)	(0.2, 0.6)	(0.6, -0.5)	(0.4, -0.6)
	P_4	(0.5, -0.5)	(0.6, -0.4)	(0.5, -0.5)	(0.5, -0.5)	(0.6, -0.4)
	P_5	(1.0, 0.0)	(0.5, -0.5)	(0.2, -0.6)	(0.4, 0.6)	(1.0, 0.0)
	P_6	(0.5, -0.5)	(0.0, -0.4)	(0.6, -0.5)	(0.4, -0.5)	(0.6, -0.4)
D_2		Q_1	Q_2	Q_3	Q_4	Q_5
	P_1	(0.5, -0.5)	(0.6, -0.6)	(0.8, -0.2)	(1.0, -0.4)	(0.5, -0.2)
	P_2	(0.3, -0.5)	(0.2, -0.4)	(0.6, 0.0)	(0.4, -0.6)	(0.8, 0.0)
	P_3	(0.4, -0.6)	(0.5, 0.0)	(0.5, -0.5)	(0.6, -0.5)	(1.0, -0.6)
	P_4	(0.6, -0.5)	(0.5, -1.0)	(0.6, -0.5)	(1.0, -0.5)	(0.5, -0.4)
	P_5	(0.5, -0.5)	(0.4, -0.4)	(1.0, -0.2)	(0.6, -0.5)	(0.6, -0.4)
	P_6	(0.4, -0.6)	(0.5, -0.6)	(0.6, -0.5)	(0.5, -0.6)	(0.5, 0.0)
D_3		Q_1	Q_2	Q_3	Q_4	Q_5
	P_1	(0.4, -0.4)	(0.5, -0.5)	(0.5, -0.5)	(0.2, -0.5)	(0.4, -0.4)
	P_2	(0.6, -0.5)	(0.5, -0.6)	(0.6, -0.2)	(0.6, -0.4)	(0.4, -0.6)
	P_3	(0.5, -0.5)	(0.4, -0.4)	(1.0, 0.0)	(0.8, -0.6)	(1.0, -0.5)
	P_4	(0.4, -0.5)	(0.5, -0.4)	(0.6, 0.0)	(0.6, -0.5)	(0.4, -0.4)
	P_5	(0.5, -0.6)	(0.2, 0.0)	(0.4, -0.5)	(1.0, 0.0)	(0.6, -0.6)
	P_6	(0.3, -0.5)	(0.6, -0.4)	(0.6, 0.0)	(0.8, -0.4)	(1.0, -0.6)

Step 3: In order to obtain better accuracy for selecting the best project proposal, second order bipolar fuzzy membership values are calculated by multiplying the expectation levels of the evaluators with the score given in table 2.

Table 3. Second order bipolar fuzzy decision matrix

D_1		Q_1	Q_2	Q_3	Q_4	Q_5
	P_1	(0.64, -0.16)	(0.3, -0.3)	(0.0, -0.4)	(0.54, -0.45)	(0.36, -0.54)
	P_2	(0.4, -0.4)	(0.36, -0.3)	(0.24, -0.16)	(0.72, -0.18)	(0.9, 0.0)
	P_3	(0.8, 0.0)	(0.24, -0.36)	(0.08, -0.24)	(0.54, -0.45)	(0.36, -0.54)
	P_4	(0.4, -0.4)	(0.36, -0.24)	(0.2, -0.2)	(0.45, -0.45)	(0.54, -0.36)
	P_5	(0.8, 0.0)	(0.3, -0.3)	(0.08, -0.24)	(0.36, -0.54)	(0.9, 0.0)
	P_6	(0.4, -0.4)	(0.0, -0.24)	(0.24, -0.2)	(0.36, -0.45)	(0.54, -0.36)
D_2		Q_1	Q_2	Q_3	Q_4	Q_5
	P_1	(0.35, -0.35)	(0.36, -0.36)	(0.56, -0.14)	(0.8, -0.32)	(0.4, -0.16)
	P_2	(0.21, -0.35)	(0.12, -0.24)	(0.42, 0.0)	(0.32, -0.48)	(0.64, 0.0)
	P_3	(0.28, -0.42)	(0.3, 0.0)	(0.35, -0.35)	(0.48, -0.4)	(0.8, -0.48)
	P_4	(0.42, -0.35)	(0.3, -0.6)	(0.42, -0.35)	(0.8, -0.4)	(0.4, -0.32)
	P_5	(0.35, -0.35)	(0.24, -0.24)	(0.7, -0.14)	(0.48, -0.4)	(0.48, -0.32)
	P_6	(0.28, -0.42)	(0.3, -0.36)	(0.42, -0.35)	(0.4, -0.48)	(0.4, 0.0)
D_3		Q_1	Q_2	Q_3	Q_4	Q_5
	P_1	(0.32, -0.32)	(0.35, -0.35)	(0.25, -0.25)	(0.16, -0.4)	(0.28, -0.28)
	P_2	(0.48, -0.4)	(0.35, -0.42)	(0.3, -0.1)	(0.48, -0.32)	(0.28, -0.42)
	P_3	(0.4, -0.4)	(0.28, -0.28)	(0.5, 0.0)	(0.64, -0.48)	(0.7, -0.35)
	P_4	(0.32, -0.4)	(0.35, -0.28)	(0.3, 0.0)	(0.48, -0.4)	(0.28, -0.28)
	P_5	(0.4, -0.48)	(0.14, 0.0)	(0.2, -0.25)	(0.8, 0.0)	(0.42, -0.42)
	P_6	(0.24, -0.4)	(0.42, -0.28)	(0.3, 0.0)	(0.64, -0.32)	(0.7, -0.42)

Step 4: Calculate the average of second order bipolar fuzzy decision matrix

Table 4. Average of second order bipolar fuzzy decision matrix

	Q_1	Q_2	Q_3	Q_4	Q_5
P_1	(0.437, -0.277)	(0.337, -0.337)	(0.27, -0.263)	(0.5, -0.390)	(0.347, -0.327)
P_2	(0.363, -0.383)	(0.277, -0.32)	(0.32, -0.087)	(0.507, -0.327)	(0.607, -0.14)
P_3	(0.493, -0.273)	(0.273, -0.213)	(0.31, -0.197)	(0.553, -0.443)	(0.62, -0.457)
P_4	(0.38, -0.383)	(0.337, -0.373)	(0.307, -0.183)	(0.577, -0.417)	(0.407, -0.32)
P_5	(0.517, -0.277)	(0.277, -0.18)	(0.327, -0.21)	(0.547, -0.313)	(0.6, -0.247)
P_6	(0.307, -0.407)	(0.24, -0.293)	(0.32, -0.183)	(0.467, -0.417)	(0.547, -0.26)
W	0.184	0.152	0.120	0.200	0.192

Step 5: Calculate the weighted second order bipolar fuzzy decision matrix

Table 5. Weighted second order bipolar fuzzy decision matrix

	Q_1	Q_2	Q_3	Q_4	Q_5
P_1	(0.080, -0.051)	(0.051, -0.051)	(0.032, -0.032)	(0.1, -0.078)	(0.067, -0.063)
P_2	(0.067, -0.070)	(0.042, -0.049)	(0.038, -0.010)	(0.101, -0.065)	(0.117, -0.027)
P_3	(0.091, -0.050)	(0.041, -0.032)	(0.037, -0.024)	(0.111, -0.089)	(0.119, -0.088)
P_4	(0.070, -0.070)	(0.051, -0.057)	(0.037, -0.022)	(0.115, -0.089)	(0.078, -0.061)
P_5	(0.095, -0.051)	(0.035, -0.027)	(0.039, -0.025)	(0.109, -0.063)	(0.115, -0.047)
P_6	(0.056, -0.075)	(0.036, -0.045)	(0.038, -0.022)	(0.093, -0.083)	(0.105, -0.050)

Step 6: Determine the positive part and negative part of second order bipolar fuzzy set and the distance $d_{bp} = \left(d_{bp}{}^{p}, d_{bp}{}^{n}\right)$ of each alternative $\dot{\hat{A}}_{bp}{}^{p}$ and $\dot{\hat{A}}_{bp}{}^{n}$

Table 6. Positive and negative part of second order bipolar fuzzy decision matrix

	$\left(\dot{\hat{A}}_{bp}{}^{p}, \ddot{\hat{A}}_{bp}{}^{p}\right)$	$\left(\dot{\hat{A}}_{bp}{}^{n}, \ddot{\hat{A}}_{bp}{}^{n}\right)$
P_1	(0.1, -0.078)	(0.032, -0.032)
P_2	(0.117, -0.070)	(0.038, -0.010)
P_3	(0.119, -0.089)	(0.037, -0.024)
P_4	(0.115, -0.089)	(0.037, -0.022)
P_5	(0.115, -0.063)	(0.035, -0.025)
P_6	(0.105, -0.083)	(0.036, -0.022)

Table 7. Distance of each alternative

	$\dot{d}_{bp}{}^{p}$	$\ddot{d}_{bp}{}^{p}$	$\dot{d}_{bp}{}^{n}$	$\ddot{d}_{bp}{}^{n}$
P_1	0.092	0.092	0.062	0.062
P_2	0.121	0.105	0.077	0.092
P_3	0.117	0.123	0.095	0.095
P_4	0.117	0.095	0.082	0.098
P_5	0.112	0.124	0.056	0.051
P_6	0.109	0.128	0.079	0.089

Step 7: Calculate the score function

Table 8. Score function of each alternative

	$(\mathcal{C}_{bp})_i^{p}$	$(\mathcal{C}_{bp})_i^{n}$	$\zeta(\mathcal{C}_{bp})_i$	$\mathcal{I}(\mathcal{C}_{bp})_i$	$\mathfrak{D}(\mathcal{C}_{bp})_i$	$\mathfrak{B}(P_i)$
P_1	0.5	-0.5	0	0	0	4
P_2	0.465	-0.544	-0.079	-0.06	-0.139	5
P_3	0.513	-0.5	0.013	0.009	0.022	2
P_4	0.448	-0.544	-0.096	-0.07	-0.166	6
P_5	0.525	-0.477	0.048	0.036	0.084	1
P_6	0.540	-0.530	0.01	0.009	0.019	3

After obtaining the score function, $\mathfrak{D}(\mathcal{C}_{bp})_i$ the best alternative is given by, $\mathfrak{B}(P_i) = \max\left\{\mathfrak{D}(\mathcal{C}_{bp})_i, i = 1,2,\dots,5\right\}$ where $\mathfrak{B}(P_i)$ is given in table 8. Therefore, the best project proposal among the six project proposals submitted is P_5.