APPENDIX – 1

The following are the identities and the results used in the analytical solutions. Let m and N be non negative integers.

A. Identities

1. $$\sum_{n=2}^{\infty} z^n \left(\sum_{k=1}^{n-1} P_{n-k}^*(\theta)\, g_k \right) = \left(\sum_{k=1}^{\infty} g_k\, z^k \right) \left(\sum_{n=1}^{\infty} P_n^*(\theta)\, z^n \right) = X(z)\; P^*(z,\theta)$$

2. $$\sum_{n=m+1}^{\infty} z^n \left(\sum_{k=1}^{n-m} SE_{n-k}^*(\theta)\, g_k \right) = \left(\sum_{k=1}^{\infty} g_k\, z^k \right) \left(\sum_{k=m}^{\infty} SE_n^*(\theta)\, z^n \right) = X(z)\, SE^*(z,\theta)$$

3. $$\sum_{n=1}^{\infty} z^n \left(\sum_{k=1}^{n} Q_{n-k}^*(\theta)\, g_k \right) = \left(\sum_{k=1}^{\infty} g_k\, z^k \right) \left(\sum_{n=0}^{\infty} Q_n^*(\theta)\, z^n \right) = X(z)\, Q^*(z,\theta)$$

m = 1 and 0 in (2) give the results in (1) and (3) respectively.

4. $$\sum_{n=1}^{m-1} z^n \left(\sum_{k=1}^{n} PI_{n-k}\, g_k \right) + \sum_{n=m}^{\infty} z^n \left(\sum_{k=n-m+1}^{\infty} PI_{n-k}\, g_k \right) = \left(\sum_{k=1}^{\infty} g_k\, z^k \right) \left(\sum_{n=0}^{m-1} PI_n\, z^n \right) = X(z) PI(z)$$

5. $$\left(\sum_{k=1}^{\infty} g_k\, z^k \right) \left(\sum_{n=m}^{N-1} U_n\, z^n \right) = \sum_{n=m+1}^{N-1} z^n \left(\sum_{k=1}^{n-m} U_{n-k}\, g_k \right) + \sum_{n=N}^{\infty} z^n \left(\sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k \right)$$

6. $$\left(\sum_{k=0}^{\infty} h_k\, z^k \right) \left(\sum_{n=m}^{\infty} \xi_n\, z^n \right) = \sum_{n=m}^{\infty} z^n \left(\sum_{i=m}^{n} \xi_i\, h_{n-i} \right)$$

7. $$\left(\sum_{n=0}^{\infty} \alpha_n\, z^n \right) \left(\sum_{n=0}^{m-1} \delta_n\, z^n \right) = \sum_{n=0}^{m-1} z^n \left(\sum_{i=0}^{n} \alpha_i\, \delta_{n-i} \right) + \sum_{n=m}^{\infty} z^n \left(\sum_{i=0}^{m-1} \delta_i\, \alpha_{n-i} \right)$$

8. $$\sum_{r=m}^{n} \delta_r \left(\sum_{k=0}^{n-r} h_k\, \pi_{n-r-k} \right) = \sum_{k=m}^{n} \pi_{n-k} \left(\sum_{i=m}^{k} \delta_i\, h_{k-i} \right)$$

9. $$\sum_{k=1}^{n} g_k \left(\sum_{i=0}^{n-k} \alpha_i\, \pi_{n-k-i} \right) = \sum_{k=0}^{n-1} \alpha_k \left(\sum_{i=1}^{n-k} g_i\, \pi_{n-k-i} \right)$$

(8) and (9) give the re-arrangement of the summation.

B. Results using L′ Hospital Rule

If f(1) = g(1) = 0, then $\frac{d}{dz}\left(\frac{f(z)}{g(z)} \right)_{z=1} = \frac{g'(1)\, f''(1) - f'(1)\, g''(1)}{2\,(g'(1))^2}$, where the dashes represent the derivatives of the functions.

Let S, D, R, H be any random variables with LST $S^*(\theta)$, $D^*(\theta)$, $R^*(\theta)$ and $H^*(\theta)$ respectively and let $w_X(z) = \lambda\,(1 - X(z))$, $g_a(w_X(z)) = a + w_X(z)$ and $h_a(w_X(z)) = w_X(z) + a\,(1 - R^*(w_X(z)))$, then

$$\lim_{z\to 1} \frac{1-S^*(g_a(w_X(z)))}{g_a(w_X(z))} = \frac{1-S^*(a)}{a}$$

$$\frac{d}{dz}\left(\frac{1-S^*(g_a(w_X(z)))}{g_a(w_X(z))}\right)_{z=1} = \lambda\,E(X)\,\frac{S^{*'}(a)}{a} + \frac{1-S^*(a)}{a^2}$$

$$\lim_{z\to 1} \frac{-w_X(z)}{D(z)} = \frac{\lambda\,E(X)}{D'(1)}$$

$$\frac{d}{dz}\left(\frac{-w_X(z)}{D(z)}\right) = \frac{\lambda\,E(X\,(X-1))\,D'(1) + \lambda\,E(X)\,(-D''(1))}{2(D'(1))^2}$$

(for any denominator D(z) with D(1) = 0).

$$\lim_{z\to 1}\left(\frac{1-S^*(h_a\,(w_X(z)))}{h_a(w_X(z))}\right) = E(S)\,(1 + a\,E(R))$$

$$\frac{d}{dz}\left(\frac{1-S^*\,(h_a\,(w_X(z)))}{h_a\,(w_X(z))}\right)_{z=1} = \frac{\lambda\,E(X)E(S^2)}{2}(1 + a\,E(R))$$

$$\lim_{z\to 1}\left(\frac{-w_X(z)}{z - H^*(w_X(z))}\right) = \frac{\lambda\,E(X)}{1-\rho_H}$$

$$\frac{d}{dz}\left(\frac{-w_X(z)}{z - H^*(w_X(z))}\right)_{z=1} = \frac{\lambda\,E(X(X-1)) + (\lambda\,E(X))^3\,E(H^2)}{2\,(1-\rho_H)^2}$$

where $\rho_H = \left(\frac{d}{dz}H^*(w_X(z))\right)_{z=1} = \lambda\,E(X)\,E(H)$

$$\lim_{z\to 1}\left(\frac{1-R^*(w_X(z))D^*(w_X(z))}{w_X(z)}\right)_{z=1} = E(D) + E(R)$$

$$\frac{d}{dz}\left(\frac{1-R^*(w_X(z))D^*(w_X(z))}{w_X(z)}\right)_{z=1} = \frac{\lambda\,E(X)}{2}(E(D^2) + E(R^2) + 2\,E(R)\,E(D))$$

$$\lim_{z\to 1} \frac{z-1}{z-H^*\,(w_X(z))} = \frac{1}{1-\rho_H}$$

$$\frac{d}{dz}\left(\frac{z-1}{z-H^*\,(w_X(z))}\right) = \frac{\lambda\,E(X\,(X-1))\,E(H) + (\lambda\,E(X))^2\,E(H^2)}{2\,(1-\rho_H)^2}$$

APPENDIX – 2

VARIOUS DISTRIBUTIONS USED IN NUMERICAL ANALYSIS

Discrete Distributions :

Name	Distribution	Probability mass function
Geometric distribution (X ~ Geo(p))	$(1 - p)\, p^{k-1}, k \geq 1$	$\frac{(1-p)\,z}{1-pz}$
Binomial (N, p)	$NC_k\, p^k\, q^{N-k} \quad k = 0, 1, \ldots N$	pz + q

Continuous Distributions :

Name	Density function (f(x))	LST
k-stage hyper exponential distribution	$\sum_{i=1}^{k} a_i\, \mu_i\, e^{-\mu_i x}, 0 \leq x \leq 1$ $\sum_{i=1}^{\infty} a_i = 1$	$\sum_{i=1}^{k} \frac{a_i\, \mu_i}{\mu_i + \theta}$
when k = 1	$\mu\, e^{-\mu x}$ (exponential distribution)	$\frac{\mu}{\mu + \theta}$
Uniform distribution	1, $0 \leq x \leq 1$ 0, otherwise	$\frac{e^{-as} - e^{-bs}}{b - s}$
Gamma distribution G(k, λ)	$\frac{\lambda^k\, x^{k-1}\, e^{-\lambda x}}{\Gamma k}, x > 0$ 0, $x < 0$	$\left(\frac{\lambda}{\lambda + \theta}\right)^k$
Erlang-k-distribution E_k	$\frac{(\lambda k)^k\, x^{k-1}\, e^{-\lambda kx}}{\Gamma k}, x > 0$	$\left(\frac{k\lambda}{k\lambda + \theta}\right)^k$

APPENDIX – 3

Applications – Examples

Example – 1

The following quality control problem (constructed by Kella (1989)) satisfies the criteria of the (m, N) policy of the queueing systems analysed in chapters II to V :

A manufacturing plant, produces certain items, that are occasionally defective. The good items produced are marketed, while the defective ones are kept in storage until they can be reworked to meet specifications. One of the machines in the plant may be converted as needed (at some cost), from production mode to a repair mode in order to perform the rework. Instead of starting the rework process as soon as the defective items are produced, an appropriate cut off number N is suggested to start the rework. Since setup time may be required before starting the process, the special machine is released from production mode when the number of items accrued to at least **m** ($m \leq N$)so that during the setup operation more defective items may be accumulated and the number may raise to N at the end of the setup period. Even if N defective items are not gathered at the end of the setup work, the special machine continue to stay in the repair mode till the number of items raised to at least N. Because of the cost involved in switching modes, after conversion to repair mode, the machine will rework all the defective items (including the new arrivals) exhaustively, and then switch back to the production mode.

In this example, the defective items may be interpreted as the arriving units and the special machine as the server. The server is said to be on a series of vacations, when it is in the production mode .Thus the vacation time is the time required to produce the products. At the end of each vacation, the management takes decision whether to keep the machine in productive mode or change it to the repair mode ((i.e.) to take another vacation).The defective items are reworked one by one. Locating the kinds of defect may be considered as first phase service. The second phase consists of multi-optional facilities and the items may be treated accordingly.

Some unpredictable interruptions may occur during the re-work period and the special machine may itself undergo repair process and the rework may be resumed or repeated when the machine is fixed.

At the end of each rework the machine may need time to rework the next item. This can be referred as vacation between services.

If the rework is not properly done, it may be repeated again and this may correspond to feedback service.

Example – 2

Operational states of a Heterogeneous Sensor Network (HSN) node particularly in Vehicle Tracking System can be considered as an example for the model of chapter II. HSN is used in Wireless Sensor Networks (WSN) to improve sensor network performance in terms of energy consumption. A HSN model consists of two physically different types of sensor nodes. A small number of powerful high-end sensors (H-sensors) and a large number of low-end sensors (L-sensors) uniformly distributed in the field. After deployment, clusters are formed and H-sensor in each cluster serves as cluster head (CH). In some of the existing Vehicle Tracking Systems, data packets are directly sent from L-sensors to the Base station (BS). So the Base station should always be listening to the arrival of data, which necessitates investment on high end servers. Instead, if a cluster head (CH) collects data from multiple L-sensors and sends data to the BS using (m, N)-policy as proposed by our model, the contention in the data cables can be reduced and the sensor nodes may be efficiently used. In the model proposed, the data packets arrive in batches to form a queue and are processed one by one. (Batch arrival and single service).

The two major operational states of a sensor node (L-sensor and H-sensor node) are sleep state and active state. In sleep state, a node cannot interact with the external world. In active state, a sensor node may be in idle mode or it may generate data or transmit and /or receive data packets. An **H-**sensor node in a cluster during its period of active time, usually remains in IDLE state. When **m** number of data packets accumulate, the sensors are kept in the ready state to operate (setup) and switches to busy state when the node's buffer is filled at least with threshold number of packets (N). During busy state, the sensor node will be processing the data and it can be considered as First Essential State (FES). The data can be transmitted as received or can be manipulated for further actions. The different types of manipulation can be considered as Second Optional Services (SOS). The node switches back from BUSY state to vacation state when there are no packets in the buffer. In vacation state the sensor node can be used for some other purposes like comparing the data with the previous data so that if there is no variation in the data stored, that data need not be saved.

After comparing one set of data, if the node finds **m** new data packets in the queue then the node will switch to operate state (set up operations). The nodes thus repeat the process of comparing the data with the previous data until **m** number of new packets arrive or the number of consecutive vacations taken by the node reaches a fixed number J, whichever occurs earlier. Unpredictable breakdowns may occur while different H-sensors send data to BS at the same time and stop the transmissions. The remaining data can be sent to the BS as soon as the node is fixed.