Batch Arrival Retrial Queue with Fluctuating Modes of Service, Randomized Vacations and Orbital Search

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INTRODUCTION

Queueing theory provides models to predict the behavior of systems that attempt to render service for randomly arising demands. The study of queueing systems finds application in a variety of real life situations like regulating traffic flow scheduling and facility design.

QUEUEING SYSTEM

A queue or a waiting line is formed when units needing some kind of service, arrive at the facility which provides the service they seek. A queueing system can be described by the flow of units for service, waiting for service if it is not immediate and leaving the system after receiving service or sometimes even without receiving service.

CHARACTERISTICS OF QUEUEING PROCESS

The basic characteristics of a queueing system which provide an adequate description are arrival pattern, service pattern, queue discipline, system capacity and number of service channels.

Arrival Pattern

In a queueing system the process of arrival is stochastic. Arrival may be either single or in batches. Thus it is necessary to know the probability distributions describing the times between successive arrivals and describing the size of the batch.

It is also necessary to know the reaction of the customer upon entering the system. A customer may decide to wait in the queue or not to enter the system. If a customer decides not to enter the queue upon arrival, the customer is said to have balked. If a customer who waits for service becomes impatient and leaves the system without being served, the customer is said to have reneged. If a customer who waits for service may switch from one waiting line to another in the event of the system having two or more parallel waiting lines, then the customer is said to have jockey from position.

The arrival pattern of customers that does not change with time is called stationary arrival pattern otherwise it is called non-stationary.

Service pattern

Service pattern describes the manner in which the service is rendered to the arrivals. The customer may be served either singly or in batches. The time required for serving a unit is called a service time. The service pattern of customers may be stationary or non-stationary with respect to time and state dependent or independent with respect to number of customers waiting for service.

Queue Discipline

Queue discipline refers to the manner in which customers are selected for service from the queue. In the most common disciplines are First Come First Served (FCFS) and Last Come First Served (LCFS) which are based on the arrivals of customers into the system. Customers may also be served randomly irrespective of their arrivals to the system called Service In Random Order (SIRO).

Another discipline is priority queue discipline, which allows service to be offered to customer depending on their priority in relation to another customer. There are two types of priority disciplines, that is preemptive priority and non preemptive priority. In the preemptive case, the customer with high priority is allowed to enter service immediately suspending the service in progress to a customer with lower priority. In non preemptive case the high priority goes to the head of the queue but gets into service only after the completion of the service in progress to the customer with lower priority.

System capacity

In some queueing processes there is a physical limitation to the amount of waiting rule so that when the line reaches the certain length, no further customers are allowed to enter until space becomes available as the result of a service completion. These are referred to as finite queueing situations. A queue with limited waiting room can be viewed as one with forced balking.

Service channel

The number of services in a queueing model may be finite or infinite. The number of services may be arranged in series, parallel or a combination of both, depending upon the nature of the services required. In parallel channels, all the channels provide identical services so that several customer may be served simultaneously. In series channels, a customer must pass through successively in several ordered channels before service is completed.

KENDALL'S NOTATION

Kendall's notation describes the basic characteristics of a queueing system. This notation has considerable extension to allow it to represent a wide variety of queueing models. A queueing process is described by a series of symbols and slashes as A/B/X/Y/Z.

A : represents the nature of arrival process to the queue. Letters used to symbolize the nature of inter arrival time distributions are:

M - Exponential distribution

D - Deterministic

E_k - Erlang distribution of order K

G - General distribution

B : represents the nature of service time distribution for the customers getting served in the queue. Here, the symbols are similar to inter-arrival time distribution.

X : Number of servers in the queue.

Y : System capacity. The default is infinity.

Z : Queue discipline. The default is first come first served.

For example, a queueing system with bulk arrival Poisson process, general service distribution, single server with infinite capacity and first in first out queue discipline is denoted as $M^X/G/1$.

MARKOVIAN AND NON MARKOVIAN QUEUEING MODELS

The queueing model with exponential inter-arrival time and service time is known as Markovian queueing models. Markovian queueing models can be analyzed by difference differential equation method or Neuts matrix – geometric algorithm.

If the inter-arrival times and/or service times, are not exponentially distributed, then the queueing system is known as non-Markovian queueing system. The methods generally used in studying non-Markovian queues are Embedded Markov chain method and supplementary variable technique.

SERVER VACATIONS

Queueing system with server vacations arises as models of many diverse fields such as computer, communication and production systems. The non availability of server at the system is termed as server's vacation. The purpose of leaving the system is manifold. The server may want to utilize his idle time for another task or may leave to attend an emergency call.

(i)Multiple Vacation

As soon as the system becomes empty the server leaves the system for a vacation. If, on return from vacation, the server finds one or more customers waiting, he works until the system is empty and then goes on another vacation. Otherwise the server takes repeated vacations until he finds the customer again in the system.

(ii)Single Vacation

On return from vacation, even if no customer is waiting, the server stays in the system and waits for at least one service completion. The server takes only one vacation at a time.

(iii)Random Vacation

The non availability of server irrespective of the number in the waiting line is considered as server's random vacation.

RETRIAL QUEUES

In conventional queueing theory it is usually assumed that a customer who cannot get service immediately after arrival either joins the waiting line or leaves the system forever. Sometimes impatient customers leave the queue but it is also assumed that they are leaving the system forever. However, as a matter of fact, the assumption about loss of customers which elected to leave the system is just a first order approximation to a real situation. Usually such a customer after some random period of time returns to the system and tries to get service again. The standard queueing models do not consider the phenomenon of retrials and therefore cannot be applied in solving a number of practically important problems. Retrial queues (or queues with returning customers, repeated orders, etc) have been introduced to meet this inadequacy. Retrial queueing systems are those such that arriving customers who find the server busy, join the retrial queue (orbit) to try again for their request after sometime.

Structure of Retrial Queue

The general structure of a retrial queue is shown in the following figure.

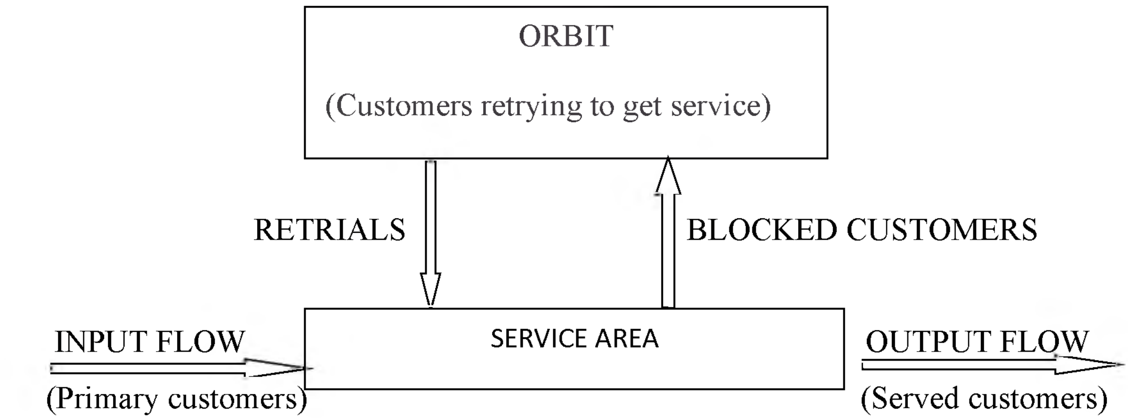

Fig.(1.1) Structure of retrial queue

REVIEW OF LITERATURE

Now a days, retrial queueing systems form an active research area due to its wide applicability in telephone switching system, telecommunication networks and computer networks. There is extensive literature on this topic; recent bibliographies include Yang and Templeton (1987), Falin (1990), Falin and Templeton (1997) and Artalejo (1999, 1999a). Retrial queueing systems are introduced extensively by Artalejo and Gomez-Corral (2008).

In recent years, queueing systems with server vacation have drawn more attention. In a vacation queueing system, the server may not be available for a period of time due to many reasons like, being checked for maintenance, working at other queues, scanning for new work or simply taking break. This period of time, when the server is unavailable for primary customers is referred as a **vacation**.

Levy and Yechiali (1976) investigated an M/M/S queue with server vacations. The queue with Bernoulli schedule vacation was first studied by Keilson and Servi (1986). Doshi (1986) presented a survey on queueing systems with vacations. Li and Yang (1995) studied an M/G/1 retrial queue with a finite number of input sources and Bernoulli vacations by the method of supplementary variables. Since the work of Li and Yang, the retrial queues with Bernoulli vacations were studied by many authors. Madan (2000) analysed on a single server queue with two-stage heterogeneous service and binomial schedule server vacations. Krishna Kumar et al. (2002) presented the steady state analysis of batch arrival retrial queue with Bernoulli Vacation and general retrial time. Choudhury and Madan (2005) studied on two stage batch arrival queueing systems with a modified Bernoulli schedule vacation under N-policy. Zhou (2005) derived the ergodic condition and probability generating function of the system size for M/G/1 retrial queue with Bernoulli vacation. Choudhury (2007) discussed an M^X/G/1 queueing system with two phases of heterogeneous service and Bernoulli vacation which operates under classical retrial policy. Choudhury (2008) extended the same model by considering linear retrial policy. Zhou et al. (2008) analysed a retrial queueing model with Bernoulli Vacations and server breakdowns. Senthil Kumar and Arumuganathan (2008) investigated a single server retrial queue with batch arrivals, two phases of heterogeneous service and vacation and discussed the application of the model of the analysis of a communication protocol. Wang and Li (2008) obtained the steady state solution for both queueing and reliability measures of a repairable M/G/1 retrial queue with general retrial time, Bernoulli vacation, setup time and two phase service.

A detailed study on batch arrival queue under both classical and constant retrial policies was done by Jain et al. (2008). A batch arrival retrial queue with starting failures, two phases of heterogeneous service and Bernoulli vacation was proposed by Ke and Chang (2009a). Boualem et al. (2009) derived stochastic inequalities for M/G/1 retrial queue with vacations and constant retrial policy. Explicit expressions for average system size and average orbit size of the state dependent batch arrival retrial queueing system with Bernoulli vacation and two phases of service were determined by Shweta (2010). Wang (2012) extended the continuous time M/G/1 retrial queue with Bernoulli Vacation to discrete time. Ebenesar Anna Begyam and Udaya Chandrika (2013) analysed batch arrival retrial queueing system with state dependent admission and Bernoulli vacation. Maurya (2013) presented a paper on maximum entropy analysis of $M^{[X]}/(G1,G2)/1$ retrial queueing model with second phase optional service and Bernoulli vacation schedule. Rajadurai et al. (2014) discussed on an $M^{[X]}/(G1,G2)/1$ retrial queueing system with balking,optional re-service under modified vacation policy and service interruption. Zhang and Liu (2014) analysed an M/G/1 G-queue with server breakdown, working vacations and vacation interruption. Jain and Bhagat (2014) considered unreliable bulk retrial queues with delayed repairs and modified vacation policy.

As a generalization of the single and multiple vacation policy, Ke and Chang (2009) introduced the concept of **J Vacations** in M/G/1 retrial queueing system. Chang and Ke (2009) extended the model to batch arrival retrial queues. Jain and Charu Bhargava (2009) suggested an M/G/1 retrial queueing system with modified vacation and K phase repair and obtained the probability generating function of the steady state queue size at random epoch using supplementary variable technique. Chen et al. (2010) discussed retrial queue with modified vacation and server breakdown which has potential applications in a packet switched network. Zhang and Zhu (2013) considered a discrete time Geo/G/1 retrial queue with J vacations and two types of breakdown. Rajadurai et al. (2015) studied in the analysis of repairable $M^{[X]}/(G1,G2)/1$ feedback retrial G-queue with balking and starting failures under atmost J vacations.

In many real life situation there could be variation in the mean service rate due to a variety of reasons such as fluctuation in internet speed in case of online server efficiency in case of a human server or fluctuations due to some extraneous factors including climatic conditions or intermittently available power supply etc. Recently, Baruah et al. (2014) have studied a queueing system in which the server provides general service in two fluctuating modes. Madan (2014) have studied a queueing system in which the server provides general service to customers in three fluctuating modes with different service rates.

In the retrial setup, each service is preceded followed by the server's idle time because of the ignorance of the status of the server and orbital customers by each other. Server's idle time is reduced by the introduction of **search of orbital customers** immediately after a service completion. Artalejo et al. (2002) considered a retrial queue in which immediately after a service completion the server searches for customers from the orbit or remains idle. Dudin et al. (2004) extended the model to a batch arrival retrial queue and performed the steady state analysis of the queueing system.

Krishnamoorthy et al. (2005) analysed M/G/1 retrial queue with non persistent customers and orbital search using supplementary variable method and discussed the structure of the busy period and its analysis in terms of Laplace transform. Chakravarthy et al. (2006) studied a multi-server retrial queueing model with orbital search using direct truncation and matrix geometric approximation. Sumitha and Udaya Chandrika (2011) considered a single server batch arrival retrial queueing system with additional optional service and orbital search and obtained the steady state distributions of the server state and the number of customers in the orbit. Sumitha and Udaya Chandrika (2012a) investigated a repairable M/G/1 retrial queue with Bernoulli vacation and orbital search and derived the queueing and reliability indices to predict the system behaviour. Sumitha and Udaya Chandrika (2012) discussed on the retrial queueing system with starting failure, single vacation and orbital search. Deepak et al. (2013) obtained expected queue length of a batch arrival retrial queueing system with two types of search of customers from the orbit. Rajadurai et al. (2015a) investigated $M^{[X]}/G/1$ feedback retrial queue with two phase service, Bernoulli vacation, delayed repair and orbital search. Rajadurai et al. (2015b) studied $M^{[X]}/G/1$ unreliable retrial G-queue with orbital search and feedback under Bernoulli vacation schedule.

PROFILE OF PRESENT WORK

The main objective of the dissertation is to analyse the steady state behavior of single and batch arrival retrial queueing systems by considering the concepts like fluctuating modes of service, vacation and orbital search.

The content of the thesis is presented in four chapters.

- Chapter one gives the basic concepts of queueing theory and review of literature related to the thesis topic.
- Chapter two deals with single server retrial queue with fluctuating modes of service.

- Single server retrial queue with fluctuating modes of service and randomized vacation is considered in Chapter three.
- In Chapter four, batch arrival retrial queue with fluctuating modes of service, randomized vacations and orbital search is analysed. (The content of this Chapter is submitted for publication.)

2. RETRIAL QUEUE WITH FLUCTUATING MODES OF SERVICE

Single server retrial queue with fluctuating modes of service is considered in this chapter. Server provides general service in M fluctuating modes with different service rates. If the server is idle upon the arrival of a customer, then the customer receives any one of the types immediately. Otherwise he joins the orbit. Using supplementary variable method the steady state probability generating function for the orbit size and system size are obtained. System Performance measures are derived and Stochastic Decomposition law is verified.

MODEL DESCRIPTION

Consider a single server retrial queueing system in which new customers arrive according to a Poisson process with the rate. The server provides M types of service, the customer opts type i service with probability $p_i (1 \leq i \leq M)$. If the arriving customer finds the server free, then the customer joins the server immediately, otherwise he joins the retrial queue. The retrial time is generally distributed with distribution function A(x), density function a(x), Laplace Stieltjes transform $A^*(s)$ and conditional completion rate $\eta(x) = \frac{a(x)}{1-A(x)}$.

The service time of type i (i=1,2,3….M) follows a general distribution with distribution function $B_i(x)$, density function $b_i(x)$, Laplace Stieltjes transform $B_i{}^*(s)$, n^{th} factorial moments $\mu_{i,n}$ and conditional completion rate $\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}$. Interarrival times, Retrial times, Service times are assumed to be mutually independent.

GOVERNING EQUATIONS

The state of the system at time t can be described by the Markov process {N(t);t≥0}={C(t),X(t),$\xi_0(t), \xi_1(t); t \geq 0$} where C(t) denote the server state 0 or i $(1 \leq i \leq M)$ accordingly as the server is free or busy in i^{th} type service respectively. X(t) corresponding to the number of customers in the orbit at time t. If C(t)=0 and X(t)>0 then $\xi_0(t)$ represent the elapsed retrial time and C(t)=i then $\xi_1(t)$ corresponds to the elapsed service time$(1 \leq i \leq M)$.

For the process $\{N(t); t \geq 0\}$,we define the probability

$I_0(t) = P\{C(t) = 0, X(t) = 0\}$ and the probability densities

$I_n(x,t) = P\{C(t) = 0, X(t) = n, x \leq \xi_0(t) < x + dx\}$ for $x \geq 0$ and $n \geq 1$.

$P_{i,n}(x,t) = P\{C(t) = i, X(t) = n, x \leq \xi_1(t) < x + dx\}$for $x \geq 0$ and $n \geq 0$,

$1 \leq i \leq M$

By the method of supplementary variable technique we obtain the following system of equations that governs the dynamics of the system behavior.

$$\frac{d}{dt} I_0(t) = -\lambda I_0(t) + \sum_{i=1}^{M} \int_0^\infty P_{i,0}(x,t)\mu_i(x)dx \tag{2.1}$$

$$\frac{d}{dx} I_n(x,t) = -(\lambda + \eta(x))I_n(x,t), \qquad n \geq 1 \tag{2.2}$$

$$\frac{d}{dx} P_{i,n}(x,t) = -(\lambda + \mu_i(x))P_{i,n}(x,t) + \lambda P_{i,n-1}(x,t), n \geq 0, 1 \leq i \leq M \tag{2.3}$$

with boundary conditions

$$I_n(0,t) = \sum_{i=1}^{M} \int_0^\infty P_{i,n}(x,t)\mu_i(x)\, dx\,, \qquad n \geq 1 \tag{2.4}$$

$$P_{i,0}(0,t) = p_i \left[\lambda I_0(t) + \int_0^\infty I_1(x,t)\eta(x)dx \right] 1 \leq i \leq M \tag{2.5}$$

$$P_{i,n}(0,t) = p_i \left[\lambda \int_0^\infty I_n(x,t)dx + \int_0^\infty I_{n+1}(x,t)\eta(x)dx \right], n \geq 1, 1 \leq i \leq M \tag{2.6}$$

STEADY STATE ANALYSIS

Let $I_n(x)$ and $P_{i,n}(x)$ be the steady state probabilities of $I_n(x,t)$ and $P_{i,n}(x,t)$.

Then the steady state equations corresponding to the equations (2.1) to (2.6) are

$$\lambda I_0 = \sum_{i=1}^{M} \int_0^{\infty} P_{i,0}(x)\mu_i(x)\, dx \qquad (2.7)$$

$$\frac{d}{dx} I_n(x) = -\big(\lambda + \eta(x)\big) I_n(x), n \geq 1 \qquad (2.8)$$

$$\frac{d}{dx} P_{i,n}(x) = -\big(\lambda + \mu_i(x)\big) P_{i,n}(x) + \lambda P_{i,n-1}(x)\, , n \geq 0{,}1 \leq i \leq M \qquad (2.9)$$

with boundary conditions

$$I_n(0) = \sum_{i=1}^{M} \int_0^{\infty} P_{i,n}(x)\mu_i(x) dx, n \geq 1 \qquad (2.10)$$

$$P_{i,0}(0) = p_i \left[\lambda I_0 + \int_0^{\infty} I_1(x)\eta(x) dx\right], 1 \leq i \leq M \qquad (2.11)$$

$$P_{i,n}(0) = p_i \left[\lambda \int_0^{\infty} I_n(x) dx + \int_0^{\infty} I_{n+1}(x)\eta(x) dx \right], n \geq 1{,}1 \leq i \leq M \qquad (2.12)$$

The normalizing conditions is

$$I_0 + \sum_{n=1}^{\infty} \int_0^{\infty} I_n(x) dx + \sum_{n=0}^{\infty} \int_0^{\infty} P_{i,n}(x) dx = 1 \qquad (2.13)$$

Define the probability generating functions

$$I(x, z) = \sum_{n=1}^{\infty} I_n(x) z^n \quad \text{and} \quad P_i(x, z) = \sum_{n=0}^{\infty} P_{i,n}(x) z^n$$

Multiplying equation (2.8) by z^n and summing over n for n=1, 2, 3,..., we get

$$\frac{d}{dx} \sum_{n=1}^{\infty} I_n(x) z^n = -\sum_{n=1}^{\infty} \big(\lambda + \eta(x)\big) I_n(x) z^n$$

$$\frac{d}{dx} I(x, z) = -\big(\lambda + \eta(x)\big) I(x, z)$$

$$\left[\frac{d}{dx} + \lambda + \eta(x)\right] I(x, z) = 0$$

Solution of the above equation is given by

$$I(x,z) = C_1 e^{-\int[\lambda+\eta(x)]dx}$$

$$= C_1 e^{-\lambda x} e^{-\int \eta(x)dx}$$

$$= C_1 e^{-\lambda x} e^{-\int \frac{a(x)}{1-A(x)}dx}$$

$$= C_1 e^{-\lambda x} e^{\log[1-A(x)]}$$

$$= C_1 e^{-\lambda x}[1 - A(x)]$$

Put x=0 then $C_1 = I(0,z)$

Accordingly we have

$$I(x,z) = I(0,z)e^{-\lambda x}[1 - A(x)] \tag{2.14}$$

Multiplying equation (2.9) by z^n and, summing over n for n=1, 2, 3,…, we get

$$\frac{d}{dx}P_i(x,z) = -(\lambda + \mu_i(x))P_i(x,z) + \lambda z P_i(x,z)$$

$$\left[\frac{d}{dx} + \lambda + \mu_i(x) - \lambda z\right]P_i(x,z) = 0$$

Solving we obtain

$$P_i(x,z) = P_i(0,z)e^{-\lambda(1-z)x}[1 - B(x)] \tag{2.15}$$

Equation (2.10) with the use of equations (2.7) and (2.15) gives

$$I(0,z) = \sum_{i=1}^{M}\int_0^\infty \sum_{n=0}^{\infty} P_{i,n}(x)\, z^n \mu_i(x)dx - \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\int_0^\infty P_{i,0}(x)\mu_i(x)dx$$

$$= \sum_{i=1}^{M}\int_0^\infty P_i(x,z)\mu_i(x)dx - \lambda I_0$$

$$= \sum_{i=1}^{M}\int_0^\infty P_i(0,z)e^{-\lambda(1-z)x}[1 - B(x)]\mu_i(x)dx - \lambda I_0$$

$$= \sum_{i=1}^{M} P_i(0,z) \int_0^\infty e^{-\lambda(1-z)x} b_i(x)dx - \lambda I_0$$

$$= \sum_{i=1}^{M} P_i(0,z) B_i^*(\lambda - \lambda z) - \lambda I_0 \tag{2.16}$$

Equations (2.11) and (2.12) yield

$$P_i(0,z) = p_i[\lambda \int_0^\infty I(x,z)dx + \frac{1}{z}\int_0^\infty I(x,z)\eta(x)dx + \lambda I_0]$$

$$= p_i[\lambda \int_0^\infty I(0,z)e^{-\lambda x}[1 - A(x)]dx + \frac{1}{z}\int_0^\infty I(0,z)e^{-\lambda x}a(x)dx + \lambda I_0]$$

$$= p_i[I(0,z)\left(1 - A^*(\lambda)\right) + \frac{1}{z} I(0,z)A^*(\lambda) + \lambda I_0]$$

$$= \frac{p_i}{z}\left[\lambda I_0 z + I(0,z)\left(z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right)\right] \tag{2.17}$$

Solving equations (2.16) and (2.17) , we obtain

$$I(0,z) = \frac{\lambda I_0 z[\sum p_i B_i^*(\lambda - \lambda z) - 1]}{z - \left[z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right]\sum p_i B_i^*(\lambda - \lambda z)} \tag{2.18}$$

$$P_i(0,z) = \frac{p_i}{z}\left[\lambda I_0 z + \frac{\left[z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right]\lambda I_0 z \sum p_i B_i^*(\lambda - \lambda z)}{z - \left[z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right]\sum p_i B_i^*(\lambda - \lambda z)}\right]$$

$$= \frac{p_i \lambda I_0 [z-1] A^*(\lambda)}{z - \left[z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right]\sum p_i B_i^*(\lambda - \lambda z)} \tag{2.19}$$

Now the expressions in equations (2.14) and (2.15) become

$$I(x,z) = \frac{\lambda I_0 z[\sum p_i B_i^*(\lambda - \lambda z) - 1]}{z - \left[z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right]\sum p_i B_i^*(\lambda - \lambda z)} e^{-\lambda x}[1 - A(x)] \tag{2.20}$$

$$P_i(x,z) = \frac{p_i \lambda I_0 [z-1] A^*(\lambda)}{z - \left[z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right]\sum p_i B_i^*(\lambda - \lambda z)} e^{-\lambda(1-z)x}[1 - B_i(x)] \tag{2.21}$$

Define the partial generating function $\psi(z) = \int_0^{\infty} \psi(x, z)dx$ for any generating function $\psi(x, z)$ then we have the following results.

The probability generating function of the orbit size when the server is idle is given by

$$I(z) = \int_0^{\infty} I(x, z)dx$$

$$= \int_0^{\infty} \frac{\lambda I_0 z[\sum p_i B_i^*(\lambda - \lambda z) - 1]}{z - [z(1 - A^*(\lambda)) + A^*(\lambda)] \sum p_i B_i^*(\lambda - \lambda z)} e^{-\lambda x}[1 - A(x)]dx$$

$$= \frac{I_0 z[\sum p_i B_i^*(\lambda - \lambda z) - 1][1 - A^*(\lambda)]}{z - [z(1 - A^*(\lambda)) + A^*(\lambda)] \sum p_i B_i^*(\lambda - \lambda z)} \tag{2.22}$$

The probability generating function of the orbit size when the server is busy is given by

$$P_i(z) = \int_0^{\infty} P_i(x, z)dx$$

$$= \int_0^{\infty} \frac{p_i \lambda I_0 [z - 1] A^*(\lambda)}{z - [z(1 - A^*(\lambda)) + A^*(\lambda)] \sum p_i B_i^*(\lambda - \lambda z)} e^{-\lambda(1-z)x}[1 - B_i(x)]dx$$

$$= \frac{I_0 p_i A^*(\lambda)(B_i^*(\lambda - \lambda z) - 1)}{z - [z(1 - A^*(\lambda)) + A^*(\lambda)] \sum p_i B_i^*(\lambda - \lambda z)} \tag{2.23}$$

Now we calculate $I = \lim_{z\to 1} I(z)$ and $P_i = \lim_{z\to 1} P_i(z)$ to find I_0 .

As numerator and denominator of $I(z)$ and $P_i(z)$ at z=1 are zero , by applying L'Hospital 's rule we have the following results.

$$I = \lim_{z\to 1} I(z)$$

$$= \frac{I_0(1 - A^*(\lambda))\lambda \sum p_i \mu_{i,1}}{A^*(\lambda) - \lambda \sum p_i \mu_{i,1}} \tag{2.24}$$

$$P = \sum_{i=1}^{M} P_i = \lim_{z\to 1} \sum_{i=1}^{M} P_i(z)$$

$$= \lim_{z \to 1} \frac{I_0 p_i A^*(\lambda)(B_i^*(\lambda - \lambda z) - 1)}{z - \left[z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right] \sum p_i B_i^*(\lambda - \lambda z)}$$

$$= \frac{I_0 A^*(\lambda) \lambda p_i \mu_{i,1}}{A^*(\lambda) - \lambda \sum p_i \mu_{i,1}} \tag{2.25}$$

The normalizing condition in (2.13) is equivalent to

$$I_0 + I + P = 1. \tag{2.26}$$

Using equations (2.24) and (2.25) in (2.26), we get

$$I_0 = \frac{A^*(\lambda) - \lambda \sum p_i \mu_{i,1}}{A^*(\lambda)} \tag{2.27}$$

The probability generating function for the number of customers in the orbit is

$$P_q(z) = I_0 + I(z) + \sum_{i=1}^{M} P_i(z)$$

$$= \frac{I_0 A^*(\lambda)(z - 1)}{z - \left[z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right] \sum p_i B_i^*(\lambda - \lambda z)} \tag{2.28}$$

The probability generating function for the number of customer in the system is

$$P_s(z) = I_0 + I(z) + z \sum_{i=1}^{M} P_i(z)$$

$$= \frac{I_0 A^*(\lambda)(z - 1) \sum p_i B_i^*(\lambda - \lambda z)}{z - \left[z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right] \sum p_i B_i^*(\lambda - \lambda z)} \tag{2.29}$$

PERFORMANCE MEASURES

The steady state probability that the server is idle during the retrial time, when the system is not empty is

$$I = \lim_{z \to 1} I(z)$$

$$= \frac{(1 - A^*(\lambda)) \lambda \sum p_i \mu_{i,1}}{A^*(\lambda)}$$

The steady state probability that the server is busy is

$$P = \sum_{i=1}^{M} P_i = \lim_{z \to 1} \sum_{i=1}^{M} P_i(z)$$

$$= \lambda \sum p_i \mu_{i,1}$$

The mean number of customers in the orbit L_q and in the system L_s are obtained by repeated application of L'Hospital's rule

The mean number of customers in the orbit is

$$L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z)$$

$$= \frac{I_0 A^*(\lambda)\lambda[\lambda \sum p_i \mu_{i,2} + 2(1 - A^*(\lambda)) \sum p_i \mu_{i,1}]}{2(A^*(\lambda) - \lambda \sum p_i \mu_{i,1})^2} \tag{2.30}$$

The mean number of customers in the system is

$$L_s = \lim_{z \to 1} \frac{d}{dz} P_s(z)$$

$$= \frac{I_0 A^*(\lambda)\lambda[\lambda \sum p_i \mu_{i,2} + 2 \sum p_i \mu_{i,1}[1 - \lambda \sum p_i \mu_{i,1}]]}{2(A^*(\lambda) - \lambda \sum p_i \mu_{i,1})^2} \tag{2.31}$$

It can be observed that

$$L_s = L_q + P$$

STOCHASTIC DECOMPOSITION

Theorem

The number of customers in the system in steady state at a random point of time L_s can be expressed as the sum of two independent random variable, one of which is L- The mean number of customers in the classical queue with fluctuating modes of service and the other is L_I- The mean number of customers in the orbit given that the server is idle.

Proof:

The probability generating function $\pi(z)$ of the system size in the classical queue with fluctuating modes of service is given by

$$\pi(z) = \frac{1-(z-1)\lambda \sum p_i \, \mu_{i,1} \sum p_i \, B_i^{*}(\lambda-\lambda z)}{z-\sum p_i B_i^{*}(\lambda-\lambda z)} \tag{2.32}$$

The probability generating function $\psi(z)$ of the number of customers in the orbit when the system is idle is given by

$$\psi(z) = \frac{I_0 + I(z)}{I_0 + I}$$

$$= \frac{z-\sum p_i \, B_i^{*}(\lambda-\lambda z)\left(A^{*}(\lambda)-\lambda \sum p_i \mu_{i,1}\right)}{z-\left[z\left(1-A^{*}(\lambda)\right)+A^{*}(\lambda)\right]\sum p_i B_i^{*}(\lambda-\lambda z)\left(1-\lambda \sum p_i \, \mu_{i,1}\right)} \tag{2.33}$$

From equations (2.29), (2.31) and (2.32) it is observed that $P_s(z) = \pi(z).\ \psi(z)$

$$\lim_{z\to 1} \frac{d}{dz} P_s(z) = \lim_{z\to 1} \frac{d}{dz} \pi(z).\ \psi(z)$$

$$= \lim_{z\to 1} \pi'(z).\psi(z) + \lim_{z\to 1} \pi(z).\psi'(z)$$

$$= L + L_I$$

Hence, $L_s = L + L_I$

NUMERICAL RESULTS

Assume that the retrial time and service times follow exponential distribution with respective rates η, μ_i $(1 \le i \le M)$.

For computation we choose arbitrarily the parameters

$$\lambda = 0.1, M = 2, \eta = 5, p_1 = 0.6, p_2 = 0.4, \mu_1 = 0.95, \mu_2 = 0.05$$

Performance measures are calculated by varying η and μ_1 and presented in Table 2.1.

In Table 2.1, we observe that

- Increase in μ_1 increases I_0 and decreases P, L_q and L_s .

- Increase in η decreases L_q and L_s and has no effect on I_0 and P.

η	μ_1	I_0	I	P	L_q	L_s
5.0000	0.9600	0.1203	0.0172	0.8625	13.7704	14.6329
	0.9800	0.1216	0.0172	0.8612	13.6206	14.4818
	1.0000	0.1228	0.0172	0.8600	13.4798	14.3398
5.0100	0.9600	0.1203	0.0172	0.8625	13.7656	14.6281
	0.9800	0.1216	0.0172	0.8612	13.6159	14.4772
	1.0000	0.1228	0.0172	0.8600	13.4752	14.3352
5.0200	0.9600	0.1203	0.0172	0.8625	13.7609	14.6234
	0.9800	0.1216	0.0172	0.8612	13.6113	14.4725
	1.0000	0.1229	0.0171	0.8600	13.4707	14.3307
5.0300	0.9600	0.1204	0.0171	0.8625	13.7562	14.6187
	0.9800	0.1217	0.0171	0.8612	13.6067	14.4679
	1.0000	0.1229	0.0171	0.8600	13.4662	14.3262
5.0400	0.9600	0.1204	0.0171	0.8625	13.7515	14.6140
	0.9800	0.1217	0.0171	0.8612	13.6021	14.4633
	1.0000	0.1229	0.0171	0.8600	13.4617	14.3217
5.0500	0.9600	0.1204	0.0171	0.8625	13.7468	14.6093
	0.9800	0.1217	0.0171	0.8612	13.5975	14.4587
	1.0000	0.1230	0.0171	0.8600	13.4572	14.3172

Table 2.1 Performance measures versus η and μ_1

3. RETRIAL QUEUE WITH FLUCTUATING MODES OF SERVICE AND RANDOMIZED VACATIONS

In this chapter, single server retrial queue with fluctuating modes of service and vacation is analyzed. The server provides M fluctuating modes of service. If the server is idle upon the arrival of a customer, then the customer receives any one of the types of service immediately. Otherwise he joins the orbit. Whenever the orbit becomes empty, the server immediately takes a vacation. At the end of a vacation if the system is still empty, the server leaves for another vacation or remains idle in the system. This pattern continues until the server finds at least one customer in the orbit or the number of vacations reaches a preassigned number J. At the end of J^{th} vacation even if the orbit is empty the server remains in the system for new arrival. Using supplementary variable technique expected number of customers in the orbit and expected numbers of customers in the system are derived. Stochastic law is verified. Numerical illustrations are obtained to see the effect of system parameters.

MODEL DESCRIPTION

Consider infinite capacity queueing systems with single sever. Assume that customers arrive singly according to Poisson process with rate λ. The server provides M heterogeneous type of service. Customer opt type i service with probability $p_i (1 \leq i \leq M)$. If the arriving customer finds the server free, then the customer receives any one of the types of service immediately. Otherwise he joins the retrial queue. The retrial time is generally distributed with distribution function A(x), density function a(x), Laplace Stieljes transform $A^*(s)$ and conditional completion rate $\eta(x) = \frac{a(x)}{1-A(x)}$.

The service time of type i (i=1,2,…,M) follows a general distribution with distribution function $B_i(x)$, density function $b_i(x)$, Laplace Stieltje's transform $B_i^*(s)$, n^{th} factorial moments $\mu_{i,n}$ and conditional rate

$\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}$.

At a service completion epoch if the orbit becomes empty, the server leaves for a vacation of random length V. At the vacation completion epoch, if the orbit is still empty the server either remains idle in the system with probability q or leaves for another vacation of another vacation with probability $\bar{q}$(=1-q). This pattern continues until the server finds at least one customer in the orbit or number of vacations reaches J. At the end of J^{th} vacation, even if the orbit is empty the server remains in the system. The random variable V has distribution function V(x), Laplace Stieljes transform $V^{*}(s)$, n^{th} factorial moment v_n and conditional completion rate function $\gamma(x) = \frac{v(x)}{1-V(x)}$.

The state of the system at a time t can be described by the Markov process

$\{N(t);t\geq 0\}=\{C(t),X(t), \xi_0(t), \xi_1(t), \xi_2(t); t \geq 0\}$

where C(t) denote the server state 0, i or M+j accordingly as the server being idle, busy in type i $(1 \leq i \leq M)$ service or the server is on j^{th} vacation. X(t) corresponds to the number of the customers in the orbit. If C(t)=0, then $\xi_0(t)$ represent the elapsed retrial time. If C(t)=i$(1 \leq i \leq M)$, then $\xi_1(t)$ represents the elapsed service time.If C(t)= M+j(1≤j≤J)then $\xi_2(t)$ represent elapsed vacation time.

DEFINITIONS AND NOTATIONS

For the process {N (t); t≥0}, define the probability

$I_0(t) = P\{C(t) = 0, X(t) = 0\}$ and the probability densities for t>0;x>0 as

$I_n(x,t)dx = P\{C(t) = 0, X(t) = n, x < \xi_0(t) < x + dx\}$, n ≥1.

$P_n(x,t)dx = P\{C(t) = i, X(t) = n, x < \xi_1(t) < x + dx\}, n \geq 0, (1 \leq i \leq M)$

$V_n(x,t)dx = P\{C(t) = M + j, X(t) = n, x < \xi_2(t) < x + dx\}, n \geq 0, j = 1,2, \ldots, J$

STEADY STATE EQUATIONS

The steady state equations that governs the model under consideration using supplementary variable method are given below

$$\lambda I_0 = \int_0^\infty V_{J,0}(x)\gamma(x)dx + q\sum_{j=1}^{J-1}\int_0^\infty V_{j,0}(x)\,\gamma(x)dx \tag{3.1}$$

$$\frac{d}{dx}I_n(x) = -(\lambda + \eta(x))I_n(x), \qquad n \geq 1 \tag{3.2}$$

$$\frac{d}{dx}P_{i,n}(x) = -(\lambda + \mu_i(x))P_{i,n}(x) + \lambda P_{i,n-1}(x), n \geq 0, 1 \leq i \leq M \tag{3.3}$$

$$\frac{d}{dx}V_{j,n}(x) = -(\lambda + \gamma(x))V_{j,n}(x) + \lambda V_{j,n-1}(x), n \geq 0, j = 1,2,\dots,J \tag{3.4}$$

with boundary conditions

$$I_n(0) = \sum_{i=1}^{M}\int_0^\infty P_{i,n}(x)\mu_i(x)\,dx + \sum_{j=1}^{J}\int_0^\infty V_{j,n}(x)\gamma(x)dx, n \geq 1 \tag{3.5}$$

$$P_{i,0}(0) = p_i\left[\lambda I_0 + \int_0^\infty I_1(x)\eta(x)dx\right] \tag{3.6}$$

$$P_{i,n}(0) = p_i\left[\lambda\int_0^\infty I_n(x)dx + \int_0^\infty I_{n+1}(x)\eta(x)dx\right], n \geq 1, 1 \leq i \leq M. \tag{3.7}$$

$$V_{j,n}(0) = \begin{cases} \bar{q}\int_0^\infty V_{j-1,n}(x)\gamma(x)dx, j = 2,3,\dots,J, n = 0 \\ 0 \qquad\qquad\qquad\qquad\qquad , n \neq 0 \end{cases} \tag{3.8}$$

$$V_{1,n}(0) = \begin{cases} \sum_{i=1}^{M}\int_0^\infty P_{i,n}(x)\mu_i(x)dx, n = 0 \\ \qquad\qquad 0 \quad , n \neq 0 \end{cases} \tag{3.9}$$

The normalizing condition is

$$I_0 + \sum_{n=1}^{\infty}\int_0^\infty I_n(x)dx + \sum_{n=0}^{\infty}\int_0^\infty P_{i,n}(x)dx + \sum_{n=0}^{\infty}\int_0^\infty V_{j,n}(x)dx = 1 \tag{3.10}$$

Define the probability generating function

$$I(x,z) = \sum_{n=1}^{\infty} I_n(x) z^n$$

$$P_i(x,z) = \sum_{n=0}^{\infty} P_{i,n}(x) z^n \qquad \text{and}$$

$$V_j(x,z) = \sum_{n=0}^{\infty} V_{j,n}(x) z^n$$

STEADY STATE PROBABILITY GENERATING FUNCTION

Multiplying equation (3.2) to (3.4) by z^n and summing over all possible n, we get the following differential equations

$$\left[\frac{d}{dx} + \left(\lambda + \eta(x)\right)\right] I(x,z) = 0 \tag{3.11}$$

$$\left[\frac{d}{dx} + \lambda + \mu_i(x) - \lambda z\right] P_i(x,z) = 0 \tag{3.12}$$

$$\left[\frac{d}{dx} + \lambda(1-z) + \gamma(x)\right] V_j(x,z) = 0 \tag{3.13}$$

Solving the partial differential equations (3.11) ,(3.12) and (3.13) we get respectively

$$I(x,z) = I(0,z) e^{-\lambda x}[1 - A(x)] \tag{3.14}$$

$$P_i(x,z) = P_i(0,z) e^{-[\lambda(1-z)]x}[1 - B_i(x)] \tag{3.15}$$

$$V_j(x,z) = V_j(0,z) e^{-\lambda(1-z)x}[1 - V(x)] \tag{3.16}$$

Solving equation (3.4) at n=0, we get

$$V_{j,0}(x) = V_{j,0}(0) e^{-\lambda x}[1 - V(x)], j = 1,2,\ldots,J \tag{3.17}$$

Multiplying equation (3.17) by $\gamma(x)$ and integrating with respect to x from 0 to ∞, we have

$$\int_0^{\infty} V_{j,0}(x)\, \gamma(x) dx = V_{j,0}(0) V^*(\lambda)\, , j = 1,2,\ldots,J \tag{3.18}$$

Equation (3.9) gives

$$V_{j,0}(0) = \bar{q}V_{j-1,0}(0)V^*(\lambda), j = 1,2,3, \dots, J \tag{3.19}$$

From equation (3.8) and (3.9) it is clear that

$$V_j(0, z) = V_{j,0}(0).$$

Applying equation (3.19) repeatedly for j=J, J-1, J-2,…, we get

$$V_j(0, z) = \frac{V_{J,0}(0)}{(\bar{q}V^*(\lambda))^{J-j}}, j = 1,2,3, \dots, J-1 \tag{3.20}$$

Substituting the expressions of equations (3.18) and (3.20) in equation (3.1) and after some algebraic manipulations, we get

$$V_{J,0}(0) = \frac{\lambda I_0}{V^*(\lambda)\left[1 + \frac{q\left(\left(\bar{q}V^*(\lambda)\right)^{J-1} - 1\right)}{(\bar{q}V^*(\lambda) - 1)\left(\left(\bar{q}V^*(\lambda)\right)^{J-1}\right)}\right]} \tag{3.21}$$

Using equation (3.21) in (3.20) we get

$$\sum_{j=1}^{J} V_j(0, z) = \frac{\lambda I_0\left[\left(\bar{q}V^*(\lambda)\right)^J - 1\right]}{\left[\left(\left(\bar{q}V^*(\lambda)\right)^{J-1}\left(\left(\bar{q}V^*(\lambda)\right) - 1\right) + q\left(\left(\bar{q}V^*(\lambda)\right)^{J-1} - 1\right)\right)\right]}$$

$$= \lambda I_0 T \tag{3.22}$$

where

$$T = \frac{[(\bar{q}V^*(\lambda))^J - 1]}{\left[\left(\left(\bar{q}V^*(\lambda)\right)^{J-1}\left(\left(\bar{q}V^*(\lambda)\right) - 1\right) + q((\bar{q}V^*(\lambda))^{J-1} - 1)\right)\right]}$$

Multiplying equation (3.5) by z^n and summing over possible values of n, we get

$$I(0, z) = \sum_{i=1}^{M}\left[\int_0^\infty P_i(x, z)\mu_i(x)dx - \int_0^\infty P_{i,0}\mu_i(x)dx\right] +$$

$$\sum_{j=1}^{J}\int_0^\infty V_j(x,z)\gamma(x)dx - \sum_{j=1}^{J}\int_0^\infty V_{j,0}(x)\gamma(x)dx$$

Using equation (3.9) we get

$$I(0,z) = \sum_{i=1}^{M}\int_0^\infty P_i(x,z)\,\mu_i(x)dx - V_{1,0}(0) + \sum_{j=1}^{J}\int_0^\infty V_j(x,z)\gamma(x)dx -$$

$$(q+\bar{q})\sum_{j=1}^{J-1}\int_0^\infty V_{j,0}(x)\gamma(x)dx \;-\int_0^\infty V_{J,0}(x)\,\gamma(x)dx$$

Using (3.1) and (3.8) we get

$$I(0,z) = \sum_{i=1}^{M}\int_0^\infty P_i(x,z)\,\mu_i(x)dx - \sum_{j=1}^{J} V_{j,0}(0) + \sum_{j=1}^{J}\int_0^\infty V_j(x,z)\gamma(x)dx - \lambda I_0 \quad (3.23)$$

Substituting the expressions of $P_i(x,z)$ and $V_j(x,z)$ interms of $P_i(0,z)$ and $V_j(0,z)$ respectively in equation (3.23) we get

$$I(0,z) = \sum_{i=1}^{M} P_i(0,z)B_i^{*}(\lambda-\lambda z) + \sum_{j=1}^{J} V_j(0,z)V^{*}(\lambda-\lambda z) - \sum_{j=1}^{J} V_{j,0}(0) - \lambda I_0$$

$$= \sum_{i=1}^{M} P_i(0,z)B_i^{*}(\lambda-\lambda z) + \lambda I_0 T V^{*}(\lambda-\lambda z) - \lambda I_0 T - \lambda I_0 \qquad (3.24)$$

Multiplying equations (3.6) and (3.7) by appropriate powers of z and summing we get

$$P_i(0,z) = p_i\left[\lambda\int_0^\infty I(x,z)dx + \frac{1}{z}\int_0^\infty [I(x,z)]\eta(x)dx + \lambda I_0\right] \qquad (3.25)$$

Using equation (3.14) we obtain

$$P_i(0,z) = \frac{p_i}{z}\left[I(0,z)\left(z\big(1-A^{*}(\lambda)\big) + A^{*}(\lambda)\right) + \lambda I_0 z\right] \qquad (3.26)$$

Solving equations (3.24) and (3.26) for $I(0,z)$ and $P_i(0,z)$ we get

$$I(0,z)=\frac{z\lambda I_0[(\sum p_i B_i^*(\lambda-\lambda z)-1)+T(V^*(\lambda-\lambda z)-1)]}{\left[z-\left(z\left(1-A^*(\lambda)\right)+A^*(\lambda)\right)\sum p_i B_i^*(\lambda-\lambda z)\right]} \tag{3.27}$$

$$P_i(0,z)=\frac{p_i\lambda I_0\left[A^*(\lambda)(z-1)+T(V^*(\lambda-\lambda z)-1)\left(z\left(1-A^*(\lambda)\right)+A^*(\lambda)\right)\right]}{\left[z-\left(z\left(1-A^*(\lambda)\right)+A^*(\lambda)\right)\sum p_i B_i^*(\lambda-\lambda z)\right]} \tag{3.28}$$

Substituting the expressions of equations (3.27) and (3.28) in equations(3.14) and (3.15) we obtain

$$I(x,z)=\frac{z\lambda I_0[(\sum p_i B_i^*(\lambda-\lambda z)-1)+T(V^*(\lambda-\lambda z)-1)]}{\left[z-\left(z\left(1-A^*(\lambda)\right)+A^*(\lambda)\right)\sum p_i B_i^*(\lambda-\lambda z)\right]}e^{-\lambda x}[1-A(x)] \tag{3.29}$$

$$P_i(x,z)=\frac{p_i\lambda I_0\left[A^*(\lambda)(z-1)+T(V^*(\lambda-\lambda z)-1)\left(z\left(1-A^*(\lambda)\right)+A^*(\lambda)\right)\right]}{\left[z-\left(z\left(1-A^*(\lambda)\right)+A^*(\lambda)\right)\sum p_i B_i^*(\lambda-\lambda z)\right]}$$

$$e^{-[\lambda(1-z)]x}[1-B_i(x)] \tag{3.30}$$

PERFORMANCE MEASURES

Define the partial generating function $\psi(z)=\int_0^\infty \psi(z,x)dx$ for any generating function $\psi(z,x)$.

The probability generating function of the orbit size when the server is idle is given by

$$I(z)=\int_0^\infty I(x,z)dx$$

$$=\frac{z\,I_0[(\sum p_i B_i^*(\lambda-\lambda z)-1)+T(V^*(\lambda-\lambda z)-1)][1-A^*(\lambda)]}{\left[z-\left(z\left(1-A^*(\lambda)\right)+A^*(\lambda)\right)\sum p_i B_i^*(\lambda-\lambda z)\right]} \tag{3.31}$$

$$I=\lim_{z\to 1} I(z)$$

$$= \frac{I_0\left(1 - A^*(\lambda)\right)(\lambda \sum p_i \mu_{i,1} + T\lambda\gamma_1)}{A^*(\lambda) - \lambda \sum p_i \mu_{i,1}} \tag{3.32}$$

The probability generating function of the orbit size when the server is busy is given by

$$P(z) = \sum_{I=1}^{M} P_i(z) = \sum_{i=1}^{M} \int_0^{\infty} P_i(x, z)dx$$

$$= \frac{p_i I_0 \left[A^*(\lambda)(z-1) + T(V^*(\lambda - \lambda z) - 1)\left(z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right)\right]}{[1 - z]\left[z - \left(z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right) \sum p_i B_i{}^*(\lambda - \lambda z)\right]}$$

$$[1 - B_i{}^*(\lambda - \lambda z)] \tag{3.33}$$

$$P = \sum_{I=1}^{M} P_i = \lim_{z \to 1} P_i(z)$$

$$= \frac{I_0 \lambda p_i \mu_{i,1}[A^*(\lambda) + T\lambda\gamma_1]}{A^*(\lambda) - \lambda \sum p_i \mu_{i,1}} \tag{3.34}$$

The probability generating function of the orbit size when the server on vacation is given by

$$V(z) = \sum_{j=1}^{J} V_j(z) = \frac{I_o[1 - V^*(\lambda - \lambda z)]T}{(1 - z)} \tag{3.35}$$

$$V = \sum_{j=1}^{J} V_j = \lim_{z \to 1} \sum_{j=1}^{J} V_j(z)$$

$$= \frac{I_0 T(-V^{*'}(\lambda - \lambda z))}{-1}$$

$$= I_0 T\lambda\gamma_1 \tag{3.36}$$

The normalizing condition (3.10) becomes

$I_0 + I + P + V = 1$

Using equations (3.32), (3.34), (3.36) and solving for I_0 we get

$$I_0 = \frac{A^*(\lambda) - \lambda \sum p_i\, \mu_{i,1}}{A^*(\lambda) + T\lambda\gamma_1} \tag{3.37}$$

Now equations (3.32), (3.34) and (3.36) yield the following results

The steady state probability that the server is idle, when the system is not empty is

$$I = \frac{\left(1 - A^*(\lambda)\right)(\lambda \sum p_i \mu_{i,1} + T\lambda\gamma_1)}{A^*(\lambda) + T\lambda\gamma_1}$$

The steady state probability that the server is busy is

$$P = \lambda p_i \mu_{i,1}$$

The steady state probability that the server on vacation is

$$V = \frac{T\lambda\gamma_1 (A^*(\lambda) - \lambda \sum p_i\, \mu_{i,1})}{A^*(\lambda) + T\lambda\gamma_1}$$

MEAN ORBIT SIZE AND SYSTEM SIZE

The probability generating function for the number of customers in the orbit is

$$P_q(z) = I_0 + I(z) + P(z) + V(z)$$

$$= \frac{I_0 \left[A^*(\lambda)(z-1) + \left(T\left(V^*(\lambda - \lambda z) - 1\right)\right)\left(z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right)\right]}{\left[z - \left(z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right) \sum p_i\, B_i^*(\lambda - \lambda z)\right]} \tag{3.38}$$

Let N (z) and D (z) are the numerator and denominator of $P_q(z)$.

The mean number of customers in the orbit is

$$L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z)$$

$$= \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \tag{3.39}$$

$$N'(1) = I_0(A^*(\lambda) + T\lambda\gamma_1)$$

$$N''(1) = I_0(2T\lambda\gamma_1 \left(1 - A^*(\lambda)\right) + T\lambda^2\gamma_2)$$

$$D'(1) = A^*(\lambda) - \lambda \sum p_i\, \mu_{i,1}$$

$$D''(1) = -\left[\lambda^2 \sum p_i \mu_{i,2} + 2\lambda(1 - A^*(\lambda)) \sum p_i \mu_{i,1}\right]$$

The probability generating function for the number of customers in the system is

$$P_s(z) = I_0 + I(z) + zP(z) + V(z)$$

$$= \frac{I_0 \sum p_i B_i^*(\lambda - \lambda z)\left[A^*(\lambda)(z-1) + \left(T(V^*(\lambda - \lambda z) - 1)\right)\left(z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right)\right]}{\left[z - \left(z\left(1 - A^*(\lambda)\right) + A^*(\lambda)\right)\sum p_i B_i^*(\lambda - \lambda z)\right]}$$

(3.40)

The mean number of customers in the system is

$$L_s = \lim_{z \to 1} \frac{d}{dz} P_s(z)$$

$$= L_q + P$$

STOCHASTIC DECOMPOSITION PROPERTY

Theorem:

The number of customers in the system L_s can be expressed as the sum of two independent random variables one of which is the mean number of customers(L) in the classical queue with fluctuating modes of service and other is the mean number of customers in the orbit (L_I) given that the server is idle or on vacation.

Proof:

The probability generating function $\pi(z)$ of the system size in the classical queue with fluctuating modes of service is given by

$$\pi(z) = \frac{(1 - \lambda \sum p_i \mu_{i,1}) \sum p_i B_i^*(\lambda - \lambda z)(z-1)}{(1 + T\lambda\gamma_1)(z - \sum p_i B_i^*(\lambda - \lambda z))} \qquad (3.41)$$

The probability generating function $\psi(z)$ of the number of customers in the orbit when the system is idle or on vacation is given by

$$\psi(z) = \frac{I_0 + I(z) + V(z)}{I_0 + I + V}$$

$$= \frac{T(z)\left[A^*(\lambda)(z-1) + \left(T(V^*(\lambda-\lambda z)-1)\right)\left(z\left(1-A^*(\lambda)\right)+A^*(\lambda)\right)\right]}{T_1(z)\left[z-\left(z\left(1-A^*(\lambda)\right)+A^*(\lambda)\right)\sum p_i B_i^*(\lambda-\lambda z)\right]} \quad (3.42)$$

where

$$T(z) = [z - \sum p_i B_i^*(\lambda - \lambda z)]\left(A^*(\lambda) - \lambda \sum p_i \mu_{i,1}\right)$$

$$T_1(z) = (z-1)\left(1 - \lambda \sum p_i \mu_{i,1}\right)(A^*(\lambda) + T\lambda\gamma_1)$$

From equations (3.40), (3.41) and (3.42), we see that

$$P_s(z) = \pi(z).\ \psi(z) \quad (3.43)$$

Differentiating (3.43) with respect to z and taking limit as z→1 we get,

$$L_s = L + L_I$$

NUMERICAL RESULTS

By assuming the retrial time, service times and vacation follow exponential distribution with respective rates η, μ_i $(1 \le i \le M)$ and γ performance measures are calculated for various parametric values.

For Computation we choose arbitrarily the parameters

$\lambda = 0.1, \eta = 5, p_1 = 0.6, p_2 = 0.4, \ \gamma = 0.11, J = 7, q = 0.4, \mu_1 = 0.95, \mu_2 = 0.05$
Effect of parameters (q, μ_1), (η, γ) and (λ, μ_2) on the performance measures are obtained and presented respectively in Tables 3.1 to 3.3.

In Table 3.1, we observe the effect of q and μ_1.

- Increase in μ_1 increases I_0 and V and decreases P, L_q and L_s .
- Increase in q increases I_0 and decreases I, V, L_q and L_s and has no effect on P.

In Table 3.2, we observe the effect of η and γ.

- Increase in γ increases I_0 , decreases V, L_q and L_s , slightly decreases the value of I and has no effect on P.
- Increase in η decreases L_q and L_s and has no effect on I_0, P and V.

In Table 3.3, we observe the effect of λ and μ_2.

- Increase in μ_2 increases I_0 and V and decreases I, P, L_q and L_s.

- Increase in λ increases I, P, V, L_q and L_s and decreases I_0.

q	μ_1	I_0	I	P	V	L_q	L_s
0.4	0.95	0.0131	0.0194	0.8632	0.1044	14.6757	15.5388
	0.96	0.0132	0.0193	0.8625	0.1049	14.5971	15.4596
	0.97	0.0133	0.0193	0.8619	0.1055	14.5210	15.3829
0.6	0.95	0.0366	0.0189	0.8632	0.0813	14.4932	15.3563
	0.96	0.0368	0.0189	0.8625	0.0818	14.4146	15.2771
	0.97	0.0370	0.0189	0.8619	0.0822	14.3385	15.2004
0.8	0.95	0.0516	0.0186	0.8632	0.0666	14.3766	15.2398
	0.96	0.0519	0.0186	0.8625	0.0670	14.2981	15.1606
	0.97	0.0522	0.0186	0.8619	0.0673	14.2220	15.0838

Table 3.1 Performance measures versus q and μ_1

Table 3.2 Performance measures versus η and γ

η	γ	I_0	I	P	V	L_q	L_s
5.0000	0.1100	0.0131	0.0194	0.8632	0.1044	14.6757	15.5388
	0.1200	0.0156	0.0193	0.8632	0.1019	14.5907	15.4539
	0.1300	0.0180	0.0193	0.8632	0.0996	14.5188	15.3820
5.0100	0.1100	0.0131	0.0193	0.8632	0.1044	14.6708	15.5340
	0.1200	0.0156	0.0193	0.8632	0.1020	14.5858	15.4490
	0.1300	0.0180	0.0192	0.8632	0.0996	14.5140	15.3771
5.0200	0.1100	0.0131	0.0193	0.8632	0.1044	14.6660	15.5291
	0.1200	0.0156	0.0192	0.8632	0.1020	14.5810	15.4442
	0.1300	0.0180	0.0192	0.8632	0.0996	14.5092	15.3723

Table 3.3 Performance measures versus λ and μ_2

λ	μ_2	I_0	I	P	V	L_q	L_s
0.0200	0.0500	0.5210	0.0019	0.1726	0.3044	0.1476	0.3202
	0.1500	0.5885	0.0016	0.0660	0.3439	0.0769	0.1429
	0.2500	0.6020	0.0016	0.0446	0.3517	0.0719	0.1165
0.0300	0.0500	0.3692	0.0038	0.2589	0.3681	0.3384	0.5973
	0.1500	0.4495	0.0033	0.0989	0.4482	0.1588	0.2577
	0.2500	0.4656	0.0032	0.0669	0.4643	0.1469	0.2138
0.0400	0.0500	0.2588	0.0059	0.3453	0.3900	0.6257	0.9710
	0.1500	0.3442	0.0052	0.1319	0.5187	0.2596	0.3915
	0.2500	0.3612	0.0051	0.0893	0.5444	0.2374	0.3266

4. BATCH ARRIVAL RETRIAL QUEUE WITH FLUCTUATING MODES OF SERVICE, RANDOMIZED VACATIONS AND ORBITAL SEARCH

Single server retrial queue with arrivals in batches of variable size and general service in fluctuating modes with different mean service rates is considered. Whenever the system becomes empty the server takes randomized vacations. During the idle period, the server searches for customer in the orbit with certain probability. Using supplementary variable technique, average number of customers in the orbit and average number of customers in the system are derived. Stochastic decomposition law is verified. Numerical results are obtained.

MODEL DESCRIPTION

Consider a single server retrial queueing system with fluctuating modes of service in which customers arrive in batches according to Poisson process with rate λ . It is assumed that at every arrival epoch, a batch of K customers arrive with probability c_k and the first two moments of the batch size are m_1 and m_2.

The server provides M heterogeneous types of service and a customer opts one of the types i with probability $p_i (1 \leq i \leq M)$. If the arriving batch finds the server free, then one of the customers in the batch receives service immediately and others join the orbit. Otherwise all the customers join the retrial queue. The retrial time is generally distributed with distribution A(x), density function a(x), Laplace Stieltjes transform $A^*(s)$ and conditional completion rate $\eta(x) = \frac{a(x)}{1-A(x)}$.

The service time of type i (i=1,2,...M) follows general distribution with distribution function $B_i(x)$, density function $b_i(x)$, Laplace Stieltje's transform $B_i^*(s)$, n^{th} factorial moments $\mu_{i,n}$ and conditional completion rate $\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}$.

At a service completion epoch if the orbit becomes empty, the server leaves for a vacation of random length V. At the vacation completion epoch, if the orbit is still empty the server either remains idle in the system with probability q or leaves for another vacation with probability $\bar{q}(=1-q)$. This pattern continues until the number of vacations taken reaches a preassigned number J. If the system is empty at the end of the J^{th} vacation, the server is dormant idly in the system. During the idle time if the orbit is non-empty, the server may search for customer from the orbit. Vacation time is generally distributed with distribution function V(x), Laplace Stieltjes transform $V^*(s)$, n^{th} factorial moment v_n and conditional completion rate $\gamma(x)=\frac{v(x)}{1-V(x)}$. During the idle period, if the orbit is non-empty, server searches for customers in the orbit with probability θ or remains idle with probability $\bar{\theta}(=1-\theta)$.

The state of the system at time t can be described by the Markov process$\{N(t);t\geq 0\}=\{C(t),X(t),\xi_0(t),\xi_1(t),\xi_2(t);t\geq 0\}$ where C(t) denotes the server state 0, i or M+j accordingly as the server being idle, busy in type i service or the server is on j^{th} vacation. X(t) corresponds to the number of the customers in the orbit. If C(t)=0 and X(t)>0 then $\xi_0(t)$ represent the elapsed retrial time. If C(t)=i$(1\leq i\leq M)$, then $\xi_1(t)$ represents the elapsed service time. If C(t)= M+j (1≤j≤J) then $\xi_2(t)$ represents elapsed vacation time.

DEFINITIONS AND NOTATIONS

For the process $\{N(t); t\geq 0\}$, define the probability

$I_0(t)=P\{C(t)=0,X(t)=0\}$ and

the probability densities for t>0; x>0 as

$I_n(x,t)=P\{C(t)=0,X(t)=n,x<\xi_0(t)<x+dx\}$, $n\geq 1$.

$P_{i,n}(x,t)=P\{C(t)=i,X(t)=n,x<\xi_1(t)<x+dx\}n\geq 0,1\leq i\leq M$.

$V_{j,n}(x,t)=P\{C(t)=M+j,X(t)=n,x<\xi_2(t)<x+dx\}n\geq 0,j=1,2,3,\dots,J$.

STEADY STATE EQUATIONS

Let I_0, $I_n(x)$, $P_{i,n}(x)$, $V_{j,n}(x)$ be the steady state probabilities of $I_0(t)$, $I_n(x,t)$, $P_{i,n}(x,t)$, $V_{j,n}(x,t)$.

The system of steady state equations that governs the model under consideration is given below.

$$\lambda I_0 = \int_0^\infty V_{J,0}(x)\gamma(x)dx + q\sum_{j=1}^{J-1}\int_0^\infty V_{j,0}(x)\,\gamma(x)dx \tag{4.1}$$

$$\frac{d}{dx}I_n(x) = -(\lambda+\eta(x))I_n(x), \qquad n \geq 1 \tag{4.2}$$

$$\frac{d}{dx}P_{i,n}(x) = -(\lambda+\mu_i(x))P_{i,n}(x) + \lambda\sum_{k=1}^{n} C_k\, P_{i,n-k}(x), n \geq 0, 1 \leq i \leq M \tag{4.3}$$

$$\frac{d}{dx}V_{j,n}(x) = -(\lambda+\gamma(x))V_{j,n}(x) + \lambda\sum_{k=1}^{n} C_k\, V_{j,n-k}(x), n \geq 0,1 \leq j \leq J \tag{4.4}$$

with boundary conditions

$$I_n(0) = \sum_{i=1}^{M}\int_0^\infty P_{i,n}(x)\mu_i(x)\,dx + \bar{\theta}\sum_{j=1}^{J}\int_0^\infty V_{j,n}(x)\gamma(x)dx, n \geq 1 \tag{4.5}$$

$$P_{i,0}(0) = p_i\left[\lambda C_1 I_0 + \int_0^\infty I_1(x)\eta(x)dx + \theta\sum_{j=1}^{J}\int_0^\infty V_{j,1}(x)\gamma(x)dx\right] \tag{4.6}$$

$$P_{i,n}(0) = p_i\left[\lambda C_{n+1}I_0 + \int_0^\infty I_{n+1}(x)\eta(x)dx + \right.$$

$$\left.\lambda\sum_{k=1}^{n} C_k\int_0^\infty I_{n-k+1}(x)dx \; + \theta\sum_{j=1}^{J}\int_0^\infty V_{j,n+1}(x)\gamma(x)dx\right], n \geq 1 \tag{4.7}$$

$$V_{1,n}(0) = \begin{cases} \displaystyle\sum_{i=1}^{M}\int_0^\infty P_{i,n}(x)\mu_i(x)dx, n = 0 \\ \qquad\qquad 0 \qquad , n \neq 0 \end{cases} \tag{4.8}$$

$$V_{j,n}(0) = \begin{cases} \bar{q}\displaystyle\int_0^\infty V_{j-1,n}(x)\gamma(x)dx, n = 0, j = 2,3,\ldots,J. \\ \qquad\qquad 0 \;, n \neq 0 \end{cases} \tag{4.9}$$

Define the probability generating functions

$$I(x,z)=\sum_{n=1}^{\infty} I_n(x)z^n \ , \ P_i(x,z)=\sum_{n=0}^{\infty} P_{i,n}(x)z^n \ ,$$

$$V_j(x,z)=\sum_{n=0}^{\infty} V_{j,n}(x)z^n \text{ and } C(z)=\sum_{k=1}^{\infty} c_k z^k$$

STEADY STATE PROBABILITY GENERATING FUNCTION

Multiplying equations (4.2) to (4.4) by z^n and summing over all possible values of n, we get the following differential equations.

$$\left[\frac{d}{dx}+\big(\lambda+\eta(x)\big)\right] I(x,z)=0 \tag{4.10}$$

$$\left[\frac{d}{dx}+\lambda(1-C(z))+\mu_i(x)\right] P_i(x,z)=0, 1\le i\le M \tag{4.11}$$

$$\left[\frac{d}{dx}+\lambda(1-C(z))+\ \gamma(x)\right] V_j(x,z)=0, 1\le j\le J \tag{4.12}$$

Solving the partial differential equations (4.10), (4.11) and (4.12), we get respectively

$$I(x,z)=I(0,z)\ e^{-\lambda x}[1-A(x)] \tag{4.13}$$

$$P_i(x,z)=P_i(0,z)e^{-[\lambda(1-C(z))]x}[1-B_i(x)] \tag{4.14}$$

$$V_j(x,z)=V_j(0,z)e^{-\lambda(1-C(z))x}[1-V(x)] \tag{4.15}$$

Solving equation (4.4) at n=0, we get

$$V_{j,0}(x)=V_{j,0}(0)e^{-\lambda x}[1-V(x)], j=1,2,\dots,J \tag{4.16}$$

Multiplying equation (4.16) by $\gamma(x)$ and intrgrating with respect to x from 0 to ∞, we have

$$\int_0^{\infty} V_{j,0}(x)\ \gamma(x)dx=V_{j,0}(0)V^*(\lambda)\ , j=1,2,\dots,J \tag{4.17}$$

Equation (4.9) gives

$$V_{j,0}(0)=\bar{q}V_{j-1,0}(0)V^*(\lambda), j=1,2,3,\dots,J \tag{4.18}$$

From equation (4.8) and (4.9) it is clear that

$$V_j(0,z)=V_{j,0}(0).$$

Applying equation (4.18) repeatedly for j=J,J-1,J-2,…,we get

$$V_j(0,z)=\frac{V_{J,0}(0)}{(\bar{q}V^*(\lambda))^{J-j}}, j=1,2,3,\dots,J-1 \tag{4.19}$$

Substituting the expressions of equations (4.17) and (4.19) in equation (4.1) and after some algebraic manipulations, we get

$$V_{J,0}(0)=\frac{\lambda I_0}{V^*(\lambda)\left[1+\frac{q\left(\left(\bar{q}V^*(\lambda)\right)^{J-1}-1\right)}{(\bar{q}V^*(\lambda)-1)\left(\left(\bar{q}V^*(\lambda)\right)^{J-1}\right)}\right]} \tag{4.20}$$

Using equation (4.20) in (4.19), we get

$$\sum_{j=1}^{J}V_j(0,z)=\frac{\lambda I_0\left[\left(\bar{q}V^*(\lambda)\right)^{J}-1\right]}{\left[\left(\left(\bar{q}V^*(\lambda)\right)^{J-1}\left(\left(\bar{q}V^*(\lambda)\right)-1\right)+q\left(\left(\bar{q}V^*(\lambda)\right)^{J-1}-1\right)\right)\right]}$$

$$=\lambda I_0 T \tag{4.21}$$

where

$$T=\frac{[(\bar{q}V^*(\lambda))^{J}-1]}{\left[\left(\left(\bar{q}V^*(\lambda)\right)^{J-1}\left(\left(\bar{q}V^*(\lambda)\right)-1\right)+q((\bar{q}V^*(\lambda))^{J-1}-1)\right)\right]}$$

Multiplying equation (4.5) by z^n and summing over possible values of n ,we get

$$I(0,z)=\left[\sum_{n=1}^{\infty}\int_0^{\infty}\sum_{i=1}^{M}P_{i,n}(x)\mu_i(x)z^n dx+\bar{\theta}\sum_{j=1}^{J}\sum_{n=1}^{\infty}\int_0^{\infty}V_{j,n}(x)z^n\gamma(x)dx\right]$$

$$= \sum_{i=1}^{M} \int_0^\infty P_i(x,z)\,\mu_i(x)dx - \int_0^\infty P_{i,0}\mu_i(x)dx + \bar{\theta}\sum_{j=1}^{J}\int_0^\infty V_j(x,z)\gamma(x)dx -$$

$$\bar{\theta}\sum_{j=1}^{J}\int_0^\infty V_{j,0}(x)\gamma(x)dx$$

$$= \sum_{i=1}^{M} \int_0^\infty P_i(x,z)\,\mu_i(x)dx - \int_0^\infty P_{i,0}\mu_i(x)dx + \bar{\theta}\sum_{j=1}^{J}\int_0^\infty V_j(x,z)\gamma(x)dx -$$

$$(q+\bar{q})\sum_{j=1}^{J-1}\int_0^\infty V_{j,0}(x)\gamma(x)dx \;\; - \int_0^\infty V_{J,0}(x)\,\gamma(x)dx$$

Using equation (4.9), we get

$$I(0,z) = \sum_{i=1}^{M} \int_0^\infty P_i(x,z)\,\mu_i(x)dx - V_{1,0}(0) + \bar{\theta}\sum_{j=1}^{J}\int_0^\infty V_j(x,z)\gamma(x)dx -$$

$$(q+\bar{q})\sum_{j=1}^{J-1}\int_0^\infty V_{j,0}(x)\gamma(x)dx \;\; - \int_0^\infty V_{J,0}(x)\,\gamma(x)dx$$

Using (4.1) and (4.8), we have

$$I(0,z) = \sum_{i=1}^{M} \int_0^\infty P_i(x,z)\,\mu_i(x)dx - \sum_{j=1}^{J} V_{j,0}(0) + \bar{\theta}\sum_{j=1}^{J}\int_0^\infty V_j(x,z)\gamma(x)dx - \lambda I_0 \tag{4.22}$$

Substituting the expressions of $P_i(x,z)$ and $V_j(x,z)$ interms of $P_i(0,z)$ and $V_j(0,z)$ respectively in equation (4.22), we obtain

$$I(0,z) = \sum_{i=1}^{M} P_i(0,z)B_i^{*}(\lambda-\lambda C(z)) + \bar{\theta}\sum_{j=1}^{J} V_j(0,z)V^{*}(\lambda-\lambda C(z)) - \sum_{j=1}^{J} V_{j,0}(0) - \lambda I_0$$

$$= \sum_{i=1}^{M} P_i(0,z)B_i^{*}(\lambda-\lambda C(z)) + \bar{\theta}\,\lambda I_0 T V^{*}(\lambda-\lambda C(z)) - \lambda I_0 T - \lambda I_0 \tag{4.23}$$

Multiplying equations (4.6) and (4.7) by appropriate powers of z and summing, we get

$$P_i(0,z) = \frac{p_i}{z}\left[\lambda I_0 C(z) + \lambda C(z)\int_0^\infty I(x,z)dx + \int_0^\infty [I(x,z)]\eta(x)dx + \theta\sum_{j=1}^{J}\int_0^\infty V_j(x,z)\gamma(x)dx\right] \quad (4.24)$$

Using equations (4.13) and (4.15), we obtain

$$P_i(0,z) = \frac{p_i}{z}\left[I(0,z)\left(C(z)\big(1-A^*(\lambda)\big)+A^*(\lambda)\right) + \lambda I_0 C(z) + \theta\lambda I_0 T V^*(\lambda-\lambda C(z))\right] \quad (4.25)$$

Solving equations (4.23) and (4.25) for $I(0,z)$ and $P_i(0,z)$, we get

$$I(0,z) = \frac{\lambda I_0\left[\begin{array}{c}\sum p_i B_i^*\big(\lambda-\lambda C(z)\big)\left(TV^*\big(\lambda-\lambda C(z)\big)\theta + C(z)\right) + \\ zT\bar{\theta}V^*\big(\lambda-\lambda C(z)\big) - z(T+1)\end{array}\right]}{\left[z-\left(A^*(\lambda)+C(z)\big(1-A^*(\lambda)\big)\right)\sum p_i B_i^*\big(\lambda-\lambda C(z)\big)\right]} \quad (4.26)$$

$$P_i(0,z) = p_i\lambda I_0 T_1(z) \quad (4.27)$$

where

$$T_1(z) = \frac{\left[\begin{array}{c}A^*(\lambda)(C(z)-1) + TV^*\big(\lambda-\lambda C(z)\big)\theta + \\ T\big(V^*(\lambda-\lambda C(z))\bar{\theta}-1\big)\big(A^*(\lambda)+C(z)\big(1-A^*(\lambda)\big)\big)\end{array}\right]}{\big(1-C(z)\big)\left[z-\left(A^*(\lambda)+C(z)\big(1-A^*(\lambda)\big)\right)\sum p_i B_i^*\big(\lambda-\lambda C(z)\big)\right]}$$

Substituting equations (4.26) and (4.27) in equations (4.13) and (4.14) , we obtain

$$I(x,z) = \frac{\lambda I_0\left[\begin{array}{c}\sum p_i B_i^*\big(\lambda-\lambda C(z)\big)\left(TV^*\big(\lambda-\lambda C(z)\big)\theta + C(z)\right) + \\ zT\bar{\theta}V^*\big(\lambda-\lambda C(z)\big) - z(T+1)\end{array}\right]}{\left[z-\left(A^*(\lambda)+C(z)\big(1-A^*(\lambda)\big)\right)\sum p_i B_i^*\big(\lambda-\lambda C(z)\big)\right]} e^{-\lambda x}[1-A(x)] \quad (4.28)$$

$$P_i(x,z) = p_i\lambda I_0 T_1(z) e^{-[\lambda(1-C(z))]x}[1-B_i(x)] \quad (4.29)$$

PERFORMANCE MEASURES

Define the partial generating function $\psi(z) = \int_0^\infty \psi(z,x)dx$ for any generating function $\psi(z,x)$.

The probability generating function of the orbit size when the server is idle is given by

$$I(z) = \int_0^\infty I(x,z)dx$$

$$= \frac{I_0[1 - A^*(\lambda)]\begin{bmatrix}\sum p_i B_i^*(\lambda - \lambda C(z))\left(TV^*(\lambda - \lambda C(z))\theta + C(z)\right) + \\ zT\bar{\theta}V^*(\lambda - \lambda C(z)) - z(T+1)\end{bmatrix}}{\left[z - \left(A^*(\lambda) + C(z)(1 - A^*(\lambda))\right)\sum p_i B_i^*(\lambda - \lambda C(z))\right]} \tag{4.30}$$

The probability generating function of the orbit size when the server is busy is given by

$$P(z) = \sum_{i=1}^{M} P_i(z) = \sum_{i=1}^{M}\int_0^\infty P_i(x,z)dx$$

$$= I_0[T_1(z)]\sum_{i=1}^{M} p_i\left[1 - B_i^*(\lambda - \lambda C(z))\right] \tag{4.31}$$

The probability generating function of the orbit size when the server on vacation is given by

$$V(z) = \sum_{j=1}^{J} V_j(z) = \frac{I_o[1 - V^*(\lambda - \lambda C(z))]T}{(1 - C(z))} \tag{4.32}$$

Let I, P and V be the limiting values of $I(z)$, $P(z)$and $V(z)$ as z tends to 1 then equations (4.30), (4.31) and (4.32) yield the following expressions

$$I = \frac{I_o[1 - A^*(\lambda)]\left[\lambda m_1(T\theta + 1)\sum p_i \mu_{i,1} + T(\lambda\gamma_1 m_1 - \theta) + m_1 - 1\right]}{1 - m_1[1 - A^*(\lambda)] - \lambda m_1 \sum p_i \mu_{i,1}} \tag{4.33}$$

$$P = \frac{I_o\lambda\sum p_i\mu_{i,1}\left[T\lambda\gamma_1 m_1 + m_1 A^*(\lambda) - T\theta m_1\left(1 - A^*(\lambda)\right)\right]}{1 - m_1[1 - A^*(\lambda)] - \lambda m_1\sum p_i\mu_{i,1}} \tag{4.34}$$

$$V = I_0 T\lambda\gamma_1 \tag{4.35}$$

Now the normalizing condition

$I_0 + I + P + V = 1$ gives

$$I_0 = \frac{1 - m_1[1 - A^*(\lambda)] - \lambda m_1 \sum p_i \mu_{i,1}}{A^*(\lambda) - T\theta + T\theta A^*(\lambda) + T\lambda\gamma_1} \tag{4.36}$$

Substituting the expression of I_0 in equations (4.33) to (4.35) we get the following results

The steady state probability that the server is idle, when the system is not empty is

$$I = \frac{[1 - A^*(\lambda)]\left[\lambda m_1(T\theta + 1)\sum p_i \mu_{i,1} + T(\lambda\gamma_1 m_1 - \theta) + m_1 - 1\right]}{A^*(\lambda) - T\theta + T\theta A^*(\lambda) + T\lambda\gamma_1} \tag{4.37}$$

The steady state probability that the server is busy is

$$P = \frac{\lambda \sum p_i \mu_{i,1}\left[T\lambda\gamma_1 m_1 + m_1 A^*(\lambda) - T\theta m_1\left(1 - A^*(\lambda)\right)\right]}{A^*(\lambda) - T\theta + T\theta A^*(\lambda) + T\lambda\gamma_1} \tag{4.38}$$

The steady state probability that the server is on vacation is given by

$$V = \frac{[1 - m_1\left(1 - A^*(\lambda)\right) - \lambda m_1 \sum p_i \mu_{i,1}]T\lambda\gamma_1}{A^*(\lambda) - T\theta + T\theta A^*(\lambda) + T\lambda\gamma_1} \tag{4.39}$$

MEAN ORBIT SIZE AND SYSTEM SIZE

The probability generating function for the number of customers in the orbit is

$$P_q(z) = I_0 + I(z) + P(z) + V(z)$$

$$= I_o(1 - z)T_1(z) \tag{4.40}$$

Let $N(z)$ and $D(z)$ are the numerator and denominator of $P_q(z)$ then the mean number of customers in the orbit is

$$L_q = \lim_{z\to 1}\frac{d}{dz}P_q(z)$$

$$= \frac{D''(1)N'''(1) - N''(1)D'''(1)}{3(D''(1))^2} \tag{4.41}$$

$$N''(1) = 2I_0 m_1 \left[T\theta\left(1 - A^*(\lambda)\right) - T\lambda\gamma_1 - A^*(\lambda)\right]$$

$$N'''(1) = 3I_0[(T\theta)\left(m_2\left(1 - A^*(\lambda)\right)\right) - A^*(\lambda)m_2 - T\lambda^2 m_1{}^2\gamma_2 - T\lambda m_2\gamma_1 -$$

$$2T\lambda\gamma_1\bar{\theta}m_1{}^2(1 - A^*(\lambda))]$$

$$D''(1) = 2m_1[m_1 \sum p_i\lambda\mu_{i,1} + m_1\left(1 - A^*(\lambda)\right) - 1]$$

$$D'''(1) = 3\{[(m_1)\left(\sum p_i\left(\lambda^2\mu_{i,2}m_1{}^2 + \lambda\mu_{i,1}m_2\right) + 2m_1{}^2\lambda\left(1 - A^*(\lambda)\right)\sum p_i\mu_{i,1}\right.$$

$$\left. m_2(1 - A^*(\lambda))\right)] + [(m_2)\left(m_1\left(1 - A^*(\lambda)\right) + m_1\lambda \sum p_i\mu_{i,1} - 1\right)]\}$$

The probability generating function for the number of customers in the system is

$$P_s(z) = I_0 + I(z) + zP(z) + V(z)$$

$$= I_0(1 - z)\sum p_i B_i{}^*(\lambda - \lambda C(z))\; T_1(z) \qquad (4.42)$$

The mean number of customers in the system is

$$L_s = \lim_{z\to 1}\frac{d}{dz}P_s(z)$$

$$= L_q + P$$

STOCHASTIC DECOMPOSITION

Theorem: The number of customer in the system (L_s) can be expressed as the sum of two independent random variables, one of which is the mean number of customers (L) in the batch arrival queue with fluctuating modes of service and orbital search and other is the mean number of customers in the orbit (L_I) given that the server is idle or on vacation.

Proof:

The probability generating function $\pi(z)$ of the system size in the batch arrival queueing system with fluctuating modes of service and orbital search is given by

$$\pi(z) = \frac{\left(1 - \lambda m_1 \sum p_i\, \mu_{i,1}\right)(1 - z)\sum p_i\, B_i{}^*\left(\lambda - \lambda C(z)\right)(C(z) - 1)}{(z - \sum p_i\, B_i{}^*\left(\lambda - \lambda C(z)\right))(1 + T\lambda\gamma_1)(1 - C(z))} \qquad (4.44)$$

The probability generating function $\psi(z)$ of number of customers in the orbit when the system is idle or on vacation is given by

$$\psi(z) = \frac{I_0 + I(z) + V(z)}{I_0 + I + V}$$

$$= [T_1(z)][T_2(z)](\sum p_i B_i^*(\lambda - \lambda C(z)) - z) \tag{4.45}$$

where

$$T_2(z) = \frac{[1 - m_1(1 - A^*(\lambda)) - \lambda m_1 \sum p_i \mu_{i,1}]}{(A^*(\lambda) - T\theta + T\theta A^*(\lambda) + T\lambda\gamma_1)(1 - \lambda m_1 \sum p_i \mu_{i,1})}$$

From equations (4.42) , (4.44) and (4.45) we see that

$$P_s(z) = \pi(z).\ \psi(z) \tag{4.46}$$

Differentiating (4.46) with respect to z and taking limit as $z \to 1$, we get

$$L_s = L + L_I$$

SPECIAL CASES

Case(i): Let $\theta \to 0$(no orbital search) then our model will be reduced to batch arrival retrial queueing system with fluctuating modes of service and randomized J vacation. In this case, the expressions of I_0, I, P, V, $P_q(z)$, $P_s(z)$ are

$$I_0 = \frac{1 - m_1[1 - A^*(\lambda)] - \lambda m_1 \sum p_i \mu_{i,1}}{A^*(\lambda) + T\lambda\gamma_1}$$

$$I = \frac{I_o[1 - A^*(\lambda)][\lambda m_1 \sum p_i \mu_{i,1} + (T\lambda\gamma_1 m_1) + m_1 - 1]}{1 - m_1[1 - A^*(\lambda)] - \lambda m_1 \sum p_i \mu_{i,1}}$$

$$P = \frac{I_o \lambda \sum p_i \mu_{i,1} \left(T\lambda\gamma_1 m_1 + m_1 A^*(\lambda)\right)}{1 - m_1[1 - A^*(\lambda)] - \lambda m_1 \sum p_i \mu_{i,1}}$$

$$V = I_0 T\lambda\gamma_1$$

$$P_q(z) = \frac{I_o(1 - z)\begin{bmatrix} A^*(\lambda)(C(z) - 1) + [A^*(\lambda) + C(z)\left(1 - A^*(\lambda)\right)] \\ T\left(V^*(\lambda - \lambda C(z)) - 1\right) \end{bmatrix}}{(1 - C(z))\left[z - \left(A^*(\lambda) + C(z)\left(1 - A^*(\lambda)\right)\right) \sum p_i B_i^*(\lambda - \lambda C(z))\right]}$$

$$P_s(z) = \frac{I_o(1-z)\sum p_i B_i^*(\lambda-\lambda C(z)) \begin{bmatrix} A^*(\lambda)(C(z)-1)+[A^*(\lambda)+C(z)(1-A^*(\lambda))] \\ T(V^*(\lambda-\lambda C(z))-1) \end{bmatrix}}{(1-C(z))\left[z-\left(A^*(\lambda)+C(z)(1-A^*(\lambda))\right)\sum p_i B_i^*(\lambda-\lambda C(z))\right]}$$

Case(ii): Let $A^*(\lambda) \to 1$ (no retrial queue) then we get the results for a batch arrival classical queue with fluctuating modes of service, randomized J vacation and orbital search.

$$I_0 = \frac{1-\lambda m_1 \sum p_i \mu_{i,1}}{1+T\lambda\gamma_1}$$

$$I = 0$$

$$P = \frac{I_o\lambda \sum p_i \mu_{i,1}(T\lambda\gamma_1 m_1 + m_1)}{1-\lambda m_1 \sum p_i \mu_{i,1}}$$

$$V = I_0 T\lambda\gamma_1$$

$$P_q(z) = \frac{I_o(1-z)\left[(C(z)-1)+T\left(V^*(\lambda-\lambda C(z))-1\right)\right]}{(1-C(z))\left[z-\sum p_i B_i^*(\lambda-\lambda C(z))\right]}$$

$$P_s(z) = \frac{I_o(1-z)\sum p_i B_i^*(\lambda-\lambda C(z))\left[(C(z)-1)+T\left(V^*(\lambda-\lambda C(z))-1\right)\right]}{(1-C(z))\left[z-\sum p_i B_i^*(\lambda-\lambda C(z))\right]}$$

Case(iii): Let c(z)=z (single arrival) then our model is reduces to a M/G/1 retrial queue with fluctuating modes of service, randomized J vacation and orbital search. In this case I_0, I, P, V, $P_q(z)$ and $P_s(z)$ can be simplified to the following expressions.

$$I_0 = \frac{A^*(\lambda)-\lambda\sum p_i \mu_{i,1}}{A^*(\lambda)-T\theta+T\theta A^*(\lambda)+T\lambda\gamma_1}$$

$$I = \frac{I_o[1-A^*(\lambda)]\left[\lambda(T\theta+1)\sum p_i \mu_{i,1}+T(\lambda\gamma_1-\theta)\right]}{A^*(\lambda)-\lambda\sum p_i \mu_{i,1}}$$

$$P = \frac{I_o\lambda\sum p_i \mu_{i,1}(T\lambda\gamma_1+A^*(\lambda)-T\theta(1-A^*(\lambda)))}{A^*(\lambda)-\lambda\sum p_i \mu_{i,1}}$$

$$V = I_0 T\lambda\gamma_1$$

$$P_q(z) = \frac{I_o[A^*(\lambda)(z-1) + [A^*(\lambda) + z[1 - A^*(\lambda)]]T(V^*(\lambda - \lambda z)\bar{\theta} - 1) + TV^*(\lambda - \lambda z)\theta]}{\left[z - \left(A^*(\lambda) + z\left(1 - A^*(\lambda)\right)\right) \sum p_i {B_i}^*(\lambda - \lambda z)\right]}$$

$$P_s(z) = \frac{I_o \sum p_i {B_i}^*(\lambda - \lambda z) \begin{bmatrix} A^*(\lambda)(z-1) + [A^*(\lambda) + z[1 - A^*(\lambda)]]T(V^*(\lambda - \lambda z)\bar{\theta} - 1) + \\ TV^*(\lambda - \lambda z)\theta \end{bmatrix}}{\left[z - \left(A^*(\lambda) + z\left(1 - A^*(\lambda)\right)\right) \sum p_i {B_i}^*(\lambda - \lambda z)\right]}$$

Case(iv): Let C(z)=z, $\theta \to 0$, $A^*(\lambda) \to 1$(single arrival , no orbital search, no retrial queue) then we get a M/G/1 queueing system with fluctuating modes of service and randomized J vacation.

$$I_0 = \frac{1 - \lambda \sum p_i \mu_{i,1}}{1 + T\lambda\gamma_1}$$

$$I = 0$$

$$P = \frac{I_o \lambda \sum p_i \mu_{i,1} (T\lambda\gamma_1 + 1)}{1 - \lambda \sum p_i \mu_{i,1}}$$

$$V = I_0 T\lambda\gamma_1$$

$$P_q(z) = \frac{I_o[(z-1) + T(V^*(\lambda - \lambda z) - 1)]}{[z - \sum p_i {B_i}^*(\lambda - \lambda z)]}$$

$$P_s(z) = \frac{I_o \sum p_i {B_i}^*(\lambda - \lambda z) \; [(z-1) + T(V^*(\lambda - \lambda z) - 1)]}{[z - \sum p_i {B_i}^*(\lambda - \lambda z)]}$$

Case(v): Let $p_1 = p_2 = \cdots = 1$ then our model becomes bulk arrival retrial queue with randomized J vacation and orbital search. In this case,

$$I_0 = \frac{1 - m_1[1 - A^*(\lambda)] - \lambda\mu_1 m_1}{A^*(\lambda) - T\theta + T\theta A^*(\lambda) + T\lambda\gamma_1}$$

$$I = \frac{I_o[1 - A^*(\lambda)][\lambda\mu_1 m_1(T\theta + 1) + T(\lambda\mu_1 m_1 - \theta) + m_1 - 1]}{1 - m_1[1 - A^*(\lambda)] - \lambda\mu_1 m_1}$$

$$P = \frac{I_o\lambda\mu_1(T\lambda\gamma_1 m_1 + m_1 A^*(\lambda) - T\theta(1 - A^*(\lambda)))}{1 - m_1[1 - A^*(\lambda)] - \lambda\mu_1 m_1}$$

$$V = I_0 T\lambda\gamma_1$$

$$P_q(z) = \frac{I_o(1-z)\begin{bmatrix} A^*(\lambda)(C(z)-1)+[A^*(\lambda)+C(z)[1-A^*(\lambda)]] \\ T\big(V^*(\lambda-\lambda C(z))\bar{\theta}-1\big)+TV^*(\lambda-\lambda C(z))\theta \end{bmatrix}}{\big(1-C(z)\big)\Big[z-\Big(A^*(\lambda)+C(z)\big(1-A^*(\lambda)\big)\Big)B^*\big(\lambda-\lambda C(z)\big)\Big]}$$

$$P_s(z) = \frac{I_o(1-z)B^*\big(\lambda-\lambda C(z)\big)\begin{bmatrix} A^*(\lambda)(C(z)-1)+[A^*(\lambda)+C(z)[1-A^*(\lambda)]] \\ T\big(V^*(\lambda-\lambda C(z))\bar{\theta}-1\big)+TV^*(\lambda-\lambda C(z))\theta \end{bmatrix}}{\big(1-C(z)\big)\Big[z-\Big(A^*(\lambda)+C(z)\big(1-A^*(\lambda)\big)\Big)B^*\big(\lambda-\lambda C(z)\big)\Big]}$$

NUMERICAL RESULTS

Numerical results are presented to illustrate the effect of various parameters on the main performance of our system by choosing arbitrary values $\lambda = 1.7;\ \eta = 1.12;$

$p_1 = 0.62;\ p_2 = 0.38;\ m_1 = 0.09;\ m_2 = 1.22;\ \gamma = 0.11; J = 7; \theta = 0.95; q = 0.64;$

$\mu_1 = 0.3;\ \mu_2 = 0.2; M = 2$. It is assumed that the retrial time, Service times and Vacation time follow exponential distribution with respesctive rates $\eta, \mu_i\ (1 \le i \le M), \gamma$.

The combined effect of η and γ on I_0, I, P, V, L_q and L_s are presented in Fig.4.1 to 4.6. From the Figure it is observed that

i) Increase in γ increases I_0 and V, decreases I, L_s and L_q and has no effect on P.

ii)Increase in η increases V, L_s and L_q, decreases I and has no effect on I_0 and P.

Fig.4.7 to 4.12 display the effect of (λ, θ) on the performance measures.The Figures depict that

i) Increase in θ increases V, L_s and L_q and decreases I and has no effect on I_0 and P.

ii) Increase in λ increases I, P, L_s and L_q and decreases I_0 and V.

The trends of the system characteristic by varying μ_1 and q are presented in Fig.4.13 to 4.18. Figures reveal that

i) Increase in q increases I_0, decreases I, V, L_q and L_s and has no effect on P.

ii) Increase in μ_1 increases I_0 and V, decreases I, P, L_q and L_s.

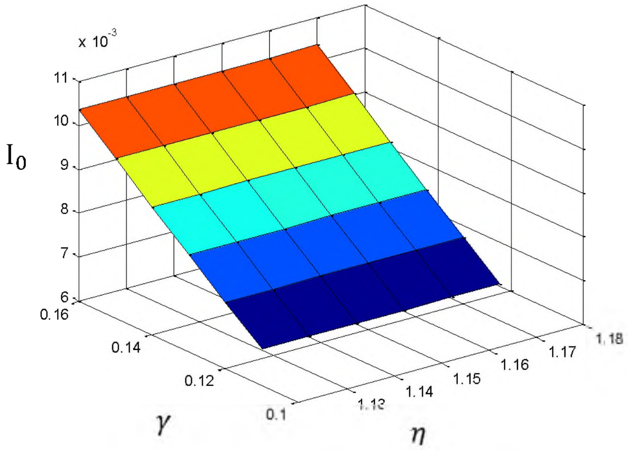

Fig.4.1 I_0 versus (η, γ)

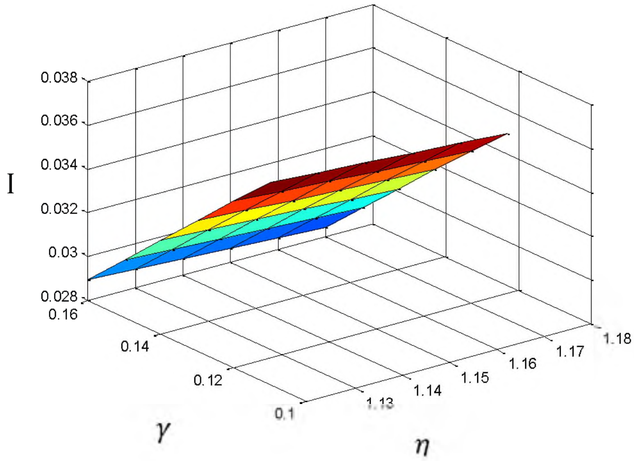

Fig.4.2 I versus (η, γ)

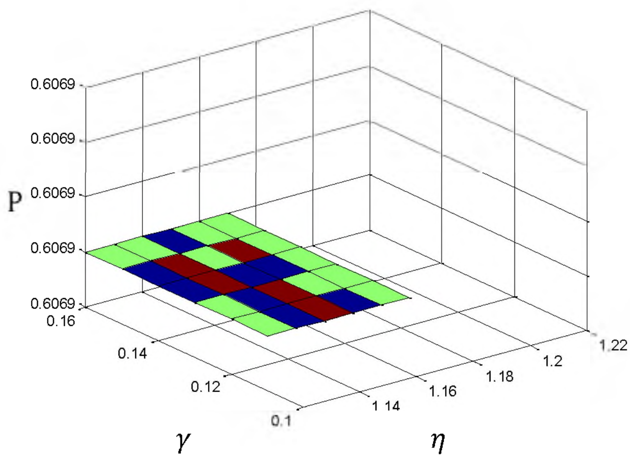

Fig.4.3 P versus (η, γ)

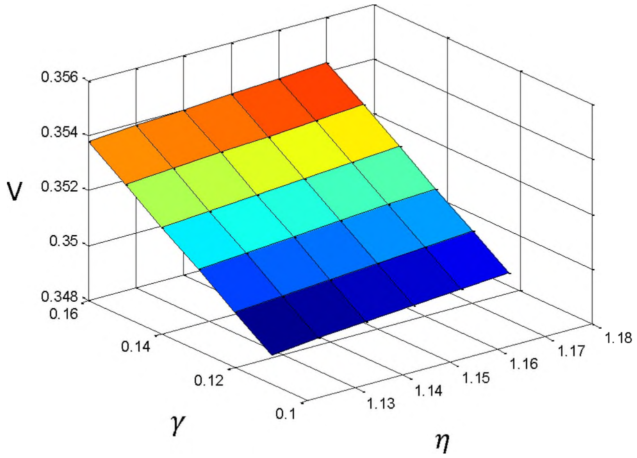

Fig.4.4 V versus (η, γ)

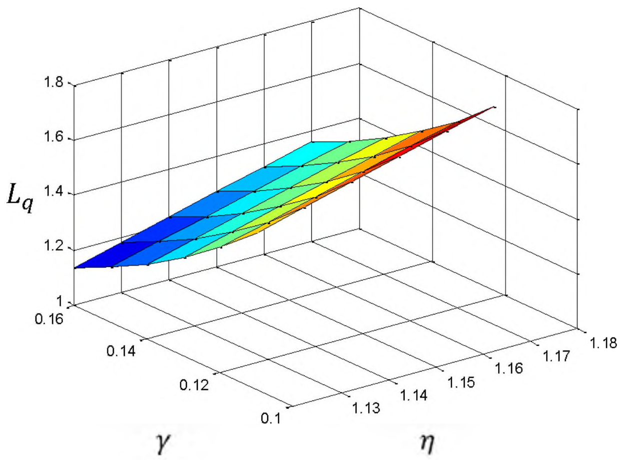

Fig.4.5 L_q versus (η, γ)

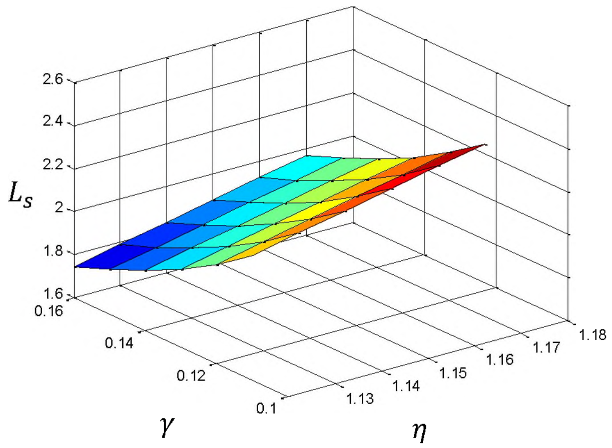

Fig.4.6 L_s versus (η, γ)

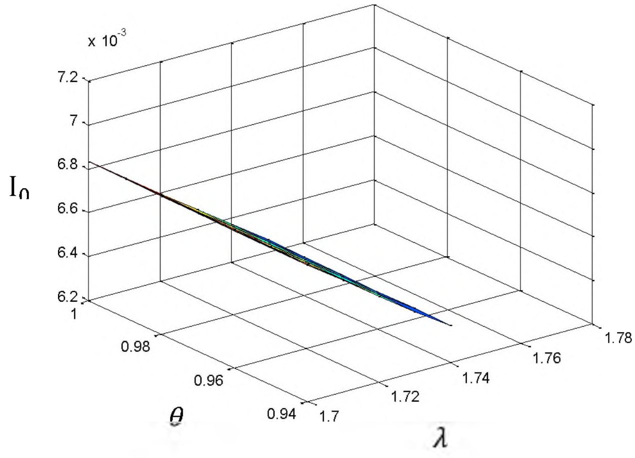

Fig.4.7 I_0 versus (λ, θ)

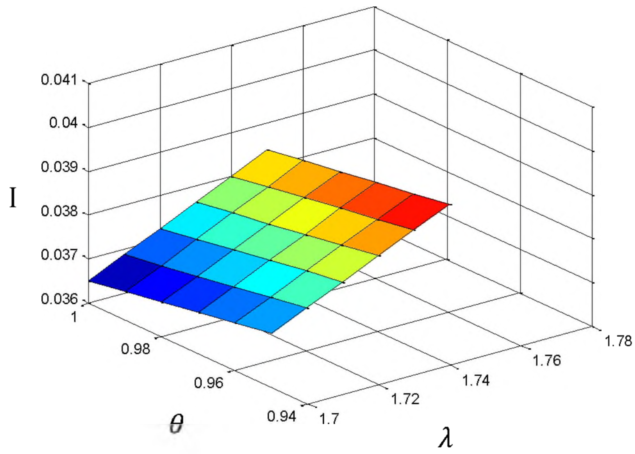

Fig.4.8 I versus (λ, θ)

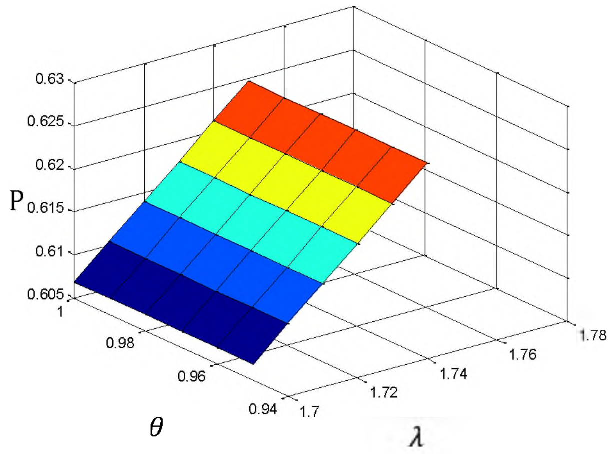

Fig.4.9 P versus (λ, θ)

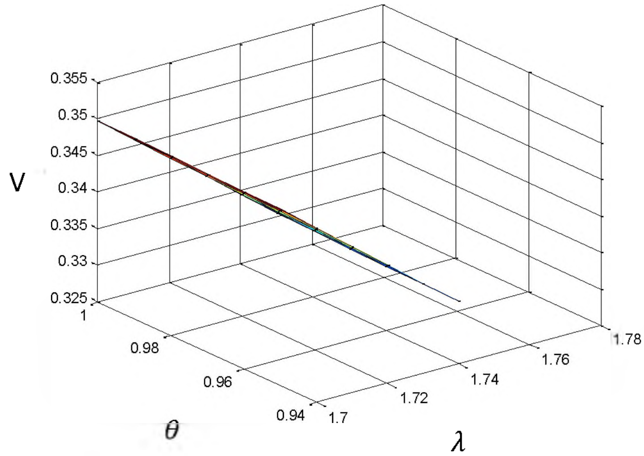

Fig.4.10 V versus (λ, θ)

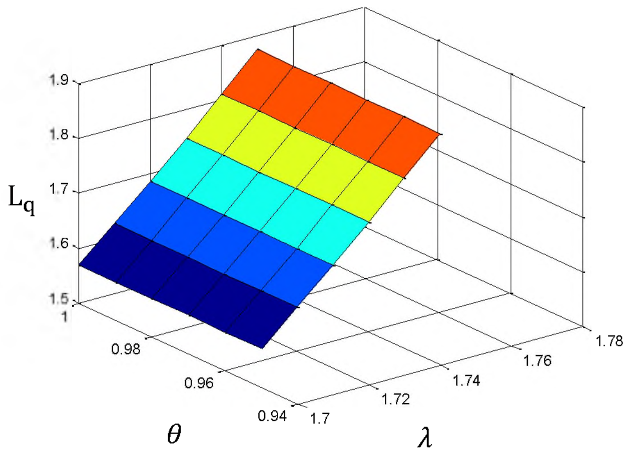

Fig.4.11 L_q versus (λ, θ)

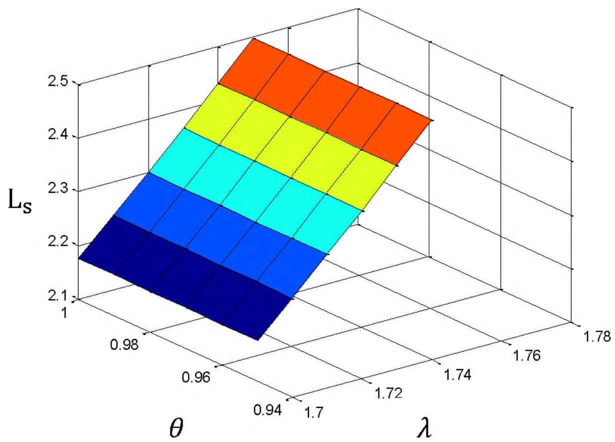

Fig.4.12 L_s versus (λ, θ)

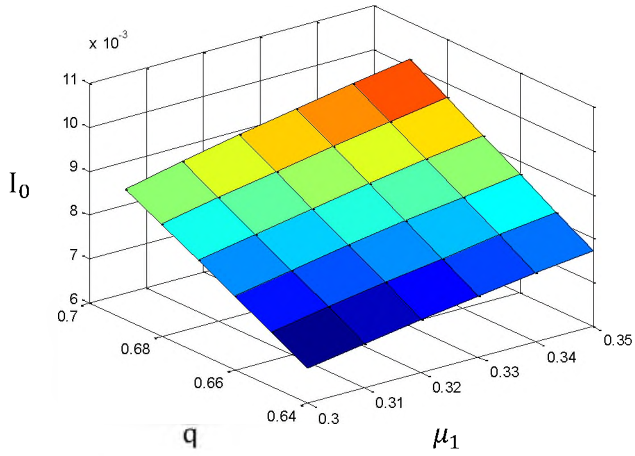

Fig.4.13 I_0 versus (μ_1, q)

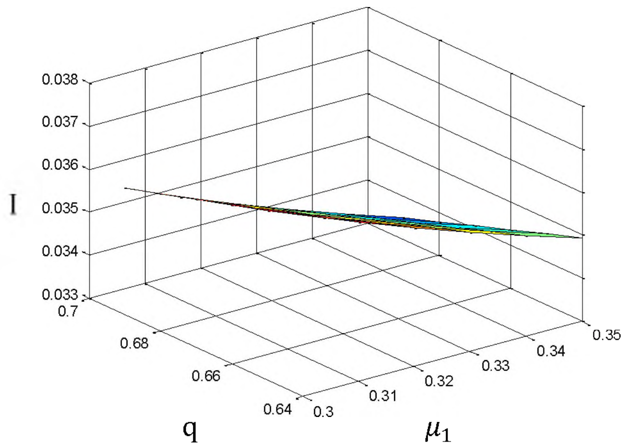

Fig.4.14 I versus (μ_1, q)

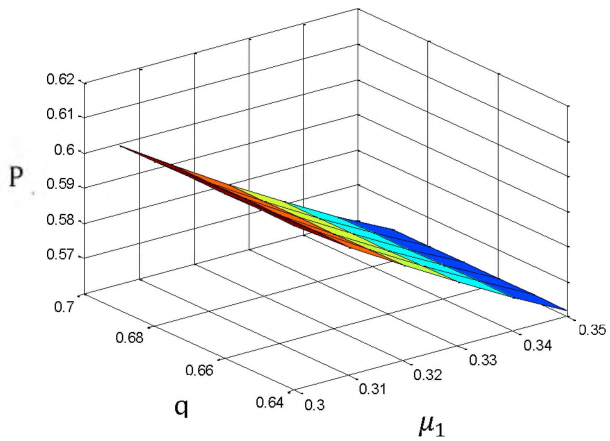

Fig.4.15 P versus (μ_1, q)

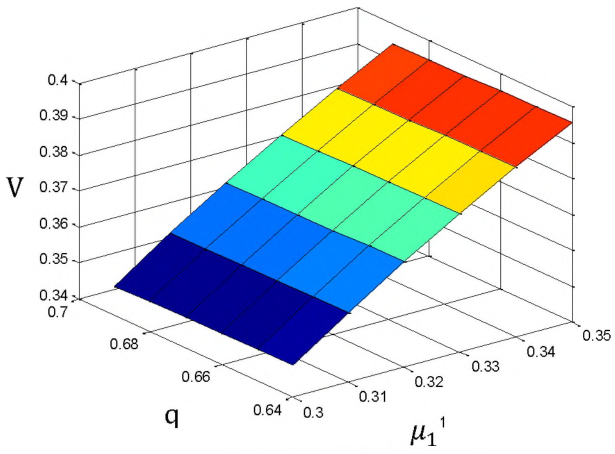

Fig.4.16 V versus (μ_1, q)

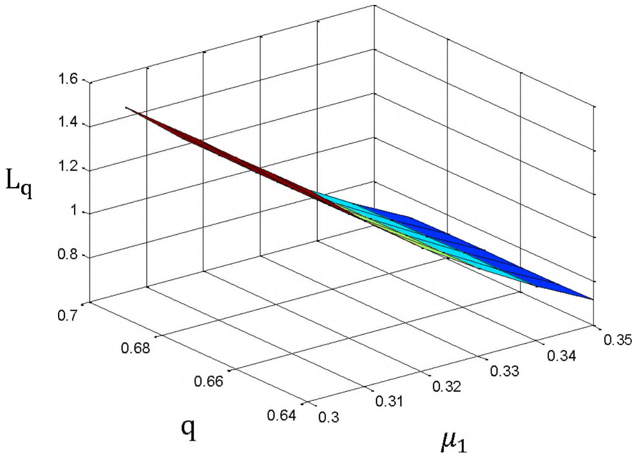

Fig.4.17 L_q versus (μ_1, q)

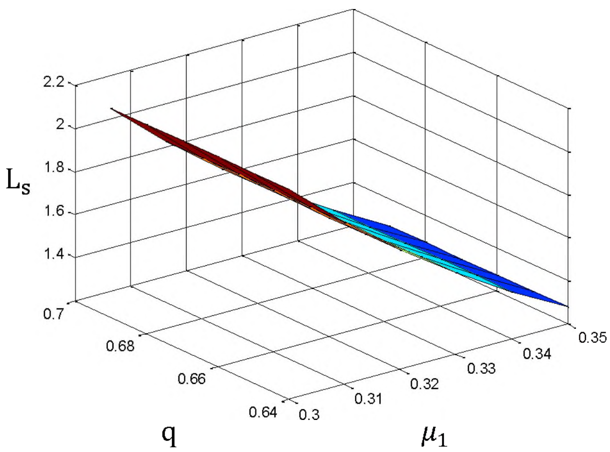

Fig.4.18 L_s versus (μ_1, q)

SUMMARY AND CONCLUSION

Single server retrial queues with fluctuating modes of service is considered in this thesis. Server provides general service in M fluctuating modes with different service rates. The models are analysed by introducing i) vacation and ii) orbital search.

Chapter one is devoted for preliminary definitions and relevant literature. In Chapter two, retrial queue with M- Fluctuating modes of service is considered. If the server is idle upon the arrival of a customer, then the customer receives any one of the types of service immediately. Otherwise he joins the orbit. Chapter three presents M/G/1 retrial queue with fluctuating modes of service and randomized vacations. Whenever the orbit becomes empty, the server immediately takes a vacation. At the end of the vacation if the system is still empty, the server leaves for another vacation or remains idle in the system. This pattern continues until the server finds atleast one customer in the orbit or the number of vacations reaches a preassigned number J. At the end of J^{th} vacation even if the orbit is empty the server remains in the system for new arrival. In the final Chapter, the model in Chapter three is investigated by incorporating bulk arrival and orbital search. If the server is idle upon the arrival of a batch, one of the customers in the batch commences any one of the service and others join the orbit. Otherwise all the customers in the batch join the orbit.

In all the models, retrial time, service time and vacation time are assumed to follow general distribution. All the models are formulated mathematically and analysed at equilibrium state using supplementary variable technique. Explicit expressions are derived for probability generating function of the server states, the number of customers in the system and in the orbit. Special cases are obtained. The decomposition law is verified. The effect of various parameters on the performance measure is illustrated numerically.

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