Introduction

INTRODUCTION

The linear algebra over semirings is a subject of intensive research because of its purely algebraic interest and its numerous applications. There have been numerous investigations into the theory of matrices over algebraic structures. A fair amount of effort has been directed towards discovering various properties of matrices when the underlying algebraic structure is a semiring.

"A semiring is a set S with two binary operations, addition and multiplication, such that:

- S is an abelian monoid under addition (identity denoted by 0);
- S is a semigroup under multiplication (identity, if any, denoted by 1);
- Multiplication is distributive over addition on both sides;
- s0 = 0s = 0 for all $s \in S$."

Some combinatorially interesting semirings are finite Boolean algebras including the two-element Boolean algebra $\mathbb{B}$ = {0, 1}, fuzzy algebra $\mathcal{F}$ = [0, 1], nonnegative integers Z^+, nonnegative real numbers R^+.

The study of matrices over semirings (matrices with entries from semirings) has a long history. In 1964, Rutherford gave a proof of Cayley-Hamiltion theorem for a commutative semiring avoiding the use of determinants. Since then, a number of works on theory of matrices over semirings were carried out by many famous researchers- Gbosh. S, Golan, J.S, Gregory, D.A, Pullman, N.J, Reutenauer, C, Straubing, H, Guterman, A.E.

Regular matrices play a central role in the theory of matrices, and they have many applications in network and switching theory and information theory. Recently a number of authors have studied regular matrices over semirings. Prime and semiprime matrices over semirings are studied by Han Hyuk Cho and Suh-Ryung Kim.

Invertible matrices are an important type of matrices. There are many research papers published on invertible matrices. In 2007, Yijia Tan studied invertible matrices over general commutative antirings. (An antiring is a semiring which is zero sum free i.e., a+b = 0 implies a = b = 0 for any a, b in this semiring).

Idempotent matrices over semirings are studied by Beasley, L.B, Pullman, N.J, [3] Bapat, R.B, et. al. [2] and Kyung-Tae Kang et. al [15].

One of the most active and fertile subjects in matrix theory during the past one hundred years is the linear preserver problem, which concerns the characterizations of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. There are many research papers published on linear preserver problem for matrices over semirings [3, 17, 18, 27, 28, 29].

This thesis is devoted to the study of some interesting results on matrices over semirings. The discussion is under the following topics:

1. Regular matrices over semirings.
2. Prime and semiprime matrices over semirings.
3. Invertible matrices over semirings.
4. Idempotent matrices over semirings.
5. Regularity preservers for matrices over semirings.
6. Determinant, singularity and non-singularity preservers for matrices over semirings.
7. Rank and perimeter preservers for matrices over semirings.

The first chapter deals with preliminary definitions and results.

Chapter II deals with regular matrices over semirings. "An $m \times n$ matrix A over a semiring S is called regular if there is an $n \times m$ matrix G over S such that AGA = A".

Some interesting examples of regular matrices over semirings are given and some interesting properties of these matrices are investigated.

Prime and semiprime matrices over semirings are studied in chapter III. In this chapter, the relationship between (semi) prime matrices and their row spaces are studied (Theorems 3.19, 3.23 and 3.25). The following factorization property of a square matrix with full semiring rank over a chain semiring is also proved:

"Let S be a chain semiring. Let A be an non-monomial matrix with full semiring rank over S. Then $A = P_d P_{d-1} \ldots\ldots\ldots P_1$ ($d \geq 1$), where each P_i is an elementary matrix or a semiprime matrix over S".

Chapter IV deals with invertible matrices over semirings. In this chapter, the complete description of the invertible matrices over a commutative antiring is given. Some necessary and sufficient conditions for a matrix over a commutative antiring to be invertible are obtained. Cramer's rule for a matrix equation over a commutative antiring is presented.

Chapter V is devoted to the study of idempotent matrices over semirings. Following are the important Theorems proved in this Chapter:

1. Let A be a matrix in $M_n(\mathbb{B}_1)$ where $\mathbb{B}_1$, is an binary Boolean algebra.

   Then A is idempotent if and only if the following two conditions are satisfied:

   (a) there exist integers $r, l \geq 0$ such that A is a sum of r disjoint rectangle parts and l line parts.

(b) if for some $i \neq j$, the i^{th} row matrix $r_i[A]$ and the j^{th} column matrix $c_j[A]$ are not (i, j)-disjoint then $E_{i,j}$ is a cell in A. (Theorem 5.17).

2. Let $A = [a_{ij}]$ be a matrix in $M_n(\mathbb{K})$, where $\mathbb{K}$ is an chain semiring. Then A is idempotent if and only if all $a_{i,j}$-patterns of A are idempotent in $M_n(\mathbb{B}_1)$. (Theorem 5.19).

Chapter VI deals with regularity preservers for matrices over semirings. In this chapter, the linear operators that strongly preserve regular matrices over semirings including the binary Boolean algebra, the nonnegative reals, the nonnegative integers and fuzzy scalars are studied. Interesting characterizations of linear operators that strongly preserve regular matrices over semirings are established in Theorems 6.30 and 6.34.

Chapter VII deals with determinant, singularity and non-singularity preservers for matrices over semirings. The concepts of $\mathcal{S}$-singularity and $\mathcal{R}$-singularity of matrices over semirings are introduced and the following interesting characterizations of linear operators preserving $\mathcal{S}$-singularity and $\mathcal{R}$-singularity are obtained.

1. For a surjective linear operator $T: M_n(S) \to M_n(S)$ the following statements are equivalent:
   (i) T preserves the set of $\mathcal{S}$-singular matrices;
   (ii) T preserves the set of $\mathcal{S}$-nonsingular matrices,
   (iii) T is a (P, Q, B)-operators and the entries of B are invertible elements from S. (Theorem 7.12).
2. For a surjective linear transformation $T: M_n(S) \to M_n(S)$. The following statements are equivalent:
   (i) T preserves the set of $\mathcal{R}$-nonsingular matrices,

(ii) T preserves the set of $\mathcal{R}$-singular matrices,

(iii) T is standard.

The development of linear algebra over semirings certainly requires such an important matrix invariant as the determinant function. However it turns out that even over commutative semirings without zero divisors the classical determinant can not be defined over fields and commutative rings. The main problem lies in the fact that in semirings which are not rings not all elements possess an additive inverse. A natural replacement of the determinant function for matrices over commutative semirings is the bideterminant.

"A bideterminant of a matrix $A = [a_{ij}] \in M_n(S)$ is a pair bidet $A = (||A||^+, ||A||^-)$, where

$||A||^+ = \sum_{\sigma \in A_n} a_{1\sigma(1)} \ldots\ldots\ldots\ldots a_{n\sigma(n)}$, $||A||^- = \sum_{\sigma \in S_n | A_n} a_{1\sigma(1)} \ldots\ldots\ldots\ldots a_{n\sigma(n)}$ where S_n denotes the symmetric group on the set {1, 2,……..n}, A_n denote its subgroup of even permutations"

Using the definition of bideterminant, an interesting characterization of determinant preservers of matrices over semirings is established in Theorem 7.24.

Chapter VIII deals with Rank and perimeter preservers for matrices over semirings. The set of linear operators preserving the rank and perimeter of every rank-1 matrix over any chain semiring is characterized as follows:

"For a linear operator T on $M_n(S)$, where S is a chain semiring, the following statements are equivalent:

(i) T is a (P, Q, B)-operator,

(ii) T preserves both rank and perimeter of rank-1 matrices.

(iii) T preserves both rank and perimeters 2 and k ($k \geq 4$, $k \neq n+1$) of rank-1 matrices".