CHAPTER – 5

CUBIC SUBALGEBRAS AND FILTERS OF CI-ALGEBRAS

SECTION 5.1

CUBIC SUBALGEBRAS OF CI-ALGEBRAS

Definition : 5.1.1

Let I be a closed unit interval [0, 1]. By an interval number, mean a closed subinterval $\overline{a} = [a^-, a^+]$ of I, where $0 \le a^- \le a^+ \le 1$.

Note

The set of all interval numbers are denoted by D[0, 1].

Definition : 5.1.2

Consider two interval numbers $\overline{a}_1 = [a_1^-, a_1^+]$ and $\overline{a}_2 = [a_2^-, a_2^+]$ in D[0, 1]. Then

(i) refined minimum of $\overline{a}_1$ and $\overline{a}_2$ is,

$r\min\{\overline{a}_1, \overline{a}_2\} = [\min\{a_1^-, a_2^-\}, \min\{a_1^+, a_2^+\}]$.

(ii) $\overline{a}_1 \succeq \overline{a}_2$ if and only if $a_1^- \ge a_2^-$ and $a_1^+ \ge a_2^+$.

(iii) $\overline{a}_1 \preceq \overline{a}_2$ if and only if $a_1^- \le a_2^-$ and $a_1^+ \le a_2^+$.

(iv) $\overline{a}_1 \succ \overline{a}_2$, mean $\overline{a}_1 \succeq \overline{a}_2$ $a_1^- \succeq a_2^-$ and $\overline{a}_1 \ne \overline{a}_2$.

(v) $\overline{a}_1 \prec \overline{a}_2$, mean $\overline{a}_1 \preceq \overline{a}_2$ and $\overline{a}_1 \ne \overline{a}_2$.

(vi) Let $\overline{a}_i \in$ D[0, 1] where $i \in \Lambda$. Define

$$r\inf_{i\in\Lambda} \overline{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \text{ and}$$

$$r\sup_{i\in\Lambda} \overline{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right]$$

Definition : 5.1.3

Let X be a CI-algebra. An **interval-valued fuzzy set (IVF set) $\tilde{\mu}_A$** defined on X is given by $\tilde{\mu}_A = \{(x, [\mu_A^-(x), \mu_A^+(x)]) / x \in X\}$ which is denoted by $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ where μ_A^- and μ_A^+ are two fuzzy sets in X such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in X$.

For any IVF set $\tilde{\mu}_A$ on X and $x \in X$, $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ is called the **degree of membership** of an element x to $\tilde{\mu}_A$, in which $\mu_A^-(x)$ and $\mu_A^+(x)$ are referred to as the lower and upper degrees, respectively of membership of x to $\tilde{\mu}_A$.

Definition : 5.1.4

Let X be a CI-algebra. A **cubic set $\mathscr{A}$ in X is a structure** $\mathscr{A} = \{(x, \tilde{\mu}_A(x), \lambda(x)) / x \in X\}$ which is denoted by $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an IVF set in X and λ is a fuzzy set in X.

Note

The family of cubic sets in a set X is denoted by $\mathcal{C}(X)$.

Definition : 5.1.5

Let $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ be a cubic set in a CI-algebra X, $r \in [0, 1]$ and $[s, t] \in D[0, 1]$. The set $\mathcal{C}(\mathscr{A}; [s, t], r) = \{\tilde{\mu}_A(x) \succeq [s, t], \lambda(x) \leq r / x \in X\}$ is called the **cubic level set of $\mathscr{A} = (\tilde{\mu}_A, \lambda)$**.

Definition : 5.1.6

Let X be a CI-algebra. A cubit set $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ is called a **cubic subalgebra of X** if it satisfies :

$\tilde{\mu}_A(x * y) \succeq r\min\{\tilde{\mu}_A(x),\ \tilde{\mu}_A(y)\},\ \forall\ x, y \in X$

$\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\},\ \forall\ x, y \in X.$

Example : 5.1.7

Let X = {1, a, b, c} be a CI-algebra as in example (4.1.18). Define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by $\tilde{\mu}_A = \begin{pmatrix} 1 & a & b & c \\ [0.6, 0.9] & [0.4, 0.8] & [0.3, 0.7] & [0.1, 0.3] \end{pmatrix}$ and

$\lambda = \begin{pmatrix} 1 & a & b & c \\ 0.2 & 0.2 & 0.6 & 0.7 \end{pmatrix}$ respectively. Then $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of X.

Proposition : 5.1.8

If $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of a CI-algebra X, then $\tilde{\mu}_A(1) \succeq \tilde{\mu}_A(x)$ and $\lambda(1) \leq \lambda(x)$ for all $x \in X$.

Proof

It is straight forward.

Theorem : 5.1.9

Let X be a CI-algebra. For a cubic set $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$, the following are equivalent :

(i) $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of X.

(ii) The nonempty cubic level set of $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a subalgebra of X.

Proof

Assume that $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of a CI-algebra X. Then

$\tilde{\mu}_A(x * y) \succeq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ (1)

$\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\}$ (2)

Let $x, y \in \mathcal{C}(\mathcal{A}; [s, t], r)$ for all $r \in [0, 1]$ and $[s, t] \in D[0, 1]$.

Then $\tilde{\mu}_A(x) \succeq [s, t]$, $\lambda(x) \leq r$, $\tilde{\mu}_A(y) \succeq [s, t]$ and $\lambda(y) \leq r$.

Then $\tilde{\mu}_A(x * y) \succeq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \succeq [s, t]$ and

$\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\} \leq r$

$\Rightarrow x * y \in \mathcal{C}(\mathcal{A}; [s, t], r)$.

Therefore the non empty cubic level set of $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a subalgebra of X.

Conversely, assume that $\mathcal{C}(\mathcal{A}; [s, t], r)$ is a subalgebra of X for all $r \in [0, 1]$ and $[s, t] \in D[0, 1]$ with $\mathcal{C}(\mathcal{A}; [s, t], r) \neq \Phi$.

Case (i)

Suppose that (1) is not true and (2) is valid. Then there exist $[s_0, t_0] \in D[0, 1]$ and $a, b \in X$ such that

$\tilde{\mu}_A(a * b) \prec [s_0, t_0] \preceq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ and $\lambda(a * b) \leq \max\{\lambda(a), \lambda(b)\}$.

It follows that $a, b \in \mathcal{C}(\mathcal{A}; [s_0, t_0], \max\{\lambda(a), \lambda(b)\})$ but

$a * b \notin \mathcal{C}(\mathcal{A}; [s_0, t_0], \max\{\lambda(a), \lambda(b)\})$. This is a contradiction.

Case (ii)

If (1) is true and (2) is not valid. Then $\tilde{\mu}_A(a * b) \succeq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ and $\lambda(a * b) > r_0 \geq \max\{\lambda(a), \lambda(b)\}$ for some $r_0 \in [0, 1]$ and $a, b \in X$.

Thus $a, b \in \mathcal{C}(\mathcal{A}; r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, r_0)$ but

$a * b \notin \mathcal{C}(\mathcal{A}; r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, r_0)$ which is a contradiction.

Case (iii)

Assume that there exist $[s_0, t_0] \in D[0, 1]$, $r_0 \in [0, 1]$ and a, b $\in$ X such that

$\tilde{\mu}_A(a * b) \prec [s_0, t_0] \preceq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ and

$\lambda(a * b) > r_0 \geq \max\{\lambda(a), \lambda(b)\}$

Then a, b $\in \mathcal{C}(\mathscr{A}; [s_0, t_0], r_0)$ but $a * b \notin \mathcal{C}(\mathscr{A}; [s_0, t_0], r_0)$

This is also a contradiction. Hence (1) and (2) are valid.

Therefore $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of X.

Theorem : 5.1.10

If $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of a CI-algebra X, then the set $\mathscr{S} = \{x \in X / \tilde{\mu}_A(x) = \tilde{\mu}_A(1), \lambda(x) = \lambda(1)\}$ is a subalgebra of X.

Proof

Let $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of a CI-algebra X.

Then $\tilde{\mu}_A(x * y) \succeq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ (1)

and $\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\}$ (2)

Let x, y $\in \mathscr{S}$

Then $\tilde{\mu}_A(x) = \tilde{\mu}_A(1) = \tilde{\mu}_A(y)$ and $\lambda(x) = \lambda(1) = \lambda(y)$.

It follows from (1) and (2) that

$\tilde{\mu}_A(x * y) \succeq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(1)$

and $\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\} = \lambda(1)$ so from proposition (5.1.8), that $\tilde{\mu}_A(x * y) = \tilde{\mu}_A(1)$ and $\lambda(x * y) = \lambda(1)$. Hence $x * y \in S$, and so $\mathscr{S}$ is a subalgebra of X.

Theorem : 5.1.11

For a subset S of a CI-algebra X, let $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ be defined by

$$\tilde{\mu}_A(x) = \begin{cases} [s, t] & \text{if } x \in S, \\ \bar{0} = [0, 0] & \text{otherwise} \end{cases}$$

and $\lambda(x) = \begin{cases} 0 & \text{if } x \in S \\ r & \text{otherwise} \end{cases}$ where $r, s, t \in (0, 1]$ with $s < t$. Then

(i) If S is a subalgebra of X, then $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of X and $\mathcal{C}(\mathscr{A}; [s, t], r) = S$.

(ii) If $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of X, then S is a subalgebra of X.

Proof

(i) Assume that S is a subalgebra of a CI-algebra X.

Claim : $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of X. Let $x, y \in X$.

Case (i)

If $x, y \in S$ then $x * y \in S$

$\tilde{\mu}_A(x * y) = [s, t] = r\min\{[s, t], [s, t]\}$

$= r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

and $\lambda(x * y) = 0 = \max\{0, 0\} = \max\{\lambda(x), \lambda(x)\}$

Case (ii)

If $x, y \notin S$, then $\tilde{\mu}_A(x) = \bar{0} = [0, 0] = \tilde{\mu}_A(y)$ and $\lambda(x) = r = \lambda(y)$.

Hence $\tilde{\mu}_A(x * y) \geq \bar{0} = [0, 0] = r\min\{\bar{0}, \bar{0}\} = r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\lambda(x * y) \leq r = \max\{r, r\} = \max\{\lambda(x), \lambda(y)\}$.

Case (iii)

If $x \in S$ and $y \notin S$, then $\tilde{\mu}_A(x) = [s, t]$, $\tilde{\mu}_A(y) = \bar{0}$, $\lambda(x) = 0$ and $\lambda(y) = r$. It follows that,

$\tilde{\mu}_A(x * y) \geq \bar{0}$ = r min {[s, t], $\bar{0}$} = r min $\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and

$\lambda(x * y) \leq r$ = max {0, r} = max $\{\lambda(x), \lambda(y)\}$.

Case (iv)

Similarly for the case $x \notin S$ and $y \in S$, we have

$\tilde{\mu}_A(x * y) \geq$ r min $\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and

$\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\}$

Therefore $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of X.

(ii) Suppose that $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic subalgebra of X. Let $x, y \in S$.

Then $\tilde{\mu}_A(x)$ = [s, t] = $\tilde{\mu}_A(y)$ and $\lambda(x) = 0 = \lambda(y)$, and so

$\tilde{\mu}_A(x * y) \geq$ r min $\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ = r min {[s, t], [s, t]} = [s, t] and

$\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\} = 0$

Thus $x * y \in S$ and therefore S is a subalgebra of X.

Definition : 5.1.12

Let X and Y be CI-algebras. A mapping $f : X \rightarrow Y$ induces two mappings $\mathcal{C}_f : \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$, $\mathscr{A} \rightarrow \mathcal{C}_f(\mathscr{A})$, and $\mathcal{C}_f^{-1} : \mathcal{C}(Y) \rightarrow \mathcal{C}(X)$, $\mathscr{B} \rightarrow \mathcal{C}_f^{-1}(\mathscr{B})$, where $\mathcal{C}_f(\mathscr{A})$ is given by

$$\mathcal{C}_f(\tilde{\mu}_A)(y) = \begin{cases} r\sup\limits_{y=f(x)} \tilde{\mu}_A(x) & \text{if } f^{-1}(y) \neq \Phi \\ \bar{0} = [0, 0] & \text{otherwise} \end{cases}$$

$$\mathcal{C}_f(\lambda)(y) = \begin{cases} \inf\limits_{y=f(x)} \lambda(x) & \text{if } f^{-1}(y) \neq \Phi \\ 1 & \text{otherwise} \end{cases}$$

for all $y \in Y$, and $\mathcal{C}_f^{-1}(\mathscr{B})$ is defined by $\mathcal{C}_f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x))$ and

$\mathcal{C}_f^{-1}(\kappa)(x) = \kappa(f(x))$ for all $x \in X$.

Then the mapping $\mathcal{C}_f$ (respectively $\mathcal{C}_f^{-1}$) is called a **cubic transformation** (respectively **inverse cubic transformation**) induced by f.

Definition : 5.1.13

A cubic set $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ in a CI-algebra X has the cubic property if for any subset T of X there exists $x_0 \in T$ such that $\tilde{\mu}_A(x_0) = \underset{x \in T}{\text{r sup}}\, \tilde{\mu}_A(x)$ and $\lambda(x_0) = \inf_{x \in T} \lambda(x)$.

Theorem : 5.1.14

Let X be a CI-algebra. For a homomorphism $f : X \to Y$ of CI-algebras, let $\mathcal{C}_f : \mathcal{C}(X) \to \mathcal{C}(Y)$ and $\mathcal{C}_f^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ be the cubic transformation and inverse cubic transformation, respectively, induced by f.

(i) If $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ is a cubic subalgebra of X which has the cubic property, then $\mathcal{C}_f(\mathscr{A})$ is a cubic subalgebra of Y.

(ii) If $\mathscr{B} = (\tilde{\mu}_B, \kappa) \in \mathcal{C}(Y)$ is a cubic subalgebra of Y, then $\mathcal{C}_f^{-1}(\mathscr{B})$ is a cubic subalgebra of X.

Proof

For a homomorphism $f : X \to Y$ of CI-algebras, let $\mathcal{C}_f : \mathcal{C}(X) \to \mathcal{C}(Y)$ and $\mathcal{C}_f^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ be the cubic transformation and inverse cubic transformation, respectively, induced by f.

To Prove (i)

Let $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ be a cubic subalgebra of X which has the cubic property.

Given $f(x), f(y) \in f(X)$, let $x_0 \in f^{-1}(f(x))$ and $y_0 \in f^{-1}(f(y))$ be such that

$\tilde{\mu}_A(x_0) = \underset{a \in f^{-1}(f(x))}{\text{r sup}}\, \tilde{\mu}_A(a)$, $\lambda(x_0) = \inf_{a \in f^{-1}(f(x))} \lambda(a)$ and

$\tilde{\mu}_A(y_0) = \underset{b \in f^{-1}(f(y))}{\text{r sup}}\, \tilde{\mu}_A(b)$, $\lambda(y_0) = \inf_{b \in f^{-1}(f(y))} \lambda(b)$ respectively.

$$\text{Then } \mathcal{C}_f(\widetilde{\mu}_A)(f(x) * f(y)) = \underset{z \in f^{-1}(f(x) * f(y))}{\text{r sup}} \widetilde{\mu}_A(z)$$

$$\succeq \widetilde{\mu}_A(x_0 * y_0) \succeq \text{r min}\{\widetilde{\mu}_A(x_0), \widetilde{\mu}_A(y_0)\}$$

$$= \text{r min}\{\underset{a \in f^{-1}(f(x))}{\text{r sup}} \widetilde{\mu}_A(a), \underset{b \in f^{-1}(f(y))}{\text{r sup}} \widetilde{\mu}_A(b)\}$$

$$= \text{r min}\{\mathcal{C}_f(\widetilde{\mu}_A)(f(x)), \mathcal{C}_f(\widetilde{\mu}_A)(f(y))\},$$

$$\mathcal{C}_f(\lambda)(f(x) * f(y)) = \inf_{z \in f^{-1}(f(x) * f(y))} \lambda(z)$$

$$\le \lambda(x_0 * y_0) \le \max\{\lambda(x_0), \lambda(y_0)\}$$

$$= \max\{\inf_{a \in f^{-1}(f(x))} \lambda(a), \inf_{b \in f^{-1}(f(y))} \lambda(b)\}$$

$$= \max\{\mathcal{C}_f(f(x)), \mathcal{C}_f(f(y))\},$$

Therefore $\mathcal{C}_f(\mathscr{A})$ is a cubic subalgebra of Y.

To Prove (ii)

Let $\mathscr{B} = (\widetilde{\mu}_B, \kappa) \in \mathcal{C}(Y)$ be a cubic subalgebra of Y. For any x, y $\in$ X we have

$$\mathcal{C}_f^{-1}(\widetilde{\mu}_B)(x * y) = \widetilde{\mu}_B(f(x * y) = \widetilde{\mu}_B(f(x) * f(y))$$

$$\succeq \text{r min}\{\widetilde{\mu}_B(f(x)), \widetilde{\mu}_B(f(y))\}$$

$$= \text{r min}\{\mathcal{C}_f^{-1}(\widetilde{\mu}_B)(x), \mathcal{C}_f^{-1}(\widetilde{\mu}_B)(y)\},$$

$$\mathcal{C}_f^{-1}(\kappa)(x * y) = \kappa(f(x * y)) = \kappa(f(x) * f(y))$$

$$\le \max\{\kappa(f(x)), \kappa(f(y))\}$$

$$= \max\{\mathcal{C}_f^{-1}(\kappa)(x), \mathcal{C}_f^{-1}(\kappa)(y)\}$$

Hence $\mathcal{C}_f^{-1}(\mathscr{B})$ is a cubic subalgebra of X.

SECTION 5.2

CUBIC FILTERS OF CI-ALGEBRAS

Definition : 5.2.1

Let X be a CI-algebra. A cubic set $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ is called a **cubic filter of X** if it satisfies : For all x, y $\in$ X.

(CF 1) $\tilde{\mu}_A(1) \succeq \tilde{\mu}_A(x)$, $\lambda(1) \leq \lambda(x)$, for all x, y $\in$ X

(CF 2) $\tilde{\mu}_A(y) \succeq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(x * y)\}$, for all x, y $\in$ X

(CF 3) $\lambda(y) \leq \max\{\lambda(x), \lambda(x * y)\}$, for all x, y $\in$ X

Example : 5.2.2

Consider a CI-algebra X = {1, a, b, c} as in example (1.2.15). Define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by

$$\tilde{\mu}_A = \begin{pmatrix} 1 & a & b & c \\ [0.5, 0.8] & [0.4, 0.7] & [0.4, 0.7] & [0.1, 0.3] \end{pmatrix} \text{ and}$$

$$\lambda = \begin{pmatrix} 1 & a & b & c \\ 0.2 & 0.2 & 0.2 & 0.6 \end{pmatrix}$$ respectively. Then $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic filter of X.

Proposition : 5.2.3

Let X be a CI-algebra. Every cubic filter $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ of a CI-algebra X satisfies the following. For all a, b, x, y, z $\in$ X,

(i) $x * y = 1 \Rightarrow \tilde{\mu}_A(y) \succeq \tilde{\mu}_A(x)$, $\lambda(y) \leq \lambda(x)$

(ii) $a * (b * x) = 1 \Rightarrow \begin{pmatrix} \tilde{\mu}_A(x) \succeq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\} \\ \lambda(x) \leq \max\{\lambda(a), \lambda(b)\} \end{pmatrix}$

(iii) $\begin{cases} \tilde{\mu}_A(x * z) \succeq r\min\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}, \\ \lambda(x * z) \leq \max\{\lambda(x * (y * z)), \lambda(y)\} \end{cases}$

(iv) $\tilde{\mu}_A(x) \preceq \tilde{\mu}_A((x * y) * y)$, $\lambda(x) \geq \lambda((x * y) * y)$

(v) $\begin{cases} \tilde{\mu}_A((a*(b*x))*x) \succeq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, \\ \lambda((a*(b*x))*x) \leq \max\{\lambda(a), \lambda(b)\} \end{cases}$

Proof

Let $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ be a cubic filter of a CI-algebra X.

To Prove (i)

Assume that $x * y = 1$ for all $x, y \in X$. Then

$\tilde{\mu}_A(x) = r\min\{\tilde{\mu}_A(1), \tilde{\mu}_A(x)\} = r\min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(x)\} \preceq \tilde{\mu}_A(y)$ and

$\lambda(x) = \max\{\lambda(1), \lambda(x)\} = \max\{\lambda(x * y), \lambda(x)\} \geq \lambda(y)$.

To Prove (ii)

Let $a, b, x \in X$ be such that $a * (b * x) = 1$. Then

$$\begin{aligned} \tilde{\mu}_A(x) &\succeq r\min\{\tilde{\mu}_A(b*x), \tilde{\mu}_A(b)\} \\ &\succeq r\min\{r\min\{\tilde{\mu}_A(a*(b*x)), \tilde{\mu}_A(a)\}, \tilde{\mu}_A(b)\} \\ &= r\min\{r\min\{\tilde{\mu}_A(1), \tilde{\mu}_A(a)\}, \tilde{\mu}_A(b)\} \\ &= r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\} \end{aligned}$$

$$\begin{aligned} \text{and } \lambda(x) &\leq \max\{\lambda(b * x), \lambda(b)\} \\ &\leq \max\{\max\{\lambda(a * (b * x)), \lambda(a)\}, \lambda(b)\} \\ &= \max\{\max\{\lambda(1), \lambda(a)\}, \lambda(b)\} \\ &= \max\{\lambda(a), \lambda(b)\} \end{aligned}$$

To Prove (iii)

Using (CF 2), (CF 3) and (CI 3) we have

$$\begin{aligned} \tilde{\mu}_A(x*z) &\succeq r\min\{\tilde{\mu}_A(y*(x*z)), \tilde{\mu}_A(y)\} \\ &= r\min\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y)\} \end{aligned}$$

and $\lambda(x * z) \leq \max\{\lambda(y * (x * z)), \lambda(y)\} = \max\{\lambda(x * (y * z)), \lambda(y)\}$ for all $x, y, z \in X$.

To Prove (iv)

If we take y = (x ∗ y) ∗ y in (CF 2) (CF 3), then

$$\tilde{\mu}_A((x * y) * y) \succeq \text{r min } \{\tilde{\mu}_A(x * ((x * y)) * y)), \tilde{\mu}_A(x)\}$$
$$= \text{r min } \{\tilde{\mu}_A((x * y) * (x * y)), \tilde{\mu}_A(x)\}$$
$$= \text{r min } \{\tilde{\mu}_A(1), \tilde{\mu}_A(x)\} = \tilde{\mu}_A(x)$$

and $\lambda((x * y) * y) \leq \max \{\lambda(x * ((x * y) * y)), \lambda(x)\}$

$$= \max \{\lambda((x * y) * (x * y)), \lambda(x)\}$$
$$= \max \{\lambda(1), \lambda(x)\} = \lambda(x)$$

[by using (CI 3), (CI 1) and (CF 1)]

To Prove (v)

Using (iii) and (iv) we get

$$\tilde{\mu}_A((a * (b * x)) * x) \succeq \text{r min } \{\tilde{\mu}_A((a * (b * x)) * (b * x)), \tilde{\mu}_A(b)\}$$
$$\succeq \text{r min } \{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\} \text{ and}$$
$$\lambda((a * (b * x)) * x) \leq \max \{\lambda((a * (b * x)) * (b * x)), \lambda(b)\}$$
$$\leq \max \{\lambda(a), \lambda(b)\}$$

for all a, b, x ∈ X.

As a generalization of proposition we have the following result.

Proposition : 5.2.4

If a cubic set $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ is a cubic filter of a CI-algebra X, then

$$\text{(CF 4)} \quad \prod_{i=1}^{n} a_i * x = 1 \Rightarrow \begin{cases} \tilde{\mu}_A(x) \succeq \text{r min } \{\tilde{\mu}_A(a_i) / i = 1, 2 \ldots n\} \\ \lambda(x) \leq \max \{\lambda(a_i) / i = 1, 2 \ldots n\} \end{cases}$$

for all x, $a_1, \ldots, a_n \in X$, where $\prod_{i=1}^{n} a_i * x = a_n * (a_{n-1} * (\ldots (a_1 * x) \ldots))$.

Proof

The proof is by induction on n. Let $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ be a cubic filter of a CI-algebra X. By (i) and (ii) of proposition (5.2.3), we know that the condition (CF 4) is valid for n = 1, 2.

Assume that $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ satisfies the condition (CF 4) for n = k, that is

$$\prod_{i=1}^{k} a_i * x = 1 \Rightarrow \begin{cases} \tilde{\mu}_A(x) \succeq r\min\{\tilde{\mu}_A(a_i) / i = 1, 2 \ldots k\} \\ \lambda(x) \leq \max\{\lambda(a_i) / i = 1, 2 \ldots k\} \end{cases}$$

for all x, $a_1, \ldots, a_k \in X$.

Suppose that $\prod_{i=1}^{k+1} a_i * x = 1$ for all x, $a_1, \ldots, a_k, a_{k+1} \in X$.

Then $\tilde{\mu}_A(a_1 * x) \succeq r\min\{\tilde{\mu}_A(a_i) / i = 2, 3 \ldots, k + 1\}$

and $\lambda(a_1 * x) \leq \max\{\tilde{\mu}_A(a_i) / i = 2, 3 \ldots, k + 1\}$

Since $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic filter of X, it follows from (CF 2), (CF 3) that

$$\begin{aligned} \tilde{\mu}_A(x) &\succeq r\min\{\tilde{\mu}_A(a_1 * x), \tilde{\mu}_A(a_1)\} \\ &\succeq r\min\{r\min\{\tilde{\mu}_A(a_i) / i = 2, 3 \ldots, k + 1\}, \tilde{\mu}_A(a_1)\} \\ &= r\min\{\tilde{\mu}_A(a_i) / i = 1, 2 \ldots, k + 1\} \end{aligned}$$

and
$$\begin{aligned} \lambda(x) &\leq \max\{\lambda(a_1 * x), \lambda(a_1)\} \\ &\leq \max\{\max\{\lambda(a_i) / i = 2, 3, \ldots, k + 1\}, \lambda(a_1)\} \\ &= \max\{\lambda(a_i) / i = 1,2, \ldots, k + 1\} \end{aligned}$$

Theorem : 5.2.5

Let X be a CI-algebra. If a cubic set $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ satisfies the conditions $\tilde{\mu}_A(1) \succeq \tilde{\mu}_A(x)$, $\lambda(1) \leq \lambda(x)$ $\forall$ x, y $\in$ X and

$$a * (b * x) = 1 \Rightarrow \begin{pmatrix} \tilde{\mu}_A(x) \succeq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\} \\ \lambda(x) \leq \max\{\lambda(a), \lambda(b)\} \end{pmatrix}$$

then $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic filter of X.

Proof

Let $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ be a cubic set such that

$$\tilde{\mu}_A(1) \succeq \tilde{\mu}_A(x),\ \lambda(1) \le \lambda(x)\ \forall\ x, y \in X \qquad (1)$$

$$a * (b * x) = 1 \Rightarrow \begin{pmatrix} \tilde{\mu}_A(x) \succeq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)) \\ \lambda(x) \le \max\{\lambda(a), \lambda(b)\} \end{pmatrix} \qquad (2)$$

and since $x * ((x * y) * y) = 1$ for all $x, y \in X$, by (2)

$\tilde{\mu}_A(y) \succeq r\min\ \{\tilde{\mu}_A(x * y), \tilde{\mu}_A(x)\}$ and

$\lambda(y) \le \max\ \{\max\ (x * y), \max(x)\}$, for all $x, y \in X$.

Therefore $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic filter of X.

Theorem : 5.2.6

Let X be a CI-algebra. If a cubic set $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ satisfies the two conditions

$$\tilde{\mu}_A(1) \succeq \tilde{\mu}_A(x),\ \lambda(1) \le \lambda(x)\ \forall\ x, y \in X \qquad (1)$$

and $\begin{cases} \tilde{\mu}_A(x * z) \succeq r\min\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}, \\ \lambda(x * z) \le \max\{\lambda(x * (y * z)), \lambda(y)\} \end{cases}$

then $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic filter of X.

Proof

Let $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ be a cubic set such that

$$\tilde{\mu}_A(1) \succeq \tilde{\mu}_A(x),\ \lambda(1) \le \lambda(x)\ \forall\ x, y \in X \qquad (1)$$

and $\begin{cases} \tilde{\mu}_A(x * z) \succeq r\min\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}, \\ \lambda(x * z) \le \max\{\lambda(x * (y * z)), \lambda(y)\} \end{cases} \qquad (2)$

put x = 1 in (2), by using (CI 2).

$$\tilde{\mu}_A(z) = \tilde{\mu}_A(1 * z) \succeq r\min\ \{\tilde{\mu}_A(1 * (y * z)), \tilde{\mu}_A(y)\}$$
$$= r\min\ \{\tilde{\mu}_A(y * z), \tilde{\mu}_A(y)\}$$

and $\lambda(z) = \lambda(1 * z) \le \max\ \{\lambda(1 * (y * z)), \lambda(y)\}$

$$= \max\ \{\lambda(y * z), \lambda(y)\},\ \forall\ y, z \in X.$$

Hence $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic filter of X.

Theorem : 5.2.7

Let X be a CI-algebra. If a cubic set $\mathscr{A} = (\tilde{\mu}_A, \lambda) \in \mathcal{C}(X)$ satisfies

(i) $\begin{cases} \tilde{\mu}_A((a * (b * x)) * x) \succeq r\min\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, \\ \lambda((a * (b * x)) * x) \leq \max\{\lambda(a), \lambda(b)\} \end{cases}$

and

(ii) $\tilde{\mu}_A(y * x) \succeq \tilde{\mu}_A(x)$, $\lambda(y * x) \leq \lambda(x)$, $\forall\, x, y \in X$.

Then $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic filter of X.

Proof

By the definition of CI-algebra (CI 1), (CI 2) and the given condition (ii),

$$\tilde{\mu}_A(y) = \tilde{\mu}_A(1 * y) = \tilde{\mu}_A(((x * y) * (x * y)) * y)$$
$$\succeq r\min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(x)\} \text{ and}$$

$$\lambda(y) = \lambda(1 * y) = \lambda(((x * y) * (x * y)) * y)$$
$$\leq \max\{\lambda(x * y), \lambda(x)\}, \quad \forall\, x, y \in X.$$

If we take y = x in the given condition (ii),

then $\tilde{\mu}_A(1) = \tilde{\mu}_A(x * x) \succeq \tilde{\mu}_A(x)$ and $\lambda(1) = \lambda(x * x) \leq \lambda(x)$ for all $x \in X$.

Consequently $\mathscr{A} = (\tilde{\mu}_A, \lambda)$ is a cubic filter of X.