Chapter 6

λ_g^δ-homeomorphism

6.1 Introduction

In this chapter, the notions of λ_g^δ-open and λ_g^δ-closed functions are taken for study and their behaviours are characterized in almost weakly Hausdorff space. Two types of homeomorphisms namely λ_g^δ-homeomorphism and $\lambda_g^{\delta *}$-homeomorphism are developed and their properties are obtained. It is observed that the set of all $\lambda_g^{\delta *}$-homeomorphisms form a group under composition of functions.

6.2 λ_g^δ-open and λ_g^δ-closed functions

Definition 6.2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is

(i) ***λ_g^δ-open*** if $f(A)$ is λ_g^δ-open for every open set A in X.

(ii) ***λ_g^δ-closed*** if $f(A)$ is λ_g^δ-closed for every closed set A in X.

Theorem 6.2.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be λ_g^δ-closed then $\lambda_g^\delta cl(f(A)) \subseteq f(cl(A))$, for each $A \subseteq X$.*

Proof. Let $A \subseteq X$. Then $cl(A)$ is closed in X. By hypothesis, $f(cl(A))$ is λ_g^δ-closed in Y. Since $f(A) \subseteq f(cl(A))$, $\lambda_g^\delta cl(f(A)) \subseteq \lambda_g^\delta cl(f(cl(A))) = f(cl(A))$. □

Theorem 6.2.4. *For topological spaces $(X, \tau), (Y, \sigma), (Z, \eta), f : (X, \tau) \to (Y, \sigma)$, $g : (Y, \sigma) \to (Z, \eta)$ and $g \circ f : (X, \tau) \to (Z, \eta)$ the following results are true.*

(i) If $g \circ f$ is λ_g^δ-open and f is continuous, surjective then g is λ_g^δ-open,

(ii) If $g \circ f$ is open and g is λ_g^δ-continuous, injective then f is λ_g^δ-open.

Proof. (i) Let B be open in Y. Then $f^{-1}(B)$ is open in X. By hypothesis, $(g \circ f)(f^{-1}(B)) = g(f(f^{-1}(B))) = g(B)$ is λ_g^δ-open in Z. Thus g is λ_g^δ-open.

(ii) Let A be open in X. By hypothesis, $(g \circ f)(A) = g(f(A))$ is open in Z. Therefore $g^{-1}(g(f(A))) = f(A)$ is λ_g^δ-open, as g is an injective λ_g^δ-continuous function. Hence f is λ_g^δ-open.

□

Theorem 6.2.5. *For a bijective map $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent.*

(i) f is λ_g^δ-open;

(ii) f is λ_g^δ-closed;

(iii) f^{-1} is λ_g^δ-continuous.

Proof. (i) ⇒ (ii) Let $A \subseteq X$ be closed. Then $X \setminus A$ is open in X. By (i), $f(X \setminus A)$ is λ_g^δ-open in Y. Since f is bijective, $f(X \setminus A) = Y \setminus f(A)$ which implies $f(A)$ is λ_g^δ-closed in Y. Hence f ia λ_g^δ-closed.
(ii) ⇒ (iii) Let $A \subseteq X$ be closed. Since f is bijective, $(f^{-1})^{-1}(A) = f(A)$ is λ_g^δ-closed in Y. Therefore f^{-1} is λ_g^δ-continuous.
(iii) ⇒ (i) Let $A \subseteq X$ be open. Since f^{-1} is λ_g^δ-continuous and f is bijective, $(f^{-1})^{-1}(A) = f(A)$ is λ_g^δ-open. Hence f is λ_g^δ-open. □

Remark 6.2.6. The above theorem gives the conditions under which λ_g^δ-open and λ_g^δ-closed maps coincide.

Theorem 6.2.7. *A bijective function $f : (X, \tau) \to (Y, \sigma)$ is λ_g^δ-closed iff for each subset B in Y and each open set U in X containing $f^{-1}(B)$, there exists a λ_g^δ-open set V in Y such that*

(i) $B \subseteq V$ *and*

(ii) $f^{-1}(V) \subseteq U$.

Proof. *Necessity :* Let f be λ_g^δ-closed and $B \subseteq Y$. Take an open set U in X such that $f^{-1}(B) \subseteq U$. Then $X \setminus U$ is closed in X. By hypothesis, $f(X \setminus U)$ is λ_g^δ-closed. Define $V = Y \setminus (f(X \setminus U))$ then V is λ_g^δ-open in Y.
(i) $B \subseteq V$. Suppose that $v \notin V$ then $v \in f(X \setminus U) \Rightarrow f^{-1}(v) \in X \setminus U \Rightarrow f^{-1}(v) \notin U$ then $f^{-1}(v) \notin f^{-1}(B) \Rightarrow v \notin B$. Hence $B \subseteq V$.
(ii) $f^{-1}(V) \subseteq U$. Now, $v \in V \Rightarrow v \notin f(X \setminus U) \Rightarrow f^{-1}(v) \notin X \setminus U$ then $f^{-1}(v) \in U$. Hence $f^{-1}(V) \subseteq U$.
Sufficiency : Let A be closed in X. Then $X \setminus A$ is open in X, $f(X \setminus A) \subseteq Y$ and $Y \setminus (f(X \setminus A)) \subseteq Y$. Now, $Y \setminus f(A) \Rightarrow f(X \setminus A) \Rightarrow f^{-1}(Y \setminus f(A)) \subseteq X \setminus A$. Then by hypothesis, there exists a λ_g^δ-open set V in Y such that $Y \setminus f(A) \subseteq V$ and $f^{-1}(V) \subseteq X \setminus A$. Now, $Y \setminus f(A) \subseteq V \Rightarrow Y \setminus V \subseteq f(A)$ and $f^{-1}(V) \subseteq X \setminus A \Rightarrow V \subseteq f(X \setminus A) \Rightarrow Y \setminus f(X \setminus A) \subseteq Y \setminus V$. Thus $Y \setminus f(X \setminus A) \subseteq Y \setminus V \subseteq f(A)$. Since f is a bijection, $Y \setminus f(X \setminus A) = f(A) \Rightarrow Y \setminus V = f(A)$. Since $Y \setminus V$ is λ_g^δ-closed, so is $f(A)$. Hence f is λ_g^δ-closed. □

Corollary 6.2.8. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is λ_g^δ-open iff for each $B \subseteq Y$ and each closed set U containing $f^{-1}(B)$, there exists a λ_g^δ-closed set $V \subseteq Y$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. Similar to previous theorem. □

Remark 6.2.9. Composition of two λ_g^δ-closed functions is not a λ_g^δ-closed function.

Example 6.2.10. Let $X = Y = Z = \{a, b, c\}, \tau = \{X, \phi, \{c\}, \{b, c\}\}, \sigma = \{Y, \phi, \{a\}\}$

and $\eta = \{Z, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be an identity function and $g : (Y, \sigma) \to (Z, \eta)$ be defined by $g(a) = b, g(b) = a$ and $g(c) = c$. Then f and g are both λ_g^δ-closed functions but $g \circ f$ is not a λ_g^δ-closed function as $g(f(\{a, b\})) = \{a, b\}$ is not λ_g^δ-closed in (Z, η).

Theorem 6.2.11. *Let $f : (X, \tau) \to (Y, \sigma)$ be any function with (Y, σ) being an almost weakly Hausdorff space. Then every λ_g^δ-closed function is a closed function.*

Proof. Let A be closed in X. Since f is λ_g^δ-closed, $f(A)$ is λ_g^δ-closed. Since (Y, σ) is an almost weakly Hausdorff space, by Lemma **??**, $f(A)$ is closed in Y. Hence f is a closed function. □

Remark 6.2.12. Composition of two λ_g^δ-open functions is not a λ_g^δ-open function.

Example 6.2.13. Let $X = Y = Z = \{a, b, c\}, \tau = \{X, \phi, \{c\}, \{b, c\}\}, \sigma = \{Y, \phi, \{a\}\}$ and $\eta = \{Z, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be identity functions. Then f and g are both λ_g^δ-open functions but $g \circ f$ is not a λ_g^δ-open function as $g(f(\{b, c\})) = \{b, c\}$ is not λ_g^δ-open in (Z, η).

Theorem 6.2.14. *For topological spaces $(X, \tau), (Y, \sigma), (Z, \eta), f : (X, \tau) \to (Y, \sigma), g : (Y, \sigma) \to (Z, \eta)$ and $g \circ f : (X, \tau) \to (Z, \eta)$ the following are true.*

(i) If f and g are both λ_g^δ-closed with (Y, σ) being an almost weakly Hausdorff space then $g \circ f$ is λ_g^δ-closed.

(ii) If f is closed and g is λ_g^δ-closed then $g \circ f$ is λ_g^δ-closed.

(iii) If f is closed and g is λ_g^δ-closed with (Y, σ) being an almost weakly Hausdorff space then $g \circ f$ is closed.

Proof. Obvious from the definitions. □

Theorem 6.2.15. *Let $f : X \to Y$ and $g : Y \to Z$ such that $g \circ f : X \to Z$ is λ_g^δ-closed. Then the following are true.*

(i) If f is continuous, surjective then g is λ_g^δ-closed,

(ii) If g is λ_g^δ-irresolute, injective then f is λ_g^δ-closed,

(iii) If f is λ_g^δ-continuous, surjective and X is an almost weakly Hausdorff space then g is λ_g^δ-closed,

(iv) If f is $g\delta$-continuous, surjective and X is an almost weakly Hausdorff space then g is λ_g^δ-closed,

(v) If g is strongly λ_g^δ-continuous, injective then f is closed.

Proof. (i) Let B be closed in Y. Since f is continuous, $f^{-1}(B)$ is closed in X. Further, since $g \circ f$ is λ_g^δ-closed, $g(f(f^{-1}(B)))$ is λ_g^δ-closed in Z. As f is surjective, $g(f(f^{-1}(B))) = g(B)$ is λ_g^δ-closed in Z. Thus g is λ_g^δ-closed.

(ii) Let A be closed in X. Since $g \circ f$ is λ_g^δ-closed, $g(f(A))$ is λ_g^δ-closed in Z. As g is λ_g^δ-irresolute, $g^{-1}(g(f(A)))$ is λ_g^δ-closed in Y. Further, $g^{-1}(g(f(A))) = f(A)$, as g is injective. This proves that f is λ_g^δ-closed.

(iii) Let B be closed in Y. Since f is λ_g^δ-continuous, $f^{-1}(B)$ is λ_g^δ-closed. Since X is an almost weakly Hausdorff space, $f^{-1}(B)$ is closed in X. As $g \circ f$ is λ_g^δ-closed, $g(f(f^{-1}(B)))$ is λ_g^δ-closed in Z. Now, $g(f(f^{-1}(B))) = g(B)$ is λ_g^δ-closed in Z, as f is surjective. Hence g is λ_g^δ-closed.

(iv) Let B be closed in Y. Since f is $g\delta$-continuous, $f^{-1}(B)$ is $g\delta$-closed. Since X is an almost weakly Hausdorff space, $f^{-1}(B)$ is closed in X. As $g \circ f$ is λ_g^δ-closed, $g(f(f^{-1}(B)))$ is λ_g^δ-closed in Z. Now, $g(f(f^{-1}(B))) = g(B)$ is λ_g^δ-closed in Z, as f is surjective. Hence g is λ_g^δ-closed.

(v) Let B be closed in Y. Since $g \circ f$ is λ_g^δ-closed, $g(f(B))$ is λ_g^δ-closed in Z. As g is strongly λ_g^δ-continuous, $g^{-1}(g(f(B)))$ is closed in X. Further $g^{-1}(g(f(B))) = f(B)$ is closed in X, as g is injective.

□

Theorem 6.2.16. *A function $f : X \to Y$ is λ_g^δ-open iff $f(int(A)) \subseteq \lambda_g^\delta int(f(A))$, for every $A \subseteq X$.*

Proof. *Necessity :* Let $A \subseteq X$. Now, $int(A)$ is open in X. Since f is λ_g^δ-open, $f(int(A))$ is λ_g^δ-open in Y. We know, $int(A) \subseteq A \Rightarrow f(int(A)) \subseteq f(A) \Rightarrow \lambda_g^\delta int(f(int(A))) \subseteq \lambda_g^\delta int(f(A)) \Rightarrow f(int(A)) \subseteq \lambda_g^\delta int(f(A))$, as $f(int(A))$ is λ_g^δ-open in Y. *Sufficiency :* Let A be an open subset of X. By hypothesis, $f(int(A)) \subseteq \lambda_g^\delta int(f(A))$. Since A is open, $f(A) \subseteq \lambda_g^\delta int(f(A))$. Thus $f(A)$ is λ_g^δ-open and hence f is λ_g^δ-open. ☐

Corollary 6.2.17. If a function $f : X \to Y$ is λ_g^δ-open then $f^{-1}(\lambda_g^\delta cl(B)) \subseteq cl(f^{-1}(B))$, for every $B \subseteq Y$.

Proof. Let f be λ_g^δ-open and $B \subseteq Y$. Then $f^{-1}(B) \subseteq cl(f^{-1}(B))$. By Corollary 6.2.8, there exists a λ_g^δ-closed set $A \subseteq Y$ such that $B \subseteq A$ and $f^{-1}(A) \subseteq cl(f^{-1}(B)) \Rightarrow f^{-1}(\lambda_g^\delta cl(B)) \subseteq f^{-1}(\lambda_g^\delta cl(A)) = f^{-1}(A) \subseteq cl(f^{-1}(B))$, since A is λ_g^δ-closed. ☐

Theorem 6.2.18. *If a function $f : X \to Y$ is λ_g^δ-open then for each $x \in X$ and for each neighborhood U of $x \in X$, there exists a λ_g^δ-neighborhood W of $f(x)$ such that $W \subseteq f(U)$.*

Proof. Let $x \in X$, and for each neighborhood U of $x \in X$, there exists an open set G such that $x \in G \subseteq U$. By hypothesis, $f(G) = f(intG) \subseteq \lambda_g^\delta int(f(G)) \Rightarrow f(G)$ is λ_g^δ-open in Y. Further $f(x) \subseteq f(G) \subseteq f(U)$. By taking $f(G) = W$, the result follows. ☐

6.3 λ_g^δ-homeomorphism and $\lambda_g^{\delta^*}$-homeomorphism

Definition 6.3.1. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called ***λ_g^δ-homeomorphism*** if f is both λ_g^δ-continuous and λ_g^δ-open.

Theorem 6.3.2. *If a bijective function $f : X \to Y$ is λ_g^δ-continuous then the following statements are equivalent.*

(i) f is λ_g^δ-open;

(ii) f is λ_g^δ-homeomorphism;

(iii) f is λ_g^δ-closed.

Proof. (i) $\Leftrightarrow$ (iii) is obvious.
(i) $\Leftrightarrow$ (ii) follows from the definition of λ_g^δ-homeomorphism. $\square$

Remark 6.3.3. Homeomorphism is independent of λ_g^δ-homeomorphism as seen from the following Example.

Example 6.3.4. Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $f : (X, \tau) \to (Y, \sigma)$ be an identity function. Then f is a λ_g^δ-homeomorphism but not a homeomorphism as f is closed but not continuous.

Example 6.3.5. Let $X = Y = \{a, b, c, d\}, \tau = \sigma = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $f : (X, \tau) \to (Y, \sigma)$ be an identity function. Then f is a homeomorphism but not a λ_g^δ-homeomorphism as f is neither λ_g^δ-closed nor λ_g^δ-continuous.

Definition 6.3.6. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called ***$\lambda_g^{\delta *}$-homeomorphism*** if both f and f^{-1} are λ_g^δ-irresolute.

Theorem 6.3.7. *Composition of two $\lambda_g^{\delta *}$-homeomorphisms is a $\lambda_g^{\delta *}$-homeomorphism.*

Proof. Let $f : X \to Y$ and $f : Y \to Z$ be two $\lambda_g^{\delta *}$-homeomorphisms and $g \circ f : X \to Z$. Let U be λ_g^δ-closed in Z. Now $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$. By hypothesis, $g^{-1}(U)$ is λ_g^δ-closed in Y. Once again by hypothesis, $f^{-1}(V)$ is λ_g^δ-closed in X. Hence $g \circ f$ is λ_g^δ-irresolute. Now, let G be λ_g^δ-closed in X. Then $(g \circ f)(G) = g(f(G)) = g(W)$, where $W = f(G)$. By hypothesis, $f(G)$ is λ_g^δ-closed in Y. Again by hypothesis, $g(f(G))$ is λ_g^δ-closed in Z. Therefore $(g \circ f)^{-1}$ is λ_g^δ-irresolute and hence $g \circ f$ is a $\lambda_g^{\delta *}$-homeomorphism. $\square$

Theorem 6.3.8. *If $f : X \to Y$ is a $\lambda_g^{\delta *}$-homeomorphism then*

1. *For $A \subseteq X$, $f(\lambda_g^\delta cl(A)) \subseteq cl_\delta(f(A))$,*

2. *For $B \subseteq Y$, $\lambda_g^\delta cl(f^{-1}(B)) \subseteq f^{-1}(cl_\delta(B))$,*

3. *For $B \subseteq Y$, $f^{-1}(\lambda_g^\delta cl(B)) \subseteq cl_\delta(f^{-1}(B))$,*

4. *For $A \subseteq X$, $\lambda_g^\delta cl(f(A)) \subseteq f(cl_\delta(A))$.*

Definition 6.3.9. The set of all $\lambda_g^{\delta^*}$-homeomorphisms of (X, τ) onto itself is denoted by $\lambda_g^{\delta^*}h(X, \tau)$.

Theorem 6.3.10. *Let $f : X \to X$ be any function. If $f \in \lambda_g^{\delta^*}h(X, \tau)$ then $f^{-1} \in \lambda_g^{\delta^*}h(X, \tau)$.*

Proof. Let $f \in \lambda_g^{\delta^*}h(X, \tau)$ then both f and f^{-1} are λ_g^{δ}-irresolute. Now $f = (f^{-1})^{-1}$ is λ_g^{δ}-irresolute. Thus both f^{-1} and $(f^{-1})^{-1}$ are λ_g^{δ}-irresolute This implies $f^{-1} \in \lambda_g^{\delta^*}h(X, \tau)$. □

Theorem 6.3.11. *The set of all $\lambda_g^{\delta^*}$-homeomorphisms of (X, τ) onto itself denoted by $\lambda_g^{\delta^*}h(X, \tau)$ forms a group under composition of functions.*

Proof. Define a binary operation $\star : \lambda_g^{\delta^*}h(X, \tau) \times \lambda_g^{\delta^*}h(X, \tau) \to \lambda_g^{\delta^*}h(X, \tau)$ defined by $f \star g = g \circ f$, for all $f, g \in \lambda_g^{\delta^*}h(X, \tau)$ and $\circ$ is the usual composition of maps. By Theorem 6.3.7, $g \circ f \in \lambda_g^{\delta^*}h(X, \tau)$. Hence the closure axiom. Since composition of functions is associative, the associative axiom holds. $I \in \lambda_g^{\delta^*}h(X, \tau)$ is the identity element, where $I : (X, \tau) \to (X, \tau)$ is the identity function. By Theorem 6.3.10, if $f \in \lambda_g^{\delta^*}h(X, \tau)$ then $f^{-1} \in \lambda_g^{\delta^*}h(X, \tau)$ in such a way that $f \circ f^{-1} = f^{-1} \circ f = I$. Hence the existence of inverse. Therefore $\lambda_g^{\delta^*}h(X, \tau)$ forms a group under composition of functions. □

Theorem 6.3.12. *Let $\star : \lambda_g^{\delta^*}h(X, \tau) \to \lambda_g^{\delta^*}h(Y, \sigma)$ be a homeomorphism then λ_g^{δ}-ker$(\star)$ is a normal subgroup of $\lambda_g^{\delta^*}h(X, \tau)$.*

Proof. Since $\star(I_x) = I_y$, $I_x \in \lambda_g^{\delta}$-ker$(\star)$ and hence λ_g^{δ}-ker$(\star) \neq \phi$. Let $h_1, h_2 \in \lambda_g^{\delta}$-ker$(\star)$ then $\star(h_1) = \star(h_2) = I_y$. This implies $\star(h_1 h_2^{-1}) = \star(h_1) \star (h_2^{-1}) = I_y$ Then $h_1 h_2^{-1} \in \lambda_g^{\delta}$-ker$(\star)$. Hence λ_g^{δ}-ker$(\star)$ is a subgroup of $\lambda_g^{\delta^*}h(X, \tau)$. Now let $h_1 \in \lambda_g^{\delta}$-ker$(\star)$ and $g \in \lambda_g^{\delta^*}h(X, \tau)$ then $\star(g h_1 g^{-1}) = I_y$ which implies $g h_1 g^{-1} \in \lambda_g^{\delta}$-ker$(\star)$. Hence λ_g^{δ}-ker$(\star)$ is a normal subgroup of $\lambda_g^{\delta^*}h(X, \tau)$. □