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## Chapter 3

### Some Concepts related to $\delta P_S$ -open sets

#### 3.1 Introduction

The aim of this chapter is to introduce a new class of neighborhoods using  $\delta P_S$ -open sets namely  $\delta P_S$ -neighborhoods in topological spaces. Further, we discuss some vital relations and interesting characterizations of  $\delta P_S$ -neighborhood. This new notion can be applied in the Geographic Information System (GIS), where the concept of nearness is studied in terms of the relation between objects rather than the distance between them. In a consequent manner, concepts of  $\delta P_S$ -Frontier,  $\delta P_S$ -Boundary,  $\delta P_S$ -Exterior and  $\delta P_S$ -Saturated Set are studied.

The idea of grills was developed by Choquet in 1974. Grill is a powerful tool in dealing with proximity spaces, closure spaces and in the theory of compactification. It is efficient in dealing with many topological situations. With this motivation,  $\delta P_S$ -open sets are analyzed through grills.

#### 3.2 $\delta P_S$ -Neighborhood

**Definition 3.2.1.** A subset  $N$  of a topological space  $(X, \tau)$  is called a  **$\delta P_S$ -neighborhood** of  $x \in X$  if there exists a  $\delta P_S$ -open set  $Q$  such that  $x \in Q \subseteq N$ . The set of all  $\delta P_S$ -neighborhood of  $x$  is denoted by  $\delta P_S N(x)$

**Proposition 3.2.2.** If a subset  $N$  of a topological space  $X$  is  $\delta P_S$ -open, then  $N$  is a  $\delta P_S$ -neighborhood of each of its points.

**Proof.** Let  $N$  be a  $\delta P_S$ -open set and  $x \in N$ . Then  $x \in N \subseteq N$ . Since  $x$  is an arbitrary point in  $N$ ,  $N$  is a  $\delta P_S$ -neighborhood of each of its points.

Conversely, If  $N$  is a  $\delta P_S$ -neighborhood of each of its points then  $N$  need not be  $\delta P_S$ -open

**Remark 3.2.3.** A  $\delta P_S$ -neighbourhood need not be a  $\delta P_S$ -open set as observed from the following example.

**Example 3.2.4.** Let  $X = \{a, b, c\}$  and  $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$  then  $\delta P_S O(X, \tau) = \tau$ . The set  $\{b, c\}$  is a  $\delta P_S$ -neighbourhood of the point  $b$ , as  $b \in \{b\} \subseteq \{b, c\}$  but  $\{b, c\}$  is not a  $\delta P_S$ -open set of  $(X, \tau)$ .

**Proposition 3.2.5.** If  $F$  is a  $\delta P_S$ -closed subset of a topological space and  $x \in X \setminus F$ , then there exists a  $\delta P_S$ -neighborhood  $N$  of  $x$  such that  $N \cap F = \emptyset$ .

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**Proof.** Let  $F$  be a  $\delta P_S$ -closed subset of  $X$ , then  $X \setminus F$  is  $\delta P_S$ -open in  $X$ . By Proposition 3.2.2,  $X \setminus F$  contains a  $\delta P_S$ -neighborhood of each of its points. Therefore, there exists a  $\delta P_S$ -open set  $N$  of  $x$  such that  $N \subseteq (X \setminus F)$  which in turn implies that  $N \cap F = \emptyset$ .

**Theorem 3.2.6.** In a topological space  $(X, \tau)$  with  $x \in X$ , the following results are true.

- a)  $\delta P_S N(x) \neq \emptyset$
- b) If  $N \in \delta P_S N(x)$  then  $x \in N$
- c) If  $N \in \delta P_S N(x)$  and  $N \subseteq M$  then  $M \in \delta P_S N(x)$
- d) If  $N \in \delta P_S N(x)$ , then there exists  $M \in \delta P_S N(x)$  such that  $M \subseteq N$  and  $M \in \delta P_S N(y)$ , for all  $y \in M$ .

**Proof.** (a) Since  $X$  is a  $\delta P_S$ -open set irrespective of the topology, it is a  $\delta P_S$ -neighborhood for every  $x \in X$ . That is  $X \in \delta P_S N(x)$ , for all  $x \in X$ . Hence  $\delta P_S N(x) \neq \emptyset$ , for all  $x \in X$ .

(b) Let  $N \in \delta P_S N(x)$  then  $N$  is a neighborhood of  $x$ . Therefore  $x \in N$  follows from the definition of  $\delta P_S$ -neighborhood.

(c) Let  $N \in \delta P_S N(x)$  and  $N \subseteq M$ . Since  $N \in \delta P_S N(x)$  there exists a  $\delta P_S$ -open set  $Q$  such that  $x \in Q \subseteq N \subseteq M$  and hence  $M$  is a  $\delta P_S$ -neighborhood of  $x$ . Hence  $M \in \delta P_S N(x)$ .

(d) Let  $N \in \delta P_S N(x)$  then exists a  $\delta P_S$ -open set  $M$  such that  $x \in M \subseteq N$ . Since  $M$  is  $\delta P_S$ -open, it is a  $\delta P_S$ -neighborhood of each of its points (By Proposition 3.2.2). Thus  $M \in \delta P_S N(y)$ , for all  $y \in M$ . Since  $x \in M$ ,  $M \in \delta P_S N(x)$ .

**Theorem 3.2.7.** Let  $(X, \tau)$  be a topological space and  $x \in X$ . Suppose that a collection  $\mathcal{A}_x$  satisfies:

- (a)  $N \in \mathcal{A}_x$  such that  $x \in N$
- (b)  $N, M \in \mathcal{A}_x \Rightarrow N \cap M \in \mathcal{A}_x$

then  $\mathcal{B}$  forms a basis for a topology where  $\mathcal{B} = \{\emptyset\} \cup \{G \subseteq X | x \in G \Rightarrow \text{there exists } N \in \mathcal{A}_x \text{ such that } x \in N \subseteq G\}$ .

**Proof.** (a) By definition of  $\mathcal{B}$ ,  $\emptyset \in \mathcal{B}$ . For all  $x \in X$ , there exists  $N \in \mathcal{A}_x$ , such that  $x \in N \subseteq \mathcal{A}_x$ , by hypothesis (a). Thus  $X \in \mathcal{B}$ .

(b) Let  $G_1, G_2 \in \mathcal{B}$  and  $x \in G_1 \cap G_2 \Rightarrow x \in G_1$  and  $x \in G_2$ .  $x \in G_1 \Rightarrow$  there exists  $N \in \mathcal{A}_x$  such that  $N \subseteq G_1$  and  $x \in G_2 \Rightarrow$  there exists  $M \in \mathcal{A}_x$  such that  $x \in M \subseteq G_2$ . Now  $x \in N \cap$

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$M \subseteq G_1 \cap G_2$ , By (b),  $N \cap M \subseteq \mathcal{A}_x$ . Therefore, for all,  $x \in G_1 \cap G_2$  there exists  $N \cap M = N'$  such that  $x \in N' \subseteq G_1 \cap G_2$ . Thus  $G_1 \cap G_2 \in \mathcal{B}$ .

**Corollary 3.2.8.** Suppose that  $\mathcal{A}_x = \delta P_S N(x)$  in Theorem 3.2.7. Then  $\mathcal{B} = \{\emptyset\} \cup \{G \subseteq X | x \in G \Rightarrow \exists N \in \delta P_S N(x) \ni x \in N \subseteq G\}$  forms a basis for a topology.

### 3.3 $\delta P_S$ -limit point

**Definition 3.3.1.** Let  $A$  be a subset of a space  $X$ . A point  $x \in A$  is said to be a  **$\delta P_S$ -limit point** of  $A$  if for each  $\delta P_S$ -open set  $U$  containing  $x$ ,  $U \cap (A - \{x\}) \neq \emptyset$ . The set of all  $\delta P_S$ -limit points of  $A$  is called the  $\delta P_S$ -derived set of  $A$  and is denoted by  $D_{\delta P_S}(A)$ .

**Proposition 3.3.2.** Every  $\delta$ -limit point is a  $\delta P_S$ -limit point.

Proof. Follows from the fact that  $\delta O(X, \tau) \subseteq \delta P_S O(X, \tau)$  and Proposition 3.2.2.

**Theorem 3.3.3.** For subsets  $A, B$  of a space  $X$ , the following statements hold:

- a)  $D_{\delta P_S}(\emptyset) = \emptyset$
- b)  $D_{\delta}(A) \subseteq D_{\delta P_S}(A)$ , where  $D_{\delta}(A)$  is the  $\delta$ -derived set of  $A$
- c) If  $A \subseteq B$ , then  $D_{\delta P_S}(A) \subseteq D_{\delta P_S}(B)$
- d)  $D_{\delta P_S}(A) \cup D_{\delta P_S}(B) \subseteq D_{\delta P_S}(A \cup B)$
- e)  $D_{\delta P_S}(A \cap B) \subseteq D_{\delta P_S}(A) \cap D_{\delta P_S}(B)$
- f)  $[D_{\delta P_S}(D_{\delta P_S}(A)) \setminus A] \subseteq D_{\delta P_S}(A)$
- g)  $D_{\delta P_S}[A \cup D_{\delta P_S}(A)] \subseteq A \cup D_{\delta P_S}(A)$

**Proof.** (a). Follows directly

(b). Proof follows from Proposition 3.3.2.

(c). Let  $x \in D_{\delta P_S}(A)$  then  $N \cap (A \setminus \{x\}) \neq \emptyset$ , for each  $\delta P_S$ -open set  $N$  of  $x$ . Since  $A \subseteq B$ ,  $N \cap (A \setminus \{x\}) \subseteq N \cap (B \setminus \{x\}) \neq \emptyset \Rightarrow x \in D_{\delta P_S}(B)$ . Therefore  $D_{\delta P_S}(A) \subseteq D_{\delta P_S}(B)$

(d) Since  $A \subseteq A \cup B$ ,  $D_{\delta P_S}(A) \subseteq D_{\delta P_S}(A \cup B)$ . Similarly  $B \subseteq A \cup B \Rightarrow D_{\delta P_S}(B) \subseteq D_{\delta P_S}(A \cup B)$ . Therefore  $D_{\delta P_S}(A) \cup D_{\delta P_S}(B) \subseteq D_{\delta P_S}(A \cup B)$ .

(e) Follows from (c) as  $A \cap B \subseteq A, B$

(f) Let  $x \in D_{\delta P_S}(D_{\delta P_S}(A)) \setminus A$  and  $U$  be a  $\delta P_S$ -open set containing  $x$ . Then  $x \in D_{\delta P_S}(D_{\delta P_S}(A)) \Rightarrow U \cap [D_{\delta P_S}(A) \setminus \{x\}] \neq \emptyset$  Now let  $y \in U \cap [D_{\delta P_S}(A) \setminus \{x\}] \Rightarrow y \in U$  and

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$y \in D_{\delta P_S}(A)$  so that  $U \cap [A \setminus \{y\}] \neq \emptyset$ . Let  $z \in U \cap [A \setminus \{y\}]$ . Then  $z \neq x$  as  $z \in A$  and  $x \notin A$ . Therefore  $U \cap [A \setminus \{x\}] \neq \emptyset$ . Hence  $x \in D_{\delta P_S}(A)$ .

(g) Let  $x \in D_{\delta P_S}(A \cup D_{\delta P_S}(A))$ . If  $x \in A$ , the result is obvious. Suppose if  $x \in D_{\delta P_S}(A \cup D_{\delta P_S}(A)) \setminus A$  and let  $U$  be a  $\delta P_S$ -open set containing  $x$ . Then  $U \cap (A \cup D_{\delta P_S}(A) \setminus \{x\}) \neq \emptyset \Rightarrow U \cap (A \setminus \{x\}) \neq \emptyset$  or  $U \cap (D_{\delta P_S}(A) \setminus \{x\}) \neq \emptyset$ . Now let  $y \in U \cap [D_{\delta P_S}(A) \setminus \{x\}] \Rightarrow y \in U$  and  $y \in D_{\delta P_S}(A)$  so that  $U \cap [A \setminus \{y\}] \neq \emptyset$ . Let  $z \in U \cap [A \setminus \{y\}]$ . Then  $z \neq x$  as  $z \in A$  and  $x \notin A$ . Therefore  $U \cap [A \setminus \{x\}] \neq \emptyset$ . Hence  $x \in D_{\delta P_S}(A)$  and thus  $x \in A \cap D_{\delta P_S}(A)$ .

**Proposition 3.3.4.** Let  $A \subseteq X$ . If  $A$  is  $\delta P_S$ -closed then  $D_{\delta P_S}(A) \subseteq A$ .

**Proof.** As  $A$  is  $\delta P_S$ -closed,  $X \setminus A$  is  $\delta P_S$ -open. For each  $x \in X \setminus A$ , there exists a  $\delta P_S$ -open set  $N$  containing  $x$  such that  $x \in N \subseteq X \setminus A$ . Since  $A \cap (X \setminus A) = \emptyset$ , the  $\delta P_S$ -open set  $N$  contains no point of  $A$  and hence  $x$  is not a  $\delta P_S$ -limit point of  $A$ . Therefore, no point of  $X \setminus A$  can be a  $\delta P_S$ -limit point of  $A$  which means  $A$  contains all its limit points. Hence  $D_{\delta P_S}(A) \subseteq A$ .

**Proposition 3.3.5.** In a topological space  $X$ , for any subset  $A$ ,  $Cl_{\delta P_S}(A) = A \cup D_{\delta P_S}(A)$ .

**Proof.** Since  $D_{\delta P_S}(A) \subseteq Cl_{\delta P_S}(A)$ ,  $A \cup D_{\delta P_S}(A) \subseteq Cl_{\delta P_S}(A)$ . On the other hand, let  $x \in Cl_{\delta P_S}(A)$ . If  $x \in A$ , then the proof is complete. If  $x \notin A$ , each  $\delta$ -open set containing  $x$  intersects  $A$  at a point distinct from  $x$ ; so  $x \in D_{\delta P_S}(A)$ . Thus,  $Cl_{\delta P_S}(A) \subseteq [A \cup D_{\delta P_S}(A)]$ , which completes the proof.

### 3.4 $\delta P_S$ - Frontier

**Remark 3.4.1.** For a subset  $A$  of a topological space  $(X, \tau)$ , the following are true.

a)  $\delta P_S Cl(A) \setminus \delta P_S Cl(B) \neq \delta P_S Cl(A \setminus B)$ .

b)  $\delta P_S Int(A) \setminus \delta P_S Int(B) \neq \delta P_S Int(A \setminus B)$ .

**Example 3.4.2.** (a) Let  $X = \{a, b, c\}$  and  $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ . Then  $\delta P_S C(X, \tau) = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$  and  $\delta P_S O(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ . Let  $A = \{b, c\}$  and  $B = \{b\}$ . Now,  $\delta P_S Cl(A) \setminus \delta P_S Cl(B) = \{b, c\} \setminus \{b, c\} = \emptyset$  but  $\delta P_S Cl(A \setminus B) = \{c\}$ . Hence  $\delta P_S Cl(A) \setminus \delta P_S Cl(B) \neq \delta P_S Cl(A \setminus B)$ .

(b) Let  $X = \{a, b, c, d\}$  and  $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ . Then  $\delta P_S O(X, \tau) = \{X, \emptyset, \{c\}, \{b, c\}, \{b, c, d\}, \{a, b, c\}\}$ . Let  $A = \{b, c\}$  and  $B = \{c\}$ . Now,

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$\delta P_S \text{Int}(A) \setminus \delta P_S \text{Int}(B) = \{b, c\} \setminus \{c\} = \{b\}$  but  $\delta P_S \text{Int}(A \setminus B) = \emptyset$ . Hence  $\delta P_S \text{Int}(A) \setminus \delta P_S \text{Int}(B) \neq \delta P_S \text{Int}(A \setminus B)$ .

**Definition 3.4.3.** For a subset  $A$  of a topological space  $(X, \tau)$ ,  $\delta P_S$ -frontier of  $A$  is denoted by  $\delta P_S F(A)$  and defined as  $\delta P_S F(A) = \delta P_S Cl(A) \setminus \delta P_S \text{Int}(A)$ .

**Example 3.4.4.** Let  $X$  and  $\tau$  be defined as in Example 3.4.2(a). Then for  $A = \{a, c\}$ ,  $\delta P_S F(\{a, c\}) = \delta P_S Cl(\{a, c\}) \setminus \delta P_S \text{Int}(\{a, c\}) = \{a, c\} \setminus \{a\} = \{c\}$ .

**Theorem 3.4.5.** For a subset  $A$  of a topological space  $(X, \tau)$ , the following results are true.

(a)  $\delta P_S Cl(A) = \delta P_S \text{Int}(A) \cup \delta P_S F(A)$ .

(b)  $\delta P_S \text{Int}(A) \cap \delta P_S F(A) = \emptyset$ .

(c)  $\delta P_S F(A) = \delta P_S Cl(A) \cap \delta P_S Cl(X \setminus A)$ .

(d)  $\delta P_S F(A)$  is  $\delta$ -closed.

(e)  $\delta P_S F(A) = \delta P_S F(X \setminus A)$ .

(f)  $Fr_\delta(\delta P_S F(A)) \subseteq \delta P_S F(A)$ .

**Proof.** (a)  $\delta P_S \text{Int}(A) \cup \delta P_S F(A) = \delta P_S \text{Int}(A) \cup [\delta P_S Cl(A) \setminus \delta P_S \text{Int}(A)] = \delta P_S Cl(A)$ .

(b)  $\delta P_S \text{Int}(A) \cap \delta P_S F(A) = \delta P_S \text{Int}(A) \cap [\delta P_S Cl(A) \setminus \delta P_S \text{Int}(A)] = \emptyset$ .

(c)  $\delta P_S Cl(A) \cap \delta P_S Cl(X \setminus A) = \delta P_S Cl(A) \cap (X \setminus \delta P_S \text{Int}(A)) = \delta P_S Cl(A) \setminus \delta P_S \text{Int}(A) = \delta P_S F(A)$ .

(d)  $Cl_\delta(Cl_\delta(\delta P_S F(A))) = Cl_\delta(\delta P_S Cl(A) \cap \delta P_S Cl(X \setminus A)) \subseteq Cl_\delta(\delta P_S Cl(A)) \cap Cl_\delta(\delta P_S Cl(X \setminus A)) \subseteq Cl_\delta(Cl_\delta(A)) \cap Cl_\delta(Cl_\delta(X \setminus A)) = Cl_\delta(A) \cap Cl_\delta(X \setminus A) = Fr_\delta(A)$ , which is  $\delta$ -closed.

(e)  $\delta P_S F(X \setminus A) = \delta P_S Cl(X \setminus A) \setminus \delta P_S \text{Int}(X \setminus A) = [X \setminus \delta P_S \text{Int}(A)] \setminus [X \setminus \delta P_S Cl(A)] = [X \setminus \delta P_S \text{Int}(A)] \cap \delta P_S Cl(A) = \delta P_S Cl(A) \cap [X \setminus \delta P_S \text{Int}(A)] = \delta P_S Cl(A) \setminus \delta P_S \text{Int}(A) = \delta P_S F(A)$ .

(f)  $Fr_\delta(\delta P_S F(A)) = \delta Cl(\delta P_S F(A) \setminus \delta \text{Int}(\delta P_S F(A))) = \delta Cl(\delta P_S F(A))(X \setminus \delta Cl(X \setminus \delta P_S F(A))) = Cl_\delta(\delta P_S F(A)) \cap Cl_\delta(X \setminus \delta P_S F(A)) \subseteq Cl_\delta(\delta P_S F(A)) = \delta P_S F(A)$ , as  $\delta P_S F(A)$  is  $\delta$ -closed.

**Proposition 3.4.6.** Let  $A \subseteq B$  and  $\delta P_S \text{Int}(B) = \emptyset$  then  $\delta P_S F(A) \subseteq \delta P_S F(B)$ .

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**Proof.** Let  $A \subseteq B$  and  $\delta P_S \text{Int}(B) = \emptyset$ . Let  $x \in \delta P_S F(A) = \delta P_S \text{Cl}(A) \setminus \delta P_S \text{Int}(B)$ . Then  $x \in \delta P_S \text{Cl}(A) \subseteq \delta P_S \text{Cl}(B) \setminus \delta P_S \text{Int}(B)$ , as  $\delta P_S \text{Int}(B) = \emptyset$ . Hence  $x \in \delta P_S F(B)$ .

### 3.5 $\delta P_S$ -Boundary, $\delta P_S$ -Exterior, $\delta P_S$ -Saturated Set

**Definition 3.5.1.** For a subset  $A$  of a topological space  $(X, \tau)$ ,  $\delta P_S$ -boundary of  $A$  is denoted by  $\delta P_S B(A)$  and defined as  $\delta P_S B(A) = A \setminus \delta P_S \text{Int}(A)$ .

**Example 3.5.2.** Let  $X, \tau$  and  $A$  be defined as in Example 3.4.4. Then  $\delta P_S B(\{a, c\}) = \{a, c\} = \{a, c, \} \setminus \{a\} = \{c\}$ .

**Theorem 3.5.3.** For a subset  $A$  of a topological space  $(X, \tau)$ , the following results are true.

- (a)  $\delta P_S B(\emptyset) = \emptyset$ .
- (b)  $\delta P_S B(X) = \emptyset$ .
- (c)  $A = \delta P_S \text{Int}(A) \cup \delta P_S B(A)$ .
- (d) If  $A$  is  $\delta P_S$ -open if and only if  $\delta P_S B(A) = \emptyset$ .
- (e)  $\delta P_S \text{Int}(A) \cap \delta P_S B(A) = \emptyset$ .
- (f)  $\delta P_S B(\delta P_S \text{Int}(A)) = \emptyset$ .

**Proof.** (a)  $\delta P_S B(\emptyset) = \emptyset \setminus \delta P_S \text{Int}(\emptyset) = \emptyset \setminus \emptyset = \emptyset$ .

(b)  $\delta P_S B(X) = X \setminus \delta P_S \text{Int}(X) = X \setminus X = \emptyset$ .

(c)  $\delta P_S \text{Int}(A) \cup \delta P_S B(A) = \delta P_S \text{Int}(A) \cup [A \setminus \delta P_S \text{Int}(A)] = A$ .

(d) Let  $A$  be  $\delta P_S$ -open then  $\delta P_S \text{Int}(A) = A$ . Now,  $\delta P_S B(A) = A \setminus \delta P_S \text{Int}(A) = A \setminus A = \emptyset$ .

(e)  $\delta P_S \text{Int}(A) \cap \delta P_S B(A) = \delta P_S \text{Int}(A) \cap [A \setminus \delta P_S \text{Int}(A)] = \emptyset$ .

(f)  $\delta P_S B(\delta P_S \text{Int}(A)) = \delta P_S \text{Int}(A) \setminus \delta P_S \text{Int}(\delta P_S \text{Int}(A)) = \delta P_S \text{Int}(A) \setminus \delta P_S \text{Int}(A) = \emptyset$ .

**Definition 3.5.4.** For a subset  $A$  of a topological space  $(X, \tau)$ ,  $\delta P_S$ -exterior of  $A$  is denoted by  $\delta P_S E(A)$  and defined as  $\delta P_S E(A) = X \setminus \delta P_S \text{Cl}(A)$ .

**Example 3.5.5.** Let  $X, \tau$  and  $A$  be defined as in Example 3.4.4. Then  $\delta P_S E(\{a, c\}) = X \setminus \delta P_S \text{Cl}(\{a, c\}) = X \setminus \{a, c\} = \{b\}$

**Theorem 3.5.6.** For a subset  $A$  of a topological space  $(X, \tau)$ , the following results are true.

- (a)  $\text{Ext}_\delta(A) \subseteq \delta P_S E(A)$ .
- (b)  $\delta P_S E(X) = \emptyset$ .

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- (c)  $\delta P_S E(\emptyset) = X$ .
- (d)  $\delta P_S E(A) = \delta P_S \text{Int}(X \setminus A)$ .
- (e) If  $A \subseteq B$  then  $\delta P_S E(B) \subseteq \delta P_S E(A)$ .
- (f)  $\delta P_S E(A \cup B) \subseteq \delta P_S E(A) \cup \delta P_S E(B)$ .
- (g)  $\delta P_S E(A \cap B) \supseteq \delta P_S E(A) \cap \delta P_S E(B)$ .
- (h)  $\delta P_S E(\delta P_S E(A)) = \delta P_S \text{Int}(\delta P_S \text{cl}(A))$ .
- (i)  $\delta P_S E(A) = \delta P_S E(X \setminus \delta P_S E(A))$ .
- (j)  $X = \delta P_S \text{Int}(A) \cup \delta P_S E(A) \cup \delta P_S F(A)$ .
- (k)  $\delta P_S \text{Int}(A) \subseteq \delta P_S E(\delta P_S E(A))$ .

**Proof.** (a) Let  $x \in \text{Ext}_\delta(A)$ . Then  $x \in X \setminus \text{Cl}_\delta(A) \Rightarrow x \notin \text{Cl}_\delta(A) \Rightarrow x \notin \delta P_S \text{Cl}(A)$ , as  $\delta P_S \text{Cl}(A) \subseteq \text{Cl}_\delta(A)$ . That is,  $x \in X \setminus \delta P_S \text{Cl}(A) \Rightarrow x \in \delta P_S E(A)$ .

(b)  $\delta P_S E(X) = X \setminus \delta P_S \text{Cl}(X) = X \setminus X = \emptyset$ .

(c)  $\delta P_S E(\emptyset) = X \setminus \delta P_S \text{cl}(\emptyset) = X \setminus \emptyset = X$ .

(d)  $\delta P_S E(A) = X \setminus \delta P_S \text{Cl}(A) = \delta P_S \text{Int}(X \setminus A)$ ,

(e) Let  $A \subseteq B$  then  $X - B \subseteq X - A$ . Now by Proposition 2.5.8(d)  $\delta P_S \text{Int}(X \setminus B) \subseteq \delta P_S \text{Int}(X \setminus A) = \delta P_S E(B)$ . Hence from(d)  $\delta P_S E(B) \subseteq \delta P_S E(A)$ .

(f) We have  $A \subseteq A \cup B$  and  $B \subseteq A \cup B$ . By (e),  $\delta P_S E(A) \supseteq \delta P_S E(A \cup B)$  and  $\delta P_S E(B) \supseteq \delta P_S E(A \cup B)$  which implies  $\delta P_S E(A) \cup \delta P_S E(B) \supseteq E(A \cup B)$ .

(g) We have  $A \cap B \subseteq A$  and  $A \cap B \subseteq B$ . By (e),  $\delta P_S E(A \cap B) \supseteq \delta P_S E(A)$  and  $\delta P_S E(A \cap B) \supseteq \delta P_S E(B)$  which implies  $\delta P_S E(A \cap B) \supseteq \delta P_S E(A) \cap \delta P_S E(B)$ .

(h)  $\delta P_S E(\delta P_S E(A)) = \delta P_S E(X \setminus \delta P_S \text{Cl}(A)) = \delta P_S E(\delta P_S \text{Int}(X \setminus A)) = X \setminus \delta P_S \text{Cl}(\delta P_S \text{Int}(X \setminus A)) = \delta P_S \text{Int}(X \setminus \delta P_S \text{Int}(X \setminus A)) = \delta P_S \text{Int}(\delta P_S \text{Cl}(X \setminus X \setminus A)) = \delta P_S \text{Int}(\delta P_S \text{Cl}(A))$ .

(i)  $\delta P_S E(X \setminus \delta P_S E(A)) = \delta P_S E(X \setminus \delta P_S \text{Int}(X \setminus A))$  from(d). Then,  $\delta P_S E(X \setminus \delta P_S \text{Int}(X \setminus A)) = \delta P_S E(\delta P_S \text{Cl}(A)) = \delta P_S \text{Int}[X \setminus \delta P_S \text{Cl}(A)] = \delta P_S \text{Int}[\delta P_S \text{Int}(X \setminus A)] = \delta P_S \text{Int}(X \setminus A) = \delta P_S E(A)$ .

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$$(j) \delta P_S \text{Int}(A) \cup \delta P_S E(A) \cup \delta P_S F(A) = \delta P_S \text{Int}(A) \cup \delta P_S \text{Int}(X \setminus A) \cup [\delta P_S \text{Cl}(A) \setminus \delta P_S \text{Int}(A)] = \delta P_S \text{Cl}(A) \cup [X \setminus \delta P_S \text{Cl}(A)] = X.$$

$$(k) \delta P_S \text{Int}(A) = \delta P_S \text{Int}(\delta P_S \text{Cl}(A)) = \delta P_S \text{Int}(X \setminus \delta P_S \text{Int}(X \setminus A)) = \delta P_S \text{Int}(X \setminus \delta P_S E(A)) = \delta P_S E(\delta P_S E(A)).$$

**Remark 3.5.7.** The reverse implications (f) and (g) are not true in general as seen from the following Example.

**Example 3.5.8.** (a) Let  $X$  and  $\tau$  be defined as in Example 3.4.2(a). Then for  $A = \{a\}$  and  $B = \{b\}$ ,  $\delta P_S E(\{a\} \cup \{b\}) = \delta P_S E(\{a, b\}) = \{c\}$  but  $\delta P_S E(\{a\}) \cup \delta P_S E(\{b\}) = \{b, c\} \cup \{a, c\} = \{a, b, c\}$ .

(b). Let  $X = \{a, b, c\}$ ,  $\tau = \{X, \emptyset, \{a\}\}$ . Then  $\delta P_S C(X) = \{X, \emptyset, \{a, c\}, \{a, b\}, \{a\}\}$ . Then for  $A = \{a, b\}$  and  $B = \{b, c\}$ ,  $\delta P_S E(\{a, b\} \cap \{b, c\}) = \delta P_S E(\{b\}) = \{c\}$  but  $\delta P_S E(\{a, b\}) \cap \delta P_S E(\{b, c\}) = \emptyset$ .

**Corollary 3.5.9.**  $\text{Ext}_\delta(A) \cup \text{Ext}_\delta(B) \subseteq \delta P_S E(A \cap B)$ .

**Definition 3.5.10.** A subset  $A$  of a topological space  $(X, \tau)$  is said to be  **$\delta P_S$ -saturated** if  $\delta P_S \text{Cl}(\{x\}) \subseteq A$  for every  $x \in A$ . The set of all  $\delta P_S$ -saturated sets in  $(X, \tau)$  is denoted by  $\delta P_S \text{Sat}(X)$ .

**Theorem 3.5.11.** Every  $\delta P_S$ -closed set is a  $\delta P_S$ -saturated set.

**Proof.** Let  $A$  be a  $\delta P_S$ -closed set in  $(X, \tau)$  with  $x \in A$ . Then  $\{x\} \subseteq A$ . Taking  $\delta P_S$ -closure on both sides,  $\delta P_S \text{Cl}\{x\} \subseteq \delta P_S \text{Cl}(A) = A$ , as  $A$  is  $\delta P_S$ -closed. Therefore,  $A$  is a  $\delta P_S$ -saturated set.

### 3.6. $\delta P_S$ -Open Sets via Grill

**Definition 3.6.1.** Let  $(X, \delta P_S O(\tau), \mathcal{G})$  be a grill  $\delta P_S$ -space. We define a function  $\psi_{\delta P_S \mathcal{G}}: P(X) \rightarrow P(X)$ , called the  **$\delta P_S$ -operator** associated with the grill  $\mathcal{G}$  and  $\delta P_S O(\tau)$ , and is defined by  $\psi_{\delta P_S \mathcal{G}}(A) = \{x \in X \mid U \cap A \in \mathcal{G}, \text{ for all } \delta P_S\text{-open set } U \text{ containing } x\}$ .

**Example 3.6.2.** Let  $X, \tau$  be defined as  $X = \{a, b, c\}$ ,  $\tau = \{X, \emptyset, \{a\}\}$  and  $\mathcal{G} = \{\{b\}, \{a, b\}, \{b, c\}\}$ . Then by

$$\begin{aligned} \psi_{\delta P_S \mathcal{G}}(\{b\}) &= \{x \in X \mid U \cap \{b\} \in \mathcal{G} \text{ for every } \delta P_S\text{-open set } U \text{ containing } x\} \\ &= \{a, b, c\} \text{ and } \psi_{\delta P_S \mathcal{G}}\{a, b\} = \{a, b\}. \end{aligned}$$

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**Theorem 3.6.3.** Let  $(X, \delta P_S O(\tau))$  be a  $\delta P_S$ -space and  $\mathcal{G}, \mathcal{H}$  be two grills on  $X$ . Then for a subset  $A$  of  $X$ , the following conditions are valid:

(a)  $\mathcal{G} \subseteq \mathcal{H} \Rightarrow \psi_{\delta P_S} \mathcal{G}(A) \subseteq \psi_{\delta P_S} \mathcal{H}(A)$

(b)  $\psi_{\delta P_S}(\mathcal{G} \cup \mathcal{H})(A) \supseteq \psi_{\delta P_S} \mathcal{G}(A) \cup \psi_{\delta P_S} \mathcal{H}(A)$

**Proof.** (a) Let  $x \in \psi_{\delta P_S} \mathcal{G}(A)$ . Then  $U \cap A \in \mathcal{G}$ , for every  $\delta P_S$ -open set containing  $x$ . Since  $\mathcal{G} \subseteq \mathcal{H}$ ,  $U \cap A \in \mathcal{H}$ , for every  $\delta P_S$ -open set containing  $x$ . Therefore  $x \in \psi_{\delta P_S} \mathcal{H}(A)$  and hence  $\psi_{\delta P_S} \mathcal{G}(A) \subseteq \psi_{\delta P_S} \mathcal{H}(A)$ .

(b)  $\psi_{\delta P_S} \mathcal{G}(A) \cup \psi_{\delta P_S} \mathcal{H}(A) \subseteq \psi_{\delta P_S}(\mathcal{G} \cup \mathcal{H})(A)$  is obvious. Converse need not be true.

**Theorem 3.6.4.** Let  $(X, \delta P_S O(\tau), \mathcal{G})$  be a grill  $\delta P_S$ -space. Then for any two subsets  $A$  and  $B$  of  $X$  the following conditions hold:

- a)  $\psi_{\delta P_S} \mathcal{G}(A) \subseteq \psi_{\delta P_S} \mathcal{G}(B)$ , if  $A \subseteq B$ .
- b)  $\psi_{\delta P_S} \mathcal{G}(A \cup B) = \psi_{\delta P_S} \mathcal{G}(A) \cup \psi_{\delta P_S} \mathcal{G}(B)$
- c)  $\psi_{\delta P_S} \mathcal{G}(A) = \emptyset$ , if  $A \notin \mathcal{G}$
- d)  $\psi_{\delta P_S} \mathcal{G}(A) \subseteq \psi_{\delta} \mathcal{G}(A)$
- e)  $\psi_{\delta P_S} \mathcal{G}(A) \subseteq \delta P_S Cl(A)$ .

**Proof.** (a) Suppose  $x \notin \psi_{\delta P_S} \mathcal{G}(B)$ . Then  $U \cap B \notin \mathcal{G}$ , for every  $\delta P_S$ -open set  $U$  containing  $x$ .

This implies  $U \cap A \notin \mathcal{G}$ , for every  $\delta P_S$ -open set  $U$  containing  $x$ . Thus  $x \notin \psi_{\delta P_S} \mathcal{G}(A)$ . Hence  $\psi_{\delta P_S} \mathcal{G}(A) \subseteq \psi_{\delta P_S} \mathcal{G}(B)$

(b)  $\psi_{\delta P_S} \mathcal{G}(A) \cup \psi_{\delta P_S} \mathcal{G}(B) \subseteq \psi_{\delta P_S} \mathcal{G}(A \cup B)$  is obvious. Suppose  $x \notin \psi_{\delta P_S} \mathcal{G}(A) \cup \psi_{\delta P_S} \mathcal{G}(B)$

then  $x \notin \psi_{\delta P_S} \mathcal{G}(A)$  and  $x \notin \psi_{\delta P_S} \mathcal{G}(B)$ . That is, there exists  $\delta P_S$ -open sets  $U_1, U_2$  containing  $x$  such that  $(U_1 \cap A) \notin \mathcal{G}$  and  $(U_2 \cap B) \notin \mathcal{G}$ . Therefore  $(A \cap U_1) \cup (B \cap U_2) \notin \mathcal{G}$ . Now  $(U_1 \cap U_2)$  is a  $\delta P_S$ -open set containing  $x$  and  $(A \cup B) \cap (U_1 \cap U_2) \notin \mathcal{G}$ . Hence  $x \notin \psi_{\delta P_S} \mathcal{G}(A \cup B)$ . Thus  $\psi_{\delta P_S} \mathcal{G}(A \cup B) \subseteq \psi_{\delta P_S} \mathcal{G}(A) \cup \psi_{\delta P_S} \mathcal{G}(B)$ .

(c) Suppose there exists  $x \in \psi_{\delta P_S} \mathcal{G}(A)$  then for all  $\delta P_S$ -open set  $U$  containing  $x$ ,  $U \cap A \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ , which is a contradiction to  $A \notin \mathcal{G}$ .

(d) Let  $x \in \psi_{\delta P_S} \mathcal{G}(A)$ . Then for every  $\delta P_S$ -open set  $U$  containing  $x$ ,  $U \cap A \in \mathcal{G}$ . Let  $U'$  be a  $\delta$ -open set then it is a  $\delta P_S$ -open which implies a  $U' \cap A \in \mathcal{G}$ . Therefore  $x \in \psi_{\delta P_S} \mathcal{G}(A)$ .

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(e) Let  $x \in \psi_{\delta P_S} \mathcal{G}(A)$  then for all  $\delta P_S$ - open set  $U$  containing  $x$ ,  $U \cap A \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ . Since  $\emptyset \in \mathcal{G}$ ,  $U \cap A \neq \emptyset \Rightarrow x \in \delta P_S Cl(A)$ .

**Theorem 3.6.5.** Let  $(X, \delta P_S O(\tau), \mathcal{G})$  be a grill  $\delta P_S$ -space. Then for any two subsets  $A$  and  $B$  of  $X$ , the following conditions hold:

- a)  $\psi_{\delta P_S} \mathcal{G}(A) \setminus \psi_{\delta P_S} \mathcal{G}(B) \subseteq \psi_{\delta P_S} \mathcal{G}(A \setminus B)$
- b) If  $B \notin \mathcal{G}$ ,  $\psi_{\delta P_S} \mathcal{G}(A \cup B) = \psi_{\delta P_S} \mathcal{G}(A) = \psi_{\delta P_S} \mathcal{G}(A \setminus B)$
- c) If  $(A \setminus B) \cup (B \setminus A) \notin \mathcal{G}$ , then  $\psi_{\delta P_S} \mathcal{G}(A) = \psi_{\delta P_S} \mathcal{G}(B)$

**Proof.** (a) Let  $A$  and  $B$  be subsets of  $X$  and  $A = (A \setminus B) \cup (A \cap B)$ . By Theorem 3.6.4(b),  $\psi_{\delta P_S} \mathcal{G}(A) = \psi_{\delta P_S} \mathcal{G}(A \setminus B) \cup \psi_{\delta P_S} \mathcal{G}(A \cap B)$ . Therefore, by(c),  $\psi_{\delta P_S} \mathcal{G}(A) \subseteq \psi_{\delta P_S} \mathcal{G}(A \setminus B) \cup \psi_{\delta P_S} \mathcal{G}(B)$ . Now,  $\psi_{\delta P_S} \mathcal{G}(A) \setminus \psi_{\delta P_S} \mathcal{G}(B) \subseteq \psi_{\delta P_S} \mathcal{G}(A \setminus B) \cup \psi_{\delta P_S} \mathcal{G}(B) \setminus \psi_{\delta P_S} \mathcal{G}(B) = \psi_{\delta P_S} \mathcal{G}(A \setminus B)$ . Hence  $\psi_{\delta P_S} \mathcal{G}(A) \setminus \psi_{\delta P_S} \mathcal{G}(B) \subseteq \psi_{\delta P_S} \mathcal{G}(A \setminus B)$

(b) Let  $B \notin \mathcal{G}$ . Then  $\psi_{\delta P_S} \mathcal{G}(A \cup B) = \psi_{\delta P_S} \mathcal{G}(A) \cup \psi_{\delta P_S} \mathcal{G}(B) = \psi_{\delta P_S} \mathcal{G}(A \cup B) = \psi_{\delta P_S} \mathcal{G}(A)$ , as  $B \notin \mathcal{G} \Rightarrow \psi_{\delta P_S} \mathcal{G}(B) = \emptyset$ . In general,  $A \setminus B \subseteq A \cup B$ . By Theorem 3.6.4(a),  $\psi_{\delta P_S} \mathcal{G}(A \setminus B) \subseteq \psi_{\delta P_S} \mathcal{G}(A \cup B) = \psi_{\delta P_S} \mathcal{G}(A)$ . Further by hypothesis,  $\psi_{\delta P_S} \mathcal{G}(A) \setminus \psi_{\delta P_S} \mathcal{G}(B) \subseteq \psi_{\delta P_S} \mathcal{G}(A \setminus B) \Rightarrow \psi_{\delta P_S} \mathcal{G}(A) \subseteq \psi_{\delta P_S} \mathcal{G}(A \setminus B)$ . Therefore,  $\psi_{\delta P_S} \mathcal{G}(A \cup B) = \psi_{\delta P_S} \mathcal{G}(A) = \psi_{\delta P_S} \mathcal{G}(A \setminus B)$ .

(c) Let  $V = (A \setminus B) \cup (B \setminus A) \notin \mathcal{G}$ . Then  $A = (V \setminus B) \cup (B \setminus V)$ . By Theorem 3.6.4(b)  $\psi_{\delta P_S} \mathcal{G}(A) = \psi_{\delta P_S} \mathcal{G}(V \setminus B) \cup \psi_{\delta P_S} \mathcal{G}(B \setminus V)$ . By (b),  $\psi_{\delta P_S} \mathcal{G}(A) = \psi_{\delta P_S} \mathcal{G}((A \setminus B) \cup (B \setminus A)) = \psi_{\delta P_S} \mathcal{G}(V \setminus B) \cup \psi_{\delta P_S} \mathcal{G}(B) = \psi_{\delta P_S} \mathcal{G}(V \cup B) = \psi_{\delta P_S} \mathcal{G}(B)$ , again by (b)

**Corollary 3.6.6.** Let  $(X, \delta P_S O(\tau), \mathcal{G})$  be a grill  $\delta P_S$ -space and  $A \subseteq X$ . Then

- a)  $\emptyset \notin \mathcal{G} \Rightarrow \psi_{\delta P_S} \mathcal{G}(\emptyset) = \emptyset$
- b)  $\psi_{\delta P_S} \mathcal{G}(A) \setminus \psi_{\delta P_S} \mathcal{G}(\psi_{\delta P_S} \mathcal{G}(A)) \subseteq \psi_{\delta P_S} \mathcal{G}(A \setminus \psi_{\delta P_S} \mathcal{G}(A))$
- c)  $\psi_{\delta P_S} \mathcal{G}(A) \setminus \psi_{\delta P_S} \mathcal{G}(B) = \psi_{\delta P_S} \mathcal{G}(A \setminus B) \setminus \psi_{\delta P_S} \mathcal{G}(B)$

**Lemma 3.6.7.** Let  $(X, \delta P_S O(\tau), \mathcal{G})$  be a grill  $\delta P_S$ -space and  $A \subseteq X$  be  $\delta P_S$ -closed. Then  $\psi_{\delta P_S} \mathcal{G}(A) \subseteq A$ .

**Proof.** Suppose  $x \notin A$ , where  $A$  is  $\delta P_S$ -closed. Then  $X \setminus A$  is a  $\delta P_S$ -open set containing  $x$  and  $(X \setminus A) \cap A = \emptyset \notin \mathcal{G}$ . Therefore  $x \notin \psi_{\delta P_S} \mathcal{G}(A)$  and hence  $\psi_{\delta P_S} \mathcal{G}(A) \subseteq A$ .

**Corollary 3.6.8.** Let  $(X, \delta P_S O(\tau), \mathcal{G})$  be a grill  $\delta P_S$ -space and  $A \subseteq X$  be  $\delta P_S$ -closed. Then  $\psi_{\delta P_S} \mathcal{G}(\psi_{\delta P_S} \mathcal{G}(A)) \subseteq \psi_{\delta P_S} \mathcal{G}(A)$ .

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**Proof.** Follows from Theorem 3.6.4(a).

**Proposition 3.6.9.** Let  $(X, \delta P_S O(\tau), \mathcal{G})$  be a grill  $\delta P_S$ -space and  $A \subseteq X$ . If  $V$  is  $\delta P_S$ -open set containing  $x$  then  $\psi_{\delta P_S} \mathcal{G}(A) = \psi_{\delta P_S} \mathcal{G}(V \cap A)$ .

**Proof.** Since  $V \cap A \subseteq A$ , by Theorem 3.6.4 (a), we have  $\psi_{\delta P_S} \mathcal{G}(V \cap A) \subseteq \psi_{\delta P_S} \mathcal{G}(A)$ .

Suppose  $x \notin \psi_{\delta P_S} \mathcal{G}(V \cap A)$ , then there exists  $\delta P_S$ -open set  $U$  containing  $x$  such that  $U \cap (V \cap A) \notin \mathcal{G}$ . Now  $W = U \cap V$ , is a  $\delta P_S$ -open set containing  $x$  such that  $W \cap A \notin \mathcal{G}$ .

Therefore  $x \notin \psi_{\delta P_S} \mathcal{G}(A)$ . Hence  $\psi_{\delta P_S} \mathcal{G}(A) \subseteq \psi_{\delta P_S} \mathcal{G}(V \cap A)$ .