4

Fuzzy α –Translations and Fuzzy β – Multiplications of Z-Algebras

In the year 2009, fuzzy translations and fuzzy multiplications in BCK/BCI-algebras have been discussed by Lee et al.[44]. In the first two sections, fuzzy α-translations, fuzzy β-multiplications of fuzzy Z-Subalgebras (fuzzy Z-ideals) in Z-algebras and fuzzy Z-Subalgebra (Z-ideal) extension are introduced and obtained relations among them. In the third section, we discuss Z-homomorphism on fuzzy α-translations and fuzzy β-multiplications of Z-algebras and obtain certain results on the basis of fuzzy Z-Subalgebras and fuzzy Z-ideals of Z-algebras. In the fourth section, we define the cartesian product on fuzzy α-translations and fuzzy β-multiplications of Z-algebras and establish some of their properties in detail on the basis of fuzzy Z-Subalgebras and fuzzy Z-ideals of Z-algebras.

4.1 Fuzzy α-Translations and Fuzzy β-Multiplications of Fuzzy Z-Subalgebras in Z-Algebras

In this section, the notions of fuzzy α-translation, (normalized, maximal) fuzzy Z-Subalgebra extension and fuzzy β-multiplication of fuzzy Z-Subalgebras of Z-algebra X have been introduced and studied their properties.

For any fuzzy set A in a Z-algebra $(X,*,0)$, we denote $T = 1 - \sup\{\mu_A(x) \mid x \in X\}$ unless otherwise specified.

Definition 4.1.1: Let A be a fuzzy set of a Z-algebra X and let $\alpha \in [0,T]$. A **fuzzy** α**-translation** A_α^T of A with membership function $\mu_{A_\alpha^T} : X \to [0,1]$ is defined by $\mu_{A_\alpha^T}(x) = \mu_A(x) + \alpha$, for all $x \in X$.

Example 4.1.2: Consider a Z-algebra X= {0,a,b,c} with the following Cayley table 1.

Table 1

*	**0**	**a**	**b**	**c**
0	0	a	b	c
a	0	a	c	b
b	0	c	b	a
c	0	b	a	c

Table 2

X	**0**	**a**	**b**	**c**
μ_A	0.8	0.6	0.5	0.5

Define a fuzzy set A in X given by Table 2.

Here $T = 1 - \sup\{\mu_A(x) \mid x \in X\} = 1 - 0.8 = 0.2$. Choose $0.1 \in [0,0.2]$. Then the mapping $\mu_{A^T_{0.1}} : X \to [0,1]$ is defined by

X	**0**	**a**	**b**	**c**
$\mu_{A^T_\alpha}$	0.8+0.1	0.6+0.1	0.5+0.1	0.5+0.1

is a fuzzy 0.1- translation of A.

Theorem 4.1.3: Let X be a Z-algebra and A be a fuzzy Z-Subalgebra of X and $\alpha \in [0, T]$. Then the fuzzy α- translation A^T_α of A is a fuzzy Z-Subalgebra of X.

Proof: Let $x, y \in X$. Then,

$$\mu_{A^T_\alpha}(x * y) = \mu_A(x * y) + \alpha \geq \min\{\mu_A(x), \mu_A(y)\} + \alpha = \min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\}$$

$$= \min\{\mu_{A^T_\alpha}(x), \mu_{A^T_\alpha}(y)\}$$

$$\Rightarrow \quad \mu_{A^T_\alpha}(x * y) \geq \min\{\mu_{A^T_\alpha}(x), \mu_{A^T_\alpha}(y)\}$$

Hence A^T_α is a fuzzy Z-Subalgebra of X.

Theorem 4.1.4: Let X be a Z-algebra and A be a fuzzy set of X such that the fuzzy α- translation A_{α}^{T} of A is a fuzzy Z-Subalgebra of X for some $\alpha \in [0,T]$. Then A is a fuzzy Z-Subalgebra of X.

Proof : Let $x, y \in X$ then, $\mu_A(x*y)+\alpha = \mu_{A_{\alpha}^{T}}(x*y) \geq \min\{\mu_{A_{\alpha}^{T}}(x), \mu_{A_{\alpha}^{T}}(y)\}$

$$= \min\{\mu_A(x)+\alpha, \mu_A(y)+\alpha\}$$

$$= \min\{\mu_A(x), \mu_A(y)\} + \alpha$$

and so $\mu_A(x*y) \geq \min\{\mu_A(x), \mu_A(y)\}$

Hence A is a fuzzy Z-Subalgebra of X.

Definition 4.1.5: Let A_1 and A_2 be fuzzy sets of a Z-algebra X. If $\mu_{A_1}(x) \leq \mu_{A_2}(x)$ for all $x \in X$, then, we say that A_2 is a **fuzzy extension** of A_1.

Definition 4.1.6: When A_1 and A_2 are fuzzy sets of a Z-algebra X, A_2 is called a **fuzzy Z-Subalgebra extension** of A_1 if the following assertions are valid:

(i) A_2 is a fuzzy extension of A_1

(ii) If A_1 is a fuzzy Z-Subalgebra of X, then A_2 is a fuzzy Z-Subalgebra of X.

It follows from the definition of fuzzy α-translation, $\mu_{A_{\alpha}^{T}}(x) \geq \mu_A(x)$ for all $x \in X$. This proves the following proposition.

Proposition 4.1.7: Let A be a fuzzy Z-Subalgebra of a Z-algebra X and $\alpha \in [0,T]$. Then the fuzzy α-translation A_{α}^{T} of A is a fuzzy Z-Subalgebra extension of A.

Note : In general, the converse of Proposition 4.1.7 is not true as seen in the following Example 4.1.8.

Example 4.1.8: Consider a Z-algebra X={0,a,b,c} with the following Cayley table:

*	**0**	**a**	**b**	**c**
0	0	a	b	c
a	0	a	c	a
b	0	c	b	b
c	0	a	b	c

The fuzzy set A in X is defined by:

X	0	a	b	c
μ_A	0.7	0.5	0.4	0.4

is a fuzzy Z-Subalgebra of X and T=0.3. For $\alpha = 0.02$, A_α^T given by:

X	0	a	b	c
$\mu_{A_\alpha^T}$	0.72	0.52	0.42	0.42

is a fuzzy Z-Subalgebra of X, which is a fuzzy Z-Subalgebra extension of A for all $\alpha \in [0,T]$.

But A is not a fuzzy Z-Subalgebra extension of A_α^T.

Proposition 4.1.9: Arbitrary intersection of fuzzy Z-Subalgebra extensions of a fuzzy Z-Subalgebra A of a Z-algebra X is a fuzzy Z-Subalgebra extension of A.

Proof: Let $\{A_i \mid i \in \Omega\}$ be a family of fuzzy Z-Subalgebra extensions of a fuzzy Z-Subalgebra A of a Z-algebra X.

Then $\mu_{A_i}(x) \geq \mu_A(x)$ for all $x \in X$ and for all i .

Since A is a fuzzy Z-Subalgebra of a Z-algebra X , each A_i is a fuzzy Z-Subalgebra of a Z-algebra X. Then by Theorem 2.1.3, $\underset{i\in\Omega}{\cap} A_i$ is also a fuzzy Z-Subalgebra of a Z-algebra X. (1)

Also, $\mu_{\underset{i\in\Omega}{\cap} A_i}(x) = \inf_{i\in\Omega}(\mu_{A_i}(x)) \geq \mu_A(x)$ for all $x \in X$. (2)

From (1) and (2) we get, $\underset{i\in\Omega}{\cap} A_i$ is a fuzzy Z-Subalgebra extension of A.

Clearly, the union of fuzzy Z-Subalgebra extensions of a fuzzy Z-Subalgebra A of a Z-algebra X, is not a fuzzy Z-Subalgebra extension of A as shown in the following example.

Example 4.1.10: Consider a Z-algebra X and the fuzzy Z-Subalgebra A of X as in Example 4.1.8.

The fuzzy sets B and C of X defined by

X	0	a	b	c
μ_B	0.8	0.8	0.6	0.6
μ_C	0.9	0.7	0.8	0.7
$\mu_{B\cup C}$	0.9	0.8	0.8	0.7

are fuzzy Z-Subalgebras of X. Then B and C are fuzzy Z-Subalgebra extension of A. But the union $B \cup C$ is not a fuzzy Z-Subalgebra extension of A, since

$\mu_{B\cup C}(a * b) = \mu_{B\cup C}(c) = 0.7 < 0.8 = \min\{\mu_{B\cup C}(a), \mu_{B\cup C}(b)\}$.

Definition 4.1.11: For a fuzzy set A of a Z-algebra X, $\alpha \in [0, T]$ and $t \in [0,1]$ with $t \geq \alpha$, we define the upper level subset of A_α^T as $U_\alpha(\mu_A; t) = \{x \in X \mid \mu_A(x) \geq t - \alpha\}$.

Proposition 4.1.12: Let A be a fuzzy set of a Z-algebra X and $\alpha \in [0, T]$. Then the fuzzy α- translation A_α^T of A is a fuzzy Z-Subalgebra of X if and only if, $U_\alpha(\mu_A; t)$is a Z-Subalgebra of X, for all $t \in Im(A)$ with $t \geq \alpha$.

Proof : To prove necessity, let $t \in Im(A)$ be such that $t \geq \alpha$.

Let $x, y \in U_\alpha(\mu_A; t)$ $\Rightarrow \mu_A(x) \geq t - \alpha$ and $\mu_A(y) \geq t - \alpha$.

$$\Rightarrow \mu_A(x) + \alpha \geq t \text{ and } \mu_A(y) + \alpha \geq t$$

$$\Rightarrow \mu_{A_\alpha^T}(x) \geq t \text{ and } \mu_{A_\alpha^T}(y) \geq t$$

Now, $\mu_{A_\alpha^T}(x * y) \geq \min\{\mu_{A_\alpha^T}(x), \mu_{A_\alpha^T}(y)\} \geq \min\{t, t\} = t$.

Hence $\mu_A(x * y) + \alpha \geq t \Rightarrow \mu_A(x * y) \geq t - \alpha \Rightarrow x * y \in U_\alpha(\mu_A; t)$.

Thus $U_\alpha(\mu_A; t)$ is a Z-Subalgebra of a Z-algebra X.

To prove the sufficiency, assume that there exist $x, y \in X$, $t \in Im(A)$ with $t \geq \alpha$ such that

$$\mu_{A_\alpha^T}(x * y) < t \leq \min\{\mu_{A_\alpha^T}(x), \mu_{A_\alpha^T}(y)\}$$

$$= \min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\}$$

Then $\mu_A(x) \geq t-\alpha$ and $\mu_A(y) \geq t-\alpha$ but $\mu_A(x*y) < t-\alpha$.

This shows that $x, y \in U_\alpha(\mu_A; t)$ and $x*y \notin U_\alpha(\mu_A; t)$.

This is a contradiction, and so $\mu_{A_\alpha^T}(x*y) \geq \min\{\mu_{A_\alpha^T}(x), \mu_{A_\alpha^T}(y)\}$, for all $x, y \in X$. Hence, the fuzzy α- translation A_α^T of A is a fuzzy Z-Subalgebra of X.

Proposition 4.1.13: Let A be a fuzzy Z-Subalgebra of a Z-algebra X and $\alpha, \lambda \in [0,T]$. If $\alpha \geq \lambda$, then the fuzzy α-translation A_α^T of A is a fuzzy Z-Subalgebra extension of the fuzzy λ- translation A_λ^T of A.

Proof: For every $x \in X$ and $\alpha, \lambda \in [0,T]$ and $\alpha \geq \lambda$,

We have, $\mu_{A_\alpha^T}(x) = \mu_A(x) + \alpha \geq \mu_A(x) + \lambda = \mu_{A_\lambda^T}(x)$

$$\Rightarrow \quad \mu_{A_\alpha^T}(x) \geq \mu_{A_\lambda^T}(x)$$

Therefore A_α^T is a fuzzy extension of A_λ^T.Since A is a fuzzy Z-Subalgebra of X then A_α^T and A_λ^T of A are fuzzy Z-Subalgebras of X (by Theorem 4.1.3) . Hence A_α^T of A is a fuzzy Z-Subalgebra extension of A_λ^T of A.

Proposition 4.1.14: Let A be a fuzzy Z-Subalgebra of a Z-algebra X and $\lambda \in [0,T]$. For every fuzzy Z-Subalgebra extension B of the fuzzy λ- translation A_λ^T of A, there exist $\alpha \in [0,T]$ such that $\alpha \geq \lambda$ and B is a fuzzy Z-Subalgebra extension of the fuzzy α- translation A_α^T of A.

Proof: Since A is a fuzzy Z-subalgebra of a Z-algebra X and $\lambda \in [0,T]$, the fuzzy λ- translation A_λ^T of A is a fuzzy Z-Subalgebra of X by Theorem 4.1.3.

Let B be a fuzzy Z-Subalgebra extension of A_λ^T. Choose $\alpha = \lambda + \min_{x \in X}\{\mu_B(x) - \mu_{A_\lambda^T}(x)\}$. Clearly $\alpha \in [0,T]$ such that $\alpha \geq \lambda$ and $\mu_B(x) \geq \mu_{A_\alpha^T}(x)$ for all $x \in X$. Hence B is a fuzzy Z-Subalgebra extension of the fuzzy α- translation A_α^T of A.

The following example illustrates Proposition 4.1.14.

Example 4.1.15: Consider a fuzzy Z-Subalgebra A of a Z-algebra X as in Example 4.1.8. Here T=0.3. If we take $\lambda = 0.2$, then the fuzzy λ - translation A_λ^T of A is given by:

X	0	a	b	c
$\mu_{A_\lambda^T}$	0.9	0.7	0.6	0.6

The fuzzy set B of X defined by

X	0	a	b	c
μ_B	0.94	0.76	0.63	0.63

is a fuzzy Z-Subalgebra extension of fuzzy λ-translation A_λ^T of A. Take $\alpha = 0.23 > 0.2 = \lambda$.

Then, B is a fuzzy Z-Subalgebra extension of the fuzzy α-translation A_α^T of A given by

X	0	a	b	c
$\mu_{A_\alpha^T}$	0.93	0.73	0.63	0.63

Definition 4.1.16: A fuzzy Z-Subalgebra extension B of a fuzzy Z-Subalgebra A in a Z-algebra X is said to be **normalized** if there exists $x_0 \in X$ such that $\mu_B(x_0) = 1$.

Definition 4.1.17: Let A be a fuzzy Z-Subalgebra of a Z-algebra X. A fuzzy set B of X is called a **maximal fuzzy Z-Subalgebra extension** of A if it satisfies the following conditions:

(i) B is a fuzzy Z-Subalgebra extension of A.

(ii) there does not exist another fuzzy Z-Subalgebra of a Z-algebra X which is a fuzzy extension of B.

Example 4.1.18: Let X= {0,a,b,c} with the following Cayley table:

*	**0**	**a**	**b**	**c**
0	0	a	b	c
a	0	a	b	a
b	0	b	b	c
c	0	a	c	c

Then $(X,*,0)$ is a Z-algebra. Let A and B be fuzzy sets of X which are defined by $\mu_A(x) = \frac{1}{5}$ for all $x \in X$ and $\mu_B(x) = 1$ for all $x \in X$. Clearly A and B are fuzzy Z-Subalgebras of X. It is easy to verify that B is a maximal fuzzy Z-Subalgebra extension of A.

Proposition 4.1.19: If a fuzzy set B of a Z-algebra X is a normalized fuzzy Z-Subalgebra extension of a fuzzy Z-Subalgebra A of X, then $\mu_B(0)=1$.

Proof : It is clear, because $\mu_B(0)=\mu_B(x_0)$ for all $x_0 \in X$.

Proposition 4.1.20: Let A be a fuzzy Z-Subalgebra of a Z-algebra X. Then every maximal fuzzy Z-Subalgebra extension of A is normalized.

Proof : This follows from the definitions of the maximal fuzzy Z-Subalgebra extensions and normalized.

Definition 4.1.21: Let A be a fuzzy set of a Z-algebra X and $\beta \in (0,1]$. A **fuzzy** β**-multiplication** A_β^M of A with membership function $\mu_{A_\beta^M}: X \to [0,1]$ is defined by $\mu_{A_\beta^M}(x)=\beta.\mu_A(x)$ for all $x \in X$.

Example 4.1.22: Consider a Z-algebra X={0,a,b,c} with the following Cayley table:

*	0	a	b	c
0	0	a	b	c
a	0	a	c	b
b	0	c	b	a
c	0	b	a	c

Define a fuzzy set A in X by:

X	0	a	b	c
μ_A	0.8	0.6	0.5	0.5

is a fuzzy Z-Subalgebra of X.

Here $T=1-\sup\{\mu_A(x) \mid x \in X\}=1-0.8=0.2$. Choose $\beta = 0.1 \in (0,1]$. Then the mapping $\mu_{A_{0.1}^M}: X \to [0,1]$ is given by

X	0	a	b	c
$\mu_{A_{0.1}^M}$	0.08	0.06	0.05	0.05

is a fuzzy 0.1- multiplication of A.

Proposition 4.1.23: If A is a fuzzy Z-Subalgebra of a Z-algebra X, then the fuzzy β - multiplication A_β^M of A is a fuzzy Z-Subalgebra of X for all $\beta \in (0,1]$.

Proof : For all $x, y \in X$ and $\beta \in (0,1]$,

$$\mu_{A_\beta^M}(x*y) = \beta.\mu_A(x*y) \geq \beta.\min\{\mu_A(x), \mu_A(y)\} = \min\{\beta.\mu_A(x), \beta.\mu_A(y)\}$$

$$= \min\{\mu_{A_\beta^M}(x), \mu_{A_\beta^M}(y)\}$$

$$\Rightarrow \quad \mu_{A_\beta^M}(x*y) \geq \min\{\mu_{A_\beta^M}(x), \mu_{A_\beta^M}(y)\}$$

Hence A_β^M is a fuzzy Z-Subalgebra of X for all $\beta \in (0,1]$.

Proposition 4.1.24: For any fuzzy set A of Z-algebra X, the following are equivalent:

(a) A is a fuzzy Z-Subalgebra of X.

(b) For all $\beta \in (0,1]$, A_β^M is a fuzzy Z-Subalgebra of X.

Proof : Necessity follows from Proposition 4.1.23.

Let $\beta \in (0,1]$ be such that A_β^M is a fuzzy Z-Subalgebra of X.

Then for all $x, y \in X$,

$$\beta.\mu_A(x*y) = \mu_{A_\beta^M}(x*y) \geq \min\{\mu_{A_\beta^M}(x), \mu_{A_\beta^M}(y)\} = \min\{\beta.\mu_A(x), \beta.\mu_A(y)\}$$

$$= \beta.\min\{\mu_A(x), \mu_A(y)\}$$

and so $\mu_A(x*y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.

Hence A is a fuzzy Z-Subalgebra of X.

Proposition 4.1.25: Let A be a fuzzy set of a Z-algebra X, $\alpha \in [0, T]$ and $\beta \in (0,1]$. Then every fuzzy α-translation A_α^T of A is a fuzzy Z-Subalgebra extension of the fuzzy β -multiplication A_β^M of A.

Proof : For every $x \in X$, we have

$$\mu_{A_\alpha^T}(x) = \mu_A(x) + \alpha \geq \mu_A(x) \geq \beta.\mu_A(x) = \mu_{A_\beta^M}(x)$$

and so A_α^T is a fuzzy extension of A_β^M .

Assume that A_β^M is a fuzzy Z-Subalgebra of X. Then A is a fuzzy Z-Subalgebra of X by Proposition 4.1.24. It follows from Theorem 4.1.3 that A_α^T is a fuzzy Z-Subalgebra of X for all

$\alpha \in [0, T]$. Hence every fuzzy α - translation A_{α}^{T} is a fuzzy Z-Subalgebra extension of the fuzzy β - multiplication A_{β}^{M}.

4.2 Fuzzy α-Translations and Fuzzy β-Multiplications of Fuzzy Z – Ideals in Z-Algebras

In this section, the notions of fuzzy α- translation, fuzzy extension and fuzzy β - multiplication of fuzzy Z-ideals of Z-algebra X have been introduced and their properties are studied.

Theorem 4.2.1: Let A be a fuzzy Z-ideal of a Z-algebra X , then the fuzzy α- translation A_α^T of A is a fuzzy Z-ideal of X, for all $\alpha \in [0, T]$.

Proof : For all $x, y \in X$,

(i) $\mu_{A_\alpha^T}(0) = \mu_A(0) + \alpha \geq \mu_A(x) + \alpha = \mu_{A_\alpha^T}(x)$

(ii) $\mu_{A_\alpha^T}(x) = \mu_A(x) + \alpha \geq \min\{\mu_A(x*y), \mu_A(y)\} + \alpha$

$$= \min\{\mu_A(x*y) + \alpha, \mu_A(y) + \alpha\} = \min\{\mu_{A_\alpha^T}(x*y), \mu_{A_\alpha^T}(y)\}$$

$$\Rightarrow \quad \mu_{A_\alpha^T}(x) \geq \min\{\mu_{A_\alpha^T}(x*y), \mu_{A_\alpha^T}(y)\}$$

From (i) and (ii) we get, A_α^T is a fuzzy Z-ideal of X.

Theorem 4.2.2: Let A be a fuzzy set of Z-algebra X such that the fuzzy α- translation A_α^T of A is a fuzzy Z-ideal of X for some $\alpha \in [0, T]$. Then A is a fuzzy Z-ideal of X .

Proof : Let $x, y \in X$. Then,

(i) $\mu_A(0) + \alpha = \mu_{A_\alpha^T}(0) \geq \mu_{A_\alpha^T}(x) = \mu_A(x) + \alpha \qquad \Rightarrow \quad \mu_A(0) \geq \mu_A(x)$

(ii) $\mu_A(x) + \alpha = \mu_{A_\alpha^T}(x) \geq \min\{\mu_{A_\alpha^T}(x*y), \mu_{A_\alpha^T}(y)\} = \min\{\mu_A(x*y) + \alpha, \mu_A(y) + \alpha\}$

$$= \min\{\mu_A(x*y), \mu_A(y)\} + \alpha$$

$$\Rightarrow \quad \mu_A(x) \geq \min\{\mu_A(x*y), \mu_A(y)\}$$

From (i) and (ii) we get A is a fuzzy Z-ideal of X.

Definition 4.2.3: When A_1 and A_2 are fuzzy sets of a Z-algebra X, A_2 is called a **fuzzy Z-ideal extension** of A_1 if the following assertions are valid:

(i) A_2 is a fuzzy extension of A_1

(ii) If A_1 is a fuzzy Z-ideal of X, then A_2 is a fuzzy Z-ideal of X.

Proposition 4.2.4: Let A be a fuzzy Z-ideal of a Z-algebra X and let $\alpha, \gamma \in [0, T]$. If $\alpha \geq \gamma$, then the fuzzy α- translation A_α^T of A is a fuzzy Z-ideal extension of the fuzzy γ-translation A_γ^T of A.

Proof : For all $x \in X$, $\mu_A(x) + \alpha \geq \mu_A(x) + \gamma \quad \Rightarrow \quad \mu_{A_\alpha^T}(x) \geq \mu_{A_\gamma^T}(x)$

Hence A_α^T of A is a fuzzy extension of A_γ^T of A. (1)

By Theorem 4.2.1, the fuzzy α- translation A_α^T and fuzzy γ- translation A_γ^T of A are fuzzy Z-ideals of X. (2)

From (1) and (2), A_α^T of A is a fuzzy Z-ideal extension of A_γ^T of A.

Proposition 4.2.5: Let A be a fuzzy Z-ideal of a Z-algebra X and $\gamma \in [0,T]$. For every fuzzy Z-ideal extension B of the fuzzy γ-translation A_γ^T of A, there exists $\alpha \in [0,T]$ such that $\alpha \geq \gamma$ and B is a fuzzy Z-ideal extension of the fuzzy α-translation A_α^T of A .

Proof: Let A be fuzzy Z-ideal of a Z-algebra X and $\gamma \in [0,T]$.

Then by Theorem 4.2.1, the fuzzy γ-translation A_γ^T of A is a fuzzy Z-ideal of X.

Let B be a fuzzy Z-ideal extension of A_γ^T. Therefore, $\mu_B(x) \geq \mu_{A_\gamma^T}(x)$ for all $x \in X$.

Then choose $\alpha = \gamma + \min_{x \in X}\{\mu_B(x) - \mu_{A_\gamma^T}(x)\}$.

Clearly $\alpha \in [0,T]$ such that $\alpha \geq \gamma$.

Then, the fuzzy α-translation A_α^T of A is a fuzzy Z-ideal of a Z-algebra X and $\mu_B(x) \geq \mu_{A_\alpha^T}(x)$ for all $x \in X$.

Hence B is a fuzzy Z-ideal extension of A_α^T of A.

The following example illustrates Proposition 4.2.5.

Example 4.2.6: Consider a Z-algebra X= {0,a,b,c} as in Example 4.1.8.

A fuzzy set A in X defined by : $\mu_A(x) = 0.8 \ \forall\, x \in X_S$ is a fuzzy Z-ideal of X and T= 0.2.

If we take $\gamma = 0.14$, then the fuzzy γ - translation A_γ^T of A is given by : $\mu_{A_\gamma^T}(x) = 0.94$ $\forall\ x \in X$.

Let B be a fuzzy set of X defined by: $\mu_B(x) = 0.98\ \ \forall\ x \in X$ is a fuzzy Z-ideal extension of the fuzzy γ - translation A_γ^T of A, for all $\gamma \in [0, T]$. Take $\alpha = 0.18 > 0.14 = \gamma$. Clearly B is a fuzzy Z-ideal extension of the fuzzy α - translation A_α^T of A given by $\mu_{A_\alpha^T}(x) = 0.98\ \ \forall\ x \in X$.

Proposition 4.2.7: Let A be a fuzzy Z-ideal of a Z-algebra X and $\alpha \in [0, T]$. Then the fuzzy α - translation A_α^T of A is a fuzzy Z-ideal extension of A.

Proof : Since $\mu_A(x) + \alpha = \mu_{A_\alpha^T}(x) \geq \mu_A(x)$ for all $x \in X$ and $\alpha \in [0, T]$ and by Theorem 4.2.1 we get the fuzzy α - translation A_α^T of A is a fuzzy Z-ideal extension of A.

Proposition 4.2.8: Arbitrary intersection of fuzzy Z-ideal extensions of a fuzzy Z-ideal A of a Z-algebra X is a fuzzy Z-ideal extension of A.

Proof: Let $\{A_i \mid i \in \Omega\}$ be a family of fuzzy Z-ideal extensions of a fuzzy Z-ideal A of a Z-algebra X. Then $\mu_{A_i}(x) \geq \mu_A(x)$ for all $x \in X$ and for all i.

Since A is a fuzzy Z-ideal of X , each A_i is a fuzzy Z-ideal of a Z-algebra X.

Then by Theorem 2.2.3, $\bigcap_{i \in \Omega} A_i$ is also a fuzzy Z-ideal of a Z-algebra X. (1)

Also, $\mu_{\bigcap_{i\in\Omega} A_i}(x) = \inf_{i \in \Omega}(\mu_{A_i}(x)) \geq \mu_A(x)$ for all $x \in X$. (2)

From (1) and (2) we get, $\bigcap_{i \in \Omega} A_i$ is a fuzzy Z-ideal extension of A.

Theorem 4.2.9: For $\alpha \in [0, T]$, A_α^T be the fuzzy α -translation of a fuzzy set A of a Z-algebra X. Then the following are equivalent :

(i) A_α^T is a fuzzy Z-ideal of X.

(ii) $\forall\ \ t \in \text{Im}(A),\ \ t > \alpha \ \ \Rightarrow U_\alpha(\mu_A; t)$ is an Z-ideal of X.

Proof :

To prove: (i) $\Rightarrow$(ii)

Let $t \in \text{Im}(A)$ be such that $t > \alpha$.

Since $\mu_{A_\alpha^T}(0) \geq \mu_{A_\alpha^T}(x)$ for all $x \in X$,we have $\mu_A(0) + \alpha \geq \mu_A(x) + \alpha$

$\Rightarrow \quad \mu_A(0) \geq \mu_A(x)$ for all $x \in X$.

Let $x \in U_\alpha(\mu_A; t)$ then $\mu_A(x) \geq t - \alpha \quad \Rightarrow \mu_A(0) \geq \mu_A(x) \geq t - \alpha$

Hence $0 \in U_\alpha(\mu_A; t)$.

Let $x, y \in X$ be such that $x * y \in U_\alpha(\mu_A; t)$ and $y \in U_\alpha(\mu_A; t)$.

Then $\mu_A(x * y) \geq t - \alpha$ and $\mu_A(y) \geq t - \alpha$

That is , $\mu_A(x * y) + \alpha \geq t$ and $\mu_A(y) + \alpha \geq t$

$\Rightarrow \quad \mu_{A_\alpha^T}(x * y) \geq t$ and $\mu_{A_\alpha^T}(y) \geq t$

Since A_α^T is a fuzzy Z-ideal of X, it follows that

$\mu_A(x) + \alpha = \mu_{A_\alpha^T}(x) \geq \min\{\mu_{A_\alpha^T}(x * y), \mu_{A_\alpha^T}(y)\} \geq \min\{t, t\} = t$.

That is , $\mu_A(x) \geq t - \alpha$ so that $x \in U_\alpha(\mu_A; t)$.

Therefore, $U_\alpha(\mu_A; t)$ is an Z-ideal of X.

To prove: (ii) $\Rightarrow$(i)

Suppose that $U_\alpha(\mu_A; t)$ is an Z-ideal of X for every $t \in Im(A)$ with $t > \alpha$. If there exists $x \in X$ such that $\mu_{A_\alpha^T}(0) < t \leq \mu_{A_\alpha^T}(x)$, then $\mu_A(x) \geq t - \alpha$ but $\mu_A(0) < t - \alpha$. This shows that $x \in U_\alpha(\mu_A; t)$ and $0 \notin U_\alpha(\mu_A; t)$.

This is a contradiction, and so $\mu_{A_\alpha^T}(0) \geq \mu_{A_\alpha^T}(x)$ for all $x \in X$.

Now assume that there exists $x, y \in X$ such that

$\mu_{A_\alpha^T}(x) < t \leq \min\{\mu_{A_\alpha^T}(x * y), \mu_{A_\alpha^T}(y)\}$ where $t = \frac{1}{2}\left[\mu_{A_\alpha^T}(x) + \min\{\mu_{A_\alpha^T}(x * y), \mu_{A_\alpha^T}(y)\}\right]$

$\Rightarrow \mu_A(x) + \alpha < t \leq \min\{\mu_A(x * y) + \alpha, \mu_A(y) + \alpha\}$

Then $\mu_A(x * y) \geq t - \alpha$ and $\mu_A(y) \geq t - \alpha$, but $\mu_A(x) < t - \alpha$.

Hence $x * y \in U_\alpha(\mu_A; t)$ and $y \in U_\alpha(\mu_A; t)$, but $x \notin U_\alpha(\mu_A; t)$.

This is a contradiction, and therefore $\mu_{A_\alpha^T}(x) \geq \min\{\mu_{A_\alpha^T}(x * y), \mu_{A_\alpha^T}(y)\}$ for all $x, y \in X$.

Hence A_α^T is a fuzzy Z-ideal of X.

Proposition 4.2.10: For any fuzzy set A of a Z-algebra X, the following are equivalent

(i) A is a fuzzy Z-ideal of X.

(ii) For all $\beta \in (0,1]$, the fuzzy β-multiplication A_β^M of A is a fuzzy Z-ideal of X.

Proof :

To prove: (i)$\Rightarrow$ (ii)

Let $x, y \in X$. For all $\beta \in (0,1]$, $\mu_A(0) \geq \mu_A(x) \quad \Rightarrow \beta.\mu_A(0) \geq \beta.\mu_A(x)$

$$\Rightarrow \mu_{A_\beta^M}(0) \geq \mu_{A_\beta^M}(x) \qquad (1)$$

And, $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\} \Rightarrow \beta.\mu_A(x) \geq \beta.\min\{\mu_A(x * y), \mu_A(y)\}$

$$\Rightarrow \mu_{A_\beta^M}(x) \geq \min\{\beta.\mu_A(x * y), \beta.\mu_A(y)\}$$

$$\Rightarrow \mu_{A_\beta^M}(x) \geq \min\{\mu_{A_\beta^M}(x * y), \mu_{A_\beta^M}(y)\} \qquad (2)$$

From (1) and (2), A_β^M is a fuzzy Z-ideal of X.

To prove: (ii)$\Rightarrow$(i)

Let $\beta \in (0,1]$. Then for all $x, y \in X$, $\mu_{A_\beta^M}(0) \geq \mu_{A_\beta^M}(x) \Rightarrow \beta.\mu_A(0) \geq \beta.\mu_A(x)$

$$\Rightarrow \mu_A(0) \geq \mu_A(x) \qquad (3)$$

And, $\beta.\mu_A(x) = \mu_{A_\beta^M}(x) \geq \min\{\mu_{A_\beta^M}(x * y), \mu_{A_\beta^M}(y)\} = \min\{\beta.\mu_A(x * y), \beta.\mu_A(y)\}$

$$= \beta.\min\{\mu_A(x * y), \mu_A(y)\}$$

$$\Rightarrow \quad \mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\} \qquad (4)$$

From (3) and (4), A is a fuzzy Z-ideal of X.

Proposition 4.2.11: Let A be a fuzzy set of a Z-algebra X, $\alpha \in [0, T]$ and $\beta \in (0,1]$. Then every fuzzy α- translation A_α^T of A is a fuzzy Z-ideal extension of the fuzzy β -multiplication A_β^M of A.

Proof : For every $x \in X$ we have $\mu_{A_\alpha^T}(x) = \mu_A(x) + \alpha \geq \mu_A(x) \geq \beta.\mu_A(x) = \mu_{A_\beta^M}(x)$

$$\Rightarrow \quad \mu_{A_\alpha^T}(x) \geq \mu_{A_\beta^M}(x)$$

$\Rightarrow$ A_α^T is a fuzzy extension of A_β^M. (1)

Assume that A_β^M is a fuzzy Z-ideal of X. Then A is a fuzzy Z-ideal of X by Proposition 4.2.10.

By Theorem 4.2.1 we get A_α^T is a fuzzy Z-ideal of X for all $\alpha \in [0, T]$. (2)

From (1) and (2) we get every fuzzy α-translation A_α^T of A is a fuzzy Z-ideal extension of the fuzzy β - multiplication A_β^M of A .

The following example illustrates Proposition 4.2.11.

Example 4.2.12: Consider a Z-algebra X= {0, a , b, c} with the Example 4.1.8. Define a fuzzy set A of X by: $\mu_A(x) = 0.6 \quad \forall \quad x \in X$.

Then A is a fuzzy Z-ideal of X. If we take $\lambda = 0.3$, then the fuzzy 0.3-multiplication $A^M_{0.3}$ of A is given by: $\mu_{A^M_{0.3}}(x) = 0.18 \ \forall \ x \in X$.

Also, for any $\alpha \in [0,0.4]$, the fuzzy $\alpha -$translation A^T_α of A is given by $\mu_{A^T_\alpha}(x) = 0.6 + \alpha \ \forall \ x \in X$ is a fuzzy Z-ideal extension of the fuzzy 0.3-multiplication of $A^M_{0.3}$ of A.

4.3 Z-Homomorphism on Fuzzy α-Translations and Fuzzy β-Multiplications of Z-algebras

In this section, we discuss Z-homomorphism on fuzzy α- translations and fuzzy β-multiplications of Z-algebras and obtain certain results on the basis of fuzzy Z-Subalgebras and fuzzy Z-ideals of Z-algebras.

Definition 4.3.1: Let $h:(X,*,0)\rightarrow(Y,*',0')$ be an Z-homomorphism of a Z-algebra X and A_α^T be a fuzzy α-translation of a fuzzy set A in Y. We define a new fuzzy set $(A_\alpha^T)_h$ in X by $\mu_{(A_\alpha^T)_h}(x)=\mu_{A_\alpha^T}(h(x))=\mu_A(h(x))+\alpha$, for all $x\in X$.

Theorem 4.3.2: Let $h:(X,*,0)\rightarrow(Y,*',0')$ be a Z-homomorphism of Z-algebras. If A is a fuzzy Z-Subalgebra of a Z-algebra Y, then $(A_\alpha^T)_h$ is also a fuzzy Z-Subalgebra of X.

Proof: Let $x,y\in X$.

Now, $\mu_{(A_\alpha^T)_h}(x*y) =\mu_{A_\alpha^T}(h(x*y))=\mu_A(h(x*y))+\alpha =\mu_A(h(x)*'h(y))+\alpha$

$$\geq \min\{\mu_A(h(x)),\mu_A(h(y))\}+\alpha$$
$$=\min\{\mu_A(h(x))+\alpha,\mu_A(h(y))+\alpha\}$$
$$=\min\{\mu_{(A_\alpha^T)_h}(x),\mu_{(A_\alpha^T)_h}(y)\}$$

$$\Rightarrow \quad \mu_{(A_\alpha^T)_h}(x*y)\geq \min\{\mu_{(A_\alpha^T)_h}(x),\mu_{(A_\alpha^T)_h}(y)\}$$

Hence $(A_\alpha^T)_h$ is a fuzzy Z-Subalgebra of a Z-algebra X.

Theorem 4.3.3: Let $h:(X,*,0)\rightarrow(Y,*',0')$ be a Z-homomorphism of Z-algebras. If A is a fuzzy Z-ideal of Y, then $(A_\alpha^T)_h$ is also fuzzy Z-ideal of X.

Proof : Let $x,y\in X$,

(i) $\mu_{(A_\alpha^T)_h}(0)=\mu_{A_\alpha^T}(h(0))=\mu_A(h(0))+\alpha\geq\mu_A(h(x))+\alpha=\mu_{A_\alpha^T}(h(x))=\mu_{(A_\alpha^T)_h}(x)$

(ii) $\mu_{(A_\alpha^T)_h}(x)=\mu_{A_\alpha^T}(h(x))=\mu_A(h(x))+\alpha\geq\min\{\mu_A(h(x)*'h(y)),\mu_A(h(y))\}+\alpha$

$$=\min\{\mu_A(h(x*y)),\mu_A(h(y))\}+\alpha$$
$$=\min\{\mu_A(h(x*y))+\alpha,\mu_A(h(y))+\alpha\}$$

$$= \min\{\mu_{(A_\alpha^T)_h}(x * y), \mu_{(A_\alpha^T)_h}(y)\}$$

$$\Rightarrow \quad \mu_{(A_\alpha^T)_h}(x) \geq \min\{\mu_{(A_\alpha^T)_h}(x * y), \mu_{(A_\alpha^T)_h}(y)\}$$

From (i) and (ii), $(A_\alpha^T)_h$ is a fuzzy Z-ideal of a Z-algebra X.

Theorem 4.3.4: Let $h:(X,*,0) \to (Y,*',0')$ be an Z-epimorphism of Z-algebras. If $(A_\alpha^T)_h$ is a fuzzy Z-ideal of X, then A is also fuzzy Z-ideal of Y.

Proof: Let $y \in Y$ then there exist $x \in X$ such that $h(x) = y$.

(i) $\mu_A(0') + \alpha = \mu_{A_\alpha^T}(0') = \mu_{A_\alpha^T}(h(0)) = \mu_{(A_\alpha^T)_h}(0) \geq \mu_{(A_\alpha^T)_h}(x) = \mu_{A_\alpha^T}(h(x))$

$$= \mu_A(h(x)) + \alpha = \mu_A(y) + \alpha$$

(ii) Let $y_1, y_2 \in Y$ then there exists $x_1, x_2 \in X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

$$\mu_A(y_1) + \alpha = \mu_A(h(x_1)) + \alpha = \mu_{(A_\alpha^T)_h}(x_1) \geq \min\{\mu_{(A_\alpha^T)_h}(x_1 * x_2), \mu_{(A_\alpha^T)_h}(x_2)\}$$

$$= \min\{\mu_A(h(x_1 * x_2)) + \alpha, \mu_A(h(x_2)) + \alpha\}$$

$$= \min\{\mu_A(h(x_1) *' h(x_2)) + \alpha, \mu_A(h(x_2)) + \alpha\}$$

$$= \min\{\mu_A(y_1 *' y_2) + \alpha, \mu_A(y_2) + \alpha\}$$

$$= \min\{\mu_A(y_1 *' y_2), \mu_A(y_2)\} + \alpha$$

$$\Rightarrow \quad \mu_A(y_1) \geq \min\{\mu_A(y_1 *' y_2), \mu_A(y_2)\}$$

From (i) and (ii), A is a fuzzy Z-ideal of a Z-algebra Y.

Theorem 4.3.5: Let $h:(X,*,0) \to (X,*,0)$ be an Z-endomorphism of a Z-algebra X. If A is a fuzzy Z-ideal of X, then $(A_\alpha^T)_h$ is also a fuzzy Z-ideal of X.

Proof: Let $x, y \in X$.

(i) $\mu_{(A_\alpha^T)_h}(0) = \mu_{A_\alpha^T}(h(0)) = \mu_A(h(0)) + \alpha \geq \mu_A(h(x)) + \alpha = \mu_{A_\alpha^T}(h(x)) = \mu_{(A_\alpha^T)_h}(x)$

(ii) $\mu_{(A_\alpha^T)_h}(x) = \mu_{A_\alpha^T}(h(x)) = \mu_A(h(x)) + \alpha \geq \min\{\mu_A(h(x) * h(y)), \mu_A(h(y))\} + \alpha$

$$= \min\{\mu_A(h(x * y)), \mu_A(h(y))\} + \alpha$$

$$= \min\{\mu_A(h(x * y)) + \alpha, \mu_A(h(y)) + \alpha\}$$

$$= \min\{\mu_{(A_\alpha^T)_h}(x * y), \mu_{(A_\alpha^T)_h}(y)\}$$

$$\Rightarrow \quad \mu_{(A_\alpha^T)_h}(x) \geq \min\{\mu_{(A_\alpha^T)_h}(x * y), \mu_{(A_\alpha^T)_h}(y)\}$$

From (i) and (ii), $(A_\alpha^T)_h$ is a fuzzy Z-ideal of a Z-algebra X.

Theorem 4.3.6: Let $h:(X,*,0)\to(Y,*',0')$ be a Z-homomorphism of Z-algebras and A_α^T be a fuzzy α-translation of a fuzzy set A in Y. The pre-image of A_α^T denoted by $h^{-1}(A_\alpha^T)$ is defined as $\mu_{h^{-1}(A_\alpha^T)}(x) = \mu_{A_\alpha^T}(h(x))$, for all $x \in X$. If A is a fuzzy Z-Subalgebra (fuzzy Z-ideal) of Y then $h^{-1}(A_\alpha^T)$ is a fuzzy Z-Subalgebra (fuzzy Z-ideal) of X.

Proof: Let $x, y \in X$.

Now, $\mu_{h^{-1}(A_\alpha^T)}(x * y) = \mu_{A_\alpha^T}(h(x * y)) = \mu_A(h(x * y)) + \alpha = \mu_A(h(x) *' h(y)) + \alpha$

$$\geq \min\{\mu_A(h(x)), \mu_A(h(y))\} + \alpha$$
$$= \min\{\mu_A(h(x)) + \alpha, \mu_A(h(y)) + \alpha\}$$
$$= \min\{\mu_{A_\alpha^T}(h(x)), \mu_{A_\alpha^T}(h(y))\}$$
$$= \min\{\mu_{h^{-1}(A_\alpha^T)}(x), \mu_{h^{-1}(A_\alpha^T)}(y)\}$$

$$\Rightarrow \mu_{h^{-1}(A_\alpha^T)}(x * y) \geq \min\{\mu_{h^{-1}(A_\alpha^T)}(x), \mu_{h^{-1}(A_\alpha^T)}(y)\}$$

Therefore, $h^{-1}(A_\alpha^T)$ is a fuzzy Z-Subalgebra of a Z-algebra X.

Also, $\mu_{h^{-1}(A_\alpha^T)}(x) = \mu_{A_\alpha^T}(h(x)) \leq \mu_{A_\alpha^T}(0') = \mu_{A_\alpha^T}(h(0)) = \mu_{h^{-1}(A_\alpha^T)}(0)$ (1)

$$\min\{\mu_{h^{-1}(A_\alpha^T)}(x * y), \mu_{h^{-1}(A_\alpha^T)}(y)\} = \min\{\mu_{A_\alpha^T}(h(x * y)), \mu_{A_\alpha^T}(h(y))\}$$
$$= \min\{\mu_{A_\alpha^T}(h(x) *' h(y)), \mu_{A_\alpha^T}(h(y))\}$$
$$= \min\{\mu_A(h(x) *' h(y)) + \alpha, \mu_A(h(y)) + \alpha\}$$
$$= \min\{\mu_A(h(x) *' h(y)), \mu_A(h(y))\} + \alpha$$
$$\leq \mu_A(h(x)) + \alpha$$
$$= \mu_{A_\alpha^T}(h(x))$$
$$= \mu_{h^{-1}(A_\alpha^T)}(x) \quad (2)$$

From (1) and (2), $h^{-1}(A_\alpha^T)$ is a fuzzy Z-ideal of a Z-algebra X.

Definition 4.3.7: Let $h:(X,*,0)\to(Y,*',0')$ be a Z-homomorphism of a Z-algebra X and A_β^M be a fuzzy β - multiplication of a fuzzy set A in Y. We define a new fuzzy set by $\left(A_\beta^M\right)_h$ in X as $\mu_{\left(A_\beta^M\right)_h}(x)=\mu_{A_\beta^M}(h(x))=\beta\cdot\mu_A(h(x))$ for all $x\in X$.

Theorem 4.3.8: Let $h:(X,*,0)\to(Y,*',0')$ be a Z-homomorphism of Z-algebras. If A be a fuzzy Z-Subalgebra of a Z-algebra Y, then $\left(A_\beta^M\right)_h$ is also a fuzzy Z-Subalgebra of X.

Proof: Let $x,y\in X$.

Now,

$$\mu_{\left(A_\beta^M\right)_h}(x*y)=\mu_{A_\beta^M}(h(x*y))=\beta\cdot\mu_A(h(x*y))=\beta\cdot\mu_A(h(x)*'h(y))\geq\beta\cdot\min\{\mu_A(h(x)),\mu_A(h(y))\}$$

$$=\min\{\beta\cdot\mu_A(h(x)),\beta\cdot\mu_A(h(y))\}$$

$$=\min\{\mu_{A_\beta^M}(h(x)),\mu_{A_\beta^M}(h(y))\}$$

$$=\min\{\mu_{\left(A_\beta^M\right)_h}(x),\mu_{\left(A_\beta^M\right)_h}(y)\}$$

$\therefore$ $\left(A_\beta^M\right)_h$ is a fuzzy Z-Subalgebra of a Z-algebra X.

Theorem 4.3.9: Let $h:(X,*,0)\to(Y,*',0')$ be a Z-homomorphism of Z-algebras. If A is a fuzzy Z-ideal of Y then $\left(A_\beta^M\right)_h$ is a fuzzy Z-ideal of X.

Proof: Let $x,y\in X$.

(i) $\mu_{\left(A_\beta^M\right)_h}(0)=\mu_{A_\beta^M}(h(0))=\beta\cdot\mu_A(h(0))\geq\beta\cdot\mu_A(h(x))=\mu_{\left(A_\beta^M\right)_h}(x)$

(ii) $\mu_{\left(A_\beta^M\right)_h}(x)=\mu_{A_\beta^M}(h(x))=\beta\cdot\mu_A(h(x))\geq\beta\cdot\min\{\mu_A(h(x)*'h(y)),\mu_A(h(y))\}$

$$=\beta\cdot\min\{\mu_A(h(x*y)),\mu_A(h(y))\}$$

$$=\min\{\beta\cdot\mu_A(h(x*y)),\beta\cdot\mu_A(h(y))\}$$

$$=\min\{\mu_{\left(A_\beta^M\right)_h}(x*y),\mu_{\left(A_\beta^M\right)_h}(y)\}$$

$$\Rightarrow\ \mu_{\left(A_\beta^M\right)_h}(x)\geq\min\{\mu_{\left(A_\beta^M\right)_h}(x*y),\mu_{\left(A_\beta^M\right)_h}(y)\}$$

From (i) and (ii), $\left(A_\beta^M\right)_h$ is a fuzzy Z-ideal of a Z-algebra X.

Theorem 4.3.10: Let $h:(X,*,0)\to(Y,*',0')$ be an Z-epimorphism of Z-algebras. If $\left(A_\beta^M\right)_h$ is a fuzzy Z-ideal of X then A is a fuzzy Z-ideal of Y.

Proof: Let $y \in Y$. Then there exist $x \in X$ such that $h(x) = y$.

(i) $\beta \cdot \mu_A(0') = \mu_{A_\beta^M}(0') = \mu_{A_\beta^M}(h(0)) = \mu_{(A_\beta^M)_h}(0) \geq \mu_{(A_\beta^M)_h}(x) = \mu_{A_\beta^M}(h(x)) = \beta \cdot \mu_A(h(x))$

$$= \beta \cdot \mu_A(y)$$

$\Rightarrow \quad \mu_A(0') \geq \mu_A(y)$

(ii) Let $y_1, y_2 \in Y$ then there exists $x_1, x_2 \in X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

$$\beta \cdot \mu_A(y_1) = \mu_{A_\beta^M}(y_1) = \mu_{A_\beta^M}(h(x_1)) = \mu_{(A_\beta^M)_h}(x_1)$$

$$\geq \min\{\mu_{(A_\beta^M)_h}(x_1 * x_2), \mu_{(A_\beta^M)_h}(x_2)\}$$

$$= \min\{\mu_{A_\beta^M}(h(x_1 * x_2)), \mu_{A_\beta^M}(h(x_2))\}$$

$$= \min\{\beta \cdot \mu_A(h(x_1 * x_2)), \beta \cdot \mu_A(h(x_2))\}$$

$$= \min\{\beta \cdot \mu_A(h(x_1) *' h(x_2)), \beta \cdot \mu_A(h(x_2))\}$$

$$= \beta \cdot \min\{\mu_A(y_1 *' y_2), \mu_A(y_2)\}$$

$\Rightarrow \quad \mu_A(y_1) \geq \min\{\mu_A(y_1 *' y_2), \mu_A(y_2)\}$

From (i) and (ii) , A is a fuzzy Z-ideal of a Z-algebra Y.

Theorem 4.3.11: Let h be an Z-endomorphism of Z-algebra X. If A is a fuzzy Z-ideal of X, then $(A_\beta^M)_h$ is also a fuzzy Z-ideal of X.

Proof: Let $x, y \in X$.

(i) $\mu_{(A_\beta^M)_h}(0) = \mu_{A_\beta^M}(h(0)) = \beta \cdot \mu_A(h(0)) \geq \beta \cdot \mu_A(h(x)) = \mu_{A_\beta^M}(h(x)) = \mu_{(A_\beta^M)_h}(x)$

(ii) $\mu_{(A_\beta^M)_h}(x) = \mu_{A_\beta^M}(h(x)) = \beta \cdot \mu_A(h(x)) \geq \beta \cdot \min\{\mu_A(h(x) * h(y)), \mu_A(h(y))\}$

$$= \beta \cdot \min\{\mu_A(h(x * y)), \mu_A(h(y))\}$$

$$= \min\{\beta \cdot \mu_A(h(x * y)), \beta \cdot \mu_A(h(y))\}$$

$$= \min\{\mu_{A_\beta^M}(h(x * y)), \mu_{A_\beta^M}(h(y))\}$$

$$= \min\{\mu_{(A_\beta^M)_h}(x * y), \mu_{(A_\beta^M)_h}(y)\}$$

$\Rightarrow \mu_{(A_\beta^M)_h}(x) \geq \min\{\mu_{(A_\beta^M)_h}(x * y), \mu_{(A_\beta^M)_h}(y)\}$

From (i) and (ii), $(A_\beta^M)_h$ is a fuzzy Z-ideal of a Z-algebra X.

Theorem 4.3.12: Let $h:(X,*,0)\to(Y,*',0')$ be a Z-homomorphism of Z-algebra X into a Z-algebra Y and A_β^M be a fuzzy β -multiplication of a fuzzy set A in Y. The pre-image of A_β^M denoted by $h^{-1}(A_\beta^M)$ is defined by $\mu_{h^{-1}(A_\beta^M)}(x)=\mu_{A_\beta^M}(h(x))$ for all $x\in X$. If A is a fuzzy Z-Subalgebra (fuzzy Z-ideal) of Y, then $h^{-1}(A_\beta^M)$ is a fuzzy Z-Subalgebra (fuzzy Z-ideal) of X.

Proof: Let $x, y \in X$.

Now, $\mu_{h^{-1}(A_\beta^M)}(x*y)=\mu_{A_\beta^M}(h(x*y))=\beta\cdot\mu_A(h(x*y))=\beta\cdot\mu_A(h(x)*'h(y))$

$$\geq\beta\cdot\min\{\mu_A(h(x)),\mu_A(h(y))\}$$
$$=\min\{\beta\cdot\mu_A(h(x)),\beta\cdot\mu_A(h(y))\}$$
$$=\min\{\mu_{A_\beta^M}(h(x)),\mu_{A_\beta^M}(h(y))\}$$
$$=\min\{\mu_{h^{-1}(A_\beta^M)}(x),\mu_{h^{-1}(A_\beta^M)}(y)\}$$

$\Rightarrow\ \mu_{h^{-1}(A_\beta^M)}(x*y)\geq\min\{\mu_{h^{-1}(A_\beta^M)}(x),\mu_{h^{-1}(A_\beta^M)}(y)\}$

Therefore $h^{-1}(A_\beta^M)$ is a fuzzy Z-Subalgebra of a Z-algebra X.

Also, $\mu_{h^{-1}(A_\beta^M)}(x)=\mu_{A_\beta^M}(h(x))\leq\mu_{A_\beta^M}(0')=\mu_{A_\beta^M}(h(0))=\mu_{h^{-1}(A_\beta^M)}(0)$ (1)

$$\min\{\mu_{h^{-1}(A_\beta^M)}(x*y),\mu_{h^{-1}(A_\beta^M)}(y)\}=\min\{\mu_{A_\beta^M}(h(x*y)),\mu_{A_\beta^M}(h(y))\}$$
$$=\min\{\mu_{A_\beta^M}(h(x)*'h(y)),\mu_{A_\beta^M}(h(y))\}$$
$$=\min\{\beta\cdot\mu_A(h(x)*'h(y)),\beta\cdot\mu_A(h(y))\}$$
$$=\beta\cdot\min\{\mu_A(h(x)*'h(y)),\mu_A(h(y))\}$$
$$\leq\beta\cdot\mu_A(h(x))$$
$$=\mu_{A_\beta^M}(h(x))$$
$$=\mu_{h^{-1}(A_\beta^M)}(x) \qquad (2)$$

From (1) and (2), $h^{-1}(A_\beta^M)$ is a fuzzy Z-ideal of a Z-algebra X.

4.4 Cartesian Product on Fuzzy α-Translations and Fuzzy β-Multiplications of Z-algebras

In this section, we define the Cartesian product on fuzzy α- translations and fuzzy β-multiplications of Z-algebras and establish some of their properties in detail on the basis of fuzzy Z-Subalgebras and fuzzy Z-ideals of Z-algebras.

Definition 4.4.1: Let A_α^T and B_α^T be fuzzy α- translations of fuzzy sets A and B in a Z-algebra X. The **Cartesian product** $A_\alpha^T x B_\alpha^T$ with membership function $\mu_{A_\alpha^T x B_\alpha^T} : X x X \to [0,1]$ is defined by $\mu_{A_\alpha^T \times B_\alpha^T}(x,y) = \min\{\mu_{A_\alpha^T}(x), \mu_{B_\alpha^T}(y)\}$, for all $x, y \in X$.

Theorem 4.4.2: Let A and B are fuzzy Z-Subalgebras of Z-algebra X, then $A_\alpha^T x B_\alpha^T$ is also a fuzzy Z-Subalgebra of $X x X$.

Proof : For any $(x_1, y_1), (x_2, y_2) \in X \times X$,

$$\mu_{A_\alpha^T \times B_\alpha^T}((x_1, y_1) * (x_2, y_2)) = \mu_{A_\alpha^T \times B_\alpha^T}((x_1 * x_2), (y_1 * y_2))$$

$$= \min\{\mu_{A_\alpha^T}(x_1 * x_2), \mu_{B_\alpha^T}(y_1 * y_2)\}$$

$$= \min\{\mu_A(x_1 * x_2) + \alpha, \mu_B(y_1 * y_2) + \alpha\}$$

$$= \min\{\mu_A(x_1 * x_2), \mu_B(y_1 * y_2)\} + \alpha$$

$$\geq \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\} + \alpha$$

$$= \min\{\min\{\mu_A(x_1), \mu_A(x_2)\} + \alpha, \min\{\mu_B(y_1), \mu_B(y_2)\} + \alpha\}$$

$$= \min\{\min\{\mu_A(x_1) + \alpha, \mu_A(x_2) + \alpha\}, \min\{\mu_B(y_1) + \alpha, \mu_B(y_2) + \alpha\}\}$$

$$= \min\{\min\{\mu_{A_\alpha^T}(x_1), \mu_{A_\alpha^T}(x_2)\}, \min\{\mu_{B_\alpha^T}(y_1), \mu_{B_\alpha^T}(y_2)\}\}$$

$$= \min\{\min\{\mu_{A_\alpha^T}(x_1), \mu_{B_\alpha^T}(y_1)\}, \min\{\mu_{A_\alpha^T}(x_2), \mu_{B_\alpha^T}(y_2)\}\}$$

$$= \min\{\mu_{A_\alpha^T \times B_\alpha^T}(x_1, y_1), \mu_{A_\alpha^T \times B_\alpha^T}(x_2, y_2)\}$$

$$\Rightarrow \quad \mu_{A_\alpha^T \times B_\alpha^T}((x_1, y_1) * (x_2, y_2)) \geq \min\{\mu_{A_\alpha^T \times B_\alpha^T}(x_1, y_1), \mu_{A_\alpha^T \times B_\alpha^T}(x_2, y_2)\}$$

Hence $A_\alpha^T x B_\alpha^T$ is a fuzzy Z-Subalgebra of $X x X$.

Theorem 4.4.3: If A and B are fuzzy Z-ideals in a Z-algebra X, then $A^T_\alpha x B^T_\alpha$ is a fuzzy Z-ideal in $X \times X$.

Proof : Let $(x_1, x_2) \in X x X$.

(i) $\mu_{A^T_\alpha \times B^T_\alpha}(0,0) = \min\{\mu_{A^T_\alpha}(0), \mu_{B^T_\alpha}(0)\} = \min\{\mu_A(0) + \alpha, \mu_B(0) + \alpha\}$

$$= \min\{\mu_A(0), \mu_B(0)\} + \alpha$$

$$\geq \min\{\mu_A(x_1), \mu_B(x_2)\} + \alpha$$

$$= \min\{\mu_A(x_1) + \alpha, \mu_B(x_2) + \alpha\}$$

$$= \min\{\mu_{A^T_\alpha}(x_1), \mu_{B^T_\alpha}(x_2)\}$$

$$= \mu_{A^T_\alpha \times B^T_\alpha}(x_1, x_2)$$

$$\Rightarrow \mu_{A^T_\alpha \times B^T_\alpha}(0,0) \geq \mu_{A^T_\alpha \times B^T_\alpha}(x_1, x_2)$$

(ii) Let $(x_1, x_2), (y_1, y_2) \in X \times X$

$$\mu_{A^T_\alpha \times B^T_\alpha}(x_1, x_2) = \min\{\mu_{A^T_\alpha}(x_1), \mu_{B^T_\alpha}(x_2)\}$$

$$= \min\{\mu_A(x_1) + \alpha, \mu_B(x_2) + \alpha\}$$

$$= \min\{\mu_A(x_1), \mu_B(x_2)\} + \alpha$$

$$\geq \min\{\min\{\mu_A(x_1 * y_1), \mu_A(y_1)\}, \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\} + \alpha$$

$$= \min\{\min\{\mu_A(x_1 * y_1), \mu_A(y_1)\} + \alpha, \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\} + \alpha\}$$

$$= \min\{\min\{\mu_A(x_1 * y_1) + \alpha, \mu_A(y_1) + \alpha\}, \min\{\mu_B(x_2 * y_2) + \alpha, \mu_B(y_2) + \alpha\}\}$$

$$= \min\{\min\{\mu_{A^T_\alpha}(x_1 * y_1), \mu_{A^T_\alpha}(y_1)\}, \min\{\mu_{B^T_\alpha}(x_2 * y_2), \mu_{B^T_\alpha}(y_2)\}\}$$

$$= \min\{\min\{\mu_{A^T_\alpha}(x_1 * y_1), \mu_{B^T_\alpha}(x_2 * y_2)\}, \min\{\mu_{A^T_\alpha}(y_1), \mu_{B^T_\alpha}(y_2)\}\}$$

$$= \min\{\mu_{A^T_\alpha \times B^T_\alpha}((x_1 * y_1), (x_2 * y_2)), \mu_{A^T_\alpha \times B^T_\alpha}(y_1, y_2)\}$$

$$= \min\{\mu_{A^T_\alpha \times B^T_\alpha}((x_1, x_2) * (y_1, y_2)), \mu_{A^T_\alpha \times B^T_\alpha}(y_1, y_2)\}$$

$$\Rightarrow \mu_{A^T_\alpha \times B^T_\alpha}(x_1, x_2) \geq \min\{\mu_{A^T_\alpha \times B^T_\alpha}((x_1, x_2) * (y_1, y_2)), \mu_{A^T_\alpha \times B^T_\alpha}(y_1, y_2)\}$$

From (i) and (ii), $A^T_\alpha x B^T_\alpha$ is a fuzzy Z-ideal in $X \times X$.

Theorem 4.4.4: Let A and B be fuzzy sets in a Z-algebra X such that $A^T_\alpha \times B^T_\alpha$ is a fuzzy Z-ideal of $X \times X$. Then,

(i) Either $\mu_{A^T_\alpha}(0) \geq \mu_{A^T_\alpha}(x)$ (or) $\mu_{B^T_\alpha}(0) \geq \mu_{B^T_\alpha}(x)$ for all $x \in X$.

(ii) If $\mu_{A^T_\alpha}(0) \geq \mu_{A^T_\alpha}(x)$ for all $x \in X$, then either $\mu_{B^T_\alpha}(0) \geq \mu_{A^T_\alpha}(x)$ (or) $\mu_{B^T_\alpha}(0) \geq \mu_{B^T_\alpha}(x)$.

(iii) If $\mu_{B^T_\alpha}(0) \geq \mu_{B^T_\alpha}(x)$ for all $x \in X$, then either $\mu_{A^T_\alpha}(0) \geq \mu_{A^T_\alpha}(x)$ (or) $\mu_{A^T_\alpha}(0) \geq \mu_{B^T_\alpha}(x)$.

Proof :

(i) Suppose that $\mu_{A^T_\alpha}(0) < \mu_{A^T_\alpha}(x_1)$ and $\mu_{B^T_\alpha}(0) < \mu_{B^T_\alpha}(x_2)$ for some $x_1, x_2 \in X$.

$$\mu_{A^T_\alpha \times B^T_\alpha}(x_1, x_2) = \min\{\mu_{A^T_\alpha}(x_1), \mu_{B^T_\alpha}(x_2)\} > \min\{\mu_{A^T_\alpha}(0), \mu_{B^T_\alpha}(0)\}$$

$$= \mu_{A^T_\alpha \times B^T_\alpha}(0,0)$$, which is a contradiction.

Therefore, $\mu_{A^T_\alpha}(0) \geq \mu_{A^T_\alpha}(x)$ (or) $\mu_{B^T_\alpha}(0) \geq \mu_{B^T_\alpha}(x)$ for all $x \in X$.

(ii) Assume that $\mu_{B^T_\alpha}(0) < \mu_{A^T_\alpha}(x_1)$ and $\mu_{B^T_\alpha}(0) < \mu_{B^T_\alpha}(x_2)$ for some $x_1, x_2 \in X$.

Then $\mu_{A^T_\alpha \times B^T_\alpha}(0,0) = \min\{\mu_{A^T_\alpha}(0), \mu_{B^T_\alpha}(0)\} = \mu_{B^T_\alpha}(0)$

and hence $\mu_{A^T_\alpha \times B^T_\alpha}(x_1, x_2) = \min\{\mu_{A^T_\alpha}(x_1), \mu_{B^T_\alpha}(x_2)\}$

$> \mu_{B^T_\alpha}(0) = \mu_{A^T_\alpha \times B^T_\alpha}(0,0)$,which is a contradiction.

Hence, if $\mu_{A^T_\alpha}(0) \geq \mu_{A^T_\alpha}(x)$ for all $x \in X$, then either $\mu_{B^T_\alpha}(0) \geq \mu_{A^T_\alpha}(x)$ (or) $\mu_{B^T_\alpha}(0) \geq \mu_{B^T_\alpha}(x)$.

(iii) will obtain by interchanging the roles of A and B in part (ii).

Theorem 4.4.5: Let A and B be fuzzy sets in a Z-algebra X such that $A^T_\alpha \times B^T_\alpha$ is a fuzzy Z-ideal of $X \times X$. Then either A or B is a fuzzy Z-ideal of X.

Proof : First we prove that B is a fuzzy Z-ideal of a Z-algebra X.

Since by Theorem 4.4.4(i), either $\mu_{A^T_\alpha}(0) \geq \mu_{A^T_\alpha}(x)$ (or) $\mu_{B^T_\alpha}(0) \geq \mu_{B^T_\alpha}(x)$ for all $x \in X$.

Assume that $\mu_{B^T_\alpha}(0) \geq \mu_{B^T_\alpha}(x)$ for all $x \in X$.

$\Rightarrow \quad \mu_B(0)+\alpha \geq \mu_B(x)+\alpha$

Therefore $\mu_B(0) \geq \mu_B(x)$ (1)

It follows from Theorem 4.4.4 (iii) either $\mu_{A_\alpha^T}(0) \geq \mu_{A_\alpha^T}(x)$ (or) $\mu_{A_\alpha^T}(0) \geq \mu_{B_\alpha^T}(x)$.

If $\mu_{A_\alpha^T}(0) \geq \mu_{B_\alpha^T}(x)$ for any $x \in X$, then $\mu_{B_\alpha^T}(x) = \min\{\mu_{A_\alpha^T}(0), \mu_{B_\alpha^T}(x)\} = \mu_{A_\alpha^T \times B_\alpha^T}(0, x)$

Now, $\mu_B(x)+\alpha = \mu_{B_\alpha^T}(x) = \mu_{A_\alpha^T \times B_\alpha^T}(0, x)$

$$\geq \min\{\mu_{A_\alpha^T \times B_\alpha^T}((0,x)*(0,y)), \mu_{A_\alpha^T \times B_\alpha^T}(0,y)\}$$

$$= \min\{\mu_{A_\alpha^T \times B_\alpha^T}((0*0),(x*y)), \mu_{A_\alpha^T \times B_\alpha^T}(0,y)\}$$

$$= \min\{\mu_{A_\alpha^T \times B_\alpha^T}(0, x*y), \mu_{A_\alpha^T \times B_\alpha^T}(0,y)\}$$

$$= \min\{\min\{\mu_{A_\alpha^T}(0), \mu_{B_\alpha^T}(x*y)\}, \min\{\min\{\mu_{A_\alpha^T}(0), \mu_{B_\alpha^T}(y)\}\}$$

$$= \min\{\mu_{B_\alpha^T}(x*y), \mu_{B_\alpha^T}(y)\}$$

$$= \min\{\mu_B(x*y)+\alpha, \mu_B(y)+\alpha\}$$

$$= \min\{\mu_B(x*y), \mu_B(y)\}+\alpha$$

$\Rightarrow \quad \mu_B(x) \geq \min\{\mu_B(x*y), \mu_B(y)\}$ (2)

Hence from (1) and (2) we get B is a fuzzy Z-ideal of a Z-algebra X. By Theorem 4.4.4(i) and (ii), assume that $\mu_{A_\alpha^T}(0) \geq \mu_{A_\alpha^T}(x)$ for all $x \in X$ and $\mu_{B_\alpha^T}(0) \geq \mu_{A_\alpha^T}(x)$ for any $x \in X$.

Then A is a fuzzy Z-ideal of a Z-algebra X.

Therefore, either A or B is a fuzzy Z-ideal of a Z-algebra X.

Definition 4.4.6: Let A_β^M and B_β^M be fuzzy β-multiplications of A and B in a Z-algebra X. The **Cartesian product** $A_\beta^M \times B_\beta^M$ with membership function $\mu_{A_\beta^M x B_\beta^M} : X x X \to [0,1]$ is defined by $\mu_{A_\beta^M x B_\beta^M}(x, y) = \min\{\mu_{A_\beta^M}(x), \mu_{B_\beta^M}(y)\}$ for all $x, y \in X$.

Theorem 4.4.7: Let A and B are fuzzy Z-Subalgebras of Z-algebra X, then A^M_β x B^M_β is also a fuzzy Z-Subalgebra of X x X.

Proof : For any $(x_1, y_1),(x_2, y_2) \in X \times X$,

$$\begin{aligned}
\mu_{A^M_\beta \times B^M_\beta}((x_1, y_1) * (x_2, y_2)) &= \mu_{A^M_\beta \times B^M_\beta}((x_1 * x_2),(y_1 * y_2)) \\
&= \min\{\mu_{A^M_\beta}(x_1 * x_2), \mu_{B^M_\beta}(y_1 * y_2)\} \\
&= \min\{\beta \cdot \mu_A(x_1 * x_2), \beta \cdot \mu_B(y_1 * y_2)\} \\
&= \beta \cdot \min\{\mu_A(x_1 * x_2), \mu_B(y_1 * y_2)\} \\
&\geq \beta \cdot \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\} \\
&= \min\{\beta \cdot \min\{\mu_A(x_1), \mu_A(x_2)\}, \beta \cdot \min\{\mu_B(y_1), \mu_B(y_2)\}\} \\
&= \min\{\min\{\beta \cdot \mu_A(x_1), \beta \cdot \mu_A(x_2)\}, \min\{\beta \cdot \mu_B(y_1), \beta \cdot \mu_B(y_2)\}\} \\
&= \min\{\min\{\mu_{A^M_\beta}(x_1), \mu_{A^M_\beta}(x_2)\}, \min\{\mu_{B^M_\beta}(y_1), \mu_{B^M_\beta}(y_2)\}\} \\
&= \min\{\min\{\mu_{A^M_\beta}(x_1), \mu_{B^M_\beta}(y_1)\}, \min\{\mu_{A^M_\beta}(x_2), \mu_{B^M_\beta}(y_2)\}\} \\
&= \min\{\mu_{A^M_\beta \times B^M_\beta}(x_1, y_1), \mu_{A^M_\beta \times B^M_\beta}(x_2, y_2)\}
\end{aligned}$$

$$\Rightarrow \quad \mu_{A^M_\beta \times B^M_\beta}((x_1, y_1) * (x_2, y_2)) \geq \min\{\mu_{A^M_\beta \times B^M_\beta}(x_1, y_1), \mu_{A^M_\beta \times B^M_\beta}(x_2, y_2)\}$$

Hence A^M_β x B^M_β is a fuzzy Z-subalgebra of X x X.

Theorem 4.4.8: If A and B are fuzzy Z-ideals in a Z-algebra X then A^M_β x B^M_β is a fuzzy Z-ideal in $X \text{ x } X$.

Proof : Let $(x_1, x_2) \in X \text{ x } X$.

(i) $$\begin{aligned}
\mu_{A^M_\beta \times B^M_\beta}(0,0) = \min\{\mu_{A^M_\beta}(0), \mu_{B^M_\beta}(0)\} &= \min\{\beta \cdot \mu_A(0), \beta \cdot \mu_B(0)\} \\
&= \beta \cdot \min\{\mu_A(0), \mu_B(0)\} \\
&\geq \beta \cdot \min\{\mu_A(x_1), \mu_B(x_2)\} \\
&= \min\{\beta \cdot \mu_A(x_1), \beta \cdot \mu_B(x_2)\} \\
&= \min\{\mu_{A^M_\beta}(x_1), \mu_{B^M_\beta}(x_2)\} \\
&= \mu_{A^M_\beta \times B^M_\beta}(x_1, x_2)
\end{aligned}$$

$$\Rightarrow \mu_{A_\beta^M \times B_\beta^M}(0,0) \geq \mu_{A_\beta^M \times B_\beta^M}(x_1, x_2)$$

(ii) Let $(x_1, x_2), (y_1, y_2) \in X \times X$

$$\mu_{A_\beta^M \times B_\beta^M}(x_1, x_2) = \min\{\mu_{A_\beta^M}(x_1), \mu_{B_\beta^M}(x_2)\}$$

$$= \min\{\beta \cdot \mu_A(x_1), \beta \cdot \mu_B(x_2)\}$$

$$= \beta \cdot \min\{\mu_A(x_1), \mu_B(x_2)\}$$

$$\geq \beta \cdot \min\{\min\{\mu_A(x_1 * y_1), \mu_A(y_1)\}, \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\}$$

$$= \min\{\beta \cdot \min\{\mu_A(x_1 * y_1), \mu_A(y_1)\}, \beta \cdot \min\{\mu_B(x_2 * y_2), \mu_B(y_2)\}\}$$

$$= \min\{\min\{\beta \cdot \mu_A(x_1 * y_1), \beta \cdot \mu_A(y_1)\}, \min\{\beta \cdot \mu_B(x_2 * y_2), \beta \cdot \mu_B(y_2)\}\}$$

$$= \min\{\min\{\mu_{A_\beta^M}(x_1 * y_1), \mu_{A_\beta^M}(y_1)\}, \min\{\mu_{B_\beta^M}(x_2 * y_2), \mu_{B_\beta^M}(y_2)\}\}$$

$$= \min\{\min\{\mu_{A_\beta^M}(x_1 * y_1), \mu_{B_\beta^M}(x_2 * y_2)\}, \min\{\mu_{A_\beta^M}(y_1), \mu_{B_\beta^M}(y_2)\}\}$$

$$= \min\{\mu_{A_\beta^M \times B_\beta^M}((x_1 * y_1), (x_2 * y_2)), \mu_{A_\beta^M \times B_\beta^M}(y_1, y_2)\}$$

$$= \min\{\mu_{A_\beta^M \times B_\beta^M}((x_1, x_2) * (y_1, y_2)), \mu_{A_\beta^M \times B_\beta^M}(y_1, y_2)\}$$

$$\Rightarrow \mu_{A_\beta^M \times B_\beta^M}(x_1, x_2) \geq \min\{\mu_{A_\beta^M \times B_\beta^M}((x_1, x_2) * (y_1, y_2)), \mu_{A_\beta^M \times B_\beta^M}(y_1, y_2)\}$$

From (i) and (ii), $A_\beta^M \times B_\beta^M$ is a fuzzy Z-ideal in $X \times X$.

Theorem 4.4.9: Let A and B be fuzzy sets in Z-algebra X such that $A_\beta^M \times B_\beta^M$ is a fuzzy Z-ideal of $X \times X$. Then,

(i) Either $\mu_{A_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ (or) $\mu_{B_\beta^M}(0) \geq \mu_{B_\beta^M}(x)$ for all $x \in X$.

(ii) If $\mu_{A_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ for all $x \in X$, then either $\mu_{B_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ (or) $\mu_{B_\beta^M}(0) \geq \mu_{B_\beta^M}(x)$.

(iii) If $\mu_{B_\beta^M}(0) \geq \mu_{B_\beta^M}(x)$ for all $x \in X$, then either $\mu_{A_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ (or) $\mu_{A_\beta^M}(0) \geq \mu_{B_\beta^M}(x)$.

Proof :

(i) Suppose that $\mu_{A_\beta^M}(0) < \mu_{A_\beta^M}(x_1)$ and $\mu_{B_\beta^M}(0) < \mu_{B_\beta^M}(x_2)$ for some $x_1, x_2 \in X$.

Then $\mu_{A_\beta^M \times B_\beta^M}(x_1, x_2) = \min\{\mu_{A_\beta^M}(x_1), \mu_{B_\beta^M}(x_2)\} > \min\{\mu_{A_\beta^M}(0), \mu_{B_\beta^M}(0)\}$

$= \mu_{A_\beta^M \times B_\beta^M}(0,0)$, which is a contradiction.

Therefore, $\mu_{A_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ (or) $\mu_{B_\beta^M}(0) \geq \mu_{B_\beta^M}(x)$ for all $x \in X$.

(ii) Assume that $\mu_{B_\beta^M}(0) < \mu_{A_\beta^M}(x_1)$ and $\mu_{B_\beta^M}(0) < \mu_{B_\beta^M}(x_2)$ for some $x_1, x_2 \in X$.

Then $\mu_{A_\beta^M \times B_\beta^M}(0,0) = \min\{\mu_{A_\beta^M}(0), \mu_{B_\beta^M}(0)\} = \mu_{B_\beta^M}(0)$

and hence $\mu_{A_\beta^M \times B_\beta^M}(x_1, x_2) = \min\{\mu_{A_\beta^M}(x_1), \mu_{B_\beta^M}(x_2)\}$

$> \mu_{B_\beta^M}(0) = \mu_{A_\beta^M \times B_\beta^M}(0,0)$, which is a contradiction.

Hence, if $\mu_{A_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ for all $x \in X$, then either $\mu_{B_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ (or) $\mu_{B_\beta^M}(0) \geq \mu_{B_\beta^M}(x)$.

(iii) will obtain by interchanging the roles of A and B in part (ii).

Theorem 4.4.10: Let A and B be fuzzy sets in a Z-algebra X such that A_β^M x B_β^M is a fuzzy Z-ideal of $X \times X$. Then either A or B is a fuzzy Z-ideal of X.

Proof : First we prove that B is a fuzzy Z-ideal of a Z-algebra X.

Since by Theorem 4.4.9(i), either $\mu_{A_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ (or) $\mu_{B_\beta^M}(0) \geq \mu_{B_\beta^M}(x)$ for all $x \in X$.

Assume that $\mu_{B_\beta^M}(0) \geq \mu_{B_\beta^M}(x)$ for all $x \in X$.

$\Rightarrow \beta \cdot \mu_B(0) \geq \beta \cdot \mu_B(x)$

Therefore $\mu_B(0) \geq \mu_B(x)$ (1)

It follows from Theorem 4.4.9(iii) either $\mu_{A_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ (or) $\mu_{A_\beta^M}(0) \geq \mu_{B_\beta^M}(x)$.

If $\mu_{A_\beta^M}(0) \geq \mu_{B_\beta^M}(x)$ for any $x \in X$, then

$\mu_{B_\beta^M}(x) = \min\{\mu_{A_\beta^M}(0), \mu_{B_\beta^M}(x)\} = \mu_{A_\beta^M \times B_\beta^M}(0, x)$

Now, $\beta \cdot \mu_B(x) = \mu_{B_\beta^M}(x) = \mu_{A_\beta^M \times B_\beta^M}(0, x)$

$\geq \min\{\mu_{A_\beta^M \times B_\beta^M}((0, x) * (0, y)), \mu_{A_\beta^M \times B_\beta^M}(0, y)\}$

$= \min\{\mu_{A_\beta^M \times B_\beta^M}((0 * 0), (x * y)), \mu_{A_\beta^M \times B_\beta^M}(0, y)\}$

$= \min\{\mu_{A_\beta^M \times B_\beta^M}(0, x * y), \mu_{A_\beta^M \times B_\beta^M}(0, y)\}$

$$= \min\{\min\{\mu_{A_\beta^M}(0), \mu_{B_\beta^M}(x * y)\}, \min\{\min\{\mu_{A_\beta^M}(0), \mu_{B_\beta^M}(y)\}\}$$

$$= \min\{\mu_{B_\beta^M}(x * y), \mu_{B_\beta^M}(y)\}$$

$$= \min\{\beta \cdot \mu_B(x * y), \beta \cdot \mu_B(y)\}$$

$$= \beta \cdot \min\{\mu_B(x * y), \mu_B(y)\}$$

$$\Rightarrow \quad \mu_B(x) \geq \min\{\mu_B(x * y), \mu_B(y)\} \qquad (2)$$

Hence from (1) and (2) , B is a fuzzy Z-ideal of a Z-algebra X.

By Theorem 4.4.9 (i) and (ii) , assume that $\mu_{A_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ for all $x \in X$ and $\mu_{B_\beta^M}(0) \geq \mu_{A_\beta^M}(x)$ for any $x \in X$.

Then A is a fuzzy Z-ideal of a Z-algebra X.

Therefore, either A or B is a fuzzy Z-ideal of a Z-algebra X.