A STUDY ON M/M/1 QUEUE WITH WORKING VACATION AND M/M/1 RETRIAL QUEUEING SYSTEM WITH TWO TYPES OF VACATION POLICIES UNDER ERLANG-K TYPE SERVICE

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DEEPIKA A. M.
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CERTIFIED AS BONAFIDE RESEARCH WORK

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ACKNOWLEDGEMENT

ACKNOWLEDGEMENT

First and foremost, the investigator is extremely thankful to the **LORD ALMIGHTY** for his graces and blessings showered on her.

The author takes immense pleasure in thanking **Thiru T.S.K.MEENAKSHI SUNDARAM**, M.A., M.Phil. Chancellor, Avinashilingam Deemed University for Women, Coimbatore, for providing the conducive infrastructure for the conduct of the research study.

The author expresses her sincere thanks to Late **Thiru T.K.SHANMUGANANDAM**, B.A., B.L., Former Chancellor, Avinashilingam Deemed University for Women, Coimbatore, for giving an opportunity to do research work.

The author would like to thank **Dr.(Mrs.) SHEELA RAMACHANDRAN** M.Sc., P.G. Dip. Ed., Ph.D. (Avinashilingam), Vice Chancellor, Avinashilingam Deemed University for Women, Coimbatore, for the support extended by her throughout the study.

The author records her deep sense of gratitude and indebtedness to **Hony. Col. Dr.(Tmt.) SAROJA PRABHAKARAN**, M.A., Dip. Ed., (Madras), Ph.D. (Mother Teresa), Former Vice Chancellor, Avinashilingam Deemed University for Women, Coimbatore, for providing adequate help towards the completion of the study.

The author extends her heartfelt thanks to **Dr.(Tmt.) GOWRI RAMAKRISHNAN**, M.Sc. (Madras), M.Phil., Ph.D. (Avinashilingam), Registrar, Avinashilingam Deemed University for Women, Coimbatore, for the encouragement given by her during the investigation.

The author is immensely pleased to express her deep sense of gratitude to **Dr.(Tmt.) R.PARVATHAM**, M.Sc., Dip. Ed., M.Phil. (Madras), Ph.D. (Avinashilingam), Dean, Faculty of Science, Avinashilingam Deemed University for Women, Coimbatore, for all the encouragement.

The investigator would like to thank **Dr.(Tmt.) A.PARVATHI**, M.Sc., Dip. Ed., M.Phil. (Madras), Ph.D. (Bharathiar), Professor and Head of the Department of Mathematics, Avinashilingam Deemed University for Women, Coimbatore, for her excellent advice and valuable guidance extended throughout the study.

The author express her sincere gratitude to her thesis advisor **Dr.(Tmt.) K.UDAYA CHANDRIKA**, M.Sc., M.Phil. (Madras), Ph.D. (Bharathiar), Professor Department of Mathematics, Avinashilingam Deemed University for Women, Coimbatore, for her inspiring guidance, encouragement to carry out this project work successfully.

The author expresses her sincere thanks to **Dr. (Tmt.) S.MUTHULAKSHMI,** M.Sc., Dip.H.Ed., (Madras), M.Phil., Ph.D., (Bharathiar), Associate Professor,Department of Mathematics, Avinashilingam Deemed University for Women, Coimbatore, for the invaluable help, guidance and persistent efforts.

The author would like to thank her thesis advisor **Mrs. L. SUDHA M.Sc., M.Phil.,** (Avinashilingam), Assistant Professor, Department of Mathematics, Avinashilingam Deemed University for women, Coimbatore, for her guidance.

The investigator would like to express her sincere thanks to all the **STAFF MEMBERS OF DEPARTMENT OF MATHEMATICS** who were responsible for the good finish of this dissertation.

Words fail to express her deep indebtedness to her **LOVING PARENTS, FRIENDS** and **ALL WELL WISHERS** for being the motivating forces behind this dissertation and the providing moral support and encouragement in carrying out work.

CONTENTS

CONTENTS

CHAPTER I

INTRODUCTION

CHAPTER I

INTRODUCTION

1.1 QUEUEING SYSTEM

A queueing system may be described as one, having a service facility, at which customers arrive for service and whenever there are more customers in the system than the service facility can handle simultaneously, a queue or waiting line is developed. The waiting customers take their turn for service according to a pre-assigned rule and after service they leave the system. Thus the input to the system consists of the customers demanding service and the output is the serviced customers. There are many valuable applications of the queueing system most of which have been well document in the literature of probability, operations research, management science and industrial engineering, some examples are traffic flow (vehicles, aircraft, and communications), scheduling (patients in hospitals, jobs on machines) and facility design (bank, post office, amusement parks).

1.2 Characteristics of Queueing System

The basic characteristics of a queueing system are the following

1. **Arrival pattern of customers**

   Arrival pattern means the manner in which customers arrive and join the system. This arrival may occur in single or in groups.

2. **Service pattern**

   The service describes the manner in which service is rendered. The customers may be served either singly or in batches, the time required for serving a unit or a batch is called service time.

3. Queue discipline

The queue discipline is the method by which the customers are selected for service from the set of customers waiting for service. The various types of queue disciplines are tabulated as follows

No	**Queue discipline**	**Description**
1	FIFO (or) FCFS	First In first Out (or) First come first serve. This is the most commonly used procedure in servicing customers.
2	LIFO (or) LCFS	Last In first Out (or) Last come first serve This procedures is used in inventory systems.
3	PIR	Priority in selection. Customers are prioritized upon arrival. This procedure is used in manual transmission managing systems.

The most common discipline observed in everyday life is first in first out.

4. Capacity of the system

The number of customers in the queue and in service put together is called system capacity.

5. Service channels

Queueing system may have several service channels to provide service. These service channels may be arranged in parallel or in series or combination of both, depending on the design of the systems service mechanism.

1.3 Notations for queues

Kendall's Notation for Queues A / B / C / D / E

$$\left.\begin{array}{ll} \text{A} & \text{Inter – arrival time distribution} \\ \text{B} & \text{Service time distribution} \end{array}\right\} \Rightarrow \begin{array}{l} \text{M exponential} \\ \text{D deterministic} \\ \text{Ek Erlangian (order k)} \\ \text{G general} \end{array}$$

C Number of servers

D Maximum number of jobs they can be there in the system (waiting and in service) Default is for infinite number of waiting positions.

E Queueing Disciplines (FCFS, LCFS, SIRO etc) Default is FCFS.

M / M / 1 or M / M / 1 / ∞ represents single server queue with Poission arrivals, exponentially distributed service times and infinite number of waiting positions.

1.4 Transient and steady state queueing system

A queueing system is said to be in transient state when its operating characteristics are dependent on time. Otherwise the system is said to be in steady state or equilibrium state. Solution of a queueing system depending upon time is called transient solution and independent of time is called steady state solution. Most of the analysis of queueing models is confined to steady state results.

1.5 Distributions

1. Geometric Distribution

If X is a discrete Random Variable that can assume the values 1, 2, 3,…, such that its probability mass function is given by

$P(x = r) = q^{r-1} p$; $r = 1, 2, \ldots$, where $p + q = 1$ then X is said to follow a geometric distribution.

2. Exponential distribution

A continuous random variable X is said to follow an exponential distribution or negative exponential distribution with parameter $\lambda > 0$, if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

3. Erlang Distribution

The Erlang distribution is a special case of the Gamma distribution where the shape parameter *k* is an integer. In the Gamma distribution, this parameter is not restricted only to the integers.

The probability density function of the Erlang distribution is

$$f(x; k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \quad \text{for } x, \lambda \geq 0.$$

The cumulative distribution function of the Erlang distribution is

$$F(x; k, \lambda) = 1 - \sum_{n=0}^{k-1} e^{-\lambda x} (\lambda x)^n / n!$$

1.6 Retrial queues

Queueing system in which arriving customers who find all servers and waiting positions (if any) occupied may retry for service after a period of time are called retrial queues.

1.7 Servers vacations

The periods for which the server is unavailable are said to be server vacation periods.

After completing service, if server finds an empty queue, he leaves for a vacation if on return from a vacation (at the end of a busy period), the server finds no customer waiting, he waits for the arrival of a customer. This is called a single vacation system and is denoted by Vs.If the server finds no customer waiting, he goes on taking vacations, until he finds at least one customer waiting. This is called a multiple vacation system and is denoted by V_m.

The server goes for vacation compulsorily after servicing to all customers and the system becomes empty. This type of vacation in queueing theory is called exhaustive service type single vacation.

REVIEW OF LITERATURE

1.8 REVIEW OF LITERATURE

The queueing systems with vacations have been well studied because of their applications in modeling the computer networks, communication and manufacturing service systems. These are all discussed by Fuhrmann and Cooper (1995). The details of various vacations policies which provide more flexibility for optimal design can be seen in monographs of Takagi (2006) and Tian and Zhang (2006). Working vacations is a kind of semi-vacation policy and is also a new vacation policy that was first introduced by Servi and Finn (2003) : a customer is served at a lower rate rather than completely stopping the service during a vacation. In the classical vacation queueing models, the server completely stops the service (during the vacation period) and such a policy may cause the loss or dissatisfaction of the customers. But under working vacation policy, the server can still work during the vacation and may accomplish other assistant work simultaneously. So, the working vacation is more reasonable than the classical vacation in some cases.

Servi and Finn (2006) studied M / M / 1 queue with multiple working vacations, and obtained the probability generating function of the number of customers in the system and the Laplace Stieltjies transform of the waiting time distribution, and applied the results to perform analysis of gateway router in fiber communication networks. Later Liu et al (2007) gave simple explicit expressions of distributions for the stationary queue length, discussed the stochastic decomposition structures of stationary indices and derived expected regular busy period and expected busy cycle. Kim et al (2003) and Takagi (2006) generalized the work of Servi and Finn's model to an M / G / 1 queue with multiple working vacations. Recently Liu et al (2007) analyzed an M / G / 1 queue with exponentially distributed working vacation and obtained the joint distribution of stationary queue length and service status at the arbitrary epoch by considering semi markov-process. Baba (2005) investigated a GI/ M / 1 queue working vacatins. Banik et al (2007) studied a GI / M / 1 / N working vacation queue with limit waiting space.

Queueing systems in which arriving customers who find all servers and waiting positions (if any) occupied may retry for service after a period of time are called Retrial queues, given by the authors Artalejo, J.R.(1995), Falin, G.I and Templeton, J.G.C.(1997). Because of the complexity of the retrial queueing models, analystic results are generally difficult to obtain. There are a great number of numerical and approximations methods available. In this paper we will place more emphasis on the solutions by Matrix geometric method. In the literature, the analysis for queueing systems with vacations has been discussed through a considerable amount of work in recent years. Queues with server vacations occur in many engineering systems such as data switching systems, computer communication networks and telecommunication systems. Doshi, B.T. (1990) has recorded Keilson, J and Servi, L.D (1986) have introduced a queueing system with Bernoulli vacation scheduling service that is clearly applicable to queueing systems involving communication systems. Many examples such as production system, bank services, computer and communication networks, etc., work with different vacation policies.

WOKD DONE IN THE THESIS

1.9 WORK DONE IN THE THESIS

Preliminary definitions are presented in the introduction chapter.

In chapter II the paper entitled M / M /1 queue with working vacation presented by M. Jemila Parveen, M.I.Afthab Begum, K.Julia Rose Mary (2010) is discussed.

In chapter III the paper entitled M /M /1retrial queueing system with two types of vacation policies under Erlang-k type service is considered by G.Ayyappan, Gopal Sekar, A.Muthu Ganapathi (2010).

CHAPTER II

M/M/1 QUEUE WITH WORKING VACATION

CHAPTER II

M / M / 1 QUEUE WITH WORKING VACATION

Two models of M /M/ 1 queueing system with working vacation are considered. The server takes multiple working vacation in model I and single working vacation in model II. By modeling the queue as a two dimensional Markov chain and writing the Chapman-Kolmovgrov equations satisfied by the system size probabilities. The steady state system size probability distribution and mean system length are obtained.

2.1 MODEL I: M / M / 1 QUEUEING MODEL WITH MULTIPLE WORKING VACATION

In this section single server Markovian queueing system with multiple working vacation is analyzed.

2.1.1 Model Description

Consider a classical M / M / 1 queue with arrival rate λ and regular service rate μ_0. The server begins a working vacation of random length V at the instant when the queue becomes empty. The vacation duration follows an exponential distribution with parameter η. During a working vacation an arriving unit is served at the rate of μ_v. On returning from vacation, if the server finds customers waiting in the system, he switches service rate from μ_v to μ_0 and then starts regular busy period otherwise the server goes again for multiple working vacation. This queueing model is denoted by M / M / 1 MWV when MWV denotes multiple working vacation. In this model it is assumed that the inter arrival time, regular service time, service time during vacation and working vacation follow exponential distributions. The queue discipline is assumed to be FIFO.

Let N_v (t) denote the number of customers in the system at time t and

$$J(t) = \begin{cases} 0 \text{ when the system is on working vacation period at time t.} \\ 1 \text{ when the system is on busy period at time t.} \end{cases}$$

Then $\{N_v(t), J(t)\}$ is a Quasi birth and death process. Hence the system is studied as a Markov process on the state space $S = \{(n, 0): n \geq 0\} \cup \{(n, 1): n \geq 1\}$. The process is said to be in the state $(n, 0): n > 0$, when there are n customers in the system and the server is on vacation and working at service rate μ_v. The process is said to be in the state $(n, 1) : n \geq 1$, when there are n customers in the system and the server is busy with regular rate μ_0. Further the state $(0, 0)$ represents the server is on vacation and idle.

By defining $P_{nj}(t)$ as the probability that the system is in the state $(n, j): n \geq 0$, $j = 0, 1$ at time t, and assuming the existence steady state probabilities $P_{nj} = \lim_{t \to \infty} P_{nj}(t)$ exist, the steady state equations satisfied by P_{nj}'s are obtained as

$$\lambda P_{00} = \mu_v P_{00} + \mu_0 P_{10} \tag{1}$$

$$(\lambda + \mu_v + \eta) P_{no} = \lambda P_{n-10} + \mu_v P_{n+10} \qquad n \geq 1 \tag{2}$$

$$(\lambda + \mu_0) P_{11} = \mu_0 P_{21} + \eta P_{10} \tag{3}$$

$$(\lambda + \mu_0) P_{n1} = \lambda P_{n-11} + \mu_0 P_{n+11} + \eta P_{n0} \qquad n > 1 \tag{4}$$

2.1.2 Steady state solutions

To obtain the expressions for P_{n0} and P_{n1}, we define the forward shifting operator E such that,

$$E(P_{n0}) = P_{n+1\,0}, \qquad n \geq 0$$

$$E(P_{n1}) = P_{n+1\,1}, \qquad n \geq 1$$

With the definition of E the equations (2) and (4) can be written as

$(\mu_v E^2 - (\lambda + \mu_v + n) E + \lambda) P_{n0} \quad = \quad 0 \qquad \forall\ n \geq 0 \qquad (5)$

$(\mu_0 E^2 - (\lambda + \mu_0) E + \lambda) P_{n1} \quad = \quad -\eta P_{n+10} \qquad \forall\ n \geq 1 \qquad (6)$

The characteristic equation of the homogenous differential equations (5) is

$f(z) = \mu_v z^2 - (\lambda + \mu_v + \eta) z + \lambda = 0. \qquad (7)$

This has two roots r_0 and r_1 given by

$$r_0 = \frac{(\lambda + n + \mu_v) + \sqrt{(\lambda + \eta + \mu_v)^2 - 4\mu_v\lambda}}{2\mu_v} \qquad \text{and} \qquad r_0 > 1$$

$$r_1 = \frac{(\lambda + \eta + \mu_v) - \sqrt{(\lambda + n + \mu_v)^2 - 4\mu_v\lambda}}{2\mu_v} \qquad \text{and} \qquad r_1 < 1$$

Expanding the expression of r_1 using Binomial theorem and neglecting higher powers, we have

$$r_1 = \frac{(\lambda + n + \mu_v) - (\lambda + \eta + \mu_v)\left(1 - \frac{1}{2}\left(\frac{4\mu_v\lambda}{(\lambda + \eta + \mu_v)^2}\right)\right)}{2\mu_v}$$

$$= \frac{\lambda}{(\lambda + \eta + \mu_v)}$$

$r_1 < 1$ implies $\frac{\lambda}{(\lambda + \eta + \mu_v)} < 1$

Now from the expression of r_0 we have

$$r_0 = \frac{(\lambda + \eta + \mu_v) + (\lambda + \eta + \mu_v)\left(1 - 2\left(\frac{\mu_v\lambda}{(\lambda + \eta + \mu_v)^2}\right)\right)}{2\mu_v}$$

$$= \frac{(\lambda + n + \mu_v) - \dfrac{\mu_v \lambda}{(\lambda + n + \mu_v)}}{\mu_v}$$

$$= \frac{\lambda + \eta + \mu_v}{\mu_v} - \frac{\lambda}{\lambda + \eta + \mu_v}$$

$$= 1 + \left(\frac{\lambda + \eta}{\mu_v} - \frac{\lambda}{\lambda + \eta + \mu_v} \right)$$

$$= 1 + \left[\frac{\lambda(\lambda + \eta) + \eta(\lambda + \eta + \mu_v) + \mu_v}{\mu_v(\lambda + \eta + \mu_v)} \right] > 1$$

Since the series $\sum_{n=0}^{\infty} P_{n0}$ is convergent and $r_1 < 1$ the solution of the equation (5) is given by

$P_{n0} = Ar_1^{\,n}$ $\qquad n \geq 0$

Taking $n = 0$, we get $P_{00} = A$.

Hence $P_{n0} = P_{00}\, r_1^{\,n}$ $\qquad n \geq 0$ $\qquad$ (8)

We calculate P_{n1} $(n \geq 1)$ by solving the non homogeneous difference equation (6),

$$(\mu_0 E^2 - (\lambda + \mu_0)\, E + \lambda)\, P_{n1} = \eta\, P_{n+1\,0} \qquad n \geq 1$$

$$= (\eta\, P_{00}) r_1^{\,n+1} \qquad n \geq 1 \qquad (9)$$

The roots of the characteristic equation

$\mu_0 z^2 - (\lambda + \mu_0)\, z + \lambda = 0$ are $1, \dfrac{\lambda}{\mu_0}$

By assuming $\rho_0 = \dfrac{\lambda}{\mu_0} < 1,$ we get the steady state solution of the non homogeneous difference equation (9) as

$$P_{n1} = \left[A\rho_0^{\;n} - \frac{\eta r_1^{\;n+1}}{\mu_0 r_1^{\;2} - (\lambda + \mu_0) r_1 + \lambda} \right] P_{00}$$

$$= \left(A\rho_0^{\;n} + B r_1^{\;n} \right) P_{00} \qquad \forall\ n \ge 1 \tag{10}$$

where $$B = \frac{-\eta r_1}{\mu_0 r_1^{\;2} - (\lambda + \mu_0) r_1 + \lambda}$$

$$= \frac{-\eta r_1}{\mu_0 r_1^{\;2} - \lambda r_1 - \mu_0 r_1 + \lambda}$$

$$= \frac{-\eta r_1}{\mu_0 (1 - r_1)\left(\frac{\lambda}{\mu_0} - r_1 \right)}$$

$$= \frac{\eta r_1}{\mu_0 (1 - r_1)(r_1 - \rho_0)} \tag{11}$$

Substituting for P_{10} and P_{11}in equation $\lambda P_{00} = \mu_v P_{10} + \mu_0 P_{11}$, we get,

$$\lambda = \mu_v r_1 + \mu_0 (A\rho_0 + B r_1)$$

$$A\lambda = \lambda - \mu_v r_1 + \frac{\eta r_1^{\;2}}{(1 - r_1)(\rho_0 - r_1)}$$

by simplification we get, $A = -B$ (12)

Thus the system size probabilities P_{n0} and P_{n1} are expressed in terms of P_{00} as given by,

$P_{no} = r_1^{\;n}\ P_{00} \qquad n \ge 0$

$P_{n1} = B\,(r_1^{\;n} - \rho_0^{\;n})\,P_{00} \qquad n \ge 0$ (13)

2.1.3 Total PGF for the model

Now we derive the total PGF P(z) for the model. Let $P(z) = P_0(z) + P_1(z)$.

where $P_0(z)=\sum_{n=0}^{\infty} P_{n0}Z^n$ and $P_1(z)=\sum_{n=1}^{\infty} P_{n1}Z^n$

Then $P_0(z)=\dfrac{P_{00}}{1-r_1 z}$ and $P_1(z)=\dfrac{Bz(r_1-\rho_0)P_{00}}{(1-r_1 z)(1-\rho_0 z)}$

$$P_1(z)=\frac{\eta r_1 z P_{00}}{\mu_0(1-r_1)(1-r_1 z)(1-P_0 z)}$$

$$=\frac{\eta r_1 z P_{00}}{(1-r_1)(1-r_1 z)(\mu_0-\lambda z)} \tag{14}$$

$$=\frac{P_{00}}{1-r_1 z}\left[1+\frac{\eta r_1 z}{\mu_0(1-r_1)(\mu_0-\lambda z)}\right]$$

Hence $P(z)=\dfrac{P_{00}}{1-r_1 z}+\dfrac{\eta r_1 z P_{00}}{(1-r_1)(1-r_1 z)(\mu_0-\lambda z)}$

$$=\frac{P_{00}}{1-r_1 z}\left[\frac{\eta r_1 z}{(1-r_1)(\mu_0-\lambda z)}\right]$$

$$=\frac{P_{00}}{1-r_1 z}\left[\frac{(1-r_1)(\mu_0-\lambda_2)+\eta r_1 z}{(1-r_1)(\mu_0-\lambda_2)}\right]$$

$$=P_{00}\left[\frac{(1-r_1)(\mu_0-\lambda_2)+\eta r_1 z}{(1-r_1 z)(1-r_1)(\mu_0\lambda_2)}\right]$$

$$=\left[\frac{\mu_0-r_1\mu_0-\lambda z+r_1\lambda z+\eta r_1 z}{(\mu_0\lambda_2)(1-r_1)(1-r_1 z)}\right]P_{00}$$

$$=\left[\frac{\mu_0(1-r_1)+z(r_1\lambda+\eta r_1-\lambda)}{(\mu_0\lambda_2)(1-r_1)(1-r_1 z)}\right]P_{00}$$

$$=\left[\frac{\mu_0(1-r_1)+z((\lambda+\eta)r_1-\lambda)}{(\mu_0\lambda_2)(1-r_1)(1-r_1 z)}\right]P_{00}$$

Since r_1 is a root of the equation $\mu_v z^2-(\lambda+\mu_v+\eta)z+\lambda=0$, we have

$$\mu_v r_1^2-(\lambda+\mu_v+\eta)r_1+\lambda=0.$$

$\mu_v r_1^2 - \lambda r_1 - \mu_v r_1 - \eta r_1 + \lambda = 0.$

$\mu_v r_1 (r_1 - 1) - r_1(\lambda + \eta) + \lambda = 0.$

$\mu_v r_1 (r_1 - 1) = r_1 (\lambda + \eta) - \lambda.$

Using the above equation, P(z) becomes

$$P(z) = \frac{\mu_0(1-r_1) + z\mu_v r_1(r_1 - 1)}{(\mu_0 \lambda_2)(1-r_1)(1-r_1 z)} P_{00}$$

$$= \frac{(\mu_0 + z\mu_v r_1)(1-r_1)}{\mu_0 \lambda_2 (1-r_1 z)(1-r_1)} P_{00}$$

$$= \frac{(\mu_0 + z\mu_v r_1)}{(\mu_0 \lambda_2)(1-r_1 z)} P_{00}$$

The value of P_{00} can be calculated by using the normalizing condition

$$1 = P(1) = \underset{z \to 1}{lt} P(z) = \underset{z \to 1}{lt} \frac{(\mu_0 - \mu_v r_1 z)}{(1-r_1 z)(\mu_0 - \lambda z)} P_{00}$$

$$= \frac{(\mu_0 - \mu_v r_1)}{(1-r_1)(\mu_0 - \lambda)} P_{00}$$

$$P_{00} = \frac{(1-r_1)(\mu_0 - \lambda)}{(\mu_0 - \mu_v r_1)}$$

$$= \frac{(1-r_1)(\mu_0 - \lambda)}{\left(1 - \frac{\mu_v}{\mu_0} r_1\right)} = \frac{(1-r_1)\left(1 - \lambda/\mu_0\right)}{1 - \frac{\mu_v}{\mu_0} r_1}$$

$$= \frac{(1-r_1)(1-\rho_0)}{\left(1 - \frac{\mu_v}{\mu_0} r_1\right)}$$

$$= \frac{(1-r_1)(1-\rho_0)}{(1-\rho_0)\left(\frac{\lambda-\mu_v}{\mu_0}r_1\right)}$$

Hence $$P(z) = \left(\frac{1-\rho_0}{1-\rho_0 z}\right)\left[\frac{1-\frac{\mu_v}{\mu_0}r_1 z}{1-r_1 z}\right]\left[\frac{(1-r_1)}{1-\rho_0+\frac{\eta r_1}{\mu_0(1-r_1)}}\right] \tag{15}$$

2.1.4 Expected system length

In this section we derive the mean system size of the model and mean queue size of the model.

If L denotes the mean system size of the model

$$L = \sum_{n=1}^{\infty} nP_{n0} + \sum_{n=1}^{\infty} nP_{n1}$$

$$= \sum_{n=1}^{\infty} n\left[r_1^{\,n} + B\left(r_1^{\,n} - \rho_0^{\,n}\right)\right]P_{00}$$

$$= \left[\frac{\rho_0}{1-\rho_0} + \frac{r_1}{1-r_1} - \frac{\mu_v r_1}{\mu_0 - \mu_v r_1}\right] \tag{16}$$

If Lq denotes the mean queue size then

$$Lq = \sum_{n=1}^{\infty}(n-1)P_{no} + \sum_{n=1}^{\infty}(n-1)P_{n1}$$

$$= \sum_{n=1}^{\infty} n\left(P_{n0} + P_{n1}\right) - \sum_{n=1}^{\infty}\left(P_{n0} + P_{n1}\right)$$

$$= L - (1 - P_{00}) \tag{17}$$

2.1.5 Decomposition Property

It is well known that stochastic decomposition results have a probabilistic interpretation in a classical vacation queue where the server completely stops service during a vacation. Fuhrman and Cooper (1995) established stochastic decomposition structure for classical M / G / 1 queue with general vacation.

In the following section, we prove that the queue with working vacation (when the server serves customers at a lower rate during a vacation) has similar decomposition stochastic property. The stochastic decomposition structure of the number of customer in an M / M / 1 / MWV is demonstrated in the following theorem.

Theorem

If $\rho_0 < 1$ and $\mu_0 > \mu_v$, then the probability generating function of the steady state system size probabilities of the working vacation model is decomposed as $Q_0(z)$. $Q_d(z)$ where $Q_0(z)$ – the geometric distribution with parameter $(1 - \rho_0)$ gives the probability generating function of the system size of the classical M / M / 1 queue without vacation and $Q_d(z)$ – the modified geometrical distribution gives the probability generating function of the additional queue length Q_d.

Proof

We know that

$$P(z) = \left(\frac{1-\rho_0}{1-\rho_0 z}\right)\left(\frac{1-\frac{\mu_v}{\mu_0} r_1 z}{1-r_1 z}\right)\left(\frac{(1-r_1)}{1-\rho_0+\frac{\eta r_1}{\mu_0(1-r_1)}}\right)$$

$$= Q_0 (z)\, Q_d (z)$$

where $Q_0(z) = \left(\frac{1-\rho_0}{1-\rho_0 z}\right)$ and

$$Q_d(z) = \left(\frac{1-\frac{\mu_v}{\mu_0} r_1 z}{1-r_1 z}\right)\left(\frac{(1-r_1)}{1-\rho_0+\frac{\eta r_1}{\mu_0(1-r_1)}}\right)$$

$$= K(1-r_1)\left(\frac{1-\frac{\mu_v}{\mu_0} r_1 z}{1-r_1 z}\right)$$

where $K = \dfrac{1}{1-\rho_0+\frac{\eta r_1}{\mu_0(1-r_1)}}$

$$Q_d(z) = K(1-r_1)\left(\frac{1-r_1 z+r_1 z-\frac{\mu_v}{\mu_0} r_1 z}{1-r_1 z}\right) \tag{18}$$

$$= K\left[(1-r_1 z)+\frac{(1-r_1) r_1 z}{1-r_1 z}\left(1-\frac{\mu_v}{\mu_0}\right)\right] \tag{19}$$

If Q_d be the random variable defined by,

Prob $(Q_d = 0) = K(1- r_1)$

Prob $(Q_d = k) = K\left(1-\frac{\mu_v}{\mu_0}\right)(1-r_1) r_1^{\,k}$ K > 0.

Then the probability generating function of Q_d is given by

$$Q_d(Z) = \text{Prob } (Q_d=0) + \sum_{k=1}^{\infty} \text{Prob}(Q_d = k) Z^k$$

this will coincide with the equation (18).

2.1.6 Other performance measures

1. Let P_v denotes the probability that the server is on vacation, then $P_v = \frac{P_{00}}{(1-r_1)}$

2. Let P_{busy} denote the probability that the server is busy, then

$$P_{busy} = B\sum_{n=1}^{\infty}\left(r_1^n - \rho_0^n\right)P_{00}$$

$$= B\left(\frac{r_1}{1-r_1} - \frac{\rho_0}{1-\rho_0}\right)P_{00}$$

$$= B\left(\frac{r_1-\rho_0}{(1-r_1)(1-\rho_0)}\right)P_{00}$$

2.1.7 Waiting time distribution

We are going to derive the waiting time distribution of an arbitrary customer at a random point of time.

In this we prove the classical relation $\rho(z) = W^*(\lambda(1-z))$ which is true for M / M / 1 multiple working vacation model and we deduce the little formula.

The waiting time of an arriving customer will depend upon the state at which the customer arrivers.

Case (i)

Let $y_0(t)$ denote the probability density function of waiting time of the customer who finds the system in state (n, 1), n ≥ 1. Since the sum of (n + 1) exponential service time with rate μ_0 follows a gamma distribution.

we have,

$$y_0(t) = \sum_{n=1}^{\infty} P_{n1}\mu_0 e^{-\mu_0 t}\frac{(\mu_0 t)^n}{n!}$$

The Laplace steltjes transform $y_0(0)$ of y_0 (t) is given by

$$Y_0*(\theta)=\sum_{n=1}^{\infty}P_{n1}\int_0^{\infty}e^{-\theta t}\mu_0e^{-\mu_0t}\frac{(\mu_0t)^n}{n!}$$

$$=\sum_{n=1}^{\infty}P_{n1}\frac{\mu_0^{\,n+1}}{(\theta+\mu_0)^{n+1}}$$

$$=\frac{\mu_0}{(\theta+\mu_0)^{n+1}}P_1\left(\frac{\mu_0}{(\theta+\mu_0)}\right)$$

$$=\frac{\frac{\mu_0}{(\theta+\mu_0)}\left(\frac{\eta r_1\mu_0}{(\theta+\mu_0)}\right)}{(1-r_1)\left(1-\frac{r_1\mu_0}{(\theta+\mu_0)}\right)\left(\mu_0\frac{\lambda\mu_0}{(\theta+\mu_0)}\right)}P_{00} \tag{20}$$

$$=\frac{\mu_0\eta r_1}{(1-r_1)(\mu_0(1-r_1)+\theta)(\theta+\mu_0-\lambda)}P_{00} \tag{21}$$

Case (ii)

The waiting time of the customer who arrive at the state (n, 0) (n ≥ 0) will depend upon whether the server returns from vacation or not during his waiting time.

Let y_1 (t) denote the waiting time of the customer under the case that the server does not return from vacation during his waiting time, then,

$$Y_1(t)=\sum_{n=0}^{\alpha}P_{n0}e^{-nt}\mu_ve^{-\mu_vt}\frac{(\mu_vt)^n}{n!}$$

The waiting time is the time taken for (n + 1) service completions with rate μ_v.

The Laplace steltjes transform $Y_1(\theta)$ of $Y_1(t)$ is

$$Y_1 * (\theta) = \sum_{n=0}^{\infty} P_{n0} \int_0^{\infty} e^{-(\theta + \eta + \mu_v \ t)t} \mu_v \frac{(\mu_v t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} P_{n0} \frac{(\mu_v)^{n+1}}{(\theta + \eta + \mu_v)^{n+1}}$$

$$= \frac{\mu_v}{(\theta + \eta + \mu_v)} P_0 \left(\frac{\mu_v}{\theta + \eta + \mu_v} \right)$$

Suppose the server returns from vacation during the waiting time of the customer, then his waiting time distribution $y_2(t)$ in this is given by

$$y_2(t) = \left(\sum_{n=0}^{\infty} P_{n0} \left\{ \sum_{r=0}^{n} \int_0^{\infty} \eta e^{-\eta u} \frac{e^{-\mu_v u} (\mu_v u)^r}{r!} \mu_0 e^{-\mu_0 (t-u)} \frac{(\mu_0 (t-u))^{n-r}}{(n-r)!} \right\} \right) P_{00}$$

if the waiting time is t, and the server returns from vacation at time u and r – service completions may occur in time u ($0 \leq u \leq t$) at rate μ_v and the remaining (n – r) services occur in the remaining time (t – u) at rate μ_b.

The Laplace steltjes transform $y_2*(\theta)$ of $y_2(t)$ is given by

$$Y_2 * (\theta) = \sum_{n=0}^{\infty} P_{n0} \int_0^{\infty} e^{-\theta t} \sum_{r=0}^{n} (f_r(t) * g_r(t)) dt.$$

where $f_r(t) = \eta e^{-\eta t} \frac{e^{-\mu_v t} (\mu_v t)^r}{r!}$ and $g_r(t) = \mu_0 e^{-\mu_0 t} \frac{(\mu_0 t)^{n-r}}{(n-r)!}$

And * denotes the convolution of $f_r(t)$ and $g_r(t)$

Thus $$Y_2 * (\theta) = \sum_{n=0}^{\infty} P_{n0} \sum_{r=0}^{n} L(f_r(t) * g_r(t))$$

$$= \sum_{n=0}^{\infty} P_{n0} \sum_{r=0}^{n} L\left(f_r(t)\right) L\left(g_r(t)\right)$$

$$= \sum_{n=0}^{\infty} P_{n0} \sum_{r=0}^{n} \frac{\eta(\mu_v)^r}{(\theta+\eta+\mu_v)^{r+1}} \frac{(\mu_0)^{n-r+1}}{(\theta+\mu_0)^{n-r+1}}$$

$$= \left\{ \frac{\eta\mu_0}{(\theta+\eta+\mu_v)(\theta+\mu_0)} \sum_{n=0}^{\infty} P_{no} \left(\frac{\mu_0}{\theta+\mu_0} \right) \sum_{r=0}^{\infty} \left(\frac{\mu_v(\theta+\mu_0)}{\mu_0(\theta+\eta+\mu_v)} \right)^r \right\}$$

$$= \frac{\eta\mu_0}{(\theta+\eta+\mu_v)(\theta+\mu_0)} \sum_{n=0}^{\infty} P_{no} \left(\frac{\mu_0}{\theta+\mu_0} \right)^n \left[\frac{1-\left(\frac{\mu_v(\theta+\mu_0)}{\mu_0(\theta+\eta+\mu_v)} \right)^{n+1}}{1-\left(\frac{\mu_v(\theta+\mu_0)}{\mu_0(\theta+\eta+\mu_v)} \right)} \right]$$

$$= \left[\frac{\left(\eta\mu_0 P_0 \frac{(\mu_0)}{(\theta+\mu_0)} \right)}{((\mu_0-\mu_v)\theta+\eta\mu_0)(\theta+\mu_0)} \right] - \left[\frac{\left(\eta\mu_v\mu_0 P_0 \left(\frac{\mu_v}{\theta+\eta+\mu_v} \right) \right)}{(\theta+\eta+\mu_v)((\mu_0+\mu_v)\theta+\eta\mu_0)} \right]$$

Thus,

$$y_1{}^*(\theta)+y_2{}^*(\theta) = \left(\frac{\mu_v}{\theta+\eta+\mu_v} \right) \left(\frac{\theta(\mu_0-\mu_v)}{\theta(\mu_0-\mu_v)+\eta\mu_0} \right) P_0 \left(\frac{\mu_v}{\theta+\eta+\mu_v} \right) + \left(\frac{\eta\mu_0{}^2 P_0 \left(\frac{\mu_0}{\theta+\mu_0} \right)}{((\mu_0-\mu_v)\theta+\eta\mu_0(\theta+\mu_0))} \right)$$

on further simplification

$$y_1{}^*(\theta)+y_2{}^*(\theta) = \left\{ \frac{\mu_0(\eta+\mu_v(1-r_1))+\theta\mu_v}{(\theta+(1-r_1)\mu_0)(\theta+\eta+\mu_v(1-r_1))} \right\} P_{00} \qquad (22)$$

Thus the LT of waiting time of the customer in the system is given by

$$y_0(\theta)+y_1{}^*(\theta)+y_2{}^*(\theta) = \left\{ \frac{\mu_0(\eta+\mu_v(1-r_1))+\theta\mu_v}{(\theta+(1-r_1)\mu_0)(\theta+\eta+\mu_v(1-r_1))} + \frac{\eta r_1\mu_0}{(1-r_1)(\theta+\mu_0-\lambda)(\mu_0(1-r_1)+\theta)} \right\} P_{00}$$

$$= \frac{1}{\theta + \mu_0(1 - r_1)}\left[\frac{\mu_0(\lambda - \mu_v r_1)}{\theta + \mu_0 - \lambda} + \frac{\mu_0\lambda(1 - r_1) + \theta r_1 \mu_v}{\theta r_1 + \lambda(1 - r_1)}\right]$$

By taking θ = λ (1 – z), it is found after simplification that the classical relation

$$P(z) = W^*(\lambda(1 - z)) = \frac{1 - \frac{\mu_v r_1 z}{\mu_0}}{(1 - r_1 z)(1 - \rho_0 z)}$$

is true for the working vacation model also.

Differentiating P(z) = W* (λ(1- z)) with respect to Z, and substituting z = 1, we find that

$$\underset{z \to 1}{Lt} \text{ P (z)} = 1$$

$$= \lambda \text{ W}$$

where W is the expected waiting time of the customer.

2.2 MODEL II: M/M/1 QUEUEING MODEL WITH SINGLE WORKING VACATION

2.2.1 Model description

Consider a single server queueing system with arrival rate λ. The server begins a working vacation whenever the system becomes empty. During working vacation period an arriving customer is served at a rate of μ_v, and at the end of the vacation, if the server finds customers waiting in the queue, the server changes his service rate from μ_v to μ_0, and a regular busy period starts. Otherwise, the server enters an idle period, and then a new regular busy period starts only when an arrival occurs. Working vacation is an operation period in a lower rate. When the number of customers in the system is relatively few, we set a lower rate with serving customers. Therefore, the single working vacation policy has practical significance in optimal design of the system. Also, the interarrival time, service time and working vacation time are assumed to be mutually independent following FIFO service discipline.

The above process can be studied as a continuous time Markovian chain with state space $\{(n, 0) : n \geq 0\} \cup \{(n, 1) ; n \geq 0\}$ where n denotes the number of customers in the system and (0, 1) denotes the server is idle in the system but readily available for service. It is denoted by J (t).

$$J(t) = \begin{cases} 0 \text{ when the system is in a working vacation period at time t.} \\ 1 \text{ when the system is in a busy period at time t.} \end{cases}$$

The state (0, 0) denotes the server is idle in vacation.

Let P_{nj} represent the steady state probability when the system is in state (n, j). Based on this, the steady state equations are given by

$$(\lambda + \eta) P_{00} = \mu_v P_{10} + \mu_0 P_{11} \tag{23}$$

$$(\lambda + \eta + \mu_v) P_{n0} = \lambda P_{n-10} + \mu_v P_{n+10} \qquad n \geq 1 \tag{24}$$

$$(\lambda + \mu_0) P_{n1} = \lambda P_{n-11} + \mu_0 P_{n+11} + \eta P_{n0} \qquad n \geq 2 \tag{25}$$

$$(\lambda + \mu_0) P_{11} = \lambda P_{01} + \mu_0 P_{21} + \eta P_0 \tag{26}$$

$$\lambda P_{01} = \eta P_{00} \tag{27}$$

2.2.2 Steady State Solution

To obtain the expressions for P_{no} and P_{n1}, we are going to define the forward shifting operator E.

$$E P_{n0} = P_{n+10} \qquad n \geq 0.$$

$$E P_{n1} = P_{n+11} \qquad n \geq 1.$$

by proceeding as in the previous chapter, we get

$$P_{n0} = r_1^n P_{00}, \qquad n \geq 0.$$

where $r_1 < 1$ is the root of $\mu_0 z^2 - (\lambda + \eta + \mu_v) z + \lambda = 0$ and

$$P_{n1} = (A_s \rho_0^n + B r_1^n) P_{00} \qquad n \geq 1.$$

where $B = \frac{\eta r_1}{\mu_0 (1 - r_1)(r_1 - \rho_0)}$ and $\rho_0 = \frac{\lambda}{\mu_0}$ (28)

Now we find A_s:

Substituting for P_{10} and P_{11} in equation (26), we get

$$(\lambda + \eta) P_{00} = [\mu_v r_1 + \mu_0 (A_s \rho_0 + B r_1)] P_{00}$$

$$\Rightarrow A_s \lambda = (\lambda + \eta) - \mu_v r_1 - \mu_0 B r_1.$$

By substituting for B,

$$A_s \lambda = \eta + (\lambda - \mu_v r_1) - \mu_0 \left[\frac{\eta r_1^2}{\mu_0 (1 - r_1)(r_1 - \rho_0)} \right]$$

$$= \eta + (\lambda - \mu_v r_1) - \frac{\mu_0 \eta r_1}{\lambda - \mu_0 r_1} (\lambda - \mu_v r_1)$$

$$= \eta + \frac{(\lambda - \mu_v r_1)\lambda}{(\lambda - \mu_0 r_1)}$$

$$= \eta + \frac{(\lambda - \mu_v r_1)\lambda}{\lambda - \mu_0 r_1}$$

$$= \eta + \frac{\lambda \eta r_1}{(1 - r_1)(\lambda - \mu_0 r_1)}$$

since, $\frac{\eta r_1}{1-r_1} = \lambda - \mu_0 r_1$, we have

$A_s\lambda = \eta - B\lambda$

$A_s\lambda = \frac{\eta}{\lambda} - B$

We get,

$$P_{n0} = r_1\eta\, P_{00} \qquad n \geq 0.$$

$$P_{n1} = (A_s\, \rho_0^{\,n} + B\, r_1^{\,n})\, P_{00} \qquad n \geq 1.$$

$$= \left[\left(\frac{\eta}{\lambda} - B\right)\rho_0^{\,n} + B r_1^{\,n}\right] P_{00}$$

$$= \left[\frac{\eta}{\lambda}\rho_0^{\,n} + B\left(r_1^{\,n} - \rho_0^{\,n}\right)\right] P_0 \qquad n \geq 1.$$

From equation (27) $P_{01} = \frac{\eta}{\lambda} P_{00}$

Thus the system size probabilities are expressed in terms of P_{00} and P_{00} can be calculated by using the normalizing condition.

In the following sections we are going to calculate the probability generating function of the steady state system size probabilities of the single working vacation model and P_{00} can also be derived from it.

2.2.3 The probability generating function of steady state probabilities

Let $P_0(z)$ denote the probability generating function of the probabilities when the server is on vacation. Then

$$P_0(z) = \sum_{n=0}^{\infty} P_{n0} z^n$$

$$= \frac{P_{00}}{1 - r_1 z}$$

Let $P_1(z)$ denote the probability generating function of P_{n1}, when the system is busy with regular service rate μ_0. Then $P_1(z)$ is given by,

$$P_1(z) = \sum_{n=1}^{\alpha} P_{n1} z^n$$

$$= \left[\left(\frac{\eta}{\lambda} - B\right)\frac{\rho_0 z}{(1 - \rho_0 z)} + \frac{B r_1 z}{(1 - r_1 z)}\right] P_{00}$$

Thus the total probability generating function P (z) given by

$P(z) = P_0(z) + P_1(z) + P_{01}$

$$= \left[\frac{(1 + B r_1 z)}{(1 - r_1 z)} + \left(\frac{\eta}{\lambda} - B\right)\frac{\rho_0 z}{(1 - \rho_0 z)} + \frac{\eta}{\lambda}\right] P_{00}$$

Substituting for B and simplifying we find

$$P(z) = \left[\frac{(\mu_0 - \mu_v r_1 z)}{(1 - r_1 z)(\mu_0 - \lambda z)} + \frac{\eta}{\lambda(1 - \rho_0 z)}\right] P_{00}$$

P_{00} can be calculated from the normalizing condition $P(1) = 1$.

$$\underset{z \to 1}{Lt}\ P(z) = 1$$

$$\Rightarrow \qquad P_{00} = \left[\frac{(1-r_1)(1-\rho_0)}{\frac{\eta}{\lambda}(1-r_1) + (1-\rho_0) + \frac{\eta r_1}{\mu(1-r_1)}} \right]$$

$$= (1-\rho_0)(1-r_1)K_s$$

where $K_s = \left[\frac{\eta}{\lambda}(1-r_1) + (1-\rho_0) + \frac{\eta r_1}{\mu_0(1-r_1)} \right]^{-1}$

Now P (z) can be written as

$$P(z) = \frac{(1-\rho_0)}{(1-\rho_0 z)} \left[(1-r_1) + \frac{r_1 z \left(1 - \frac{\mu_v}{\mu_0}\right)(1-r_1)}{(1-r_1 z)} + \frac{\eta}{\lambda}(1-r) \right] K_s$$

$$= \frac{(1-\rho_0)}{(1-\rho_0 z)} \left[(1-r_1)\left(1 + \frac{\eta}{\lambda}\right) + \frac{r_1 z \left(1 - \frac{\mu_v}{\mu_0}\right)(1-r_1)}{(1-r_1 z)} \right] K_s \tag{29}$$

where $K_s = \left[(1-\rho_0) + \left(\frac{\eta}{\lambda}\right)(1-r_1) + \frac{\eta r_1}{\mu_0(1-r_1)} \right]^{-1}$

2.2.4 Decomposition Property

If $\rho_0 < 1$ and $\mu_0 > \mu_v$ then the PGF of the steady state system size probabilities of the single working vacation M / M / 1 queueing model is the product of the PGF of two random variables one of which is the system size of the classical M / M / 1 queueing model without parameter (1 - ρ_0) and the additional queue length Q_d which has a modified geometric distribution.

Proof

$$P(z) = \sum_{n=0}^{\infty} P_{n0} z^n + \sum_{n=1}^{\infty} P_{n1} z^n + P_{01}$$

by using this equation and after simplification

we find ,

$$P(z) = \left[\frac{\mu_0 - \mu_v r_1 z}{(1 - r_1 z)(\mu_0 - \lambda z)} + \frac{\eta}{\lambda(1 - P_s z)} \right] P_{00}$$

$$P(1) = 1 \Rightarrow P_{00} = \left[\frac{(1 - r_1)(1 - \rho_0)}{\frac{\eta(1 - r_1)}{\lambda} + (1 - \rho_0) + \frac{\eta r_1}{\mu_0 (1 - r_1)}} \right]$$

$$= (1 - \rho_0)(1 - r_1) K_s$$

where $K_s = \left[\left(\frac{\eta}{\lambda} \right)(1 - r_1) + (1 - \rho_0) + \frac{\eta r_1}{\mu_0 (1 - r_1)} \right]^{-1} = \left[\frac{\eta}{\lambda}(1 - r_1) + 1 - \frac{r_1 \mu_v}{\mu_0} \right]^{-1}$

Hence $P(z) = \frac{(1 - \rho_0)}{(1 - \rho_0 z)} \left[(1 - r_1) + \frac{r_1 z \left(1 - \frac{\mu_v}{\mu_0} \right)(1 - r_1)}{(1 - r_1 z)} + \frac{\eta}{\lambda}(1 - r_1) \right]$

$$= Q_0(z)\, Q_d(z).$$

Where $Q_0(z) = \frac{(1 - \rho_0)}{(1 - \rho_0 z)}$ is the probability generating function of the classical M / M / 1 queue without vacation and

$$Q_0(z)=\left[(1-r_1)+\frac{r_1z\left(1-\frac{\mu_v}{\mu_0}\right)(1-r_1)}{(1-r_1z)}+\frac{\eta(1-r_1)}{\lambda}\right]K_s$$

is the modified geometrical distribution of the random variable Q_d with,

Prob $(Q_d=0)=K_s(1-r_1)\left(\frac{\eta}{\lambda}+1\right)$ and

Prob $(Q_d=k)=K_s(1-r_1)\left(1-\frac{\mu_v}{\mu_0}\right)r_1^{\,k}$

Hence the total probability generating function of M / M / 1 queueing model with single working vacation satisfies the well known decomposition property.

2.2.5 Mean system length

The expected mean system size is given by,

$$L_s=\sum_{n=1}^{\alpha}P_{n0}+\sum_{n=1}^{\infty}nP_{n1}.$$

$$=\sum_{n=1}^{\alpha}n\left[r_1^{\,n}+\frac{\eta}{\lambda}P_0^{\,n}+B\left(r_1^{\,n}-\rho_0^{\,n}\right)\right].P_{00}$$

$$=\left[\frac{\rho_0}{1-\rho_0}+\frac{r_1}{1-r_1}-\frac{r_1\left(\mu_v+\frac{\eta\mu_0}{\lambda}\right)}{\left(\mu_v-\mu_vr_1+\frac{\eta(1-r_1)\mu_0}{\lambda}\right)}\right] \tag{30}$$

2.2.6 Expected queue length

The expected queue length Lq is given by,

$$L_q = \sum_{n=1}^{\alpha} (n-1)P_{n0} + \sum_{r=0}^{n} (n-1)P_{n1}.$$

$$= \sum_{n=1}^{\alpha} n(P_{n0} - P_{n1}) - \sum_{r=0}^{n} (P_{n0} - P_{n1}).$$

$$= L_s - (1 - \rho_{00} - \rho_{01}).$$

$$= Ls\left[1 - \left(1 + \frac{\eta}{\lambda}\right)P_{00}\right]. \tag{31}$$

2.2.7 Other Performance Measures

Let P_v, P_{busy} and P_1 denote the probability that the server is on vacation, busy with regular service rate, μ_0 and idle respectively.

Then $P_v = \frac{P_{00}}{(1-r_1)}$

$$P_1 = \frac{\eta}{\lambda}P_{00}$$

$$P_{busy} = \left[\left(\frac{\eta}{\lambda} - \beta\right)\frac{\rho_s}{(1-\rho_s)} + \frac{Br_1}{(1-r_1)}\right]P_{00} \tag{32}$$

where $P_{00} = \dfrac{(1-\rho_s)(1-r_1)}{1 - \frac{\rho_s}{\mu_0}(\lambda - \mu_v r_1) + \frac{\eta}{\lambda(1-r_1)}}$

CHAPTER III

M/M/1 RETRIAL QUEUEING SYSTEM WITH TWO TYPES OF VACATION POLICIES UNDER ERLANG-K TYPE SERVICE

CHAPTER III

M / M / 1 RETRIAL QUEUEING SYSTEM WITH TWO TYPES OF VACATION POLICIES UNDER ERLANG - k TYPE SERVICE.

Introduction

In this chapter a single server k phase retrial queueing system in which customers arrive in a Poisson process is considered.The service time has Erlang k – type distribution with service rate kμ for each phase. Two types of vacation policies are discussed (i) Bernoulli type vacation and (ii)exhaustive type vacation. The vacation rate follows an exponential distribution with parameter α. It is assumed that the services in all phases are independent and identical and only one customer at a time is in the service mechanism. The system is analyzed by using matrix geometric technique. Mean number of customers in the orbit (MNCO). Truncation level (OCUT), probabilities of server free, busy and in vacation are calculated numerically. Special cases are discussed.

3.1 MODEL I: A SINGLE SERVER RETRIAL QUEUEING SYSTEM WITH BERNOULLI TYPE VACATION

3.1.1 Model description

Consider a single server retrial queueing system with Bernoulli type vacation in which customers arrive in a Poisson process with arrival rate λ. These customers are identified as primary calls. Let k be the number of phases in the service station. Assume that the service time has Erlang – k distribution with service rate kμ for each phase. The vacation rate follows an exponential distribution with parameter α.

If the server is free at the time of a primary or repeated call arrival, then this arriving call begins to be served immediately and leaves the system after completion of the service. After completion of each service, the server has an option to go on vacation with probability p or continue to serve with probability (1 -p). This type of vacation in queueing theory in called single vacation with Bernoulli Schedule. The single vacation means after completion of vacation period he can once again go for vacation after completing at least one service. The server may return from the vacation at any time and is independent of number of customers in the system.

We assume that the services in all phases are independent and identical and only one customer at a time is in the service mechanism. If the server is free at the time of a primary call arrival, the arriving call begins to be served in phase I immediately by the server then progresses through the remaining phases and must complete the last phase and leave the system before the next customer enters the first phase. If the server is busy or on vacation, then the arriving customer goes to orbit and becomes a source of repeated calls. This pool of sources of repeated calls may be viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity σ. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service, while the source which produced this repeated call disappears. Otherwise, the system state does not change.

We assume that the access from the orbit to the service facility follows the exponential distribution with rate σ which may depend on the current number n, ($n \geq 0$) the number of customers in the orbit. That is, the probability of repeated attempt during the interval $(t, t + \Delta t)$, given that there are n customers in the orbit at time t is $n\sigma \, \Delta t$. It is called the classical retrial rate policy. The input flow of primary calls, interval between repetitions and service time in phases are mutually independent.

3.1.2 Matrix Geometric Solutions

Let N (t) be the random variable which represents the number of customers in orbit at any time t and S(t) be the random variable which represents the phase in which customer is getting the service at time t. The random process is described as $\{(N (t), S (t))\}$.

$N (t) = 0, 1, 2, 3 \ldots, \quad S (t) = 0, 1, 2, \ldots, k, k + 1$

S (t) = 0 if the server being idle.

S (t) = i for server being busy with the customer in the i^{th} phase

for i = 1, 2, 3…, k

S (t) = k + 1 for the server to be on vacation.

The possible state spaces for single server retrial queueing with Erlang - k phases service are

{ (i, j) / i = 0, 1, 2, 3 …, j = 0, 1, 2, …, k, k + 1}.

The infinitesimal generator matrix Q for this model is given below

$$Q = \begin{pmatrix} A_{00} & A_0 & 0 & 0 & 0 & \dots \\ A_{10} & A_{11} & A_0 & 0 & 0 & \dots \\ 0 & A_{21} & A_{22} & A_0 & 0 & \dots \\ 0 & 0 & A_{32} & A_{33} & A_0 & \dots \end{pmatrix}$$

The matrices A_{00}, A_{nn-1}, $A_{nn,}$ and A_{nn+1} for n = 1, 2, 3…, in the infinitesimal matrix generator Q are square matrices of order k + 1.

We denote

$S_1 = -(\lambda + k\mu)$

$S_2 = (1 - p)\, k\mu$

The matrix A_{00} is described as

$$A_{00} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & S_1 & k\mu & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & S_1 & k\mu & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & S_1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & S_1 & k\mu & 0 \\ S_2 & 0 & 0 & 0 & \dots & 0 & S_1 & pk\mu \\ \alpha & 0 & 0 & 0 & \dots & 0 & 0 & -(\lambda+\alpha) \end{bmatrix}$$

The matrix $A_{nn-1} = (a_{ij})$

where $a_{ij} = n\sigma$ if $i = 1, j = 2$

$= 0$ otherwise.

The matrix A_{nn} for $n = 1, 2, 3\ldots,$ is

$$A_{nn} = \begin{bmatrix} -(\lambda+n\sigma) & \lambda & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & S_1 & k\mu & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & S_1 & k\mu & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & S_1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & S_1 & k\mu & 0 \\ S_2 & 0 & 0 & 0 & \ldots & 0 & S_1 & pk\mu \\ \alpha & 0 & 0 & 0 & \ldots & 0 & 0 & -(\lambda+\alpha) \end{bmatrix}$$

The matrix $A_{nn-1} = A_0 = (a_{ij})$ for $n = 0, 1, 2, 3\ldots,$

where $aij = \lambda$ if $i = j$, $i = 2, 3, 4\ldots, k + 1$

$= 0$ otherwise.

If the capacity of the orbit is finite say M then

The matrix A_{MM} is given as

$$A_{MM} = \begin{bmatrix} -(\lambda+M\sigma) & \lambda & 0 & 0 & \ldots & 0 & 0 \\ 0 & -k\mu & k\mu & 0 & \ldots & 0 & 0 \\ 0 & 0 & -k\mu & k\mu & \ldots & 0 & 0 \\ 0 & 0 & 0 & -k\mu & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (1-p)k\mu & 0 & 0 & 0 & \ldots & -k\mu & pk\mu \end{bmatrix}$$

Let X be a steady state probability vector of Q and partitioned as

$X = (x(0),x(1),x(2),\ldots,)$ and X satisfies

$$X Q = 0, \; Xe = 1 \tag{1}$$

where $x(i) = (P_{i0}, P_{i1}, P_{i2}, \ldots\ldots\ldots, P_{ik}, P_{ik+1})$

In this paper we are applying the Direct Truncation Method to find the steady state probability vector X. Let M denote the cut-off point for this truncation method. The steady state probability vector $X^{(M)}$ is now partitioned as

$$X^{(M)} = (x(0), x(1), x(2), \ldots, x(M))$$

which satisfies

$$X^{(M)} Q = 0, X^{(M)} e = 1.$$

where $x(i) = (P_{i0}, P_{i1}, P_{i2}, \ldots, P_{ik+1})$ $i = 0, 1, 2, \ldots, M$.

The above system of equations is solved by exploiting the special structure of the co-efficient matrix. It is solved by Gauss Jordan elementary transformation method. Since there is no clear cut choice for M, we may start the iterative process by taking, say M = 1 and increase it until the individual elements of X do not change significantly. That is, if M* denotes the truncation point then $\left\| X^{(M)}(i) - X^{M^*-1}(i) \right\|_\infty < C,$

where c is infinitesimal quantity

3.1.3 Stability Condition

Theorem:

The inequality $\left(\frac{\lambda}{\mu} + \frac{p\lambda}{\alpha} \right) < 1$ is the necessary and sufficient condition for system to be stable.

Proof :

Let Q be an infinitesimal generator matrix for the queueing system (without retrial)

The stationary probability vector X satisfying

$$XQ = 0 \text{ and } Xe = 1 \tag{2}$$

Let R be the rate matrix and satisfying the equation

$$A_0 + RA_1 + R^2 A_2 = 0 \tag{3}$$

The system is stable if sp (R) < 1

We know that the matrix R satisfies sp (R) < 1 if and only if

$$\Pi A_0 e < 0 \Pi A_2 e \tag{4}$$

where $\Pi = (\pi_1, \pi_2, \ldots..\pi_{k+1})$ and satisfies

$$\Pi A = 0 \text{ and } \Pi_e = 1. \tag{5}$$

and

$$A = A_0 + A_1 + A_2 \tag{6}$$

Here A_0, A_1 and A_2 are square matrices of order k and

$A_0 = \lambda I$, I the identity matrix of order k +1.

The matrix A_1 is given by,

$$A_1 = \begin{bmatrix} S_1 & k\mu & 0 & \ldots & 0 & 0 \\ 0 & S_1 & k\mu & \ldots & 0 & 0 \\ 0 & 0 & S_1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & S_1 & k\mu \\ \alpha & 0 & 0 & \ldots & 0 & -(\lambda+\alpha) \end{bmatrix}$$

The matrix A_2 is described as

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ (1-p)k\mu & 0 & 0 & \dots & 0 & k\mu \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

By substituting A_0, A_1, A_2 in equations (4), (5) and (6), we get $\left(\frac{\lambda}{\mu} + \frac{p\lambda}{\alpha}\right) < 1$. The inequality $\left(\frac{\lambda}{\mu} + \frac{p\lambda}{\alpha}\right) < 1$ is also a sufficient condition for the retrial queueing system to be stable.

Let Q_n be the number of customers in the orbit after the departure of n^{th} customer from the service station. We first prove the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic if, $\left(\frac{\lambda}{\mu} + \frac{p\lambda}{\alpha}\right) < 1$. $\{Q_n, n \geq 0\}$ is irreducible and aperiodic. It remains to be proved that $\{Q_n, n \geq 0\}$ is positive recurrent. The irreducible and aperiodic Markov chain $\{Q_n, n \geq 0\}$ is positive recurrent if $|\psi_m| < \infty$ for all m and $\lim_{m\to\infty} \sup \psi_m < 0$,

where $\psi_m = E((Q_{n+1} - Q_n) / Q_n = m)$ $(m = 0, 1, 2 \ldots.)$.

$$= \left(\frac{\lambda}{\mu} + \frac{p\lambda}{\alpha}\right) - \left(\frac{m\sigma}{\lambda + m\sigma}\right)$$

If $\left(\frac{\lambda}{\mu} + \frac{p\lambda}{\alpha}\right) < 1$, then $|\psi_m| < \infty$ for all m and $\lim_{m\to\infty} \sup \psi_m < 0$, Therefore the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic.

3.1.4 Special Cases

1. If k = 1 and p = 0, then this model becomes the single server retrial queueing model and our numerical results coincide with following closed form of number of customers in the orbit in the steady state (1997)s.

   Mean number of customers in the orbit = $\dfrac{\rho(\lambda+\rho\sigma)}{(1-\rho)\sigma}$.

2. As $\sigma \to \infty$ and p = 0, the closed form of number of customers in the orbit tends to length of the queue in standard queueing system with Erlang – k service (1990).

   $$Lq = \left(\frac{k+1}{2k}\right)\left(\frac{\rho^2}{1-\rho}\right)$$

For many values of λ, μ, k and very high values of σ (> 10000), the above result coincides with our numerical results.

3. If k = 1, p = 0 and $\sigma \to \infty$, then mean number of customers in the orbit L_q coincides with,that of M / M /1

   $$Lq = \left(\frac{\rho^2}{1-\rho}\right)$$

3.1.5 System performance measure

In this section some important performance measures along with formulas and their qualitative behavior for this queueing model are studied. Numerical study has been dealt in very large scale to study these measures. Define

P (n, 0) = Probability that there are n customers in the orbit and server is free.

P (n, i) = Probability that there are n customers in the orbit, server is busy with customers in the i^{th} phase for i = 1, 2, 3….k.

P (n, k+1) = Probability that there are n customers in the orbit and server in on vacation.

a. Probability that the server is idle = $\sum_{i=0}^{\infty} p$ (i, 0)

Probability that the server is busy with customer in the j^{th} phase = $\sum_{i=0}^{\infty} p(\text{i,j})$

Probability that the server is in vacation = $\sum_{i=0}^{\infty} p$ (i, k+1).

b. Probability that no customers in the orbit = $\sum_{j=0}^{k+1} p$ (0, j)

Probability that i customers in the orbit = $\sum_{j=0}^{k+1} p$ (i, j)

c. The mean number of customers in the orbit.

$$\text{MNCO} = \sum_{i=0}^{\infty} i\left(\sum_{j=0}^{k+1} p(i,j)\right)$$

d. The probability that the orbiting customer is blocked

$$\text{Blocking probability} = \sum_{i=0}^{\infty}\sum_{j=1}^{k+1} p(i,j)$$

e. The probability that an arriving customer enter into service immediately is given by

$$\text{PSI} = \sum_{i=0}^{\infty} p(i,0)$$

3.1.6 Numerical Study

MNCO : Mean Number of Customers in the Orbit.

P_0 : Probability that the server is idle.

P_1 : Probability that the server is busy.

P_2 : Probability that the server is in vacation.

Table 1 shows the effect of number of phases (k) over the system. As k increases, mean number of customers in the orbit decreases. Probabilities P_0, P_1 and P_2 are independent of k.

Table 1: System Measures for $\lambda = 5$, $\mu = 10$, $\alpha = 100$, $p = 0.5$, $\sigma = 100$

k	**Ocut**	**MNCO**	**P_0**	**P_1**	**P_2**
1	20	0.6105	0.475	0.5	0.025
2	16	0.4789	0.475	0.5	0.025
3	15	0.4351	0.475	0.5	0.025
4	14	0.4132	0.475	0.5	0.025
5	14	0.4	0.475	0.5	0.025
6	14	0.3912	0.475	0.5	0.025
7	14	0.385	0.475	0.5	0.025
8	14	0.3803	0.475	0.5	0.025
9	13	0.3766	0.475	0.5	0.025
10	13	0.3737	0.475	0.5	0.025
11	13	0.3713	0.475	0.5	0.025
12	13	0.3693	0.475	0.5	0.025
13	13	0.3676	0.475	0.5	0.025
14	13	0.3662	0.475	0.5	0.025
15	13	0.3649	0.475	0.5	0.025
16	13	0.3638	0.475	0.5	0.025
17	13	0.3628	0.475	0.5	0.025
18	13	0.362	0.475	0.5	0.025
19	13	0.3612	0.475	0.5	0.025
20	13	0.3605	0.475	0.5	0.025
21	13	0.3599	0.475	0.5	0.025
22	13	0.3596	0.475	0.5	0.025
23	13	0.3588	0.475	0.5	0.025
24	13	0.3583	0.475	0.5	0.025
25	13	0.3579	0.475	0.5	0.025
26	13	0.3575	0.475	0.5	0.025
27	13	0.3571	0.475	0.5	0.025
28	13	0.3568	0.475	0.5	0.025
29	13	0.3564	0.475	0.5	0.025
30	13	0.3561	0.475	0.5	0.025
31	13	0.3559	0.475	0.5	0.025
32	13	0.3556	0.475	0.5	0.025
33	13	0.3553	0.475	0.5	0.025
34	13	0.3551	0.475	0.5	0.025
35	13	0.3549	0.475	0.5	0.025

Table 2 shows the effect of probability of going for vacation (p) over the system. As p increases, mean number of customers in the orbit increases.

Table 2: System Measures for $\lambda = 5$, $\mu = 10$, $\alpha = 100$, $k = 5$, $\sigma = 100$.

p	Ocut	MNCO	P_0	P_1	P_2
0.02	13	0.3519	0.499	0.5	0.001
0.04	13	0.3538	0.498	0.5	0.002
0.06	13	0.3557	0.497	0.5	0.003
0.08	13	0.3577	0.496	0.5	0.004
0.1	13	0.3596	0.495	0.5	0.005
0.12	13	0.3615	0.494	0.5	0.006
0.14	13	0.3635	0.493	0.5	0.007
0.16	14	0.3654	0.492	0.5	0.008
0.18	14	0.3674	0.491	0.5	0.009
0.2	14	0.3694	0.49	0.5	0.01
0.22	14	0.3714	0.489	0.5	0.011
0.24	14	0.3734	0.488	0.5	0.012
0.26	14	0.3754	0.487	0.5	0.013
0.28	14	0.3774	0.486	0.5	0.014
0.3	14	0.3794	0.485	0.5	0.015
0.32	14	0.3814	0.484	0.5	0.016
0.34	14	0.3834	0.483	0.5	0.017
0.36	14	0.3855	0.482	0.5	0.018
0.38	14	0.3875	0.481	0.5	0.019
0.4	14	0.3896	0.48	0.5	0.02
0.42	14	0.3916	0.479	0.5	0.021
0.44	14	0.3937	0.478	0.5	0.022
0.46	14	0.3958	0.477	0.5	0.023
0.48	14	0.3979	0.476	0.5	0.024
0.5	14	0.4	0.475	0.5	0.025
0.52	14	0.4021	0.474	0.5	0.026
0.54	14	0.4042	0.473	0.5	0.027
0.56	14	0.4064	0.472	0.5	0.028
0.58	14	0.4085	0.471	0.5	0.029
0.6	14	0.4106	0.47	0.5	0.03
0.62	14	0.4128	0.469	0.5	0.031
0.64	14	0.415	0.468	0.5	0.032
0.66	14	0.4171	0.467	0.5	0.033
0.68	14	0.4193	0.466	0.5	0.034
0.7	14	0.4251	0.465	0.5	0.035
0.72	14	0.4237	0.464	0.5	0.036
0.74	15	0.4259	0.463	0.5	0.037
0.76	15	0.4281	0.462	0.5	0.038
0.78	15	0.4304	0.461	0.5	0.039
0.8	15	0.4326	0.46	0.5	0.04

Table 3 and Table 4 show the effect of retrial rate σ over the system. As σ increases, mean number of customers in the orbit decreases and this model becomes classical queueing system with Bernoulli vacation if σ > 5,000. Probabilities P_0, P_1 and P_2 are independent of σ.

Table 3: System Measures for $\lambda = 4$, $\mu = 10$, $\alpha = 100$, $p = 0.5$.

σ	**Ocut**	**MNCO**	**P_0**	**P_1**	**P_2**
10	12	0.4703	0.58	0.4	0.02
20	11	0.3255	0.58	0.4	0.02
30	11	0.2772	0.58	0.4	0.02
40	11	0.2531	0.58	0.4	0.02
50	11	0.2386	0.58	0.4	0.02
60	11	0.229	0.58	0.4	0.02
70	11	0.2221	0.58	0.4	0.02
80	11	0.2169	0.58	0.4	0.02
90	11	0.2129	0.58	0.4	0.02
100	11	0.2097	0.58	0.4	0.02
200	11	0.1952	0.58	0.4	0.02
300	10	0.1903	0.58	0.4	0.02
400	10	0.1879	0.58	0.4	0.02
500	10	0.1865	0.58	0.4	0.02
600	10	0.1855	0.58	0.4	0.02
700	10	0.1848	0.58	0.4	0.02
800	10	0.1843	0.58	0.4	0.02
900	10	0.1839	0.58	0.4	0.02
1000	10	0.1836	0.58	0.4	0.02
2000	10	0.1821	0.58	0.4	0.02
3000	10	0.1817	0.58	0.4	0.02
4000	10	0.1841	0.58	0.4	0.02
5000	10	0.1813	0.58	0.4	0.02
6000	10	0.1812	0.58	0.4	0.02
7000	13	0.1811	0.58	0.4	0.02

Table 4: System Measures for $\lambda = 8$, $\mu = 10$, $\alpha = 100$, $p = 0.5$.

σ	**Ocut**	**MNCO**	**P_0**	**P_1**	**P_2**
10	62	6.82	0.16	0.8	0.04
20	56	4.72	0.16	0.8	0.04
30	54	4.02	0.16	0.8	0.04
40	53	3.67	0.16	0.8	0.04
50	52	3.46	0.16	0.8	0.04
60	51	3.32	0.16	0.8	0.04
70	51	3.22	0.16	0.8	0.04
80	51	3.145	0.16	0.8	0.04
90	51	3.0866	0.16	0.8	0.04
100	50	3.04	0.16	0.8	0.04
200	50	2.83	0.16	0.8	0.04
300	49	2.76	0.16	0.8	0.04
400	49	2.725	0.16	0.8	0.04
500	49	2.704	0.16	0.8	0.04
600	49	2.69	0.16	0.8	0.04
700	49	2.68	0.16	0.8	0.04
800	49	2.6725	0.16	0.8	0.04
900	49	2.6666	0.16	0.8	0.04
1000	49	2.662	0.16	0.8	0.04
2000	49	2.641	0.16	0.8	0.04
3000	49	2.634	0.16	0.8	0.04
4000	49	2.6305	0.16	0.8	0.04
5000	49	2.6284	0.16	0.8	0.04
6000	49	2.627	0.16	0.8	0.04
7000	49	2.626	0.16	0.8	0.04

3.2 MODEL IIL: A SINGLE SERVER RETRIAL QUEUEING SYSTEM WITH EXHAUSTIVE TYPE VACATION

3.2.1 Model description

Consider a single server retrial queueing system with exhaustive type vacation in which customers arrive in a Poisson process with arrival rate λ. These customers are identified as primary calls. Let k be the number of phases in the service station. Assume that the service time has Erlang – k distribution with service rate $k\mu$ for each phase. The vacation rate follows an exponential distribution with parameter α. If the server is free at the time of a primary or repeated call arrival, then this arriving call begins to be served immediately and leaves the system after completion of the service. The server goes for vacation compulsorily after servicing to all customers and the system becomes empty. This type of vacation in queueing theory is called exhaustive service type single vacation. The single vacation means after completion of vacation period he can again go for vacation once again after servicing at least one customer. The server may return from the vacation at any time and is independent of number of customers in the system. We assume that the services in all phases are independent and identical and only one customer at a time is in the service mechanism. If the server is free at the time of primary call arrival, the arriving call begins to be served in phase 1 immediately by the server then progresses through the remaining phases and must complete the last phase and leave the system before the next customer enters the first phase. If the server is busy or on vacation, then the arriving customer goes to orbit and becomes a source of repeated calls. Every such source produces a Poisson process of repeated calls with intensity σ. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service, while the source which produced this repeated call disappears.

3.2.2 Matrix Geometric Solutions

The random process is described as {(N(t), S(t)}.

N (t) = 0, 1, 2, 3…,S (t) = 0, 1, 2, …, k, k + 1

S (t) = 0 if the server being idle.

S (t) = i if the server is busy with the customer in the ith phase for

i = 1, 2, 3…..k.

S (t) = k+1 if the server is on vacation.

The possible state spaces are {(i, j) / i = 0, 1, 2, 3…, ; j = 0, 1, 2, ...,k, k + 1}.

The matrices A_{00}, A_{01}, A_{nn-1}, A_{nn} and A_{nn+1} for n = 1, 2, 3…, in the infinitesimal matrix generator Q are square matrices of order k + 1.

The matrix A_{00} is described as

$$A_{00} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & S_1 & k\mu & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & S_1 & k\mu & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & S_1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & S_1 & k\mu & 0 \\ S_2 & 0 & 0 & 0 & \dots & 0 & S_1 & k\mu \\ \alpha & 0 & 0 & 0 & \dots & 0 & 0 & -(\lambda+\alpha) \end{bmatrix}$$

$A_{nn-1} = (a_{ij})$ for n = 1, 2, 3…,

where $a_{ij} = n\,\sigma$ if i = 1, j = 2

= 0, otherwise.

The matrix A_{nn} for n = 1, 2, 3,..., is

$$\mathrm{A}_{nn} = \begin{bmatrix} -(\lambda+n\sigma) & \lambda & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & S_1 & k\mu & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & S_1 & k\mu & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & S_1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & S_1 & k\mu & 0 \\ S_2 & 0 & 0 & 0 & \ldots & 0 & S_1 & 0 \\ \alpha & 0 & 0 & 0 & \ldots & 0 & 0 & -(\lambda+\alpha) \end{bmatrix}$$

$\mathrm{A}_{nn+1} = \mathrm{A}_0 = (a_{ij})$ for n = 0, 1, 2, 3……..

where, $a_{ij} = \lambda$ if $i = 2, 3, 4\ldots, k+1$

$= 0$ otherwise.

If the capacity of the orbit is finite say M then the matrix A_{MM} is given below

$$\mathrm{A}_{MM} = \begin{bmatrix} -(\lambda+M\sigma) & \lambda & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -k\mu & k\mu & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -k\mu & k\mu & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & -k\mu & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & -k\mu & k\mu & 0 \\ k\mu & 0 & 0 & 0 & \ldots & 0 & -k\mu & 0 \\ \alpha & 0 & 0 & 0 & \ldots & 0 & 0 & -\alpha \end{bmatrix}$$

Let X be a steady – state probability vector of Q and partitioned as X = (x(0), x(1), x(2), ...) and X satisfies,

XQ = 0, Xe = 1

where $x(i) = (P_{i0}, P_{i1}, P_{i2}, \ldots\ldots P_{ik}, P_{ik+1})$.

In this paper we are applying the Direct Truncation method to find the steady state probability vector X. Let M denote the cut – off point for this truncation method. The steady state probability vector $X^{(M)}$ is now partitioned as

$X^{(M)} = (x(0), x(1),x(2), \ldots,x(M))$

which satisfies

$X^{(M)} Q = 0, X^{(M)} e = 1.$

where $x(i) = (P_{i0}, P_{i1}, P_{i2},\ldots,P_{ik}, P_{ik+1})$ i = 0, 1, 2, 3 …, M.

The above system of equations is solved by exploiting the special structure of the co-efficient matrix. It is solved by Gauss-Jordan elementary transformation method. Since there is no clear cut choice for M, we may start the iterative process by taking, say M = 1 and increase it until the individual elements of *x* do not change significantly. That is if M* denotes the truncation point then.

$\| x^{M^*} (i) - x^{M^*-1} (i) \|_\infty < C$, where C is an infinitesimal quantity.

3.2.3 Stability Condition

Theorem

The inequality $\left(\frac{\lambda}{\mu}\right) < 1$ is the necessary and sufficient condition for system to be stable.

Proof:

Let Q be an infinitesimal generator matrix for the queueing system (without retrial).

Let X be the stationary probability vector such that

$$XQ = 0 \text{ and } Xe = 1 \tag{7}$$

and R be the rate matrix with

$$A_0 + RA_1 + R^2 A_2 = 0 \tag{8}$$

The system is stable if sp (R) < 1

We know that the matrix R satisfies sp (R) < 1 if and only if.

$$\prod A_0 e < \prod A_2 e \tag{9}$$

where $\prod = (\pi_1, \pi, \ldots, \pi_k)$ with

$$\prod A = 0 \text{ and } \prod e = 1 \tag{10}$$

and $$A = A_0 + A_1 + A_2 \tag{11}$$

Here A_0, A_1, and A_2 are square matrices of order k and $A_0 = \lambda I$, I is corresponding identify matrix.

The matrix A_1,

$$A_1 = \begin{bmatrix} S_1 & k\mu & 0 & \ldots & 0 & 0 & 0 \\ 0 & S_1 & k\mu & \ldots & 0 & 0 & 0 \\ 0 & 0 & S_1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & S_1 & k\mu & 0 \\ 0 & 0 & 0 & \ldots & 0 & S_1 & 0 \\ \alpha & 0 & 0 & \ldots & 0 & 0 & -(\lambda+\alpha) \end{bmatrix}$$

and matrix A_2 is

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ k\mu & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}$$

By substituting A_0, A_1, A_2 in equation (9),(10) and (11) we get $\left(\frac{\lambda}{\mu}\right)<1$. The inequality $\left(\frac{\lambda}{\mu}\right)<1$ is also a sufficient condition for the retrial queueing system to be stable. Let Q_n be the number of customers in the orbit after the departure of n^{th} customer from the service station. We first prove the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic if $\left(\frac{\lambda}{\mu}\right) < 1$. $\{Q_n, n \geq 0\}$ is irreducible and aperiodic. It remains to be proved that $\{Q_n, n \geq 0\}$ is positive recurrent. The irreducible and aperiodic Markov chain $\{Q_n, n \geq 0\}$ is positive for all m and recurrent if $|\psi_m| < \infty \lim_{m \to \infty} \sup \psi_m < 0$, where

$$\psi_m = E\left((Q_{n+1} - Q_n) / Q_n = m\right), m = 0, 1, 2\ldots\ldots$$

$$= \left(\frac{\lambda}{\mu}\right) - \left(\frac{m\sigma}{\lambda + m\sigma}\right)$$

If $\left(\frac{\lambda}{\mu}\right) < 1$, then $\psi_m < \infty$ for all m and $\lim_{m \to \infty} \sup \psi_m < 0$,

Therefore, the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic.

3.2.4 System performance measures

In this section some important performance measures along with formulas and their qualitative behavior for this queueing model are studied. Numerical study has been dealt in very large scale to study the measures. We can find various probabilities for various values of λ, μ, α, k and σ. The formulas for system measures which are discussed in section 3.1 hold for this model.

3.2.5 Numerical study

MNCO	:	Mean Number of Customers in the Orbit
P_0	:	Probability that the server is idle.
P_1	:	Probability that the server is busy.
P_2	:	Probability that the server is in vacation.

Table 5 shows the effect number of phases (k) over the system. As k increases, mean number of customers in the orbit decreases.

Table 5: System Measures for $\lambda = 5$, $\mu = 10$, $\alpha = 100$, $\sigma = 100$.

k	**Ocut**	**MNCO**	**P_1**	**P_0**	**P_2**
1	21	0.0190	0.4827	0.5000	0.0173
2	18	0.8935	0.4831	0.5000	0.0169
3	16	0.8517	0.4833	0.5000	0.0167
4	16	0.8308	0.04834	0.5000	0.0166
5	15	0.8182	0.4834	0.5000	0.0166
6	15	0.8099	0.4835	0.5000	0.0165
7	15	0.8039	0.4835	0.5000	0.0165
8	15	0.7994	0.4835	0.5000	0.0165
9	15	0.7959	0.4835	0.5000	0.0165
10	15	0.7931	0.4835	0.5000	0.0165
11	15	0.7908	0.4835	0.5000	0.0165
12	15	0.7889	0.4835	0.5000	0.0165
13	15	0.7873	0.4836	0.5000	0.0164
14	14	0.7859	0.4836	0.5000	0.0164
15	14	0.7847	0.4836	0.5000	0.0164
16	14	0.7837	0.4836	0.5000	0.0164
17	14	0.7828	0.4836	0.5000	0.0164
18	14	0.7820	0.4836	0.5000	0.0164
19	14	0.7812	0.4836	0.5000	0.0164
20	14	0.7806	0.4836	0.5000	0.0164
21	14	0.7800	0.4836	0.5000	0.0164
22	14	0.7794	0.4836	0.5000	0.0164
23	14	0.7789	0.4836	0.5000	0.0164
24	14	0.7785	0.4836	0.5000	0.0164
25	14	0.7780	0.4836	0.5000	0.0164
26	14	0.7777	0.4836	0.5000	0.0164
27	14	0.7773	0.4836	0.5000	0.164
28	14	0.7770	0.4836	0.5000	0.0164
29	14	0.7767	0.4836	0.5000	0.0164
30	14	0.7764	0.4836	0.5000	0.0164
31	14	0.7761	0.4836	0.5000	0.0164
32	14	0.7758	0.4836	0.5000	0.0164
33	14	0.7756	0.4836	0.5000	0.0164
34	14	0.7754	0.4836	0.5000	0.0164
35	14	0.7752	0.4836	0.5000	0.0164
36	14	0.7750	0.4836	0.5000	0.0164
37	14	0.7748	0.4836	0.5000	0.0164
38	14	0.7746	0.4836	0.5000	0.0164
39	14	0.7744	0.4836	0.5000	0.0164
40	14	0.7743	0.04836	0.5000	0.0164

Table 6 shows the effect of vacation rate (α) over the system. As α increases, mean number of customers in the orbit decreases.

Table 6: System Measures for $\lambda = 5$, $\mu = 10$, k = 5, $\sigma = 100$.

σ	**Ocut**	**MNCO**	**P_0**	**P_1**	**P_2**
10	15	0.5736	0.2968	0.5000	0.2032
20	13	0.4181	0.3866	0.5000	0.1134
30	13	0.3837	0.4222	0.5000	0.0778
40	13	0.3706	0.4410	0.5000	0.0590
50	13	0.3642	0.4526	0.5000	0.0474
60	13	0.3606	0.4603	0.5000	0.0397
70	13	0.3583	0.4659	0.5000	0.0341
80	13	0.3567	0.4701	0.5000	0.0299
90	13	0.3556	0.4734	0.5000	0.0266
100	13	0.3548	0.4761	0.5000	0.0239
200	13	0.3518	0.4880	0.5000	0.0120
300	13	0.3511	0.4920	0.5000	0.0080
400	13	0.3507	0.4940	0.5000	0.0060
500	13	0.3506	0.4952	0.5000	0.0048
600	13	0.3505	0.4960	0.5000	0.0040
700	13	0.3504	0.4966	0.5000	0.0034
800	13	0.3503	0.4970	0.5000	0.0030
900	13	0.3503	0.4973	0.5000	0.0027
1000	13	0.3503	0.4976	0.5000	0.0024
2000	13	0.3501	0.4988	0.5000	0.0024
3000	13	0.3501	0.4992	0.5000	0.0012
4000	13	0.3501	0.4994	0.5000	0.0008
5000	13	0.3500	0.4995	0.5000	0.0006
6000	13	0.3500	0.4996	0.5000	0.0005
7000	13	0.3500	0.4997	0.5000	0.0004
8000	13	0.3500	0.4997	0.5000	0.0003
9000	13	0.3500	0.4997	0.5000	0.0003
10000	13	0.3500	0.4998	0.5000	0.0002

Table 7 and Table 8 show the effect of retrial rate σ over the system. As σ increases, mean number of customers in the orbit decreases and this model becomes classical queueing system with exhaustive vacation if $\sigma > 5000$.

Table 7: System Measures for $\lambda = 4$, $\mu = 10$, $\alpha = 100$, $k = 5$.

σ	**Ocut**	**MNCO**	**P_0**	**P_1**	**P_2**
10	12	0.4405	0.5811	0.4000	0.0189
20	11	0.3018	0.5787	0.4000	0.0213
30	11	0.2553	0.5779	0.4000	0.0221
40	10	0.2319	0.5774	0.4000	0.0226
50	10	0.2179	0.5772	0.4000	0.0228
60	10	0.2085	0.5770	0.4000	0.0230
70	10	0.2018	0.5768	0.4000	0.0232
80	10	0.1968	0.5767	0.4000	0.0233
90	10	0.1929	0.5767	0.4000	0.0233
100	10	0.1898	0.5766	0.4000	0.0234
200	10	0.1757	0.5763	0.4000	0.0237
300	10	0.1710	0.5762	0.4000	0.0238
400	10	0.1686	0.5762	0.4000	0.0238
500	10	0.1672	0.5762	0.4000	0.0238
600	10	0.1663	0.5761	0.4000	0.0239
700	10	0.1656	0.5761	0.4000	0.0239
800	10	0.1651	0.5761	0.4000	0.0239
900	10	0.1647	0.5761	0.4000	0.0239
1000	10	0.1644	0.5761	0.4000	0.0239
2000	10	0.1630	0.5761	0.4000	0.0239
3000	10	0.1625	0.5761	0.4000	0.0239
4000	10	0.1623	0.5761	0.4000	0.0240
5000	10	0.1622	0.5760	0.4000	0.0240
6000	10	0.1621	0.5760	0.4000	0.0240
7000	10	0.1620	0.5760	0.4000	0.0240
8000	10	0.1619	0.5760	0.4000	0.0240

Table 8: System Measures for λ = 8, μ = 10, α = 100, k = 5.

σ	Ocut	MNCO	P_0	P_1	P_2
10	49	5.1337	0.1969	0.8000	0.0031
20	44	3.5368	0.1930	0.8000	0.0070
30	43	3.0026	0.1908	0.8000	0.0092
40	42	2.7347	0.1895	0.8000	0.0105
50	41	2.5737	0.1886	0.8000	0.0114
60	41	2.4662	0.1879	0.8000	0.0121
70	40	2.3893	0.1874	0.8000	0.0126
80	40	2.3316	0.1871	0.8000	0.0129
90	40	2.2867	0.1868	0.8000	0.0132
100	40	2.2508	0.1865	0.8000	0.0135
100	40	2.2508	0.1865	0.8000	0.0135
200	39	2.0888	0.1854	0.8000	0.0146
300	39	2.0347	0.1849	0.8000	0.0151
400	39	2.0076	0.1847	0.8000	0.0153
500	39	1.9914	0.1846	0.8000	0.0154
600	39	1.9805	0.1845	0.8000	0.0155
700	39	1.9728	0.1845	0.8000	0.0155
800	39	1.9670	0.1844	0.8000	0.0156
900	39	1.9625	0.1844	0.8000	0.0156
1000	39	1.9589	0.1844	0.8000	0.0156
1000	39	1.9589	0.1844	0.8000	0.0156
2000	39	1.9426	0.1842	0.8000	0.0158
3000	39	1.9372	0.1842	0.8000	0.0158
4000	39	1.9345	0.1842	0.8000	0.0158
5000	39	1.9328	0.1841	0.8000	0.0159
6000	39	1.9318	0.1841	0.8000	0.0159
7000	39	1.9310	0.1841	0.8000	0.0159
8000	39	1.9304	0.1841	0.8000	0.0159
9000	39	1.9300	0.1841	0.8000	0.0159
10000	39	1.9296	0.1841	0.8000	0.0159

SUMMARY AND CONCLUSION

SUMMARY AND CONCLUSION

In this dissertation, M / M / 1 queue with working vacation and M / M / 1 retrial queue with two types of vacation policies under Erlang-k type service are considered.

M / M /1 queue with working vacation is analyzed and analytic expressions for stationary distribution of queue length is derived. The stochastic decomposition structure of stationary queue length and waiting time is verified. The additional queue length and additional delay are obtained.

M / M / 1 retrial queue with two types of vacation policies under Erlang-k type service with the vacation rate following an exponential distribution with parameter α is considered. Probability mass function of server state and number of customers in the orbit and mean number of customers in the orbit are derived for both queueing models.

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