

Bulk Service with Erlang Input

By

ANURADHA .M

A DISSERTATION

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MASTER OF SCIENCE IN MATHEMATICS**

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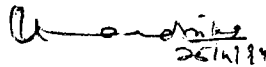
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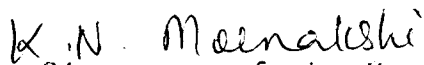
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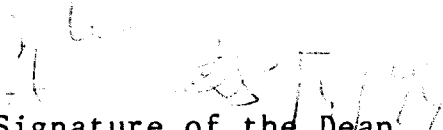
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CONTENTS

CONTENTS

	INTRODUCTION	1
1.	SINGLE SERVICE QUEUE WITH ERLANG INPUT	7
2.	BULK SERVICE QUEUE	
	STEADY STATE PROBABILITIES	16
	WAITING TIME DISTRIBUTION	32
	BIBLIOGRAPHY	42

INTRODUCTION

INTRODUCTION

Reading furnishes the mind only with materials of knowledge, It is thinking that makes what we read ours.

- JOHN LOCKE.

Queueing theory is a branch of operations research utilizing concepts from the field of stochastic processes. A queueing system can be described as composed of customers arriving for service, waiting for service if it is not immediate, and if having waiting for service, leaving the system after being served.

The basic four characteristics of queueing systems are

- i) Arrival pattern
- ii) Service mechanism
- iii) Queue discipline
- iv) Service channels.

ARRIVAL PATTERN :

The arrival pattern is measured in terms of the mean arrival rate or mean inter arrival time. Generally the arrival of customers is deterministic but in most of the cases the arrival pattern is probabilistic or stochastic. Arrival of customers may occur either singly or in batches.

Most of the arrival pattern is assumed to follow poisson process. In some cases arriving units may go through several exponential phases before entering for service. This type of arrival pattern is known as Erlang process.

SERVICE MECHANISM :

This means the arrangement of service facility to serve the customers. If there are infinite number of servers then all the customers are served instantaneously on arrival and there will be no queue.

If the number of servers is finite then the customers are served according to specific order. Further, the customers may be served in batches of fixed size or variable size rather than individually by the same server such as a computer with parallel processing or people boarding a bus. The service system in this case is called bulk service system.

QUEUE DISCIPLINE :

The queue discipline is a rule by which the customers are selected for service when a queue has been formed. The most common discipline is the "first come, first serve" (FCFS) or "first in, first out" (FIFO) rule under which the customers are served in strict order of their arrivals. Sometimes the customers are served on the basis "last in

first out" (LIFO). Another queue discipline is "service in Random order" (SIRO) rule according to which the customers are served randomly irrespective of their arrivals in the system. "Priority " is another queue discipline, where the customers are given priorities upon entering the system.

SERVICE CHANNELS :

The number of services or service channels in a queueing model may be finite or infinite. Depending on the model, if the number of servers is more than one, the customers may form a single queue waiting for service or separate queue in front of each server.

NOTATION :

Kendall (11) designed a convenient notation to denote queueing systems. A queueing process is described by a series of symbols and slashes such as $A|B|X|Y|Z$

where

- A : indicates the interarrival distribution
- B : the service pattern as described by probability distribution for service time.
- X : the number of parallel service channels.
- Y : the restriction on system capacity
- Z : the queue discipline.

The following letters are used for signifying distributions.

- M : for negative exponential inter arrival or service time distributions
- D : for constant inter arrival or service times
- E_k : for the k - Erlang distribution
- G : for general or arbitrary inter arrival on service time distribution.

The queueing theory had its origin in 1909, when the mathematician A.K.Erlang (7) published his paper relating to study of congestion in telephone traffic. Being the pioneer researcher, Erlang is called the father of queueing theory. Felin Pollaezek did some further pioneering work in early 1930's. These early works in queueing theory picked up momentum slowly and now intensive research is going on in this field. Many researchers Ledermann and Reuter (12), Bailey (2), Champernowne (4), Conolly (5), Medhi (15), Satty (21), Gross and Harris (10) soon have studied the queueing systems with the assumption of Poisson input and exponential service time. But these assumptions may not hold in many practical situations. Especially in Industrial process, units may go through several exponential phases before they are out. Griffin (9), Gross and Harris (10), Satty (21), Easton and Chaudry (6) have discussed queueing system wherein inter arrival times follow on Erlang distribution.

Bulk service queueing structure occupy an important place in our daily life, mainly at taxi stand, unscheduled car ferry, a ground floor station of an elevator etc. Arora (1), Ghare (8) have studied bulk service queue.

Neuts (19) has introduced the general bulk service rule for single channel system with capacity range (a,b) . Under this rule, the service starts only when a minimum number of customers 'a' is present in the queue and maximum service capacity is 'b'. If the server is free and queue length is 'a' or more but less than or equal to b, then entire queue is taken up for service. If the queue size is greater than 'b', only 'b' customers are taken up for service while others wait in the queue.

Borthakur (3) has discussed the $M|M(a,b)|1$ queue and obtained the steady state solution. Medhi (14) has obtained distribution of waiting time in the queue. Medhi (13,15) has also discussed a method for direct steady state solution and the busy period distribution.

Medhi and Barthakur (17) have considered the system of two channel homogenous server case under the general bulk service rule. Medhi (16) have discussed the steady state waiting time distribution of $M|M(a,b)|C$ system.

Neuts and Nadarajan (20) have considered the same model as a Markov process in a matrix geometric approach and they have expressed the stationary waiting time distribution in a usable algorithmic form.

By considering the queue with Erlang input and bulk service, Nadarajan and Sankaranarayan (18) have presented a paper. Easton and Chaudry (6) have discussed $E_k|M(a,b)|1$ they have obtained the steady state probabilities and waiting time distribution.

In this dissertation the $E_k|M|1$ and $E_k|M(a,b)|1$ queues are analysed.

The first chapter of this discussion deals with Erlang k - type input and exponential service time. The steady state probabilities are obtained.

In chapter II the paper entitled "The queueing system $E_k|M(a,b)|1$ and its numerical analysis" presented by Easton and Chaudry (6) is discussed. The first section deals with the steady state solution of the queue. In second section the waiting time distribution and expected waiting time in queue are derived.

**SINGLE SERVICE QUEUE
WITH ERLANG INPUT**

CHAPTER - I

Single Service Queue with Erlang Input

In this chapter, the single server queue in which inter arrival times follow an Erlang distribution is considered. Using generating function, the steady state probabilities are obtained.

Consider a single server queueing system by assuming that the inter arrival times have an Erlang k -type distribution with mean for each phase as $1/k\lambda$. Clearly the overall mean inter arrival time is $1/\lambda$. Also assume that the service times are exponentially distributed with parameter μ . This model can be symbolically represented as $E_k/M/1$.

We define for $0 \leq n < \infty$ and $1 \leq r \leq k$,

$Q_{n,r}(t)$ = probability that at a time t there are n units in the system, the arrival is in phase r .

The steady state equations are given by ;

$$k\lambda Q_{0,1} = \mu Q_{1,1} \quad (1)$$

$$k\lambda Q_{0,r} = k\lambda Q_{0,r-1} + \mu Q_{1,r} ; 2 \leq r \leq k \quad (2)$$

$$(k\lambda + \mu) Q_{n,1} = k\lambda Q_{n-1,k} + \mu Q_{n+1,1} \quad (3)$$

$$(k\lambda + \mu) Q_{n,r} = k\lambda Q_{n,r-1} + \mu Q_{n+1,r} ; n \geq 1, 2 \leq r \leq k \quad (4)$$

Define a generating function

$$Q_r(z) = \sum_{n=0}^{\infty} Q_{n,r} z^n ; |z| < 1 \quad (5)$$

$$Q(x,z) = \sum_{r=1}^k Q_r(z) x^r \quad (6)$$

The equations (3) and (4) yield

$$(k\lambda + \mu) (Q_1(z) - Q_{0,1}) = k\lambda z Q_k(z) + \frac{\mu}{z} (Q_1(z) - Q_{0,1} + Q_{1,1}(z)) \quad (7)$$

$$(k\lambda + \mu) (Q_r(z) - Q_{0,r}) = k\lambda (Q_{r-1}(z) - Q_{0,r-1}) + \frac{\mu}{z} (Q_r(z) - Q_{0,r} - Q_{1,r}(z)) \quad (8)$$

Multiply (1) and (7) by x and (2) and (8) by x^{μ} to obtain,

$$Q(x,z) \left(k\lambda + \mu - \frac{\mu}{z} - k\lambda x \right) = \sum_{r=1}^k Q_{0,r} x^r (1-z^{-1}) + k\lambda x Q_k(z) (z-x^k) \quad (9)$$

Substituting $x = 1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z}$ in (9), we have,

$$Q_k(z) = \frac{\mu}{k\lambda} \frac{\sum_{r=1}^k Q_{0,r} (z^{-1}-1) \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z} \right)^{r-1}}{z - \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z} \right)^k} \quad (10)$$

The zeros of denominator of (10) can be obtained from the solution of the equation.

$$z - \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z} \right)^k = 0 \quad (11)$$

Applying Rouché's theorem to (11) we find that if $\rho = \frac{\lambda}{\mu}$ is less than one, there are $(k-1)$ roots inside, one on and one root outside $|z| = 1$.

Let z_0 be the single root which lies outside the unit circle. Since $f(1) < 0$ and $f(\infty) > 0$, the root z_0 is real.

As $Q_k(z)$ exists, the zeros of the denominator make the numerator to be zero which implies k zeros of the numerator are also the zeros of the denominator. Hence by cancelling the zeros of the numerator and denominator, we have,

$$Q_k(z) = \frac{A}{z-z_0}$$

$$= \sum_{n=0}^{\infty} \frac{-A}{z_0^{n+1}} z^n$$

Where A is a constant.

since $Q_k(z) = \sum_{n=0}^{\infty} Q_{n,k} z^n$, taking $n=0$ we get,

$$Q_{0,k} = \frac{-A}{z_0}$$

Which in turn gives,

$$A = -z_0 Q_{0,k}$$

$$\text{Hence } Q_k(z) = \frac{-z_0 Q_{0,k}}{z_0 - z}$$

$$= Q_{0,k} \sum_{n=0}^{\infty} \left(\frac{z}{z_0}\right)^n \quad (12)$$

Equating the coefficient of z^n , we have

$$Q_{n,k} = \frac{Q_{0,k}}{z_0^n}; \quad n \geq 0 \quad (13)$$

Taking $r = k$ in equation (4)

$$(k\lambda + \mu) Q_{n,k} = k\lambda Q_{n,k-1} + \mu Q_{n+1,k}$$

$$k\lambda Q_{n,k-1} = (k\lambda + \mu) Q_{n,k} - \mu Q_{n+1,k}$$

$$= (k\lambda + \mu) \frac{Q_{0,k}}{z_0^n} - \frac{\mu Q_{0,k}}{z_0^{n+1}}$$

(using 13)

$$= \frac{Q_{0,k}}{z_0^n} \left(k\lambda + \mu - \frac{\mu}{z_0} \right)$$

$$Q_{n,k-1} = \frac{Q_{0,k}}{z_0^n} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0} \right)$$

Similarly by considering (4) with $r = k-1$ we get,

$$Q_{n,k-2} = \frac{Q_{0,k}}{z_0^n} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0} \right)^2$$

Proceeding like this we have generally,

$$Q_{n,r} = \frac{Q_{0,k}}{z_0^n} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0}\right)^{k-r}; n \geq 1 \quad (14)$$

Using (11)

$$Q_{n,r} = \frac{Q_{0,k}}{z_0^{n-1}} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0}\right)^{-r}; n \geq 1$$

$$1 \leq r \leq k \quad (15)$$

To find $Q_{0,r}$, we define a operation D as $DQ_{0,r} = Q_{0,r+1}$

From (2) we have,

$$(D-1) Q_{0,r} = \frac{\mu}{k\lambda} Q_{1,r}$$

$$Q_{0,r} = A + \frac{\frac{\mu}{k\lambda} Q_{1,r}}{1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0} - 1}$$

$$Q_{0,r} = A + \frac{z_0 Q_{0,k}}{z_0 - 1} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0}\right)^{-r}$$

(16)

Taking $r = k-1$ in (16) and using equation (11)

$$Q_{0,k-1} = A + \frac{Q_{0,k}}{z_0-1} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0}\right) \quad (17)$$

Putting $r = k$ in equation (2)

$$\begin{aligned} Q_{0,k-1} &= Q_{0,k} - \frac{\mu}{k\lambda} Q_{1,k} \\ &= Q_{0,k} - \frac{\mu}{k\lambda} Q_{0,k} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0}\right)^{-k} \\ &= Q_{0,k} \left(1 - \frac{\mu}{k\lambda z_0}\right) \end{aligned} \quad (18)$$

Now equating $Q_{0,k}$ in (17) and (18) we get,

$$Q_{0,k} \left(1 - \frac{\mu}{k\lambda z_0}\right) = A + \frac{Q_{0,k}}{z_0-1} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0}\right)$$

$$A = Q_{0,k} \left(1 - \frac{\mu}{k\lambda z_0} - \frac{1}{(z_0-1)} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0}\right)\right)$$

Hence

$$Q_{0,r} = Q_{0,k} \left\{ \left(1 - \frac{\mu}{k\lambda z_0} \right) - \frac{1}{(z_0-1)} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0} \right) \right. \\ \left. [1 - z_0 \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0} \right)^{r-1}] \right\} \quad 1 \leq r \leq k \quad (19)$$

All the steady state probabilities $Q_{n,r}$, $n \geq 1$, $1 \leq r \leq k$ are in terms of $Q_{0,k}$ which can be easily obtained with the use of boundary condition. The above steady state results have been verified with those of $M|M|1$ by taking $k = 1$.

BULK SERVICE QUEUE

CHAPTER - 2

BULK SERVICE QUEUE

The present chapter extends the model considered in the previous chapter to bulk service. The analytical expression for steady state probabilities are obtained in the first section. In the second section, the waiting time distribution, expected queue length and expected waiting time in queue are derived.

In this model the interarrival times have an Erlang distribution with mean for each phase $1/k\lambda$ so that the average mean interarrival is $1/\lambda$. The service pattern is assumed to follow a general bulk service rule introduced by Neuts. The server starts his service only when a minimum number of customers 'a' is present in the queue and the maximum capacity is 'b'. If the queue length is greater than 'b' then only 'b' customers are taken for service while others wait in the queue.

**STEADY STATE
PROBABILITIES**

2.1 STEADY STATE PROBABILITIES

Here we find steady state busy probabilities and idle probabilities in the model $E_k / M_{(a,b)} / 1$.

For $0 \leq n < \infty$ and $1 \leq r \leq k$, we define,

$Q_{n,r}(t)$ = propability that at time t there are n units waiting in queue, the arrival is in phase r and server is busy.

For $0 \leq n < a$ and $1 \leq r \leq k$, define,

$P_{n,r}(t)$ = probability that at time t there are n units waiting in queue, the arrival is in phase r and server is idle.

Assuming the existence of steady state, we get the following steady state equations ;

$$k\lambda P_{0,1} = \mu Q_{0,1} \tag{1}$$

$$k\lambda P_{n,1} = \mu Q_{n,1} + k\lambda P_{n-1,k} ; 1 \leq n \leq a-1 \tag{2}$$

$$k\lambda P_{n,r} = \mu Q_{n,r} + k\lambda P_{n,r-1} ; 0 \leq n \leq a-1 \\ 2 \leq r \leq k \tag{3}$$

$$(k\lambda + \mu) Q_{0,1} = k\lambda P_{a-1,k} + \mu \sum_{j=a}^b Q_{j,1} \quad (4)$$

$$(k\lambda + \mu) Q_{0,r} = k\lambda Q_{0,r-1} + \mu \sum_{j=a}^b Q_{j,r} ; 2 \leq r \leq k \quad (5)$$

$$(k\lambda + \mu) Q_{n,1} = k\lambda Q_{n-1,k} + \mu Q_{n+b,1} ; n \geq 1 \quad (6)$$

$$(k\lambda + \mu) Q_{n,r} = k\lambda Q_{n,r-1} + \mu Q_{n+b,r} ; n \geq 1, 2 \leq r \leq k \quad (7)$$

Define the generating function,

$$Q_r(z) = \sum_{n=0}^{\infty} Q_{n,r} z^n ; |z| < 1 \quad (8)$$

$$Q(x,z) = \sum_{r=1}^k Q_r(z) x^r \quad (9)$$

Now we derive the expression for $Q(x,z)$. Using the steady state equation from (4) through (7). We get,

$$(k\lambda + \mu) Q(x,z) = k\lambda P_{a-1,k} x + \mu \sum_{r=1}^k \sum_{j=a}^b Q_{j,r} x^r +$$

$$k\lambda \sum_{n=1}^{\infty} Q_{n-1,k} z^n x + \mu \sum_{r=1}^k \sum_{n=1}^{\infty} Q_{n+b,r} z^n x^r +$$

$$\begin{aligned}
& k \lambda \sum_{r=2}^k \sum_{n=0}^{\infty} Q_{n,r-1} z^n x^r \\
&= k \lambda P_{a-1,k} x + \mu \left[\sum_{r=1}^k \sum_{j=0}^b Q_{j,r} x^r - \right. \\
&\quad \left. \sum_{r=1}^k \sum_{j=0}^{a-1} Q_{j,r} x^r \right] + k \lambda z \sum_{n=0}^{\infty} Q_{n,k} z^n x + \\
&\quad \mu z^{-b} \left[\sum_{r=1}^k \sum_{n=b+1}^{\infty} Q_{n,r} z^n x^r \right] + k \lambda x \sum_{r=1}^{k-1} \sum_{n=0}^{\infty} Q_{n,r} z^n x^r \\
&= k \lambda P_{a-1,k} x + \mu [1 - z^{n-b}] \sum_{r=1}^k \sum_{n=0}^{b-1} Q_{n,r} x^r - \\
&\quad \mu \sum_{r=1}^k \sum_{n=0}^{a-1} Q_{n,r} x^r + k \lambda x [z - x^k] \sum_{n=0}^{\infty} Q_{n,k} z^n + \\
&\quad \sum_{r=1}^k \sum_{n=0}^{\infty} Q_{n,r} z^n x^r [k \lambda x + \mu z^{-b}] \tag{10}
\end{aligned}$$

Considering the equation (1), (2) and (3) and taking the sum over n from 0 to a-1 and then over r from 1 to k.

We get,

$$k \lambda P_{a-1,k} = \mu \sum_{r=1}^k \sum_{n=0}^{a-1} Q_{n,r} \quad (11)$$

Hence substituting the value of $P_{a-1,k}$ in (10) and using (8) and (9). We have,

$$Q(x,z) [k \lambda (1-x) + \mu (1-z^{-b})] = \sum_{r=1}^k \sum_{n=0}^{a-1} Q_{n,r} (x-x^r) + \mu \sum_{r=1}^k \sum_{n=0}^{b-1} Q_{n,r} (1-z^{n-b}) x^r + k \lambda x (z-x^k) Q_k(z) \quad (12)$$

Taking $x = (1 + \frac{\mu}{k \lambda} - \frac{\mu}{k \lambda z^b})$ in the above equation.

We get,

$$Q_k(z) = \frac{\mu}{k \lambda} \left\{ \sum_{r=1}^k \sum_{n=0}^{b-1} Q_{n,r} (z^{n-b-1}) \left(1 + \frac{\mu}{k \lambda} - \frac{\mu}{k \lambda z^b}\right)^{r-1} - \sum_{r=1}^k \sum_{n=0}^{a-1} Q_{n,r} \left[1 - \left(1 + \frac{\mu}{k \lambda} - \frac{\mu}{k \lambda z^b}\right)^{r-1}\right] \right\} / \left[z - \left(1 + \frac{\mu}{k \lambda} - \frac{\mu}{k \lambda z^b}\right)^k \right] \quad (13)$$

The zeros of the denominator of (13) can be obtained from the solution of the equation

$$z - \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z^b} \right)^k = 0 \quad (14)$$

Applying Rouché's theorem to (14) we find that if $\tau = \lambda/b\mu$ is less than one, there are $(bk-1)$ roots inside, one on and one root outside $|z| = 1$. Let z_0 be the root which lies outside the unit circle. Since $f(1) < 0$ and $f(\infty) > 0$, the root z_0 is real.

Since $Q_k(z)$ exists, the zeros of the denominator make the numerator to be zero which implies bk zeros of the numerator are also the zeros of the denominator. Hence by cancelling the zeros of numerator and denominator we have,

$$Q_k(z) = \frac{A}{z-z_0}$$

$$= \sum_{n=0}^{\infty} \frac{-A}{z_0^{n+1}} z^n$$

Where A is a constant

$$\text{Since } Q_k(z) = \sum_{n=0}^{\infty} Q_{n,k} z^n,$$

Taking $n=0$ we get;

$$Q_{0,k} = \frac{-A}{z_0}$$

$$A = -z_0 Q_{0,k}$$

$$\begin{aligned} \text{Hence } Q_k(z) &= \frac{z_0 Q_{0,k}}{z_0 - z} \\ &= Q_{0,k} \sum_{n=0}^{\infty} \left(\frac{z}{z_0}\right)^n \end{aligned}$$

Equating the coefficient of z^n we have,

$$Q_{n,k} = \frac{Q_{0,k}}{z_0^n} ; n \geq 0 \quad (15)$$

Taking $r=k$ in equation (7) we obtain,

$$\begin{aligned} k\lambda Q_{n,k-1} &= (k\lambda + \mu) Q_{n,k} - \mu Q_{n+b,k} \\ &= (k\lambda + \mu) \frac{Q_{0,k}}{z_0^n} - \frac{Q_{0,k}}{z_0^{n+b}} \end{aligned} \quad (\text{using 15})$$

$$= \frac{Q_{0,k}}{z_0^n} \left(k\lambda + \mu - \frac{\mu}{z_0^b} \right)$$

$$Q_{n,k-1} = \frac{Q_{0,k}}{z_0^n} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b} \right)$$

Similarly by considering (7) with $r = k-1$ we have,

$$Q_{n,k-2} = \frac{Q_{0,k}}{z_0^n} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b} \right)^2$$

Proceeding like this we find,

$$Q_{n,r} = \frac{Q_{0,k}}{z_0^n} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b} \right)^{k-r} ; n \geq 1$$

$$Q_{n,r} = \frac{Q_{0,k}}{z_0^n} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b} \right)^{-r} ; n \geq 1$$

$$1 \leq r \leq k$$

(using 14) (16)

To determine $Q_{0,r}$, we define an operator E as

$E Q_{0,r} = Q_{0,r+1}$. From (5) we get,

$$[(k\lambda + \mu) E - k\lambda] Q_{0,r} = \mu \sum_{j=a}^b Q_{j,r}$$

Using (16) we have

$$Q_{0,r} = A \left(\frac{k\lambda}{k\lambda+\mu} \right)^r + \frac{z_0 Q_{0,k} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b} \right)^{-r} (z_0^b - z_0^{a-1})}{z_0^{a-1} (z_0 - 1)} \quad (17)$$

Equating the expressions of $Q_{0,k-1}$ obtained from (17) and (5) with use of (14) we get,

$$A = \left(\frac{k\lambda}{k\lambda+\mu} \right)^{-k} Q_{0,k} \frac{(z_0^a - z_0^b)}{z_0^{a-1} (z_0 - 1)}$$

Hence

$$\begin{aligned} Q_{0,r} &= \left(\frac{k\lambda}{k\lambda+\mu} \right)^{r-k} Q_{0,k} \frac{(z_0^a - z_0^b)}{z_0^{a-1} (z_0 - 1)} + \frac{Q_{0,k}}{z_0^{a-1} (z_0 - 1)} \\ &\quad \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b} \right)^{-r} (z_0^{b+1} - z_0^a) \\ &= \frac{Q_{0,k}}{z_0^{a-1} (z_0 - 1)} \left\{ \left(\frac{k\lambda+\mu}{k\lambda} \right)^{k-r} (z_0^a - z_0^b) + \right. \\ &\quad \left. (z_0^{b+1} - z_0^a) \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b} \right)^{-r} \right\}; \quad 1 \leq r < k \end{aligned} \quad (18)$$

To find the remaining busy probabilities $Q_{0,k}$ we define,

$$P_r(z) = \sum_{n=0}^{a-1} P_{n,r} z^n$$

$$P(x,z) = \sum_{r=1}^k P_r(z) x^r$$

Then,

$$P(x,z) = \frac{\mu}{k\lambda} Q_{0,1} x + \frac{\mu}{k\lambda} \sum_{n=1}^{a-1} Q_{n,r} z^n x + \sum_{n=1}^{a-1} P_{n-1,k} z^n$$

$$+ \frac{\mu}{k\lambda} \sum_{r=2}^k Q_{0,r} x^r + \sum_{r=1}^k P_{0,r-1} x^r + \frac{\mu}{k\lambda} \sum_{r=2}^k \sum_{n=1}^{a-1}$$

$$Q_{n,r} x^r z^n + \sum_{r=2}^k \sum_{n=1}^{a-1} P_{n,r-1} z^n x^r$$

(using (1), (2), and (3))

$$= \frac{\mu}{k\lambda} \sum_{r=1}^k \sum_{n=0}^{a-1} Q_{n,r} z^n x^r + z \sum_{n=0}^{a-2} P_{n,k} z^n x +$$

$$x \sum_{r=1}^{k-1} \sum_{n=0}^{a-1} P_{n,r} z^n x^r$$

$$= \frac{\mu}{k\lambda} \sum_{r=1}^k \sum_{n=0}^{a-1} Q_{n,r} z^n x^r + zx \left(\sum_{n=0}^{a-1} P_{n,k} z^n - P_{a-1,k} z^{a-1} \right) + x [P(x,z) - P_k(z)]$$

$$(1-x) P(x,z) = \frac{\mu}{k\lambda} \sum_{r=1}^k \sum_{n=0}^{a-1} Q_{n,r} z^n x^r + x(z-1) P_k(z) - zx P_{a-1,k} z^{a-1}$$

$$= \frac{\mu}{k\lambda} \sum_{r=1}^k \sum_{n=0}^{a-1} Q_{n,r} x^r (z^n - z^a) + x(z-1) P_k(z)$$

(using (11))

Putting $x=1$, we get

$$(z-1) P_k(z) = \frac{\mu}{k\lambda} \sum_{r=1}^k \sum_{n=0}^{a-1} (z^a - z^n) Q_{n,r} \quad (19)$$

Since there are only k phases, the probability of the customer in the r th phase is $1/k$.

$$\text{Hence } \sum_{n=0}^{\infty} Q_{n,r} + \sum_{n=0}^{a-1} P_{n,r} = 1/k ; 1 \leq r \leq k.$$

Which implies

$$Q_r(1) + P_r(1) = 1/k ; 1 \leq r \leq k.$$

Consequently

$$Q_k(1) + P_k(1) = 1/k$$

Taking limit as $z \rightarrow 1$ in (19) and using L - Hospital's rule. We have,

$$\begin{aligned} \lim_{z \rightarrow 1} P_k(z) &= \lim_{z \rightarrow 1} \frac{\mu}{k\lambda} \sum_{n=0}^{a-1} \sum_{r=1}^k \frac{(z^a - z^n)}{(z-1)} Q_{n,r} \\ &= \lim_{z \rightarrow 1} \frac{\mu}{k\lambda} \sum_{r=1}^k \sum_{n=0}^{a-1} (a z^{a-1} - n z^{n-1}) Q_{n,r} \end{aligned}$$

$$P_k(1) = \frac{\mu}{k\lambda} \sum_{n=0}^{a-1} \sum_{r=1}^k (a-n) Q_{n,r}$$

$$1/k = Q_k(1) + P_k(1)$$

$$= \frac{z_0 Q_{0,k}}{z_0-1} - \frac{\mu}{k\lambda} \sum_{r=1}^k \sum_{n=0}^{a-1} (n-a) Q_{n,r} \quad (20)$$

Substituting the value of $Q_{n,r}$ in (20) we have,

$$\begin{aligned} 1/k &= \frac{z_0 Q_{0,k}}{z_0-1} + \frac{\mu a}{k\lambda} \sum_{r=1}^k Q_{0,r} - \frac{\mu}{k\lambda} \sum_{r=1}^k \sum_{n=1}^{a-1} \\ &\quad (n-a) Q_{n,r} \quad (21) \end{aligned}$$

The second term of (21) becomes

$$\begin{aligned}
& \frac{Q_{0,k}}{z_0^{a-1} (z_0 - 1)} \frac{\mu^a}{k\lambda} \sum_{r=1}^k [(z_0^a - z_0^b) (1 + \frac{\mu}{k\lambda})^{k-r} + \\
& \quad (z_0^{b+1} - z_0^a) (1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b})^{-r}] \\
& \frac{Q_{0,k}}{z_0^{a-1} (z_0 - 1)} [(z_0^a - z_0^b) (1 - (1 + \frac{\mu}{k\lambda})^k) + \\
& \quad (z_0^{b+1} - z_0^a) \frac{(z_0-1)}{z_0} \frac{z_0^b}{z_0^{b+1} - 1}] \tag{22}
\end{aligned}$$

Substituting the expression for $Q_{n,r}$ in the third term of equation (21) we get,

$$\begin{aligned}
& \frac{\mu}{k\lambda} \sum_{r=1}^k \sum_{n=1}^{a-1} (n-a) Q_{n,r} \\
& = \frac{\mu}{k\lambda} Q_{0,k} \sum_{r=1}^k \sum_{n=1}^{a-1} \frac{1}{z_0^n} (1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b})^{k-r} \\
& = Q_{0,k} \sum_{r=1}^k \frac{n-a}{z_0^n} \frac{(1-z_0) z_0^b}{z_0^b - 1} \\
& = \frac{(1-z_0) z_0^b}{z_0^b - 1} Q_{0,k} [z_0 \frac{d}{d z_0} \frac{(z_0^{-1} - z_0^{-a})}{1 - z_0^{-1}} -
\end{aligned}$$

$$\begin{aligned}
& \left. \frac{a (z_0^a - z_0)}{z_0^a (z_0 - 1)} \right] \\
= & \frac{-z_0^b}{z_0^b - 1} Q_{0k} \left\{ z_0^2 \left(\frac{a}{a+1} - \frac{1}{z_0^2} \right) - \frac{(z_0^a - z_0)}{z_0^a (z_0 - 1)} \right. \\
& \left. - \frac{a (z_0^a - z_0)}{z_0^a (z_0 - 1)} \right\} \\
= & \frac{-z_0^b}{z_0^b - 1} Q_{0k} \left\{ \frac{a(z_0 - z_0^{a-1}) - z_0^{a-1}(z_0 - 1) - z_0^{a-1} + 1}{z_0^{a-1} (z_0 - 1)} \right\} \\
= & \frac{z_0^b}{z_0^b - 1} Q_{0k} \left\{ \frac{z_0^{a-1}}{z_0^{a-1} (z_0 - 1)} - a \right\} \tag{23}
\end{aligned}$$

Using (22) and (23), (21) becomes

$$\begin{aligned}
Q_{0,k} = & \frac{1}{k} \left\{ \frac{z_0}{z_0 - 1} - \frac{z_0^b}{z_0^b - 1} \left(\frac{z_0^a - 1}{z_0^{a-1} (z_0 - 1)} - a \right) + \right. \\
& \left. + \frac{a(z_0^a - z_0^b) \left(\left(1 + \frac{\mu}{k\lambda} \right)^k - 1 \right)}{z_0^{a-1} (z_0 - 1)} + \frac{z_0^b (z_0^{b+1} - z_0^a)}{z_0^a (z_0^b - 1)} \right\}^{-1} \tag{24}
\end{aligned}$$

In the remaining section we calculate the idle steady state probabilities $P_{n,r}$ $0 \leq n \leq a-1$ and $1 \leq r \leq k$ recursively. From (4)

$$\begin{aligned}
 P_{a-1,k} &= \frac{k\lambda + \mu}{k\lambda} Q_{0,1} - \frac{\mu}{k\lambda} \sum_{j=a}^b Q_{j,1} \\
 &= \left(1 + \frac{\mu}{k\lambda}\right) Q_{0,1} - \frac{\mu}{k\lambda} Q_{0,k} \sum_{j=a}^b \frac{1}{z_0^{j-1}} \\
 &\qquad\qquad\qquad \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{-1} \\
 &= \left(1 + \frac{\mu}{k\lambda}\right) Q_{0,1} - \frac{\mu}{k\lambda} Q_{0,k} \frac{(z_0^{1-a} - z_0^{-b})z_0}{(z_0 - 1) \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)} \\
 &\qquad\qquad\qquad (25)
 \end{aligned}$$

Repeating recursively with equation (3) we obtain,

$$\begin{aligned}
 P_{n,r} &= P_{n,k} + z_0^b \left\{ \frac{\left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{-r}}{z_0^n (z_0^b - 1)} Q_{0,k} \right\}; \\
 &\qquad\qquad\qquad 1 \leq r \leq k-1 ; 0 \leq n \leq a-1 \\
 &\qquad\qquad\qquad (26)
 \end{aligned}$$

Finally by considering equation (2) and using equations (26) and (16) we get,

$$P_{n-1,k} = P_{n,k} + \frac{Q_{0,k}}{z_0^n (z_0^b - 1)} \left\{ z_0^b - \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{-1} \right. \\ \left. [z_0^{b+1} + \frac{\mu}{k\lambda} z_0 (z_0^b - 1)] \right\}$$

$$P_{n-1,k} = P_{n,k} + \frac{Q_{0,k}}{z_0^n (z_0^b - 1)} \left\{ z_0^b - \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{-1} \right. \\ \left. [z_0^{b+1} (1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b})] \right\}$$

$$= P_{n,k} + \frac{(z_0^b - z_0^{b+1})}{z_0^n (z_0^b - 1)} Q_{0,k}$$

$$= P_{n,k} + \frac{z_0^b (z_0^{-n} - z_0^{1-n})}{z_0^b - 1} Q_{0,k}$$

(27)

Taking $n=a-1$ in (27)

$$P_{a-2,k} = P_{a-1,k} + \frac{z_0^b (z_0^{1-a} - z_0^{2-a})}{z_0^b - 1} Q_{0,k}$$

Proceeding like this we get

$$\begin{aligned}
 P_{n,k} &= P_{a-1,k} + \frac{z_0^b (z_0^{1-a} - z_0^{-n})}{z_0^b - 1} Q_{0,k} \\
 &= P_{a-1,k} - \frac{z_0^b (z_0^{-n} - z_0^{1-a})}{z_0^b - 1} Q_{0,k}; 0 \leq n \leq a-2
 \end{aligned}
 \tag{28}$$

WAITING TIME DISTRIBUTION

2.2 WAITING TIME DISTRIBUTION

In this section first we state the relationship between the random point probabilities and the arrival epoch. Then we obtain the expected number in the queue at random epoch (L_q); the expected number in queue at arrival epoch (L_q^-), and waiting time.

Define Q_n^- = probability of n in queue at arrival epoch when the server is busy.

$$\begin{aligned}
 Q_n^- &= \Pr \{n \text{ in system/ arrival about to occur}\} \\
 &= \Pr \{n \text{ in system/ arrival in phase } k\} \\
 &= \frac{\Pr \{n \text{ in system and arrival in phase } k\}}{\Pr \{\text{arrival in phase } k\}}
 \end{aligned}$$

$$Q_n^- = k Q_{n,k} ; n \geq 0 \quad (29)$$

Similarly by defining

$$P_n^- = k P_{n,k} ; 0 \leq n \leq a-1 \quad (30)$$

then

$$L_q = \sum_{n=0}^{\infty} \sum_{r=1}^k n Q_{n,r} + \sum_{n=0}^{a-1} \sum_{r=1}^k n P_{n,r}$$

and

$$\begin{aligned}
 L_q^- &= \sum_{n=0}^{\infty} nQ_n^- + \sum_{n=0}^{a-1} nP_n^- \\
 &= k \left\{ \sum_{n=0}^{\infty} nQ_{n,k} + \sum_{n=0}^{a-1} nP_{n,k} \right\}
 \end{aligned}
 \tag{31}$$

An arrival unit in phase k of the arrival channel, will find the system in any one of the following four cases;

- i) $(0, a-1)$
- ii) $(0, n) ; 0 \leq n \leq a-2$
- iii) $(1, n) ; a-1 \leq m \leq b-1 ; n = ib + m$
- iv) $(1, n) ; 0 \leq m \leq a-2 ; n = ib + m$

where $i = 0, 1, 2, \dots$

$(1, n)$ refers to server state. 1 may be busy (1) or idle (0).
 n is number of units in the queue.

In case (i) the arriving unit need not wait and he can enter the service directly. In the remaining three cases, the arriving unit has to wait seeing the state of the system.

In case (ii) the arrival must wait till the arrival of $(a-1-n)$ units. It requires the completion of $k(a-1-n)$ arrival phases each with parameter $k\lambda$.

In case (iii) the arriving unit will join the (i+1)th waiting batch so that he has to wait for (i+1) service completions each with parameter μ .

The situation in case (iv) has 2 criteria which must be satisfied prior to an arriving unit proceed to service channel. The arriving unit must wait either the arrival of (a-1-m) customers to complete the quorum or the completion of service on (i+1) batches ; whichever occurs later. This situation can be described by random variable Z defined as the maximum value of two Erlang variates (Y_1, Y_2).

$$\text{Hence } Z = \text{Max } (Y_1, Y_2) \quad (32)$$

where

$$f_{Y_1}(t) = \frac{(k\lambda)^{k(a-1-n)} t^{k(a-1-n)-1} e^{-k\lambda t}}{(k(a-1-n)-1)!} \quad (33)$$

and

$$f_{Y_2}(t) = \frac{\mu^{i+1} t^i e^{-\mu t}}{i!} \quad (34)$$

Let the corresponding distribution functions be $F_{Y_1}(t)$ and $F_{Y_2}(t)$

Then

$$f_z(t) = F_z^{(0)}(t) = F_{Y_1}(t) f_{Y_2}(t) + F_{Y_2}(t) f_{Y_1}(t) \quad (35)$$

If V_q^- represents the time an arriving unit must wait in queue prior to entering service and $W_q^-(t)$ is the corresponding p.d.f.

$$\text{then } W_q^-(t) = P(V_q^- \leq t)$$

$$\text{clearly } W_q^-(0) = P_{a-1}^- = kP_{a-1,k} \quad (36)$$

For $t > 0$ we get from the above discussion

$$\begin{aligned} W_q^-(t) &= \sum_{n=0}^{a-2} P_n^- f_{Y_1}(t) + \sum_{i=0}^{\infty} \sum_{m=a-1}^{b-1} Q_{m+ib}^- f_{Y_2}(t) \\ &\quad + \sum_{i=0}^{\infty} \sum_{m=0}^{a-2} Q_{m+ib}^- f_z(t) \end{aligned} \quad (37)$$

Using (25) (28) and (33) the first term of equation (37) becomes,

$$k \sum_{n=0}^{a-2} \left\{ \left(1 + \frac{\mu}{k\lambda}\right) Q_{0,1} - \frac{\mu}{k\lambda} \left[\frac{z_0 (z_0^{1-a} - z_0^{-b})}{(z_0-1) \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)} \right] \right\}$$

$$- \frac{z_0^b (z_0^{-n} - z_0^{1-a})}{z_0^b - 1} \} Q_{0,k} \} \frac{(k\lambda)^{k(a-1-n)} t^{k(a-1-n)-1} e^{-k\lambda t}}{(k(a-1-n)-1)!}$$

$$= k \sum_{j=1}^{a-1} \frac{(k\lambda) (k\lambda)^{kj-1} e^{-k\lambda t}}{(kj-1)!} \left\{ \left(1 + \frac{\mu}{k\lambda} \right) Q_{0,1} - \right.$$

$$z_0^b \left\{ \left(\frac{z_0 (z_0^{1-a} - z_0^{-b}) \frac{\mu}{k\lambda}}{z_0^b (z_0 - 1) \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b} \right)} - \frac{z_0^{1-a}}{z_0^b - 1} \right) + \right.$$

$$\left. \left. \frac{z_0^b z_0^{j-a+1}}{z_0^b - 1} \right\} Q_{0,k} \right\}$$

(38)

Using (15), (29) and (34), the second term of equation (37) becomes

$$k Q_{0,k} \mu e^{-\mu t (1 - z_0^{-b})} z_0 \frac{(z_0^{1-a} - z_0^{-b})}{(z_0 - 1)} \quad (39)$$

From (15), (29) and (35) we have the third term of equation (37),

$$k Q_{0,k} \mu e^{-\mu t(1-z_0^{-b})} \sum_{m=0}^{a-2} z_0^{-m} \left\{ 1 - \sum_{j=0}^{k(a-1-m)-1} \right.$$

$$\left. \frac{e^{-k\lambda t} (k\lambda t)^j}{j!} \right\} + k Q_{0,k} \left\{ \frac{1 - e^{-\mu t(1-z_0^{-b})}}{1 - z_0^{-b}} \right\}$$

$$\frac{z_0^{1-a} k\lambda \sum_{j=1}^{a-1} (k\lambda t)^{kj-1} e^{-k\lambda t} z_0^j}{(kj-1)!} \quad (40)$$

$$\text{Let } A = \frac{k\lambda z_0^{1-a}}{\mu(z_0^{-b} - 1)}$$

$$B = \frac{z_0(1-z_0^{-b})}{z_0 - 1}$$

$$C = k^2 \lambda \left\{ \left(1 + \frac{\mu}{k\lambda} \right) Q_{0,1} - z_0^b \left[\frac{z_0(z_0^{1-a} - z_0^{-b})}{z_0^b(z_0 - 1) \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b} \right)} \right. \right.$$

$$\left. \left. - \frac{z_0^{1-a}}{z_0^b - 1} \right] Q_{0,k} \right\}$$

Hence using (38), (39) and (40)

$$\begin{aligned}
 W_q^-(t) &= k \mu e^{-\mu t (1-z_0^{\frac{-b}{a}})} Q_{0,k} \left\{ A \sum_{j=1}^{a-1} \frac{z_0^j (k \lambda t)^{kj-1} e^{-k \lambda t}}{(kj-1)!} \right. \\
 &+ B \sum_{m=0}^{a-2} \frac{k(a-1-m)-1}{\sum_{j=0}^{a-1} z_0^{-m} e^{-k \lambda t}} \frac{z_0^{-m} e^{-k \lambda t}}{j!} (k \lambda t)^j \\
 &+ C \sum_{j=1}^{a-1} \frac{(k \lambda t)^{kj-1}}{(kj-1)!} e^{-k \lambda t}
 \end{aligned} \tag{41}$$

Finally we derive the expected waiting time in the queue.

$$\begin{aligned}
 E(T) &= \int_0^{\infty} t W_q^-(t) dt. \\
 &= k \mu Q_{0,k} [A J_1(1) + B J_2(1) - J_4(1) + C J_3(1)]
 \end{aligned} \tag{42}$$

where

$$J_1(1) = \int_0^{\infty} t \sum_{j=1}^{a-1} \frac{z_0^j (k \lambda t)^{kj-1}}{(kj-1)!} e^{-\mu t ((1-z_0^{\frac{-b}{a}}) + k \lambda)} dt$$

$$\begin{aligned}
&= \sum_{j=1}^{a-1} \int_0^{\infty} \frac{z_0^j (k\lambda)^{kj-1} t^{kj}}{(kj-1)!} e^{-\mu t((1-z_0^{-b})+k\lambda)} dt \\
&= \sum_{j=1}^{a-1} \frac{z_0^j (k\lambda)^{kj-1}}{(kj-1)!} \frac{(kj)!}{[\mu(1-z_0^{-b})+k\lambda]^{kj+1}} \\
&= \sum_{j=1}^{a-1} \frac{kj}{(k\lambda)^2 \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda} z_0^{-b}\right)} \quad (43)
\end{aligned}$$

$$\begin{aligned}
J_2(1) &= \int_0^{\infty} t e^{-\mu t(1-z_0^{-b})} dt \\
&= \frac{1}{[\mu(1-z_0^{-b})]^2} \quad (44)
\end{aligned}$$

$$\begin{aligned}
J_3(1) &= \int_0^{\infty} t \sum_{j=1}^{a-1} \frac{e^{-k\lambda t} (k\lambda t)^{kj-1}}{(kj-1)!} dt \\
&= \sum_{j=1}^{a-1} \frac{(k\lambda)^{kj-1}}{(kj-1)!} \frac{(kj)!}{(k\lambda)^{kj+1}}
\end{aligned}$$

$$= \sum_{j=1}^{a-1} \frac{k^j}{(k\lambda)^2}$$

$$= \frac{a(a-1)}{2k\lambda^2}$$

(45)

$$J_4(1) = \int_0^{\infty} t e^{-\mu t} (1 - z_0^{-b})^{\sum_{m=0}^{a-2} k(a-1-m)-1} \frac{z_0^{-m} e^{-k\lambda t} (k\lambda t)^j}{j!} dt$$

$$= \sum_{m=0}^{a-2} \sum_{j=0}^{k(a-1-m)-1} \frac{z_0^{-m} (k\lambda)^j}{j!} \frac{(j+1)!}{(k\lambda)^{j+2} \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{j+2}}$$

$$= \sum_{m=0}^{a-2} \sum_{j=0}^{k(a-1-m)-1} \frac{z_0^{-m}}{(k\lambda)^2} \frac{(j+1)}{\left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{j+2}}$$

(46)

Hence using (43) through (46).

$$E(T) = k\mu Q_{0,k} \left\{ \frac{z_0^{1-a}}{\lambda\mu(z_0^{-b}-1)} \sum_{j=1}^{a-1} \frac{j}{\left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^j} \right\}$$

$$\begin{aligned}
& + \frac{z_0}{z_0-1} \frac{1}{\mu^2(1-z_0^{-b})} + C \frac{a(a-1)}{2k\lambda^2} \\
& - \left. \sum_{m=0}^{a-2} \sum_{j=1}^{k(a-1-m)} \frac{z_0^{-m} j}{(k\lambda)^2 \left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{j+1}} \right\} \\
& = Q_{0,k} \left\{ \frac{kz_0}{\mu(z_0-1)(1-z_0^{-b})} - \frac{kz_0^{1-a}}{\lambda(1-z_0^{-b})} \right. \\
& \left. \sum_{j=1}^{a-1} \frac{j}{\left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^j} - \frac{\mu}{k\lambda^2} \sum_{m=0}^{a-2} \sum_{j=1}^{k(a-1-m)} \right. \\
& \left. \frac{j z_0^{-m}}{\left(1 + \frac{\mu}{k\lambda} - \frac{\mu}{k\lambda z_0^b}\right)^{j+1}} + \frac{C \mu a(a-1)}{2\lambda^2} \right\} \tag{47}
\end{aligned}$$

By taking $k=1$ the above result may be verified with Medhi [15].

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BIBLIOGRAPHY

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