5. Batch Arrival Retrial G-Queue with Multistage and Multi-Optional Services, Active Breakdown, Delayed Repair and Orbital Search

Bulk arrival retrial G-queue with two phase service is considered. The server is subject to active breakdown and breakdown due to negative arrival. The server failed due to negative arrival is sent for repair immediately and the interrupted customer leaves the system. The repair of the active breakdown server starts after some random time and the interrupted customer remains in the service position for the completion of the remaining service. During idle period, the server may search for customers in the orbit. By using the supplementary variable method, the steady state solutions for the system measures are derived. Availability of the server and failure frequency are obtained. Stochastic decomposition law is verified. The effect of various parameters on the system performance is analysed numerically.

5.1 Model Description

Two phase repairable $M^X/G/1$ retrial G-queue with orbital search is considered. The diagrammatic representation of the proposed model is shown in Fig. 5.1. The assumptions of the proposed model are described below.

The arriving customers are of two types, positive and negative. The positive customers arrive in batches according to the Poisson process with rate λ^+. The batch size Y is a random variable with distribution function $P(Y=k)=C_k$, k=1,2,3,..... and probability generating function C(z) having the first two moments m_1 and m_2. Negative customers arrive in single according to the Poisson process with rate λ^-.

If the arriving batch of customers finds the server free, then one of the customers of the batch begins his service immediately and the others join the orbit. Otherwise, all the customers join the orbit and make trial at a later time. Orbital customers behave independently of each other and keep making retrials until they receive their service. Successsive inter-retrial times are generally distributed with distribution function $A(x)$, Laplace-Stieltjes transform $A^*(s)$ and the conditional completion rate $\eta(x)$.

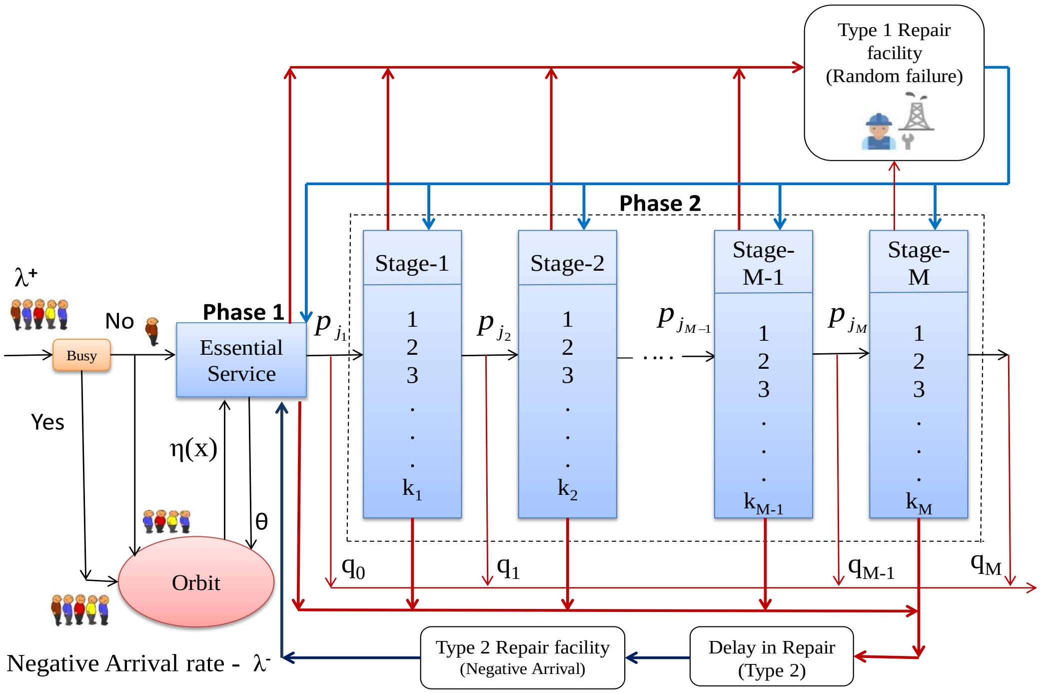


Fig. 5.1 Batch Arrival Retrial G-Queue with Multistage and Multi-Optional Services, Active Breakdown, Delayed Repair and Orbital Search

The server provides two phase service – essential and optional. The essential service time is generally distributed with distribution function $B_0(x)$, Laplace-Stieltjes transform $B_0^*(s)$, n^{th} factorial moment $\mu_0^{(n)}$ and the conditional completion rate $\mu_0(x)$.

There are M-stages in the second phase and each stage contains multi-optional services. After the completion of essential service, the customer may proceed to the first stage in second phase and opt j_1^{th} ($j_1=1,2,....,k_1$) optional service with probability p_{j_1} or leave the system with probability q_0. After the completion of i^{th} stage service, the customer may proceed to $(i+1)^{th}$ stage and opt j_{i+1}^{th} optional service with probability $p_{j_{i+1}}$ or leave the system with probability q_i. The service time of i^{th} stage optional service follows a general distribution with distribution function $B_{i,j_i}(x)$, Laplace-Stieltjes transform $B_{i,j_i}^*(s)$, n^{th} factorial moment $\mu_{i,j_i}^{(n)}$ and the conditional completion rate $\mu_{i,j_i}(x)$, $i=1,2,3,....,M$, $j_i=1,2,3,.....,k_i$.

The server is subject to two types of breakdown, active breakdown and breakdown due to negative arrival.

The lifetime of the server is assumed to be exponentially distributed with rate τ_0 during essential and τ_i (i=1,2,...,M) during optional services. Once the system fails the repair starts after some random time known as delay time. The interrupted customer remains in the service area until the remaining service gets completed. The repair time of the server failed during the essential service is generally distributed with distribution function $F_0(x)$, Laplace-Stieltjes transform $F_0^*(s)$, n^{th} factorial moment $r_0^{(n)}$, and the conditional completion rate $r_0(x)$. The repair time of the server failed during i^{th} stage j_i^{th} optional service follows general distribution with distribution function $F_{i,j_i}(x)$, Laplace-Stieltjes transform $F_{i,j_i}^*(s)$, n^{th} factorial moment $r_{i,j_i}^{(n)}$ and the conditional completion rate $r_{i,j_i}(x)$, $i=1,2,3,....,M$, $j_i=1,2,3,....,k_i$. The delay time of the server during the essential service is generally distributed with distribution function $D_0(x)$, Laplace-Stieltjes transform $D_0^*(s)$, n^{th} factorial moment $\gamma_0^{(n)}$, and the conditional completion rate $\gamma_0(x)$. The delay time of the server during i^{th} stage j_i^{th} optional service follows general distribution with distribution function $D_{i,j_i}(x)$, Laplace-Stieltjes transform $D_{i,j_i}^*(s)$, n^{th} factorial moment $\gamma_{i,j_i}^{(n)}$ and the conditional completion rate $\gamma_{i,j_i}(x)$, $i=1,2,3,....,M$, $j_i=1,2,3,....,k_i$.

The arrival of the negative customer makes the server down with the removal of positive customer being in service. As soon as the breakdown occurs, the server is sent for repair. The repair time of the server failed during the essential service is generally distributed with distribution function $R_0(x)$, Laplace-Stieltjes transform $R_0^*(s)$, n^{th} factorial moment $\beta_0^{(n)}$, and the conditional completion rate $\beta_0(x)=\dfrac{dR_0(x)}{1-R_0(x)}$. The repair time of the server failed during i^{th} stage j_i^{th} optional service follows general distribution with distribution function $R_{i,j_i}(x)$, Laplace-Stieltjes transform $R_{i,j_i}^*(s)$, n^{th} factorial moment $\beta_{i,j_i}^{(n)}$ and the conditional completion rate $\beta_{i,j_i}(x)=\dfrac{dR_{i,j_i}(x)}{1-R_{i,j_i}(x)}$, $i=1,2,3,....,M$, $j_i=1,2,3,....,k_i$.

After the completion of service or repair, the server searches for the customer in the orbit with probability θ or remains idle with complementary probability $\bar{\theta} = 1 - \theta$.

5.2 Definitions and Notations

The state of the system at time t can be described by the bivariate Markov process { S(t), N(t), $t \geq 0$ }, where S(t) denotes the state of the server at time t and N(t) denotes the number of customers in the orbit at time t. The server state at time t is given in Table 5.1.

Table 5.1 Server State

S(t)	Stage	State
0	-	Idle
1	Essential	Busy
2	i^{th} stage j_i^{th} option	Busy
3	Essential	Repair (Negative Arrival)
4	i^{th} stage j_i^{th} option	Repair (Negative Arrival)
5	Essential	Delay Time
6	i^{th} stage j_i^{th} option	Delay Time
7	Essential	Repair (Active Breakdown)
8	i^{th} stage j_i^{th} option	Repair (Active Breakdown)

Supplementary variables are defined as follows.

$\xi_1(t)$ – elapsed retrial time ; $\xi_2(t)$ – elapsed service time

$\xi_3(t)$ – elapsed repair time ; $\xi_4(t)$ – elapsed delay time

5.3 Orbit Size Distributions at Random Epoch

For the Markov process { N(t), S(t), $t \geq 0$}, the transient probabilities of server state are defined below.

$$I_0(t) = P\{S(t) = 0, \ N(t) = 0\}$$

$$I_n(x,t) = P\{S(t)=0,\ N(t)=n,\ x<\xi_1(t)\le x+dx\},\ n\ge 1$$

$$P_{0,n}(x,t)dx = P\{S(t)=1,\ N(t)=n,\ x<\xi_2(t)\le x+dx\},\ n\ge 0$$

$$P_{i,j_i,n}(x,t)dx = P\{S(t)=2,\ N(t)=n,\ x<\xi_2(t)\le x+dx\},\ n\ge 0, i=1,2,....,M,\ j_i=1,2,....,k_i$$

$$R_{0,n}(x,t)dx = P\{S(t)=3,\ N(t)=n,\ x<\xi_3(t)\le x+dx\},\ n\ge 0$$

$$R_{i,j_i,n}(x,t)dx = P\{S(t)=4, N(t)=n,\ x<\xi_3(t)\le x+dx\}, n\ge 0,\ i=1,2,....,M,\ j_i=1,2,....,k_i$$

$$D_{0,n}(x,y,t)dx\,dy = P\{S(t)=5,\ N(t)=n,\ x<\xi_2(t)\le x+dx,\ y<\xi_4(t)\le y+dy\},\ n\ge 0$$

$$D_{i,j_i,n}(x,y,t)dx\,dy = P\{S(t)=6,\ N(t)=n,\ x<\xi_2(t)\le x+dx,\ y<\xi_4(t)\le y+dy\},\ n\ge 0,\ i=1,2,....,M,\ j_i=1,2,....,k_i$$

$$F_{0,n}(x,y,t)dx\,dy = P\{S(t)=7,\ N(t)=n,\ x<\xi_2(t)\le x+dx,\ y<\xi_3(t)\le y+dy\}, n\ge 0$$

$$F_{i,j_i,n}(x,y,t)dx\,dy = P\{S(t)=8,\ N(t)=n,\ x<\xi_2(t)\le x+dx,\ y<\xi_3(t)\le y+dy\},\ n\ge 0,\ i=1,2,....,M,\ j_i=1,2,....,k_i$$

The steady state equations governing the model under consideration are given below.

$$\lambda^+ I_0 = q_0\int_0^\infty P_{0,0}(x)\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,0}(x)\mu_{i,j_i}(x)dx + \int_0^\infty R_{0,0}(x)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,0}(x)\beta_{i,j_i}(x)dx \tag{5.1}$$

$$\frac{d}{dx}I_n(x) = -(\lambda^+ + \eta(x))\,I_n(x),\ n\ge 1 \tag{5.2}$$

$$\frac{d}{dx}P_{0,n}(x) = -(\lambda^+ + \lambda^- + \mu_0(x) + \tau_0)\,P_{0,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n}C_k P_{0,n-k}(x) + \int_0^\infty F_{0,n}(x,y)r_0(y)dy, n\ge 0 \tag{5.3}$$

$$\frac{d}{dx}P_{i,j_i,n}(x) = -(\lambda^+ + \lambda^- + \mu_{i,j_i}(x))\,P_{i,j_i,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n}C_k P_{i,j_i,n-k}(x) + \int_0^\infty F_{i,j_i,n}(x,y)r_{i,j_i}(y)dy, n\ge 0,\ i=1,2,....,M,\ j_i=1,2,....,k_i \tag{5.4}$$

$$\frac{d}{dx}R_{0,n}(x) = -(\lambda^+ + \beta_0(x))\,R_{0,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n}C_k\,R_{0,n-k}(x),\ n\ge 0 \tag{5.5}$$

$$\frac{d}{dx}R_{i,j_i,n}(x) = -(\lambda^+ + \beta_{i,j_i}(x))\,R_{i,j_i,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n}C_k\,R_{i,j_i,n-k}(x),\ n\ge 0,\ i=1,2,....,M,\ j_i=1,2,....,k_i \tag{5.6}$$

$$\frac{d}{dx} D_{0,n}(x,y) = -(\lambda^{+} + \gamma_0(y))\, D_{0,n}(x,y) + \lambda^{+}(1-\delta_{0n})\sum_{k=1}^{n} C_k\, D_{0,n-k}(x,y)\ , n \geq 0 \tag{5.7}$$

$$\frac{d}{dx} D_{i,j_i,n}(x,y) = -(\lambda^{+} + \gamma_{i,j_i}(y))\, D_{i,j_i,n}(x,y) + \lambda^{+}(1-\delta_{0n})\sum_{k=1}^{n} C_k\, D_{i,j_i,n-k}(x,y)\ , n \geq 0, \quad i = 1,2,\ldots,M,\ j_i = 1,2,\ldots,k_i \tag{5.8}$$

$$\frac{d}{dx} F_{0,n}(x,y) = -(\lambda^{+} + r_0(y))\, F_{0,n}(x,y) + \lambda^{+}(1-\delta_{0n})\sum_{k=1}^{n} C_k\, F_{0,n-k}(x,y)\ , n \geq 0 \tag{5.9}$$

$$\frac{d}{dx} F_{i,j_i,n}(x,y) = -(\lambda^{+} + r_{i,j_i}(y))\, F_{i,j_i,n}(x,y) + \lambda^{+}(1-\delta_{0n})\sum_{k=1}^{n} C_k\, F_{i,j_i,n-k}(x,y)\ , n \geq 0, \quad i = 1,2,\ldots,M,\ j_i = 1,2,\ldots,k_i \tag{5.10}$$

with boundary conditions

$$I_n(0) = (1-\theta)\,[\,q_0 \int_0^{\infty} P_{0,n}(x)\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i} \int_0^{\infty} P_{i,j_i,n}(x)\mu_{i,j_i}(x)dx + \int_0^{\infty} R_{0,n}(x)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} \int_0^{\infty} R_{i,j_i,0}(x)\beta_{i,j_i}(x)dx\,],\ \ n \geq 0 \tag{5.11}$$

$$P_{0,0}(0) = \lambda^{+} C_1 I_0 + \int_0^{\infty} I_1(x)\,\eta(x)\,dx + \theta\,[\,q_0 \int_0^{\infty} P_{0,1}(x)\,\mu_0(x)\,dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i} \int_0^{\infty} P_{i,j_i,1}(x)\,\mu_{i,j_i}(x)\,dx + \int_0^{\infty} R_{0,1}(x)\,\beta_0(x)\,dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} \int_0^{\infty} R_{i,j_i,1}(x)\,\beta_{i,j_i}(x)\,dx\,] \tag{5.12}$$

$$P_{0,n}(0) = \lambda^{+} C_{n+1} I_0 + \lambda^{+} C_k \int_0^{\infty} I_{n-k+1}(x)dx + \int_0^{\infty} I_{n+1}(x)\eta(x)dx + \theta[q_0 \int_0^{\infty} P_{0,n+1}(x)\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i} \int_0^{\infty} P_{i,j_i,n+1}(x)\mu_{i,j_i}(x)dx + \int_0^{\infty} R_{0,n+1}(x,y)\beta_0(y)dy + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} \int_0^{\infty} R_{i,j_i,n+1}(x,y)\beta_{i,j_i}(y)dy\,], n \geq 1 \tag{5.13}$$

$$P_{1,j_1,n}(0) = p_{j_1} \int_0^{\infty} P_{0,n}(x)\mu_0(x)dx,\ n \geq 0,\quad j_1 = 1,2,\ldots,k_1 \tag{5.14}$$

$$P_{i,j_i,n}(0) = p_{j_i} \sum_{j_{i-1}=1}^{k_{i-1}} \int_0^{\infty} P_{i-1,j_{i-1},n}(x)\mu_{i-1,j_{i-1}}(x)dx,\ \ n \geq 0,\ i = 2,3,\ldots,M,\ \ j_i = 1,2,\ldots,k_i \tag{5.15}$$

$$R_{0,n}(0) = \lambda^{-} \int_0^{\infty} P_{0,n}(x)dx\ ,\quad n \geq 0 \tag{5.16}$$

$$R_{i,j_i,n}(0) = \lambda^{-} \int_0^{\infty} P_{i,j_i,n}(x)dx\ ,\ \ n \geq 0,\ \ i = 1,2,3,\ldots,M,\ \ j_i = 1,2,\ldots,k_i \tag{5.17}$$

$$D_{0,n}(x,0) = \tau_0\ P_{0,n}(x)\ ,\quad n \geq 0 \tag{5.18}$$

$$D_{i,j_i,n}(x,0) = \tau_i\ P_{i,j_i,n}(x)\ ,\quad n \geq 0,\ \ i = 1,2,3,\ldots,M,\ \ j_i = 1,2,\ldots,k_i \tag{5.19}$$

$$F_{0,n}(x,0) = \int_0^\infty D_{0,n}(x,y)\,\gamma_0(y)\,dy\,, \quad n \ge 0 \tag{5.20}$$

$$F_{i,j_i,n}(x,0) = \int_0^\infty D_{i,j_i,n}(x,y)\,\gamma_{i,j_i}(y)\,dy\,, \quad n \ge 0,\;\; i = 1,2,3,\ldots\ldots,M,\;\; j_i = 1,2,\ldots.,k_i \tag{5.21}$$

5.4 Generating Functions of the Orbit Length

The probability generating functions for $|z| \le 1$ are defined as

$$\left.\begin{aligned}
&I(x,z) = \sum_{n=1}^\infty I_n(x)z^n \;; \quad P_0(x,z) = \sum_{n=0}^\infty P_{0,n}(x)z^n \;; \quad P_{i,j_i}(x,z) = \sum_{n=0}^\infty P_{i,j_i,n}(x)z^n \\
&R_0(x,z) = \sum_{n=0}^\infty R_{0,n}(x)z^n \;; \quad R_{i,j_i}(x,z) = \sum_{n=0}^\infty R_{i,j_i,n}(x)z^n \;; \quad D_0(x,y,z) = \sum_{n=0}^\infty D_{0,n}(x,y)z^n \\
&D_{i,j_i}(x,y,z) = \sum_{n=0}^\infty D_{i,j_i,n}(x,y)z^n \;; F_0(x,y,z) = \sum_{n=0}^\infty F_{0,n}(x,y)z^n ; \quad F_{i,j_i}(x,y,z) = \sum_{n=0}^\infty F_{i,j_i,n}(x,y)z^n
\end{aligned}\right\} \tag{5.22}$$

Theorem 5.1

The probability generating functions of the orbit size when the system is in different states are

$$I(z) \;=\; I_0(1 - A^*(\lambda^+))[\bar{\theta}\,C(z)T_1(z) + \theta\,T_1(z) - z]\big/D(z) \tag{5.23}$$

$$P_0(z) \;=\; \lambda^+ I_0 A^*(\lambda^+)(C(z)-1)((1 - B_0^*(\omega_0(z)))/\omega_0(z))\big/D(z) \tag{5.24}$$

$$\begin{aligned} P(z) \;=\; &\lambda^+ I_0 A^*(\lambda^+)(C(z)-1)B_0^*(\omega_0(z)) \\ &\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*((1 - B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z)) \;/\; D(z) \end{aligned} \tag{5.25}$$

$$R_0(z) \;=\; -\lambda^- I_0 A^*(\lambda^+)((1 - B_0^*(\omega_0(z)))/\omega_0(z))(1 - R_0^*(h(z)))\big/D(z) \tag{5.26}$$

$$\begin{aligned} R(z) \;=\; &-\lambda^- I_0 A^*(\lambda^+)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\omega_0(z))\Lambda_{i-1}^*((1 - B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z)) \\ &(1 - R_{i,j_i}^*(h(z))) \;/\; D(z) \end{aligned} \tag{5.27}$$

$$D_0(z) \;=\; -\tau_0\, I_0 A^*(\lambda^+)((1 - B_0^*(\omega_0(z)))/\omega_0(z))(1 - D_0^*(h(z)))\big/D(z) \tag{5.28}$$

$$\begin{aligned} D_s(z) \;=\; &-\sum_{i=1}^{M}\tau_i\, I_0 A^*(\lambda^+)\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\omega_0(z))\Lambda_{i-1}^*((1 - B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z)) \\ &(1 - D_{i,j_i}^*(h(z))) \;/\; D(z) \end{aligned} \tag{5.29}$$

$$F_0(z) \;=\; -\tau_0\, I_0 A^*(\lambda^+)((1 - B_0^*(\omega_0(z)))/\omega_0(z))D_0^*(h(z))(1 - F_0^*(h(z)))\big/D(z) \tag{5.30}$$

$$F(z) = -\sum_{i=1}^{M}\tau_i I_0 A^*(\lambda^+)\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\omega_0(z))\Lambda_{i-1}^*((1-B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z))$$
$$D_{i,j_i}^*(h(z))(1-F_{i,j_i}^*(h(z)))\,/\,D(z) \tag{5.31}$$

where

$$D(z) = z-(1-\theta)[A^*(\lambda^+)+C(z)(1-A^*(\lambda^+))]T_1(z)-\theta T_1(z)$$

$$T_1(z) = q_0\, B_0^*(\omega_0(z))+q_i\,\Lambda_i^*(\omega_{i,j_i}(z))B_0^*(\omega_0(z))+\lambda^-((1-B_0^*(\omega_0(z)))/\omega_0(z))R_0^*(h(z))$$
$$+\lambda^-\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i\text{-}1}^*(\omega_{i-1,j_{i-1}}(z))((1-B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z))R_{i,j_i}^*(h(z))B_0^*(\omega_0(z))$$

$$\omega_0(z) = \lambda^+ + \lambda^- - \lambda^+ C(z)+\tau_0[1-D_0^*(h(z))F_0^*(h(z))]$$

$$\omega_{i,j_i}(z) = \lambda^+ + \lambda^- - \lambda^+ C(z)+\tau_i[1-D_{i,j_i}^*(h(z))F_{i,j_i}^*(h(z))]$$

$$h(z) = \lambda^+(1-C(z))$$

Proof.

Multiplying equations (5.2) to (5.10) by z^n and summing over n, we get

$$\left(\frac{d}{dx}+\lambda^+ +\eta(x)\right) I(x,z)=0 \tag{5.32}$$

$$\left(\frac{d}{dx}+\lambda^+(1-C(z))+\lambda^- +\tau_0+\mu_0(x)\right) P_0(x,z)=\int_0^\infty F_0(x,y,z)\, r_0(y)dy \tag{5.33}$$

$$\left(\frac{d}{dx}+\lambda^+(1-C(z))+\lambda^- +\tau_{i,j_i}+\mu_{i,j_i}(x)\right) P_{i,j_i}(x,z)=\int_0^\infty F_{i,j_i}(x,y,z)\, r_{i,j_i}(y)dy, \tag{5.34}$$
$$i=1,2,\ldots,M,\ j_i=1,2,\ldots,k_i$$

$$\left(\frac{d}{dx}+\lambda^+(1-C(z))+\beta_0(x)\right) R_0(x,z)=0 \tag{5.35}$$

$$\left(\frac{d}{dx}+\lambda^+(1-C(z))+\beta_{i,j_i}(x)\right)R_{i,j_i}(x,z)=0\,,\quad i=1,2,\ldots,M,\quad j_i=1,2,\ldots,k_i \tag{5.36}$$

$$\left(\frac{d}{dy}+\lambda^+(1-C(z))+\gamma_0(y)\right)D_0(x,y,z)=0 \tag{5.37}$$

$$\left(\frac{d}{dy}+\lambda^+(1-C(z))+\gamma_{i,j_i}(y)\right)D_{i,j_i}(x,y,z)=0\,,\quad i=1,2,\ldots,M,\quad j_i=1,2,\ldots,k_i \tag{5.38}$$

$$\left(\frac{d}{dy}+\lambda^+(1-C(z))+r_0(y)\right)F_0(x,y,z)=0 \tag{5.39}$$

$$\left(\frac{d}{dy}+\lambda^{+}(1-C(z))+r_{i,j_i}(y)\right)F_{i,j_i}(x,y,z)=0\,,\quad i=1,2,\ldots,M,\quad j_i=1,2,\ldots,k_i \qquad (5.40)$$

Solving the partial differential equations (5.32) and (5.35) to (5.40), we obtain

$$I(x,z)=I(0,z)\,e^{-\lambda^{+}x}(1-A(x)) \qquad (5.41)$$

$$R_0(x,z)=R_0(0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))x}(1-R_0(x)) \qquad (5.42)$$

$$R_{i,j_i}(x,z)=R_{i,j_i}(0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))x}(1-R_{i,j_i}(x)),\; i=1,2,\ldots,M,\; j_i=1,2,\ldots,k_i \qquad (5.43)$$

$$D_0(x,y,z)=D_0(x,0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))y}(1-D_0(y)) \qquad (5.44)$$

$$D_{i,j_i}(x,y,z)=D_{i,j_i}(x,0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))y}(1-D_{i,j_i}(y)),\quad i=1,2,\ldots,M,\quad j_i=1,2,\ldots,k_i \qquad (5.45)$$

$$F_0(x,y,z)=F_0(x,0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))y}(1-F_0(y)) \qquad (5.46)$$

$$F_{i,j_i}(x,y,z)=F_{i,j_i}(x,0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))y}(1-F_{i,j_i}(y)),\; i=1,2,\ldots,M,\; j_i=1,2,\ldots,k_i \qquad (5.47)$$

Multiplying equations (5.11) to (5.21) by z^n and summing over n, we get

$$I(0,z)=(1-\theta)[q_0\int_0^\infty P_0(x,z)\,\mu_0(x)dx+\sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i}(x,z)\,\mu_{i,j_i}(x)dx$$
$$+\int_0^\infty R_0(x,z)\beta_0(x)dx+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i}(x,z)\beta_{i,j_i}(x)dx]-\lambda^{+}I_0 \qquad (5.48)$$

$$P_0(0,z)=\frac{1}{z}[\lambda^{+}C(z)I_0+\int_0^\infty I(x,z)\eta(x)dx+\lambda^{+}C(z)\int_0^\infty I(x,z)dx]$$
$$+\theta[q_0\int_0^\infty P_0(x,z)\mu_0(x)dx+\sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i}(x,z)\,\mu_{i,j_i}(x)dx$$
$$+\int_0^\infty R_0(x,z)\beta_0(x)dx+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i}(x,z)\beta_{i,j_i}(x)dx] \qquad (5.49)$$

$$P_{1,j_1}(0,z)=p_{j_1}\int_0^\infty P_0(x,z)\mu_0(x)dx,\quad j_1=1,2,\ldots,k_1 \qquad (5.50)$$

$$P_{i,j_i}(0,z)=p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^\infty P_{i-1,j_{i-1}}(x,z)\mu_{i-1,j_{i-1}}(x)dx,\; i=2,3,\ldots,M,\; j_i=1,2,\ldots,k_i \qquad (5.51)$$

$$R_0(0,z)=\lambda^{-}\int_0^\infty P_0(x,z)dx \qquad (5.52)$$

$$R_{i,j_i}(0,z)=\lambda^{-}\int_0^\infty P_{i,j_i}(x,z)dx,\qquad i=1,2,3,\ldots,M,\quad j_i=1,2,\ldots,k_i \qquad (5.53)$$

$$D_0(x,0,z) = \tau_0 \; P_0(x,z) \tag{5.54}$$

$$D_{i,j_i}(x,0,z) = \tau_i \; P_{i,j_i}(x,z) \,, \qquad i = 1,2,3,\ldots\ldots,M, \;\; j_i = 1,2,\ldots.,k_i \tag{5.55}$$

$$F_0(x,0,z) = \int_0^\infty D_0(x,y,z) \, \gamma_0(y) \, dy \tag{5.56}$$

$$F_{i,j_i}(x,0,z) = \int_0^\infty D_{i,j_i}(x,y,z) \, \gamma_{i,j_i}(y) \, dy \,, \;\; i = 1,2,3,\ldots\ldots,M, \;\; j_i = 1,2,\ldots.,k_i \tag{5.57}$$

Using equations (5.44), (5.45), (5.54) and (5.55), the equations (5.56) and (5.57) yield

$$\begin{aligned} F_0(x,0,z) &= \int_0^\infty D_0(x,0,z) e^{-(\lambda^+ - \lambda^+ C(z))y} (1 - D_0(y)) \; \gamma_0(y) \, dy \\ &= D_0(x,0,z) \; D_0^*(\lambda^+ - \lambda^+ C(z)) \\ &= D_0^*(\lambda^+ - \lambda^+ C(z)) \, \tau_0 \; P_0(x,z) \end{aligned} \tag{5.58}$$

$$\begin{aligned} F_{i,j_i}(x,0,z) &= \int_0^\infty D_{i,j_i}(x,0,z) e^{-(\lambda^+ - \lambda^+ C(z))y} (1 - D_{i,j_i}(y)) \; \gamma_{i,j_i}(y) \, dy \,, \\ &\qquad\qquad i = 1,2,3,\ldots\ldots,M, \;\; j_i = 1,2,\ldots.,k_i \\ &= D_{i,j_i}(x,0,z) \; D_{i,j_i}^*(\lambda^+ - \lambda^+ C(z)) \\ &= D_{i,j_i}^*(\lambda^+ - \lambda^+ C(z)) \, \tau_i \; P_{i,j_i}(x,z) \end{aligned} \tag{5.59}$$

Substituting the expressions (5.58) and (5.59) in equations (5.33) and (5.34) and solving, we get

$$P_0(x,z) = P_0(0,z) \, e^{-\omega_0(z)x} (1 - B_0(x)) \tag{5.60}$$

$$P_{i,j_i}(x,z) = P_{i,j_i}(0,z) e^{-\omega_{i,j_i}(z)x} (1 - B_{i,j_i}(x)), \; i = 1,2,\ldots\ldots,M, \;\; j_i = 1,2,\ldots..,k_i \tag{5.61}$$

Using equations (5.42), (5.43), (5.60) and (5.61), the equation (5.48) gives

$$\begin{aligned} I(0,z) = (1-\theta)[&q_0 \int_0^\infty P_0(0,z) e^{-\omega_0(z)x} (1 - B_0(x)) \, \mu_0(x) dx \\ &+ \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i} \int_0^\infty P_{i,j_i}(0,z) e^{-\omega_{i,j_i}(z)x} (1 - B_{i,j_i}(x)) \, \mu_{i,j_i}(x) dx \\ &+ \int_0^\infty R_0(0,z) e^{-(\lambda^+ - \lambda^+ C(z))x} (1 - R_0(x)) \; \beta_0(x) \, dx \\ &+ \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} \int_0^\infty R_{i,j_i}(0,z) e^{-(\lambda^+ - \lambda^+ C(z))x} (1 - R_{i,j_i}(x)) \beta_{i,j_i}(x) \, dx \,] - \lambda^+ I_0 \end{aligned}$$

$$= (1-\theta)\,[\,q_0\,P_0(0,z)B_0^*(\omega_0(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} q_i\,P_{i,j_i}(0,z)B_{i,j_i}^*(\omega_{i,j_i}(z))$$

$$+R_0(0,z)R_0^*(h(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(0,z)R_{i,j_i}^*(h(z))\,] - \lambda^+ I_0 \qquad (5.62)$$

Similarly, by using equations (5.41) to (5.43), (5.60) and (5.61) and solving, equations (5.49) to (5.53) yield

$$P_0(0,z) = \frac{1}{z}[\,\lambda^+ C(z) I_0 + \left(A^*(\lambda^+) + C(z)(1-A^*(\lambda^+))\right) I(0,z)]$$

$$+\,\theta\,[\,q_0\,P_0(0,z)B_0^*(\omega_0(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} q_i\,P_{i,j_i}(0,z)B_{i,j_i}^*(\omega_{i,j_i}(z))$$

$$+R_0(0,z)R_0^*(h(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(0,z)R_{i,j_i}^*(h(z))\,] \qquad (5.63)$$

$$P_{1,j_1}(0,z) = p_{j_1}\,P_0(0,z)\,B_0^*(\omega_0(z)), \qquad j_1 = 1,2,\ldots\ldots,k_1 \qquad (5.64)$$

$$P_{i,j_i}(0,z) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}} P_{i-1,j_{i-1}}(0,z)B_{i-1,j_{i-1}}^*(\omega_{i-1,j_{i-1}}(z)),\ \ i = 2,3,\ldots\ldots,M,\ j_i = 1,2,\ldots\ldots,k_i \qquad (5.65)$$

$$R_0(0,z) = \lambda^-\,P_0(0,z)\,(1-B_0^*(\omega_0(z)))\big/\omega_0(z) \qquad (5.66)$$

$$R_{i,j_i}(0,z) = \lambda^-\,P_{i,j_i}(0,z)\,(1-B_{i,j_i}^*(\omega_{i,j_i}(z)))\big/\omega_{i,j_i}(z),\ \ i = 1,2,3,\ldots\ldots,M,\ j_i = 1,2,\ldots\ldots,k_i \qquad (5.67)$$

Using the result in the equation (5.65) repeatedly we obtain

$$P_{i,j_i}(0,z) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}}\sum_{j_{i-2}=1}^{k_{i-2}} P_{i-2,j_{i-2}}(0,z)\,B_{i-2,j_{i-2}}^*(\omega_{i-2,j_{i-2}})B_{i-1,j_{i-1}}^*(\omega_{i-1,j_{i-1}})$$

$$= p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}}\sum_{j_{i-2}=1}^{k_{i-2}} p_{j_{i-2}}\sum_{j_{i-3}=1}^{k_{i-3}} P_{i-3,j_{i-3}}(0,z)\,B_{i-3,j_{i-3}}^*(\omega_{i-3,j_{i-3}}(z))$$
$$B_{i-2,j_{i-2}}^*(\omega_{i-2,j_{i-2}}(z))B_{i-1,j_{i-1}}^*(\omega_{i-1,j_{i-1}}(z))$$

$$= p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}}\sum_{j_{i-2}=1}^{k_{i-2}} p_{j_{i-2}}\cdots\sum_{j_2=1}^{k_2} p_{j_2}\sum_{j_1=1}^{k_1} p_{j_1}P_0(0,z)\,B_0^*(\omega_0(z))\,B_{1,j_1}^*(\omega_{1,j_1}(z))$$
$$B_{2,j_2}^*(\omega_{2,j_2}(z))\ldots\ldots B_{i-2,j_{i-2}}^*(\omega_{i-2,j_{i-2}}(z))B_{i-1,j_{i-1}}^*(\omega_{i-1,j_{i-1}}(z))$$

$$= p_{j_i}\left[\sum_{j_1=1}^{k_1} p_{j_1}\,B_{1,j_1}^*(\omega_{1,j_1}(z))\sum_{j_2=1}^{k_2} p_{j_2}\,B_{2,j_2}^*(\omega_{2,j_2}(z))\ldots\ldots\sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}}\,B_{i-1,j_{i-1}}^*(\omega_{i-1,j_{i-1}}(z))\right]$$
$$B_0^*(\omega_0(z))\,P_0(0,z)$$

$$= p_{j_i}\left[\prod_{l=1}^{i-1}\sum_{j_l=1}^{k_l} p_{j_l} B^*_{l,j_l}(\omega_{1,j_1}(z))\right] B^*_0(\omega_0(z))\ P_0(0,z)$$

$$= p_{j_i}\Lambda^*_{i-1} B^*_0(\omega_0(z))\ P_0(0,z)\ ,\ i=1,2,3,......,M,\ j_i=1,2,......,k_i \tag{5.68}$$

where

$$\Lambda^*_0=1\ ,\qquad \Lambda^*_i=\prod_{l=1}^{i}\sum_{j_l=1}^{k_l} p_{j_l} B^*_{l,j_l}(\omega_{l,j_j}(z))$$

Substituting the expression (5.68) in the equation (5.67), we obtain

$$R_{i,j_i}(0,z) = \lambda^- p_{j_i}\Lambda^*_{i-1} B^*_0(\omega_0(z))\ P_0(0,z)\ (1-B^*_{i,j_i}(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z),\quad i=1,2,3,......,M,\ j_i=1,2,......,k_i \tag{5.69}$$

Inserting the expressions (5.60) and (5.61) in equations (5.54) and (5.55), we get

$$D_0(x,0,z) = \tau_0\ P_0(0,z)\,e^{-\omega_0(z)x}(1-B^*_0(\omega_0(z))) \tag{5.70}$$

$$D_{i,j_i}(x,0,z) = \tau_i\ p_{j_i}\Lambda^*_{i-1}\,e^{-\omega_{i,j_i}(z)x}\,((1-B^*_{i,j_i}(\omega_{i,j_i}(z)))P_0(0,z)B^*_0(\omega_0(z)),\quad i=1,2,3,......,M,\ j_i=1,2,......,k_i \tag{5.71}$$

Using equations (5.70) and (5.71), equations (5.58) and (5.59) yield

$$F_0(x,0,z) = \tau_0\ P_0(0,z)\,e^{-\omega_0(z)x}(1-B^*_0(\omega_0(z)))\,D^*_0(h(z)) \tag{5.72}$$

$$F_{i,j_i}(x,0,z) = \tau_i\ p_{j_i}\Lambda^*_{i-1}\,e^{-\omega_{i,j_i}(z)x}\,((1-B^*_{i,j_i}(\omega_{i,j_i}(z)))P_0(0,z)B^*_0(\omega_0(z))D^*_{i,j_i}(h(z)),\quad i=1,2,3,......,M,\ j_i=1,2,......,k_i \tag{5.73}$$

Substituting the equations (5.66) to (5.68) in equation (5.62) and simplifying, we get

$$I(0,z) = (1-\theta)\ T_1(z)\ P_0(0,z)-\lambda^+ I_0 \tag{5.74}$$

Substituting the expressions (5.66), (5.68), (5.69) and (5.74) in equation (5.63) and simplifying, we have

$$P_0(0,z) = \frac{\lambda^+ I_0 A^*(\lambda^+)(C(z)-1)}{z-(1-\theta)[A^*(\lambda^+)+C(z)(1-A^*(\lambda^+))]\ T_1(z)-\theta T_1(z)} \tag{5.75}$$

Substituting the expression of $P_0(0,z)$ in equations (5.66), (5.68) to (5.74), we obtain

$$I(0,z) = \lambda^+ I_0 [\bar{\theta} C(z) T_1(z) + \theta T_1(z) - z] / D(z) \tag{5.76}$$

$$P_0(0,z) = \lambda^+ I_0 A^*(\lambda^+)(C(z)-1) / D(z) \tag{5.77}$$

$$P_{1,j_1}(0,z) = p_{j_1} \lambda^+ I_0 A^*(\lambda^+)(C(z)-1) B_0^*(\omega_0(z)) / D(z)\ , \quad j_1 = 1,2,\ldots\ldots,k_1 \tag{5.78}$$

$$P_{i,j_i}(0,z) = p_{j_i} \lambda^+ I_0 A^*(\lambda^+)(C(z)-1) \Lambda_{i-1}^* B_0^*(\omega_0(z)) / D(z),\ i = 2,3,\ldots\ldots,M,\ \ j_i = 1,2,\ldots\ldots,k_i \tag{5.79}$$

$$R_0(x,0,z) = \lambda^- \lambda^+ I_0 A^*(\lambda^+)(C(z)-1)((1 - B_0^*(\omega_0(z)))/\omega_0(z)) / D(z) \tag{5.80}$$

$$R_{i,j_i}(x,0,z) = p_{j_i} \lambda^- \lambda^+ I_0 A^*(\lambda^+)(C(z)-1) \Lambda_{i-1}^* ((1 - B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z)) B_0^*(\omega_0(z)) / D(z),$$
$$i = 1,2,3,\ldots\ldots,M,\ \ j_i = 1,2,\ldots\ldots,k_i \tag{5.81}$$

$$D_0(x,0,z) = \tau_0 \lambda^+ I_0 A^*(\lambda^+)(C(z)-1) e^{-\omega_0(z)x} (1 - B_0^*(\omega_0(z))) / D(z) \tag{5.82}$$

$$D_{i,j_i}(x,0,z) = \tau_i p_{j_i} \lambda^+ I_0 A^*(\lambda^+)(C(z)-1) \Lambda_{i-1}^* e^{-\omega_{i,j_i}(z)x} ((1 - B_{i,j_i}^*(\omega_{i,j_i}(z))) B_0^*(\omega_0(z)) / D(z),$$
$$i = 1,2,3,\ldots\ldots,M,\ \ j_i = 1,2,\ldots\ldots,k_i \tag{5.83}$$

$$F_0(x,0,z) = \tau_0 \lambda^+ I_0 A^*(\lambda^+)(C(z)-1) e^{-\omega_0(z)x} (1 - B_0^*(\omega_0(z))) D_0^*(h(z)) / D(z) \tag{5.84}$$

$$F_{i,j_i}(x,0,z) = \tau_i p_{j_i} \lambda^+ I_0 A^*(\lambda^+)(C(z)-1) \Lambda_{i-1}^* e^{-\omega_{i,j_i}(z)} ((1 - B_{i,j_i}^*(\omega_{i,j_i}(z))) B_0^*(\omega_0(z)) D_{i,j_i}^*(h(z)) / D(z),$$
$$i = 1,2,3,\ldots\ldots,M,\ \ j_i = 1,2,\ldots\ldots,k_i \tag{5.85}$$

Integrating the equations (5.41), (5.42), (5.43), (5.60) and (5.61) with respect to x, the equations (5.44) to (5.47) with respect to both x and y and substituting the expressions in equations (5.76) to (5.85), we can obtain the probability generating functions of the server states as in equations (5.23) to (5.31).

I_0 can be obtained using normalising condition as

$$I_0 = \frac{1 - \bar{\theta} m_1 (1 - A^*(\lambda^+)) - T_1'(1)}{A^*(\lambda^+)\{1 - q_0 c_0^{(1)} - \sum_{i=1}^{M} q_i [M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^* c_0^{(1)}] - h_1 + c_0^{(1)}\}} \tag{5.86}$$

where

$$T_1'(1) = q_0 c_0^{(1)} + \sum_{i=1}^{M} q_i [M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) c_0^{(1)}] + \lambda^+ m_1 (1 - B_0^*(\lambda^-)) \beta_0^{(1)} - c_0^{(1)}$$
$$+ (\lambda^+ m_1/\lambda^-)(1 + \tau_0 (r_0^{(1)} + \gamma_0^{(1)}))\,(1 - B_0^*(\lambda^-)) + \lambda^+ m_1 B_0^*(\lambda^-) \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)$$
$$(1 - B_{i,j_i}^*(\lambda^-)) \beta_{i,j_i}^{(1)} + h_1 + (\lambda^+ m_1/\lambda^-) B_0^*(\lambda^-) h_2$$

$$h_1 = \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} [M_{i-1}^{(1)} B_0^*(\lambda^-) + c_0^{(1)} \Lambda_{i-1}^*(\lambda^-)](1 - B_{i,j_i}^*(\lambda^-)) - B_0^*(\lambda^-) \sum_{i=1}^{M} p_{j_i} \Lambda_{i-1}^*(\lambda^-)\, c_{i,j_i}^{(1)}$$

$$h_2 = \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-))(1 + \tau_i (r_{i,j_i}^{(1)} + \gamma_{i,j_i}^{(1)}))$$

$$c_0^{(1)} = \lambda^+ m_1 (1 + \tau_0 (r_0^{(1)} + \gamma_0^{(1)})) \int_0^{\infty} x\, e^{-\lambda^- x}\, b_0(x) dx$$

$$c_{i,j_i}^{(1)} = \lambda^+ m_1 \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} (1 + \tau_i (r_{i,j_i}^{(1)} + \gamma_{i,j_i}^{(1)})) \int_0^{\infty} x\, e^{-\lambda^- x}\, b_{i,j_i}(x) dx$$

$$M_i^{(1)} = \lim_{z \to 1} \Lambda_i^{*\prime}, \qquad M_i^{(2)} = \lim_{z \to 1} \Lambda_i^{*\prime\prime},$$

Corollary 5.1

The probability generating functions of the orbit size and system size are

$$P_q(z) = I_0 A^*(\lambda^+)\{z - q_0 B_0^*(\omega_0(z)) - \sum_{i=1}^{M} q_i \Lambda_i^* B_0^*(\omega_0(z)) + (\lambda^+ (C(z) - 1) - \lambda^-)$$
$$[(1 - B_0^*(\omega_0(z)))/\omega_0(z) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^* B_0^*(\omega_0(z))(1 - B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z)]$$
$$- \tau_0 (1 - B_0^*(\omega_0(z)))/\omega_0(z)[1 - D_0^*(h(z)) F_0^*(h(z))] - \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^* B_0^*(\omega_0(z))$$
$$(1 - B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z)[1 - D_{i,j_i}^*(h(z)) F_{i,j_i}^*(h(z))]\}/D(z) \qquad (5.87)$$

$$P_s(z) = I_0 A^*(\lambda^+)\{z - q_0 B_0^*(\omega_0(z)) - \sum_{i=1}^{M} q_i \Lambda_i^* B_0^*(\omega_0(z)) + (z\lambda^+ (C(z) - 1) - \lambda^-)$$
$$[(1 - B_0^*(\omega_0(z)))/\omega_0(z) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^* B_0^*(\omega_0(z))(1 - B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z)]$$
$$- z\tau_0 (1 - B_0^*(\omega_0(z)))/\omega_0(z)[1 - D_0^*(h(z)) F_0^*(h(z))] - z \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^* B_0^*(\omega_0(z))$$
$$(1 - B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z)[1 - D_{i,j_i}^*(h(z)) F_{i,j_i}^*(h(z))]\}/D(z) \qquad (5.88)$$

Proof:

The probability generating function of the number of customers in the orbit is

$$\begin{aligned} P_q(z) \;=\; & I_0 + I(z) + P_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) + D_0(z) \\ & + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} D_{i,j_i}(z) + F_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} F_{i,j_i}(z) \end{aligned} \tag{5.89}$$

The probability generating function of the number of customers in the system is

$$\begin{aligned} P_s(z) \;=\; & I_0 + I(z) + z\left[P_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + D_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} D_{i,j_i}(z) + F_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} F_{i,j_i}(z)\right] \\ & + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) \end{aligned} \tag{5.90}$$

Substituting the expressions of equations (5.23) to (5.31) in the above equations (5.89) and (5.90), the probability generating functions of the orbit size and the system size as in (5.87) and (5.88) are obtained by direct calculation.

5.5 Stability Condition

The necessary and sufficient condition for the system to be stable is

$$\bar{\theta} m_1 (1 - A^*(\lambda^+)) + T_1'(1) \;<\; 1$$

5.6 Performance Measures

- The probability that the server is idle in the non empty system and the corresponding mean number of customers in the orbit are given by

$$\begin{aligned} I \;&=\; \lim_{z\to 1} I(z) \\ &= \frac{I_0 (1 - A^*(\lambda^+))\left(\bar{\theta} m_1 + T_1'(1) - 1\right)}{1 - \bar{\theta} m_1 (1 - A^*(\lambda^+)) - T_1'(1)} \end{aligned} \tag{5.91}$$

$$\begin{aligned} L_I &= \lim_{z\to 1} \frac{d}{dz} I(z) \\ &= I_0 (1 - A^*(\lambda^+))\left\{T_2\left[\bar{\theta}(m_2 + 2m_1 T_1'(1)) + T_1''(1)\right] - T_3\left[\bar{\theta} m_1 + T_1'(1) - 1\right]\right\} \Big/ 2T_2^2 \end{aligned} \tag{5.92}$$

- The probability that the server is busy in essential service and the corresponding mean number of customers in the orbit are given by

$$P_0 = \lim_{z\to1} P_0(z)$$

$$= \frac{I_0(\lambda^+ m_1/\lambda^-)A^*(\lambda^+)(1-B_0^*(\lambda^-))}{1-\bar{\theta}m_1(1-A^*(\lambda^+))-T_1'(1)} \quad (5.93)$$

$$L_{P_0} = \lim_{z\to1}\frac{d}{dz}P_0(z)$$

$$= I_0A^*(\lambda^+)\{T_2[(\lambda^+m_2/\lambda^-)(1-B_0^*(\lambda^-))-2(\lambda^+m_1/\lambda^-)c_0^{(1)}$$
$$+2(\lambda^+m_1/\lambda^-)^2(1+\tau_0(r_0^{(1)}+\gamma_0^{(1)}))(1-B_0^*(\lambda^-))]$$
$$+T_3[(\lambda^+m_1/\lambda^-)(1-B_0^*(\lambda^-))]\}/2T_2^2 \quad (5.94)$$

- The probability that the server is busy in optional services and the corresponding mean number of customers in the orbit are given by

$$P = \lim_{z\to1}\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z)$$

$$= \frac{I_0(\lambda^+m_1/\lambda^-)A^*(\lambda^+)B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))}{1-\bar{\theta}m_1(1-A^*(\lambda^+))-T_1'(1)} \quad (5.95)$$

$$L_P = \lim_{z\to1}\frac{d}{dz}P(z)$$

$$= I_0A^*(\lambda^+)\{T_2[(\lambda^+m_2/\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}B_0^*(\lambda^-)\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))+2(\lambda^+m_1/\lambda^-)h_1$$
$$+2(\lambda^+m_1/\lambda^-)^2B_0^*(\lambda^-)h_2]+T_3[(\lambda^+m_1/\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)]\}/2T_2^2$$

(5.96)

- The probability that the server is under repair due to negative arrival and the corresponding mean number of customers in the orbit are given by

$$R = \lim_{z\to1}[R_0(z)+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}R_{i,j_i}(z)]$$

$$= \frac{I_0\lambda^+m_1A^*(\lambda^+)[(1-B_0^*(\lambda^-))\beta_0^{(1)}+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}p_{j_i}B_0^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\Lambda_{i-1}^*(\lambda^-)\beta_{i,j_i}^{(1)}]}{1-\bar{\theta}m_1(1-A^*(\lambda^+))-T_1'(1)}$$

(5.97)

$$L_R = \lim_{z\to 1}\frac{d}{dz}\left(R_0(z)+R(z)\right)$$

$$= I_0 A^*(\lambda^+)\{T_2[(\lambda^+ m_1)^2((1-B_0^*(\lambda^-))\beta_0^{(2)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(2)})$$
$$\lambda^+ m_2((1-B_0^*(\lambda^-))\beta_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)})$$
$$+2\lambda^+ m_1[-c_0^{(1)}\beta_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))c_0^{(1)}\beta_{i,j_i}^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)}(1-B_{i,j_i}^*(\lambda^-))\beta_{i,j_i}^{(1)}$$
$$-\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)B_0^*(\lambda^-))c_{i,j_i}^{(1)}\beta_{i,j_i}^{(1)}] + 2((\lambda^+ m_1)^2/\lambda^-)T_6] + T_3[\lambda^+ m_1((1-B_0^*(\lambda^-))\beta_0^{(1)}$$
$$+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)\beta_{i,j_i}^{(1)})]\}/2T_2^2 \qquad (5.98)$$

- The probability that the server is in delay time and the corresponding mean number of customers in the orbit are given by

$$D = \lim_{z\to 1} D_0(z) + \lim_{z\to 1}\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} D_{i,j_i}(z)$$

$$= \frac{I_0\ (\lambda^+ m_1/\lambda^-)A^*(\lambda^+)\,[\tau_0(1-B_0^*(\lambda^-))\,\gamma_0^{(1)} + \sum_{i=1}^{M}\tau_i\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\Lambda_{i-1}^*(\lambda^-)\,\gamma_{i,j_i}^{(1)}]}{1-\bar{\theta} m_1(1-A^*(\lambda^+)) - T_1'(1)} \qquad (5.99)$$

$$L_D = \lim_{z\to 1}\frac{d}{dz}\left(D_0(z)+D_S(z)\right)$$

$$= I_0 A^*(\lambda^+)\{T_2[((\lambda^+ m_1)^2/\lambda^-)((1-B_0^*(\lambda^-))\gamma_0^{(2)} + \sum_{i=1}^{M}\tau_i\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))$$
$$\gamma_{i,j_i}^{(2)}) + (\lambda^+ m_2/\lambda^-)((1-B_0^*(\lambda^-))\gamma_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\gamma_{i,j_i}^{(1)})$$
$$+2(\lambda^+ m_1/\lambda^-)[-c_0^{(1)}\gamma_0^{(1)} + \sum_{i=1}^{M}\tau_i\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))c_0^{(1)}\gamma_{i,j_i}^{(1)} + \sum_{i=1}^{M}\tau_i\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)}$$
$$(1-B_{i,j_i}^*(\lambda^-))\gamma_{i,j_i}^{(1)} - \sum_{i=1}^{M}\tau_i\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)B_0^*(\lambda^-))c_{i,j_i}^{(1)}\gamma_{i,j_i}^{(1)}] + 2(\lambda^+ m_1/\lambda^-)^2 T_7]$$
$$+T_3[(\lambda^+ m_1/\lambda^-)((1-B_0^*(\lambda^-))\gamma_0^{(1)}\sum_{i=1}^{M}\tau_i\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))B_0^*(\lambda^-)\,\gamma_{i,j_i}^{(1)})]\}/2T_2^2 \qquad (5.100)$$

- The probability that the server is under repair due to active breakdown and the corresponding mean number of customers in the orbit are given by

$$F = \lim_{z\to 1} [F_0(z) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} F_{i,j_i}(z)]$$

$$= \frac{I_0 \; (\lambda^+ m_1/\lambda^-) A^*(\lambda^+)[\tau_0 (1-B_0^*(\lambda^-)) \; r_0^{(1)} + \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-)) \Lambda_{i-1}^*(\lambda^-) \; r_{i,j_i}^{(1)}]}{1-\bar{\theta} m_1 (1-A^*(\lambda^+)) - T_1'(1)} \quad (5.101)$$

$$L_F = \lim_{z\to 1} \frac{d}{dz}\left(F_0(z) + F(z)\right)$$

$$= I_0 A^*(\lambda^+)\{T_2[((\lambda^+ m_1)^2/\lambda^-)(\tau_0(1-B_0^*(\lambda^-))r_0^{(2)} + \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))$$

$$r_{i,j_i}^{(2)}) + (\lambda^+ m_2/\lambda^-)(\tau_0(1-B_0^*(\lambda^-))r_0^{(1)} + \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-)) \; r_{i,j_i}^{(1)})$$

$$+ 2(\lambda^+ m_1/\lambda^-)(\lambda^+ m_1 \tau_0 \gamma_0^{(1)} \; r_0^{(1)} - \tau_0 \; c_0^{(1)} \; r_0^{(1)} + \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-)) c_0^{(1)} \; r_{i,j_i}^{(1)}$$

$$+ \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)}(1-B_{i,j_i}^*(\lambda^-)) r_{i,j_i}^{(1)} - \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) c_{i,j_i}^{(1)} \; r_{i,j_i}^{(1)} + 2(\lambda^+ m_1/\lambda^-)^2 T_8$$

$$+ \lambda^+ m_1 \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-)) \gamma_{i,j_i}^{(1)} \; r_{i,j_i}^{(1)} + T_3[(\lambda^+ m_1/\lambda^-)((1-B_0^*(\lambda^-)) r_0^{(1)}$$

$$+ \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-)) B_0^*(\lambda^-) \; r_{i,j_i}^{(1)})]\}/2T_2^2 \quad (5.102)$$

- Average orbit size is given by

$$L_q = \lim_{z\to 1} \frac{d}{dz} P_q(z)$$

$$= I_0 A^*(\lambda^+)\{T_2[-q_0 c_0^{(2)} - \sum_{i=1}^{M} q_i [M_i^{(2)} B_0^*(\lambda^-) + 2M_i^{(1)} c_0^{(1)} + \Lambda_i^*(\lambda^-) c_0^{(2)}](1-B_{i,j_i}^*(\lambda^-))$$

$$+ \lambda^+ m_2 T_5 + c_0^{(2)} - \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(2)} B_0^*(\lambda^-) + 2M_{i-1}^{(1)} c_0^{(1)} + \Lambda_{i-1}^*(\lambda^-) c_0^{(2)})(1-B_{i,j_i}^*(\lambda^-))$$

$$+ \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) c_{i,j_i}^{(2)} + 2\sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} (\Lambda_{i-1}^*(\lambda^-) c_0^{(1)} + M_{i-1}^{(1)} B_0^*(\lambda^-)) c_{i,j_i}^{(1)}] + T_4 T_3\}/2T_2^2 \quad (5.103)$$

- Average system size is given by

$$L_s = \lim_{z\to 1} \frac{d}{dz} P_s(z)$$

$$= I_0 A^*(\lambda^+)\{T_2[-q_0 c_0^{(2)} - \sum_{i=1}^{M} q_i [M_i^{(2)} B_0^*(\lambda^-) + 2M_i^{(1)} c_0^{(1)} + \Lambda_i^*(\lambda^-) c_0^{(2)}] + (2\lambda^+ m_1$$

$$+ \lambda^+ m_2) T_5 + c_0^{(2)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) c_{i,j_i}^{(2)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(2)} B_0^*(\lambda^-)$$

$$+ 2M_{i-1}^{(1)} c_0^{(1)} + \Lambda_{i-1}^*(\lambda^-) c_0^{(2)})(1 - B_{i,j_i}^*(\lambda^-)) + 2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} (\Lambda_{i-1}^*(\lambda^-) c_0^{(1)}$$

$$+ M_{i-1}^{(1)} B_0^*(\lambda^-)) c_{i,j_i}^{(1)} + 2(\lambda^+ m_1 / \lambda^-)[\tau_0 (1 - B_0^*(\lambda^-))(\gamma_0^{(1)} + r_0^{(1)})$$

$$+ \sum_{i=1}^{M} \tau_i \sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \Lambda_{i-1}^*(\lambda^-) (\gamma_{i,j_i}^{(1)} + r_{i,j_i}^{(1)})] + T_4 T_3\} / 2T_2^2 \tag{5.104}$$

where

$$T_2 = 1 - \bar{\theta} m_1 (1 - A^*(\lambda^+)) - T_1'(1)$$

$$T_3 = -\left(T_1''(1) + \bar{\theta}(1 - A^*(\lambda^+))\left(m_2 + 2m_1 T_1'(1)\right)\right)$$

$$T_4 = 1 - q_0 c_0^{(1)} - \sum_{i=1}^{M} q_i [M_i^{(1)} B_0^*(\lambda^-) + \Lambda_i^*(\lambda^-) c_0^{(1)}] - h_1 + c_0^{(1)}$$

$$T_5 = (1/\lambda^-)\,[1 - B_0^*(\lambda^-) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) B_0^*(\lambda^-)]$$

$$T_6 = (1 - B_0^*(\lambda^-))(1 + \tau_0 (r_0^{(1)} + \gamma_0^{(1)}))\beta_0^{(1)}$$
$$+ \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} (1 + \tau_i (r_{i,j_i}^{(1)} + \gamma_{i,j_i}^{(1)}))(1 - B_{i,j_i}^*(\lambda^-))\ \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) \beta_{i,j_i}^{(1)}$$

$$T_7 = (1 - B_0^*(\lambda^-))(1 + \tau_0 (r_0^{(1)} + \gamma_0^{(1)}))\gamma_0^{(1)}$$
$$+ \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} (1 + \tau_i (r_{i,j_i}^{(1)} + \gamma_{i,j_i}^{(1)}))(1 - B_{i,j_i}^*(\lambda^-))\ \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) \gamma_{i,j_i}^{(1)}$$

$$T_8 = (1 - B_0^*(\lambda^-))(1 + \tau_0 (r_0^{(1)} + \gamma_0^{(1)}))\, r_0^{(1)}$$
$$+ \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} (1 + \tau_i (r_{i,j_i}^{(1)} + \gamma_{i,j_i}^{(1)}))(1 - B_{i,j_i}^*(\lambda^-))\ \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) r_{i,j_i}^{(1)}$$

$$c_0^{(2)} = (\lambda^+ m_1 (1 + \tau_0 (r_0^{(1)} + \gamma_0^{(1)})))^2 \int_0^\infty x^2 e^{-\lambda^- x} b_0(x) dx$$
$$- (\lambda^{+^2} m_2 (r_0^{(2)} + \gamma_0^{(2)}) + 2(\lambda^+ m_1)^2 r_0^{(1)} \gamma_0^{(1)}) \int_0^\infty x\, e^{-\lambda^- x} b_0(x) dx$$

$$c_{i,j_i}^{(2)} = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} \tau_i [(\lambda^+ m_1 (1+\tau_i (r_{i,j_i}^{(1)} + \gamma_{i,j_i}^{(1)})))^2 \int_0^\infty x^2 e^{-\lambda^- x} b_{i,j_i}(x)dx$$

$$-(\lambda^{+^2} m_2 (r_{i,j_i}^{(2)} + \gamma_{i,j_i}^{(2)}) + 2(\lambda^+ m_1)^2 r_{i,j_i}^{(1)} \gamma_{i,j_i}^{(1)}) \int_0^\infty x e^{-\lambda^- x} b_{i,j_i}(x)dx]$$

$$T_1''(1) = q_0 c_0^{(2)} + \sum_{i=1}^{M} q_i [M_i^{(2)} B_0^*(\lambda^-) + 2M_i^{(1)} c_0^{(1)} + \Lambda_i^*(\lambda^-) c_0^{(2)}] + (\lambda^+)^2 m_2 T_6 + 2\lambda^+ m_1 T_9$$

$$+(\lambda^+ m_1 / \lambda^-)[-2(1+\tau_0 (r_0^{(1)} + \gamma_0^{(1)})) c_0^{(1)} + 2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} (1+\tau_i (r_{i,j_i}^{(1)} + \gamma_{i,j_i}^{(1)}))(1 - B_{i,j_i}^*(\lambda^-))$$

$$(M_{i-1}^{(1)} B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-) c_0^{(1)}) - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-)(1+\tau_i (r_{i,j_i}^{(1)} + \gamma_{i,j_i}^{(1)})) c_{i,j_i}^{(1)}]$$

$$+2(\lambda^+ m_1 / \lambda^-)^2 T_{10} + 2((\lambda^+ m_1)^2 / \lambda^-)[T_6 + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(2)} B_0^*(\lambda^-) + 2M_{i-1}^{(1)} c_0^{(1)} + \Lambda_{i-1}^*(\lambda^-) c_0^{(2)})$$

$$(1 - B_{i,j_i}^*(\lambda^-))] - 2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(1)} B_0^*(\lambda^-) + \Lambda_{i-1}^*(\lambda^-) c_0^{(1)}) c_{i,j_i}^{(1)} - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) c_{i,j_i}^{(2)}$$

$$T_9 = -c_0^{(1)} \beta_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)} B_0^*(\lambda^-)(1 - B_{i,j_i}^*(\lambda^-)) \beta_{i,j_i}^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) c_0^{(1)}$$

$$(1 - B_{i,j_i}^*(\lambda^-)) \beta_{i,j_i}^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-) c_{i,j_i}^{(1)} \beta_{i,j_i}^{(1)}]$$

$$T_{10} = [(1 - B_0^*(\lambda^-))(1+\tau_0 (r_0^{(1)} + \gamma_0^{(1)}))^2 + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^-) B_0^*(\lambda^-)$$

$$(1+\tau_i (r_{i,j_i}^{(1)} + \gamma_{i,j_i}^{(1)}))^2 (1 - B_{i,j_i}^*(\lambda^-))]$$

5.7 Stochastic Decomposition

Theorem 5.2

Average system size can be expressed as the sum of the mean number of customers in the classical batch arrival multistage and multi-optional retrial G-queue with active breakdown, delayed repair and orbital search (L_ϕ) and the mean number of customers in the orbit when the server is idle (L_ψ).

$$L_s = L_\phi + L_\psi$$

Proof:

Let $\phi(z)$ be the probability generating function of the system size for classical batch arrival multistage and multi-optional G-queue with active breakdown, delayed repair and orbital search.

$$\phi(z) = \lim_{A^*(\lambda^+)\to 1} P_s(z)$$

$$= \frac{\begin{gathered}[1-T_1'(1)]\{z - q_0 B_0^*(\omega_0(z)) - \sum_{i=1}^{M} q_i \Lambda_i^* B_0^*(\omega_0(z)) + (z\lambda^+(C(z)-1) - \lambda^-) \\ [((1-B_0^*(\omega_0(z)))/\omega_0(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^* B_0^*(\omega_0(z))((1-B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z))] \\ - z\tau_0((1-B_0^*(\omega_0(z)))/\omega_0(z))[1-D_0^*(h(z))F_0^*(h(z))] - z\sum_{i=1}^{M}\tau_i\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^* B_0^*(\omega_0(z)) \\ (1-B_{i,j_i}^*(\omega_{i,j_i}(z)))/\omega_{i,j_i}(z)[1-D_{i,j_i}^*(h(z))F_{i,j_i}^*(h(z))]\}\end{gathered}}{[z-T_1(z)]T_4}$$

(5.105)

Let $\psi(z)$ be the probability generating function of the number of customers in the orbit when the server is idle.

$$\psi(z) = \frac{I_0 + I(z)}{I_0 + I(1)}$$

$$= \frac{T_2[z - T_1(z)]}{[1 - T_1'(1)]\, D(z)} \qquad (5.106)$$

By the equations (5.90), (5.105) and (5.106), it is easy to prove

$$P_s(z) = \phi(z)\ \psi(z) \qquad (5.107)$$

On differentiating the equation (5.107) and taking limit as $z \to 1$, we obtain the result $L_s = L_\phi + L_\psi$.

5.8 Reliability Indices

In this section, the reliability indices like availability of the server and failure frequency are derived in the following theorem.

Theorem 5.3

The steady state availability ($\mathcal{A}$) and failure frequency ($\mathcal{F}$) of the server are

$$\mathcal{A} = \frac{A^*(\lambda^+)\,[\,1 - T_1'(1) + \lambda^+ m_1 T_5\,]}{T_4} \qquad (5.108)$$

$$\mathcal{F} = \frac{\lambda^+ m_1 A^*(\lambda^+)\left[\begin{array}{c}(1-B_0^*(\lambda^-))\,(\lambda^- + \tau_0) + \lambda^- B_0^*(\lambda^-)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-)) \\ + B_0^*(\lambda^-)\sum_{i=1}^{M}\tau_i\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-)(1-B_{i,j_i}^*(\lambda^-))\end{array}\right]}{\lambda^- T_4}$$

(5.109)

Proof.

The availability of the server in the system is given by

$$\mathcal{A} = I_0 + I + P_0 + P$$

Result in equation (5.108) can be obtained by using equations (5.86), (5.91), (5.93) and (5.95).

The failure frequency of the server in the system is given by

$$\mathcal{F} = \lambda^- \left(P_0 + P\right) + \tau_0 P_0 + \sum_{i=1}^{M}\tau_i \sum_{j_i=1}^{k_i} P_{i,j_i}(1)$$

Result in equation (5.109) can be obtained by using equations (5.93) and (5.95).

5.9 Special Cases

Case (i) : If M = 0, $\lambda^- = 0$ and θ = 0, then the system reduces to $M^X/G/1$ retrial queue with breakdown and delayed repair. In this case,

$$P_q(z) = \frac{I_0 A^*(\lambda^+)\,(z-1)}{z - [A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))]B_0^*(\omega_0(z))}$$

$$I_0 = \frac{1 - m_1\left(1 - A^*(\lambda^+)\right) - \lambda^+ m_1 \mu_0^{(1)}\left(1 + \tau_0(r_0^{(1)} + \gamma_0^{(1)})\right)}{A^*(\lambda^+)}$$

where $\omega_0(z) = \lambda^+ - \lambda^+ C(z) + \tau_0[1 - D_0^*(h(z))F_0^*(h(z))]$

The above results agree with the results of Choudhury and Ke (2012) with no vacation.

Case (ii) : If $\lambda^- = 0$, $\theta = 0$ and $k_1 = k_2 = ... = k_M = 1$, then the model under study becomes multistage bulk arrival retrial queue with active breakdown and delayed repair. In this case,

$$P_q(z) = \frac{I_0 A^*(\lambda^+)(z-1)}{z - [A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))] \sum_{i=0}^{M} q_i \Lambda_i^* B_0^*(\omega_0(z))}$$

$$P_s(z) = \frac{I_0 A^*(\lambda^+)(z-1) \sum_{i=0}^{M} \overline{q_i} \Lambda_i^* B_0^*(\omega_0(z))}{z - [A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))] \sum_{i=0}^{M} q_i \Lambda_i^* B_0^*(\omega_0(z))}$$

$$I_0 = \frac{1 - m_1\left(1 - A^*(\lambda^+)\right) - \sum_{i=1}^{M} \overline{q_i} \Lambda_{i-1}^* M_i^{(1)} - \overline{q_0} \lambda^+ m_1 \mu_0^{(1)}\left(1 + \tau_0(r_0^{(1)} + \gamma_0^{(1)})\right)}{A^*(\lambda^+)}$$

where

$$\Lambda_i^* = \prod_{l=1}^{i} p_l B_l^*(\omega_l(z)), \qquad \Lambda_0^* = 1$$

$$\omega_0(z) = \lambda^+ - \lambda^+ C(z) + \tau_0[1 - D_0^*(h(z))F_0^*(h(z))]$$

$$\omega_i(z) = \lambda^+ - \lambda^+ C(z) + \tau_i[1 - D_i^*(h(z))F_i^*(h(z))]$$

$$h(z) = \lambda^+(1 - C(z))$$

$$M_i^{(1)} = \sum_{m=1}^{i} p_m \mu_m^{(1)}\left(1 + \tau_i(r_i^{(1)} + \gamma_i^{(1)})\right)$$

The above results coincide with the results of Bagyam and Udaya Chandrika (2018) with no reserved time.

5.10 Practical Justification of the Model

Internet Banking System is an innovative form of advanced technology with a series of set processes (Multistage Services) that the clients (Positive Customers) logs into the bank's website through the web-browser installed on the PC by using a private username and password based on the user's selection (Essential Service). The clients try for their request are enqueued (Orbit). The data input is encrypted by SSL (Secure Socket Layer) and transmitted to the bank's server.

Once the client logged in, all the information the bank has on file can be accessed such as applying for new lines of credit, opening new accounts, view transaction details or even applying for loans or mortgages can sometimes done right from home (Multistage and Multi-Optional Services).

The quality of the internet connection is a major ingredient for any we based applications. Poor internet connection creates a risk to banking transactions. It interrupts only the current service. As a result, the program which is processing at the moment moves to the memory and makes the server down (Active breakdown). When the system experiences failure due to poor internet connection, it has to undergo repair. The repair process begins after an initial delay. Once after the recognition of the high speed internet (repair), the interrupted program which is in the memory being processed.

On the other hand, unexpected software issues (Negative Customer) on the webpage cause a heavy damage by terminating all the services and make the server breakdown. The repair of the failed browser starts immediately. After the service or repair process completed, the server searches for the clients request in the queue (Orbital Search).

5.11 Numerical Results and Discussion

Numerical examples are presented to illustrate the sensitivity of various parameters on the system measures. It is assumed that the retrial time, first phase service time, second phase service time, repair time at first phase due to negative arrival and active breakdown and repair time at second phase due to negative arrival and active breakdown are exponentially distributed with respective parameters $\eta, \mu_0, \mu_{i,j_i}, \beta_0, r_0, \beta_{i,j_i}, r_{i,j_i}, \gamma_0, \gamma_{i,j_i}$ where i=1,2,...,M and j_i = 1,2,..., k_i.

The following arbitrary values are selected for the parameters in such a way that stability condition holds $\lambda^+ = 2$, $\lambda^- = 0.2$, $\eta = 50$, M=3, k_1=2, k_2=3, k_3=2, p_{j_1}=[0.4 0.3], p_{j_2}=[0.2 0.3 0.1], p_{j_3}=[0.4 0.2], q_0=0.3, q=[0.4 0.4 1], θ=0.5, μ_0=10, β_0=7, γ_0=3, r_0=2, μ_1=[20 15], μ_2=[12 22 24], μ_3=[16 18], β_1=[7 4], β_2=[5 7 10], β_3=[2 3], γ_1=[5 6], γ_2=[1 2 3], γ_3=[4 6], r_1=[2 1], r_2=[4 5 3], r_3=[2 4], τ_0=0.5, τ_1=0.4, τ_2=0.5, τ_3=0.7.

Table 5.2 to 5.5 give the computed values of various characteristics of the system I_0 – the probability that the server is idle in the empty system, I – the probability that the server is idle in the non-empty system , P_0 – the probability that the server is busy in first phase , P – the probability that the server is busy in second phase, R – the probability that the server is under repair due to negative arrival, D – the probability that the server is in delay time, F – the probability that the server is under repair due to active breakdown, $\mathcal{A}$ – availability of the server, $\mathcal{F}$ – failure frequency of the server and L_s – expected system size by varying the rates λ^-, τ_0, r_0, γ_0. From the tables it is observed that

- I_0 increases with increase in r_0 and γ_0 but decreases with increase in λ^- and τ_0.
- I increases with increase in λ^- and τ_0 but decreases with increase in r_0 and γ_0.
- P_0 and P increases with increase in τ_0 but decreases with increase in λ^-, r_0 and γ_0
- R increases with increase in λ^- but decreases with increase in γ_0 and is independent of r_0 and τ_0.
- D increases with increase in λ^- and τ_0 but decreases with increase in r_0 and γ_0.
- F increases with increase in τ_0 but decreases with increase in λ^-, r_0 and γ_0.
- $\mathcal{A}$ increases with increase in r_0 and γ_0 but decreases with increase in λ^-, τ_0.
- $\mathcal{F}$ increases with increase in λ^-, τ_0, r_0 and γ_0.
- L_s increases with increase in λ^- and τ_0 but decreases with increase in r_0 and γ_0.

The combined effect of η and μ_0 on the performance measures are displayed in Fig. 5.2 (a) to (f). From the figures it is found that

- Increase in η increases I_0 , decreases I and has no effect on P_0, R, D and F.
- Increase in μ_0 increases I_0 and P_0 and decreases the remaining measures.

The combined effect of θ and η on the performance measures are displayed in Fig. 5.3 (a) to (d). The figures show that

- Increase in θ increases I_0 and $\mathcal{A}$ and decreases I and L_s.
- Increase in η increases I_0 , decreases I and L_s and has no effect on $\mathcal{A}$.

Table 5.2 Performance measures by varying λ^-

λ^-	I_0	I	P_0	P	R	D	F	$\mathcal{A}$	$\mathcal{F}$	L_s
0.2	0.1692	0.0140	0.2090	0.3769	0.0198	0.0696	0.1415	0.7395	0.7457	16.4917
0.3	0.1630	0.0144	0.2090	0.3739	0.0295	0.0693	0.1409	0.7311	0.8360	17.3498
0.4	0.1570	0.0149	0.2090	0.3708	0.0391	0.0690	0.1402	0.7227	0.9219	18.8160
0.5	0.1512	0.0153	0.2089	0.3677	0.0486	0.0687	0.1396	0.7146	1.0035	21.4878
0.6	0.1456	0.0157	0.2088	0.3646	0.0580	0.0684	0.1389	0.7065	1.0812	27.1586

Table 5.3 Performance measures by varying τ_0

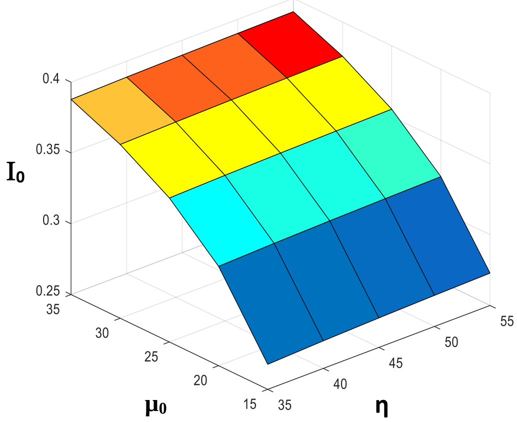
τ_0	I_0	I	P_0	P	R	D	F	$\mathcal{A}$	$\mathcal{F}$	L_s
0.2	0.2244	0.0118	0.2088	0.3765	0.0198	0.0486	0.1101	0.7899	0.6269	15.5155
0.4	0.1876	0.0133	0.2089	0.3768	0.0198	0.0626	0.1310	0.7563	0.7061	22.3952
0.6	0.1508	0.0147	0.2090	0.3771	0.0198	0.0766	0.1520	0.7227	0.7852	33.2369
0.8	0.1139	0.0161	0.2092	0.3773	0.0198	0.0906	0.1731	0.6890	0.8642	51.9011
1	0.0769	0.0176	0.2093	0.3776	0.0198	0.1046	0.1941	0.6552	0.9430	89.6774

Table 5.4 Performance measures by varying r_0

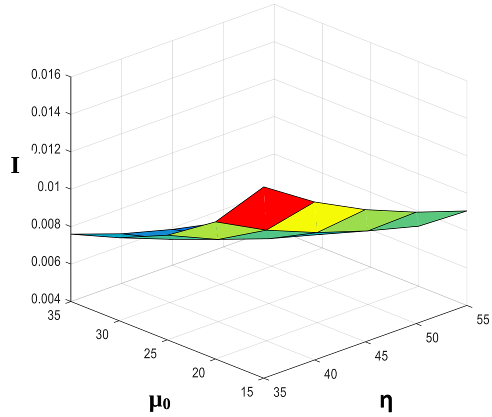
r_0	I_0	I	P_0	P	R	D	F	$\mathcal{A}$	$\mathcal{F}$	L_s
4	0.2574	0.0101	0.1936	0.3492	0.0183	0.0645	0.1069	0.7821	0.6545	13.0987
5	0.2625	0.0099	0.1936	0.3492	0.0183	0.0645	0.1021	0.7868	0.6546	12.4864
6	0.2659	0.0098	0.1936	0.3491	0.0183	0.0644	0.0988	0.7899	0.6546	12.0961
7	0.2683	0.0097	0.1935	0.3491	0.0183	0.0644	0.0965	0.7922	0.6546	11.8256
8	0.2701	0.0097	0.1935	0.3491	0.0183	0.0644	0.0948	0.7938	0.6547	11.6272

Table 5.5 Performance measures by varying γ_0

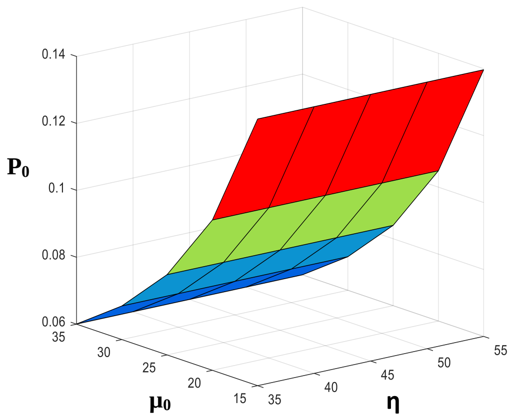
γ_0	I_0	I	P_0	P	R	D	F	$\mathcal{A}$	$\mathcal{F}$	L_s
2	0.2149	0.0116	0.1937	0.3495	0.0184	0.0807	0.1312	0.7430	0.6540	19.7494
4	0.2404	0.0107	0.1936	0.3493	0.0184	0.0564	0.1312	0.7665	0.6543	15.3936
6	0.2489	0.0104	0.1936	0.3493	0.0184	0.0483	0.1311	0.7743	0.6544	14.1911
8	0.2531	0.0103	0.1936	0.3492	0.0184	0.0443	0.1311	0.7782	0.6545	13.6296
10	0.2557	0.0102	0.1936	0.3492	0.0183	0.0419	0.1311	0.7806	0.6545	13.3046



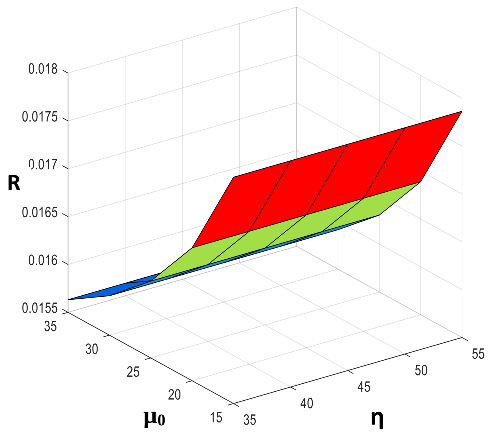
(a) I_0 versus (η , μ_0)



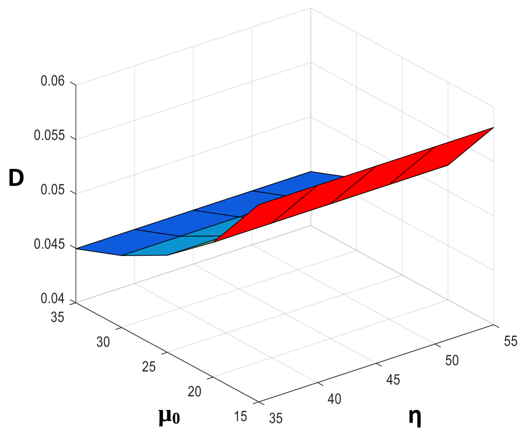
(b) I versus (η , μ_0)



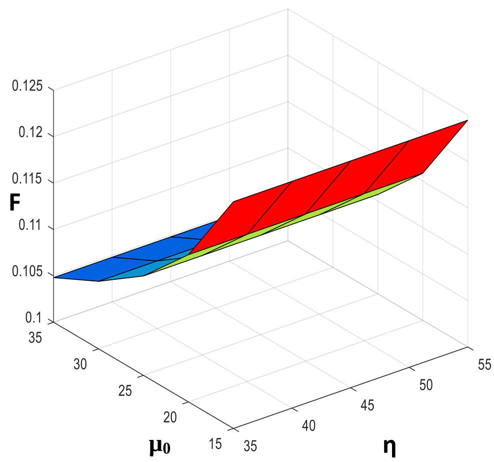
(c) P_0 versus (η , μ_0)



(d) R versus (η , μ_0)

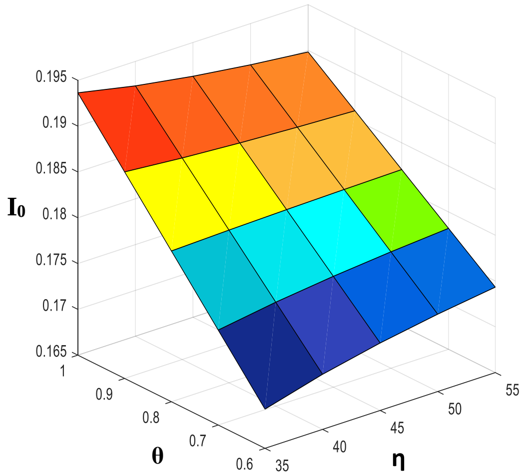


(e) D versus (η , μ_0)

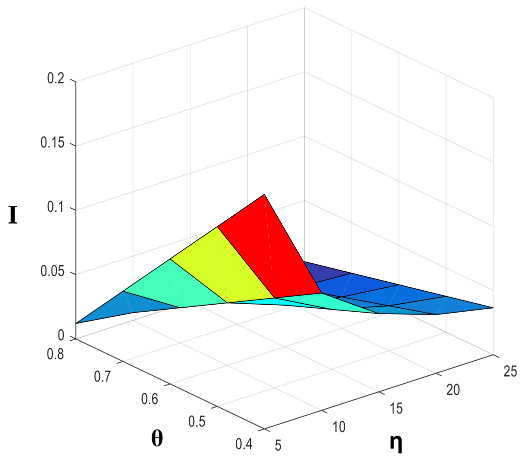


(f) F versus (η , μ_0)

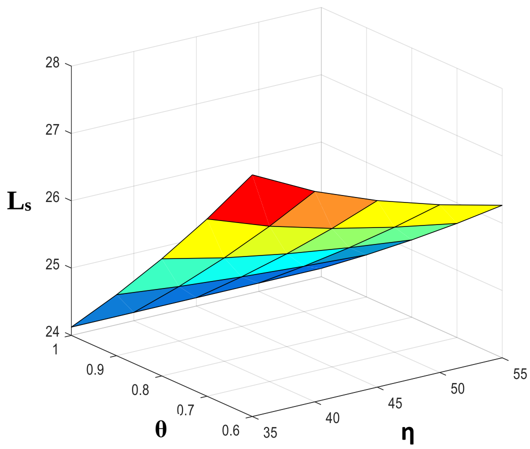
Fig. 5.2 Effect of (η , μ_0) on I_0 , I , P_0 , R , D and F



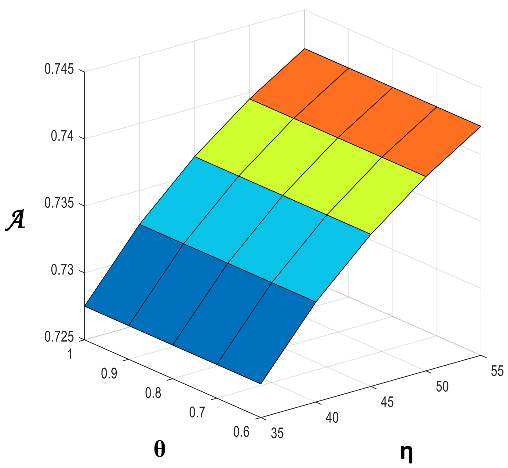
(a) I_0 versus (θ , η)



(b) I versus (θ , η)



(c) L_s versus (θ , η)



(d) $\mathcal{A}$ versus (θ , η)

Fig. 5.3 Effect of (θ , η) on I_0 , I , L_s and $\mathcal{A}$