Chapter - 1

CHAPTER 1

GRADATION OF OPENNESS

In this chapter, we discuss some basic definitions and results on fuzzy sets, fuzzy topological spaces, and gradation of openness. The results are due to Zadeh [40], Chang [8], Chattopadhyay, Hazra and Samanta [9].

SECTION: 1.1

FUZZY SETS AND FUZZY TOPOLOGICAL SPACES

Definition: 1.1.1

Let X be a nonempty set and I be the unit interval [0,1]. A **fuzzy set** in X is a function with domain X and values in I, that is an element of I^X.

Let $A, B \in \mathrm{I}^X$. We define the following fuzzy sets:

(i) **A includes B** (i.e., $B \subset A$) by $B(x) \leq A(x)$ for every $x \in X$

(ii) $\mathbf{A \cap B} \in \mathrm{I}^X$ by $(A \cap B)(x) = \min\{A(x), B(x)\}$ for every $x \in X$

(iii) $\mathbf{A \cup B} \in \mathrm{I}^X$ by $(A \cup B)(x) = \max\{A(x), B(x)\}$ for every $x \in X$

(iv) $\mathbf{A^C} \in \mathrm{I}^X$ by $A^C(x) = 1 - A(x)$ for every $x \in X$, where A^C is the complement of the fuzzy set A.

Let Δ be an indexing set and $\{A_\lambda \mid \lambda \in \Delta\}$ be a family of fuzzy sets in X. Then their **union** and **intersection** are defined as follows:

$\cup A_\lambda(x) = \sup \{A_\lambda(x) \mid \lambda \in \Delta\}$

$\cap A_\lambda(x) = \inf \{A_\lambda(x) \mid \lambda \in \Delta\}$.

Definition: 1.1.2

Let $A_1, A_2, \ldots\ldots A_n$ be fuzzy sets in X. The product fuzzy set $A = A_1 \times A_2 \times \ldots\ldots\ldots\ldots \times A_n$ in X^n is defined by

$$A(x_1, x_2, \ldots\ldots x_n) = \min\left(A_1(x_1), A_2(x_2), \ldots\ldots\ldots A_n(x_n)\right).$$

Note: 1.1.3

Ordinary subsets of X can be considered as fuzzy sets by identifying them with their characteristic functions.

Ordinary subsets are referred to as crisp sets when they are considered as fuzzy sets. Ordinary topological space is also referred to as crisp topological space.

If $A \subset X$ and if we consider A as a fuzzy set, we mean

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

Notation: 1.1.4

When an ordinary set A is considered as a fuzzy set, we write it as X_A or A itself.

In view of this, empty set ϕ and whole space X can be considered as fuzzy sets by identifying them with the constant functions 0 and 1 respectively.

Definition: 1.1.5

For $x \in X$ and $t \in I_0$, **a fuzzy point x_t** is defined by

$$x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

where $I_0 = (0, 1]$.

The **family of all fuzzy points** in X is denoted by $P_t(X)$.

Definition: 1.1.6

Let f be a function from X to Y. Let B be a fuzzy set in Y. Then **inverse image of B** or **preimage of B** written as $f^{-1}(B)$ is a fuzzy set in X defined by $\left(f^{-1}(B)\right)(x) = B\left(f(x)\right)$, for all $x \in X$.

Conversely, let A be a fuzzy set in X. **The image of A** written as **f(A)** is a fuzzy set in Y defined by

$$f(A)(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} A(z) & \text{if } f^{-1}(y) \text{ is nonempty, where } f^{-1}(y) = \{x \mid f(x) = y\} \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in Y$.

Note: 1.1.7

$$\big(f(A)\big)f(x) = \sup_{z \in f^{-1}(f(x))} A(z)$$

$$\geq A(x) \text{ as } \quad x \in f^{-1}(f(x)).$$

Proposition: 1.1.8

Let f be a function from X to Y. Then

(a) $f^{-1}(B^C) = \{f^{-1}(B)\}^C$ for any fuzzy set B in Y.

(b) $f(A^C) \supset \{f(A)\}^C$ for any fuzzy set A in X, where A^C is the complement of A.

(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$ where B_1 and B_2 are fuzzy sets in Y.

(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$ where A_1 and A_2 are fuzzy sets in X.

(e) $B \supset f(f^{-1}(B))$ for any fuzzy set B in Y.

(f) $A \subset f^{-1}(f(A))$ for any fuzzy set A in X.

(g) Let f be a function from X to Y and g be a function from Y to Z. Then $(g \circ f)^{-1}(C) = f^{-1}\big(g^{-1}(C)\big)$ for any fuzzy set C in Z, where $(g \circ f)$ is the composition of g and f.

(h) If f is onto, then $f(f^{-1}(A)) = A$.

Definition: 1.1.9

Let $\lambda, \mu \in I^X$, λ is said to be **quasi-coincident** with μ, denoted by $\lambda q \mu$ if there exists an $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise we denote it by $\lambda \overline{q} \mu$.

Definition: 1.1.10

A **fuzzy topology** on a set X is a collection of fuzzy sets in X satisfying the following axioms:

(i) ϕ, X $\in\delta$

(ii) A, B $\in\delta \Rightarrow A \cap B \in\delta$

(iii) $A_\lambda \in\delta$ for $\lambda\in\Delta \Rightarrow \underset{\lambda\in\Delta}{\cup} A_\lambda \in \delta$.

The pair (X, δ) is referred to as **fuzzy topological space**. A fuzzy topological space is denoted as **fts** in short.

Definition: 1.1.11

If (X, δ) is a fuzzy topological space, every member of δ is called an **open fuzzy set.** A fuzzy set A is called a **closed fuzzy set** iff A' $\in\delta$.

Definition: 1.1.12

Let(X, δ) be a fts. Then the **closure** and **interior** of a fuzzy set A $\in I^X$ are defined respectively as

$$\mathrm{cl}(A) = \cap\{B / B \supset A, B \in \delta\}$$

$$\mathrm{int}(A) = \cup\{B / B \subset A, B \in \delta\}$$

It is easily seen that cl(A) is the smallest closed fuzzy set larger than A and int (A) is the largest open fuzzy set smaller than A.

Definition: 1.1.13

Let δ be a fuzzy topology on a set X. A subfamily $\mathscr{B}$ of δ is a **base** for δ iff each member of δ can be expressed as the union of members of $\mathscr{B}$.

Definition: 1.1.14

Let (X, δ), (Y, δ') be two ftss. A mapping f : (X, δ) → (Y, δ') is **fuzzy continuous** iff for each open fuzzy set V in δ', the inverse image $f^{-1}(V)$ is open in δ.

Definition: 1.1.15

Let (X, δ), (Y, δ') be two ftss. A mapping f : (X, δ) → (Y, δ') is **fuzzy open** iff for each open fuzzy set V in δ, the image f(V) is open in δ'.

Definition: 1.1.16

Given two topologies δ_1, δ_2 on the same set X, we say δ_1 is **finer** than δ_2 (and that δ_2 is **coarser** than δ_1) if the identity mapping of (X, δ_1) into (X, δ_2) is fuzzy continuous.

SECTION: 1.2

GRADATION OF OPENNESS: FUZZY TOPOLOGY

Definition: 1.2.1

Let X be a nonempty set and $\tau : I^X \to I$ be a mapping satisfying the following properties:

(i) $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$

(ii) $\tau(\mu_i) \geq r$, i = 1, 2, implies $\tau(\mu_1 \cap \mu_2) \geq r$

(iii) $\tau(\mu_i) \geq r$, $i \in \Delta$, implies $\tau(\underset{i\in\Delta}{\cup} \mu_i) \geq r$

where $0 < r \leq 1$ or equivalently:

(i)' $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$

(ii)' $\tau(\mu_1 \cap \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$

(iii)' $\tau(\underset{i\in\Delta}{\cup} \mu_i) \geq \underset{i\in\Delta}{\wedge} \tau(\mu_i)$.

Then τ is called a **gradation of openness** on X.

Definition: 1.2.2

Let X be a nonempty set and $\mathscr{F}: I^X \to I$ be a mapping satisfying the following properties:

(i) $\mathscr{F}(\tilde{0}) = \mathscr{F}(\tilde{1}) = 1$

(ii) $\mathscr{F}(\mu_i) \geq r$, i = 1, 2, implies $\mathscr{F}(\mu_1 \cup \mu_2) \geq r$

(iii) $\mathscr{F}(\mu_i) \geq r$, $i \in \Delta$, implies $\tau(\underset{i\in\Delta}{\cap} \mu_i) \geq r$

where $0 < r \leq 1$ or equivalently:

(i)' $\mathcal{F}(\tilde{0}) = \mathcal{F}(\tilde{1}) = 1$

(ii)' $\mathcal{F}(\mu_1 \cup \mu_2) \geq \mathcal{F}(\mu_1) \wedge \mathcal{F}(\mu_2)$

(iii)' $\mathcal{F}(\bigcap_{i\in\Delta} \mu_i) \geq \bigwedge_{i\in\Delta} \mathcal{F}(\mu_i)$.

Then $\mathcal{F}$ is called a **gradation of closedness** on X.

Proposition: 1.2.3

Let τ be a gradation of openness on X and $\mathcal{F}_\tau: I^X \to I$ be defined by $\mathcal{F}_\tau(\mu) = \tau(\mu^c)$. Then $\mathcal{F}_\tau$ is a gradation of closedness on X.

Proposition: 1.2.4

Let $\mathcal{F}$ be a gradation of closedness on X and $\tau_{\mathcal{F}}: I^X \to I$ be defined by $\tau_{\mathcal{F}}(\mu) = \mathcal{F}(\mu^c)$. Then $\tau_{\mathcal{F}}$ is a gradation of openness on X.

Corollary: 1.2.5

Let $\mathcal{F}$ and τ be gradation of closedness and openness respectively. Then $\tau_{\mathcal{F}_\tau} = \tau$ and $\mathcal{F}_{\tau_{\mathcal{F}}} = \mathcal{F}$.

Proposition: 1.2.6

An arbitrary intersection of gradation of openness is a gradation of openness.

Definition: 1.2.7

Let (X, τ) be a fuzzy topological space and $Y \subset X$. Define a mapping $\tau_Y : I^Y \to I$ by the rule:

$\tau_Y(\mu) = V \left\{ \tau(\lambda) \, : \, \lambda \in I^X, \lambda / Y = \mu \right\}$

Then τ_Y is a gradation of openness on Y and $\tau_Y(\mu) \geq \tau_Y(\mu_X)$.

Definition: 1.2.8

Let (X, τ) and (Y, τ') be two fuzzy topological spaces and $f: X \to Y$ be a mapping. Then f is a called a **gradation preserving map (gp-map)** if for each $\mu \in I^Y$, $\tau'(\mu) \leq \tau(f^{-1}(\mu))$.

SECTION: 1.3

FUZZY CLOSURE OPERATOR

Definition: 1.3.1

Let $(X, \mathcal{F})$ be a fts with $\mathcal{F}$ being the gradation of closedness on X. For each $r \in I$, and for each $\lambda \in I^X$, define

$\mathrm{cl}(\lambda, r) = \cap \{ \lambda \in I^Y / \mu \supseteq \lambda, \mathcal{F}(\mu) \geq r \}$

Proposition: 1.3.2

Let $(X, \mathcal{F})$ be a fts with $\mathcal{F}$ being the gradation of closedness on X, and let $\mathrm{cl}: I^X \times I_0 \to I^X$ be the fuzzy closure operator in $(X, \mathcal{F})$. Then

(1) $cl(\tilde{0}, r) = \tilde{0}$, $cl(\tilde{1}, r) = \tilde{1}$, $\forall\ r \in I_0$

(2) $cl(\lambda, r) \supseteq \lambda$, $\forall\ \lambda \in I^X$

(3) $cl(\lambda, r) \subseteq cl(\lambda, r')$, if $r \leq r'$

(4) $cl(\lambda_1 \cup \lambda_2, s) = cl(\lambda_1, s) \cup cl(\lambda_2, s)\ \forall\ s \in I_0$

(5) $cl(cl(\lambda, s), s) = cl(\lambda, s)$

(6) If $r = V\{s \in I_0 / cl(\lambda, s) = \lambda\}$ then $cl(\lambda, r) = \lambda$.

Proposition: 1.3.3

Let cl: $I^X \times I_0 \rightarrow I^X$ be a mapping satisfying (1) - (4) of Proposition 1.3.2. Let $\mathscr{F}$: $I^X \rightarrow I$ be a mapping defined by

$\mathscr{F}(\lambda) = V\{r \in I_0 / cl(\lambda, r) = \lambda\}$, $\lambda \in I^X$.

Then $\mathscr{F}$ is a gradation of closedness on X such that cl = cl $\mathscr{F}$ iff (5) and (6) of Proposition 1.3.2 are satisfied by cl.

Proposition: 1.3.4

Let f : $(X, \tau) \rightarrow (Y, \tau')$ be a mapping. Then f is a gp-map iff $f(cl(\lambda, r)) \subseteq cl(f(\lambda)\ r)$, $\forall\ r \in I_0$ and $\lambda \in I^X$.