CHAPTER 5

Chapter 5

Generalized π- Closed Sets in Bi ˇCech Closure Spaces

5.1 Introduction

Cech closure spaces were introduced by ˇCech (1966) and studied by many authors. BiˇCech closure spaces were introduced by Chandrasekhara Rao, Gowri and Swaminathan (2008). Boonpok (2009) studied the concept of closed maps in BiˇCech Closure Spaces. Devi et al.(2010) introduced the concept of $\alpha\psi$- closed sets in BiˇCech closure spaces.

In this chapter , we have introduced the notion of gπ-closed sets in Bi- ˇ Cech closure space. Properties of (k_1, k_2)-gπ-open sets, (k_1, k_2)-gπ-continuous functions, (k_1, k_2)-gπ- irresolute functions, $_{g\pi}C_0$ bi-ˇCech spaces, $_{g\pi}C_1$ bi-ˇCech spaces , $_{g\pi}T^{*}_{1/2}$ bi-ˇCech spaces and $_{g\pi}T^{**}_{1/2}$ bi-ˇCech spaces are introduced and studied.

5.2. (k_1, k_2) - gπ closed sets

Definition 5.2.1. A subset A of a bi- ˇ Cech closure space (X, k_1, k_2) is said to be (k_1, k_2)- gπ closed if $k_2\pi\, cl(A) \subseteq U$ whenever $A \subseteq U$ and U is k_1- open in X.

Example: 5.2.2 Let X = {a, b, c} and let k_1 and k_2 be defined as $k_1(\{a\}) = \{a\}$ $k_1(\{b\}) = k_1(\{c\}) = k_1(\{b, c\}) = \{b, c\}$, $k_1(\{a, b\}) = k_1(\{a, c\}) = k_1(X) = X$, $k_1(\varphi) = \varphi$. and $k_2(\{a\}) = \{a\}$, $k_2(\{c\}) = k_2(\{a, c\}) = \{a, c\}$, $k_2(\{b\}) = k_2(\{a, b\}) = k_2(\{b, c\}) = k_2(X) = X$, $k_2(\varphi) = \varphi$. Now, the (k_1, k_2)- gπ closed sets are X, φ, {a}, {a, b}, {a, c}.

Theorem 5.2.3. If A and B are (k_1, k_2)- gπ closed sets then so is $A \cup B$.

Proof: Let A and B be two (k_1, k_2)- gπ closed sets. Let U be k_1-open set in X. Let $(A \cup B) \subseteq U$, $A \subseteq U$ and $B \subseteq U$. Then $k_2\, \pi\, cl(A) \subseteq U$ and $k_2\, \pi\, cl(B) \subseteq U$, and so $(k_2\, \pi\, cl(A) \cup k_2\, \pi\, cl(B)) \subseteq U$. Hence $k_2\, \pi\, cl(A \cup B) \subseteq U$. Thus $A \cup B$ is (k_1, k_2)- gπ closed set.

Theorem 5.2.4. If A is (k_1, k_2)- gπ closed set, then k_2 π cl (A) − A contains no nonempty k_1 - closed sets.

Proof: Let A be (k_1, k_2)- gπ closed. Let U be k_1 - closed contained in k_2 π cl (A) − A. Then, U ⊆ k_2 π cl (A) and U ⊆ A^c. Now, U ⊆ A^c then A ⊆ U^c. Since U is k_1-closed. k_2 π cl (A) ⊆ U^c. Consequently U ⊆ $[k_2$ π cl (A)$]^c$. As U ⊆ k_2 π cl (A) ∩ [k_2 π cl (A)$]^c$=φ, U = φ. Hence k_2 π cl (A) − A contains no non-empty k_1-closed sets.

Theorem 5.2.5. If A is a (k_1, k_2)- gπ closed set, then k_1 π cl (x)∩A≠ φ holds for each x ∈ k_2 π cl (A).

Proof: Let A be a (k_1, k_2)- gπ closed set. Let k_1 π cl (x) ∩ A=φ, for some x ∈ k_2 π cl (A). Then A ⊆ [k_1 π cl (x)$]^c$. Now, k_1 π cl (x) is k_1- π-closed. Therefore [k_1 π cl (x)$]^c$ is k_1- π open. Thus [k_1 π cl (x)$]^c$ is k_1- open. Since A is (k_1, k_2)- gπ closed set, k_2 π cl (A) ⊆ [k_1 π cl (x)$]^c$ implies k_2 π cl (A) ∩ k_1 π cl (x)=φ. Then x ∉ k_2 π cl (A) is a contradiction. Hence k_1 π cl (x) ∩ A≠ φ holds for each x ∈ k_2 π cl (A).

Theorem 5.2.6. Let (X, k_1, k_2) be a bi- ˇ Cech closure space. For each x in X, {x} is either k_1 - closed or $\{x\}^c$ is (k_1, k_2)- gπ closed.

Proof: Let (X, k_1, k_2) be a bi- ˇ Cech closure space. Suppose that {x} is not k_1- closed, then $\{x\}^c$ is not k_1- open. Therefore, the only k_1- open set containing $\{x\}^c$ is X. Thus, $\{x\}^c$ ⊆ X. Now, k_2 π cl [$\{x\}^c$] ⊆ k_2 π cl (X)=X. Hence $\{x\}^c$ is (k_1, k_2) - gπ closed set.

Theorem 5.2.7 Let A be a (k_1, k_2)- gπ closed set and if A is k_1 - open then A = k_2 π cl (A).

Proof: Let A be a (k_1, k_2)- gπ closed subset of a bi ˇ C ech closure spaces (X, k_1, k_2) and let A be k_1 - open set. Then k_2 π cl (A) ⊆ U whenever A ⊆ U and U is k_1 - open set in X. Since A is k_1 - open and A ⊆ A, k_2 π cl (A) ⊆ A. Thus, A = k_2 π cl (A).

Theorem 5.2.8. Let $A \subseteq Y \subseteq X$ and suppose that A is (k_1, k_2)- gπ closed in (X, k_1, k_2). Then A is (k_1, k_2)- gπ closed relative to Y .

Proof: Let S be any k_1 - open set in Y such that $A \subseteq S$. Then $S = U \cap Y$ for some k_1 -open set U in X. Therefore, $A \subseteq U \cap Y$. Hence $A \subseteq U$. Since A is a (k_1, k_2)- gπ closed set in X, $k_2\pi\ cl\ (A) \subseteq U$. Hence $Y \cap k_2\pi\ cl(A) \subseteq Y \cap U = S$. Thus A is (k_1, k_2)- gπ closed set relative to Y.

5.3. (k_1, k_2) - gπ open sets

Definition 5.3.1. A subset A in bi- ˇ Cech closure space (X, k_1, k_2) is called a (k_1, k_2)- gπ open set if A^c is (k_1, k_2)- gπ closed in (X, k_1, k_2).

Example 5.3.2. In example 5.2.2, when (k_1, k_2)- gπ open sets are X, φ, {b}, {c}, {b, c}.

Theorem 5.3.3. A subset A of (X, k_1, k_2) is (k_1, k_2)- gπ open set if and only if $F \subset (\text{int } k_2\ \pi\ cl\ (A))$ whenever F is a k_1- closed set and $F \subseteq A$.

Proof: Suppose A is (k_1, k_2)- gπ open in (X, k_1, k_2). Let F be a k_1- closed set and $F \subseteq A$. Then F^c is k_1- open set and $A^c \subseteq F^c$. Since A^c is a (k_1, k_2)- gπ closed set, $k_2\ \pi\ cl\ (A^c) \subseteq F^c$. Hence $F \subseteq [k_2\ \pi\ cl\ (A^c)]^c = \text{int } k_2\ \pi\ cl\ (A)$. That is, $F \subseteq \text{int } k_2\ \pi\ cl\ (A)$ whenever F is k_1- closed and $F \subseteq A$. Let V be any k_1-open set in X such that $A^c \subseteq V$. Thus $V^c \subseteq A$ and V^c is k_1- closed. Therefore $V^c \subseteq \text{int } k_2\ \pi\ cl\ (A)$. Then, $[\text{int } k_2\ \pi\ cl\ (A)]^c \subseteq V$. Implies $k_2\ \pi\ cl\ (A^c) \subseteq V$ gives A^c is (k_1, k_2)- gπ closed set. Thus A is (k_1, k_2)- gπ open set.

Corollary 5.3.4 If a subset A of (X, k_1, k_2) is (k_1, k_2)- gπ closed , then $k_2\ \pi\ cl\ (A) - A$ is (k_1, k_2)- gπ open.

Proof: Let F be a k_1- closed set such that $F \subseteq k_2\ \pi\ cl\ (A) - A$. Then, $F = \varphi$ (by Theorem 5.2.7. Therefore, $F \subseteq \text{int } k_2\ \pi\ cl\ (k_2\ \pi\ cl\ (A) - A)$. Hence $k_2\ \pi\ cl\ (A) - A$ is (k_1, k_2)- gπ open set.

Theorem 5.3.5. If A and B are (k_1, k_2)- gπ open sets, then so is $A \cap B$.

Proof: Let $(A^c \cup B^c) \subseteq U$ where U is k_1- open. This implies $A^c \subseteq U$ and $B^c \subseteq U$, since A and B are (k_1, k_2)- gπ open sets. $k_2 \pi \text{ cl } (A^c) \subseteq U$ and $k_2 \pi \text{ cl } (B^c) \subseteq U$. Thus $(k_2 \pi \text{ cl } (A^c) \cup k_2 \pi \text{ cl } (B^c)) \subseteq U$. Thus $k_2 \pi \text{ cl } (A^c \cup B^c) \subseteq U$. Therefore $A \cap B$ is (k_1, k_2)- gπ open set.

Theorem 5.3.6. Let $A \subseteq Y \subseteq X$ and suppose that Y is k_2- π closed in X and A is (k_1, k_2)- gπ open in X, then A is (k_1, k_2)- gπ open set relative to Y .

Proof. Let S be any k_1- closed set in Y such that $S \subseteq A$. Then $S = U \cap Y$ for some U is k_1- closed in X. Therefore $U \cap Y \subset A$ implies $U \subseteq A$. Since A is (k_1, k_2)- gπ open set in X, $U \subseteq \text{int } k_2\pi\text{cl}(A)$. Hence $S = Y \cap U \subseteq Y \cap \text{int } k_2\pi\text{cl } (A)$. Thus A is (k_1, k_2)- gπ open set relative to Y.

5.4 gπ Continuous Maps

Definition 5.4.1. Let (X, k_1, k_2) and (Y, l_1, l_2) be bi- ˇ Cech closure spaces. A map $f : (X, k_1, k_2) \to (Y, l_1, l_2)$ is called gπ-closed (resp. gπ-open) if f(F) is a (l_1, l_2)-gπ-closed (resp. (l_1, l_2)-gπ -open) subset of (Y, l_1, l_2) for every closed (resp. open) subset F of (X, k_1, k_2) .

Theorem 5.4.2 Let (X, k_1, k_2), (Y, l_1, l_2) and (Z, n_1, n_2) be bi- ˇ Cech closure spaces. If $g \circ f : (X, k_1, k_2) \to (Z, n_1, n_2)$ is gπ -closed and $f : (X, k_1, k_2) \to (Y, l_1, l_2)$ is surjective and continuous, then $g : (Y, l_1, l_2) \to (Z, n_1, n_2)$ is gπ -closed.

Proof: Let F be a closed subset of (Y, l_1, l_2). Then F is a closed subset of (Y, l_1) and (Y, l_2), respectively. Since f is continuous, $f^{-1}(F)$ is a closed subset of (X, k_1) and (X, k_2), respectively. Consequently, $f^{-1}(F)$ is a closed subset of (X, k_1, k_2). Since $g \circ f$ is gπ -closed and f is surjective, $g \circ f(f^{-1}(F)) = g(F)$ is a (n_1,n_2)-gπ-closed subset of (Z, n_1, n_2). Therefore, g is gπ-closed.

Definition 5.4.3. Let (X, k_1, k_2) and (Y, l_1, l_2) be bi-ˇCech closure spaces. A map $f : (X, k_1, k_2) \rightarrow (Y, l_1, l_2)$ is called gπ -continuous if f^{-1} (F) is a (k_1, k_2) -gπ -closed subset of (X, k_1, k_2) for every closed subset F of (Y, l_1, l_2).

Example 5.4.4 Let X = {a, b} = Y and define closure operators k_1 and k_2 on X by $k_1(\varphi) = \varphi$, $k_1(\{a\}) = \{a\}$, $k_1(\{b\}) = k_1(X) = X$, $k_2(\varphi) = \varphi$, $k_2(\{a\}) = k_2\{b\} = k_2(X) = X$ .Define closure operators l_1 and l_2 on Y by $l_1(\varphi) = \varphi$, $l_1(\{b\}) = \{b\}$, $l_1\{a\}) = l_1(Y) = Y$, $l_2(\varphi) = \varphi$, $l_2(\{a\}) = l_2(\{b\}) = l_2(Y) = Y$. $f : (X, k_1, k_2) \rightarrow (Y, l_1, l_2)$ be the identity map. Then f is gπ -continuous

Theorem 5.4.5. Let (X, k_1, k_2), (Y, l_1, l_2) and (Z, n_1, n_2) be bi-ˇCech closure spaces. If $g \circ f : (X, k_1, k_2) \rightarrow (Z,n_1,n_2)$ is closed and $g : (Y, l_1, l_2) \rightarrow (Z,n_1,n_2)$ is injective and gπ -continuous, then $f : (X, k_1, k_2) \rightarrow (Y, l_1, l_2)$ is gπ -closed.

Proof: Let F be a closed subset of (X, k_1, k_2). Then F is a closed subset of (X, k_1) and (X, k_2), respectively. Since $g \circ f$ is closed, $g \circ f(F)$ is a closed subset of (Z, n_1) and (Z, n_2), respectively. Consequently, $g \circ f(F)$ is a closed subset of (Z, n_1, n_2). Since g is gπ -continuous and injective, $g^{-1}(g \circ f(F)) = f(F)$ is a gπ -closed subset of (Y, l_1, l_2). Therefore, f is gπ -closed.

Definition 5.4.6. Let (X, k_1, k_2) and (Y, l_1, l_2) be bi-ˇCech closure spaces. A map $f : (X, k_1, k_2) \rightarrow (Y, l_1, l_2)$ is called gπ -irresolute if f^{-1} (F) is a (k_1, k_2)-gπ -closed subset of (X, k_1, k_2) for every (l_1, l_2)-gπ -closed subset F of (Y, l_1, l_2).

Theorem 5.4.5 Let (X, k_1, k_2) and (Y, l_1, l_2) be bi-ˇCech closure spaces. If $f : (X, k_1, k_2) \rightarrow (Y, l_1, l_2)$ is gπ -irresolute, then f is gπ -continuous.

The converse need not be true. In example 5.4.4. f is gπ -continuous but it is not gπ –irresolute, because {b} is a gπ -closed subset of (Y, l_1, l_2) but $f^{-1}(\{b\}) = \{b\}$ is not gπ -closed subset of (X, k_1, k_2).

5.5. Applications of (k_1, k_2)-gπ- bi- ˇ Cech closure spaces

Definition 5.5.1 A bi-ˇ Cech closure space (X, k_1, k_2) is said to be a $_{g\pi}C_0$ bi-ˇ Cech space if for every gπ -open subset U of (X, k_1), $x \in U$ implies $k_2(\{x\}) \subseteq U$.

Example 5.5.2 Let X = {a, b} and define a closure operator k_1 on X by $k_1(\{\varphi\}) = \varphi$, $k_1(\{a\}) = \{k_1(\{b\}) = k_1(X) = X$. Define a closure operator k_2 on X by $k_2(\{\varphi\}) = \varphi$, $k_2(\{a\}) = k_2(\{b\}) = k_2(X) = X$. Then (X, k_1, k_2) is a $_{g\pi}C_0$ bi- ˇ Cech space.

Theorem 5.5.3 A bi- ˇ Cech closure space (X, k_1, k_2) is a $_{g\pi}C_0$ bi-ˇC ech space if and only if for every gπ -closed subset F of (X, k_1) such that $x \notin F$, $k_2(\{x\}) \cap F = \varphi$.

Proof: Let F be a gπ -closed subset of (X, k_1) and let $x \notin F$. Since $x \in X-F$ and $X - F$ is a gπ -open subset of (X, k_1), $k_2(\{x\}) \subseteq X - F$. Consequently $k_2(\{x\}) \cap F = \varphi$.

Conversely, let U be a gπ -open subset of (X, k_1) and let $x \in U$. Since $X - U$ is a gπ -closed subset of (X, k_1) and $x \notin X - U$, $k_2(\{x\}) \cap (X - U) = \varphi$. Consequently $k_2(\{x\}) \subseteq U$. Hence, (X, k_1, k_2) is a $_{g\pi}C_0$ bi- ˇ Cech space.

Theorem:5.5.4 Let $\{(X_i, k_i^1, k_i^2): i \in I\}$ be a family of bi-ˇC ech closure spaces. If $\prod_{i\in I}(X_i, k_i^1, k_i^2)$ is an $_{g\pi}C_0$ bi- ˇ Cech space , then (X_i, k_i^1, k_i^2) is an $_{g\pi}C_0$ bi- ˇ Cech space for each $i \in I$.

Proof: Suppose that $\prod_{i\in I}(X_i, k_i^1, k_i^2)$ is an $_{g\pi}C_0$ bi- ˇ Cech space . Let $j \in I$ and let G be an gπ -open subset of (X_j, k_j^1) such that $x_j \in G$. Then $G \times \prod_{i\neq j, i\in I} X_i$ is an gπ -open subset of $\prod_{i\in I}(X, k_i^1)$ such that $(x_i)_{i\in I} \in G \times \prod_{i\neq j, i\in I} X_i$. Since $\prod_{i\in I}(X_i, k_i^1, k_i^2)$ is an $_{g\pi}C_0$ bi- ˇ Cech space, $\prod_{i\in I}$

$k_i^2 \pi_i(\{(x_i)_{i\in I}^{\bullet}\}) \subseteq G\times \prod_{i\neq j, i\in I} X_i$. Consequently, $k_j^2\{x_j\} \subseteq G$. Hence (X_i, k_i^1, k_i^2) is an ${}_{g\pi}C_0$ bi- ˇ Cech space.

Definition 5.5.5 A bi- ˇ Cech closure space (X, k_1, k_2) is said to be ${}_{g\pi}C_1$ bi- ˇ Cech space if for each $x, y \in X$ such that $k_1(\{x\}) \neq k_2(\{y\})$, there exist a disjoint gπ-open subset U of (X, k_2) and a gπ-open subset V of (X, k_1) such that $k_1(\{x\}) \subseteq U$ and $k_2(\{y\}) \subseteq V$.

Example 5.5.6 Let Let X = {a, b} and define a closure operator k_1 on X by $k_1(\{\varphi\}) = \varphi$ and $k_1(\{a\}) =\{a\}$, $k_1(\{b\}) =\{b\}$, $k_1(X) = X$. Define a closure operator k_2 on X by $k_2(\{\varphi\}) = \varphi$ and $k_2(\{a\})=\{a\}$, $k_2(\{b\})=\{b\}$, $k_2(X) = X$. Then (X, k_1, k_2) is a ${}_{g\pi}C_1$ bi- ˇ Cech space.

Theorem 5.5.7 Every ${}_{g\pi}C_1$ bi- ˇ Cech space is a ${}_{g\pi}C_0$ bi- ˇ Cech space.

Proof. Let (X, k_1, k_2) be a ${}_{g\pi}C_1$ bi- ˇ Cech space. Let U be a gπ -open subset of (X, k_1) and let $x \in U$. If $y \notin U$, then $k_2(\{x\}) \neq k_1(\{y\})$ because $x \notin k_1(\{y\})$.Then there exists a gπ -open subset V_y of (X, k_2) such that $k_1(\{y\}) \subseteq V_y$ and $x \notin V_y$, which implies $y \notin k_2(\{x\})$. Consequently, $k_2(\{y\}) \subseteq U$. Hence, (X, k_1, k_2) is a ${}_{g\pi}C_0$ bi- ˇ Cech space.

The converse of the above theorem need not be true by the following example.

Example 5.5.8 Let X = {a, b} and define a closure operator k_1 on X by $k_1(\{\varphi\}) = \varphi$ and $k_1(\{a\})= k_1(\{b\}) = k_1(X) = X$. Define a closure operator k_2 on X by $k_2(\{\varphi\}) = \varphi$, $k_2(\{a\}) = \{a\}$ and $k_2(\{b\}) = k_2(X) = X$. Then (X, k_1, k_2) is a ${}_{g\pi}C_0$ bi- ˇ Cech space but it is not a ${}_{g\pi}C_1$ bi- ˇ Cech space.

Theorem 5.5.9 A bi- ˇ Cech closure space (X, k_1, k_2) is a ${}_{g\pi}C_1$ bi-ˇC ech space if and only if every pair of points x, y of (X, k_1, k_2) such that $k_1(\{x\}) \neq k_2(\{y\})$, there exists a gπ -open subset U of (X, k_1) and gπ -open subset V of (X, k_2) such that $x \in V$, $y \in U$ and $U \cap V = \varphi$.

Proof: Let (X, k_1, k_2) be a $_{g\pi}$ C_1 bi- ˇ Cech space. Let x, y be points of (X, k_1, k_2) such that $k_1(\{x\}) \neq k_2(\{y\})$. There exists a gπ -open subset U of (X, k_1) and gπ -open subset V of (X, k_2) such that $x \in k_1(\{x\}) \subseteq V$ and $y \in k_2(\{y\}) \subseteq U$.

Conversely, suppose that there exist a gπ -open subset U of (X, k_1) and gπ -open subset V of (X, k_2) such that $x \in V$, $y \in U$ and $U \cap V = \varphi$. Since every $_{g\pi}$ C_1 bi- ˇ Cech space is a $_{g\pi}$ C_0 bi- ˇ Cech space, $k_1(\{x\}) \subseteq V$ and $k_2(\{y\}) \subseteq U$.

Theorem 5.5.10 Let $\{(X_i, k_i^1, k_i^2) : i \in I\}$ be a family of bi- ˇ Cech closure spaces. If (X_i, k_i^1, k_i^2) is an $_{g\pi}$ C_1 bi- ˇ Cech space for each $i \in I$, then $\prod_{i\in I}(X_i, k_i^1, k_i^2)$ is an $_{g\pi}$ C_1 bi- ˇ Cech space.

Proof: Suppose that (X_i, k_i^1, k_i^2) is an $_{g\pi}$ C_1 bi- ˇ Cech space for each $i \in I$. Let $(x_i)_{I \in I}$ and $(y_i)_{i\in I}$ be points of $\prod_{i\in I} X_i$ such that $\prod_{i\in I}(k_i^1 \pi_i(\{(x_i)_{i\in I}\}) \neq \prod_{i\in I}(k_i^1 \pi_i(\{(y_i)_{i\in I}\})$. There exists $j \in I$ such that $k_j^1\{x_j\} \neq k_j^2\{y_j\}$. Since (X_j, k_j^1, k_j^2) is a $_{g\pi}$ C_1 bi- ˇ Cech space, there exist an gπ -open subset U of (X_j, k_j^1) and an gπ -open subset V of (X_j, k_j^2) such that $U \cap V = \varphi$, $k_j^2\{y_j\} \subseteq U$ and $k_j^1\{x_j\} \subseteq V$. Consequently,

$\prod_{i\in I} k_i^2 \pi_i(\{(y_i)_{i\in I}\}) \subseteq U \times \prod_{i\neq j, i\in I} X_i$ and $\prod_{i\in I} k_i^1 \pi_I(\{(x_i)_{i\in I}\}) \subseteq V \times \prod_{i\neq j, i\in I} X_i$ such that $U \times \prod_{i\neq j, i\in I} X_i$ is an gπ-open subset of $\prod_{i\in I}(X_i, k_i^1)$, and $V \times \prod_{i\neq j, i\in I} X_i$ is an gπ -open subset of $\prod_{i\in I}(X_i, k_i^2)$ and $(U \times \prod_{i\neq j, i\in I} X_i) \cap (V \times \prod_{i\neq j, i\in I} X_i) = \varphi$.

Hence, $\prod_{i\in I}(X_i, k_i^1, k_i^2)$ is an $_{g\pi}$ C_1 bi- ˇ Cech space.

Definition 5.5.11. A bi- ˇCech closure space (X, k_1, k_2) is called a $_{g\pi}T^{*}_{1/2}$- bi- ˇCech space if every (k_1, k_2)- gπ- closed subset of (X, k_1, k_2) is a closed subset of (X, k_2).

Theorem 5.5.12 Let (X, k_1, k_2) be a bi- ˇCech closure space. If (X, k_1, k_2) is a ${}_{g\pi}T^{*}_{1/2}$- bi- ˇCech space then every singleton subset of X is either a closed subset of (X, k_1) or an open subset of (X, k_2).

Proof: Let x∈ X and suppose that {x} is not a closed subset of (X, k_1). Then X − {x} is not a open subset of (X, k_1). The only open subset of (X, k_1) containing X − {x} is X, hence X − {x} is a (k_1, k_2)- gπ- closed subset of (X, k_1, k_2). Since (X, k_1, k_2) is a ${}_{g\pi}T^{*}_{1/2}$- bi- ˇCech space, X − {x} is a closed subset of (X, k_2). Consequently, {x} is an open subset of (X, k_2).

Definition 5.5.13. A bi- ˇCech closure space (X, k_1, k_2) is called a ${}_{g\pi}T^{**}_{1/2}$- bi- ˇCech space if every (k_1, k_2)- gπ- closed subset of (X, k_1, k_2) is a g- closed subset of (X, k_2).

Theorem 5.5.14 Let (X, k_1, k_2) be a bi- ˇCech closure space. If (X, k_1, k_2) is a ${}_{g\pi}T^{**}_{1/2}$- bi- ˇCech space then every singleton subset of X is either a closed subset of (X, k_1) or an g-open subset of (X, k_2).

Proof: Let x∈ X and suppose that {x} is not a closed subset of (X, k_1). Then X − {x} is not a open subset of (X, k_1). The only open subset of (X, k_1) containing X − {x} is X, hence X − {x} is a (k_1, k_2)- gπ- closed subset of (X, k_1, k_2). Since (X, k_1, k_2) is a ${}_{g\pi}T^{**}_{1/2}$- bi- ˇCech space, X − {x} is a g-closed subset of (X, k_2). Consequently, {x} is an g-open subset of (X, k_2).