Chapter 5

Some δP_S-Continuous Functions in Topological Spaces

5.1 Introduction

Functions are imp ortant tools for studying properties of spaces and for constructing new spaces from existing spaces. The classical concept of continuity is that of metric continuity. This led to the notion of topological continuity. General topologists have introduced and investigated many different generalizations of continuous functions. In the year 1970, Norman Levine initiated the idea of continuous functions.

In this chapter, various types of continuities using δP_S-open sets namely δP_S-continuity, quasi δP_S-continuity, perfectly δP_S-continuity, totally δP_S-continuity, strongly δP_S-continuity and contra δP_S-continuity are defined and their properties are discussed.

5.2 δP_S-Continuous Functions

In this section δP_S-continuous functions in topological spaces are introduced and some interesting properties are obtained.

Definition 5.2.1. A function $f:(X,\tau) \to (Y,\sigma)$ is called **δP_S-continuous** at a point $x \in X$ if each open set V of Y containing f (x), there exists a δP_S-open set U of X containing x such that $f(U) \subseteq V$. If f is δP_S-continuous at every point of X, then it is called δP_S-continuous.

Proposition 5.2.2. Every complete continuous function is δP_S-continuous.

Proof. Let $f:(X,\tau) \to (Y,\sigma)$ be a complete continuous function. Let $x \in X$ and V be an open set containing f(x). Then $f^{-1}(V)$ is regular open in X, since f is complete continuous. Choosing $U = f^{-1}(V)$ will give $f(U) \subseteq V$ and $x \in f(U)$.

Moreover $f^{-1}(V)$ is δP_S-open from Proposition 2.2.15.

Hence f is δP_S-continuous at x. Since x is arbitrary, f is a δP_S-continuous function.

Proposition 5.2.3. Every super continuous function is δP_S-continuous function.

Proof. As in the proposition 5.2.2, $f^{-1}(V)$ is δ-open, since f is super continuous. Hence from Proposition 2.2.14, $f^{-1}(V)$ is δP_S-open which implies f is δP_S-continuous.

Proposition 5.2.4. Every P_S-continuous function is δP_S-continuous function.

Proof. As in the proposition 5.2.2, $f^{-1}(V)$ is P_S-open, since f is P_S-continuous. Hence from Proposition 2.2.13, $f^{-1}(V)$ is δP_S-open which implies f is δP_S-continuous.

Proposition 5.2.5. Every δP_S-continuous function is δ-precontinuous function.

Proof. The proof follows as in the proposition 5.2.2, $f^{-1}(V)$ is δP_S-open, since f is δP_S-continuous. Hence from Definition 1.3.2(g), f is δ-precontinuous.

Note 5.2.6. From the above results and we from Lemma 1.3.25 we obtain the following implication diagram.

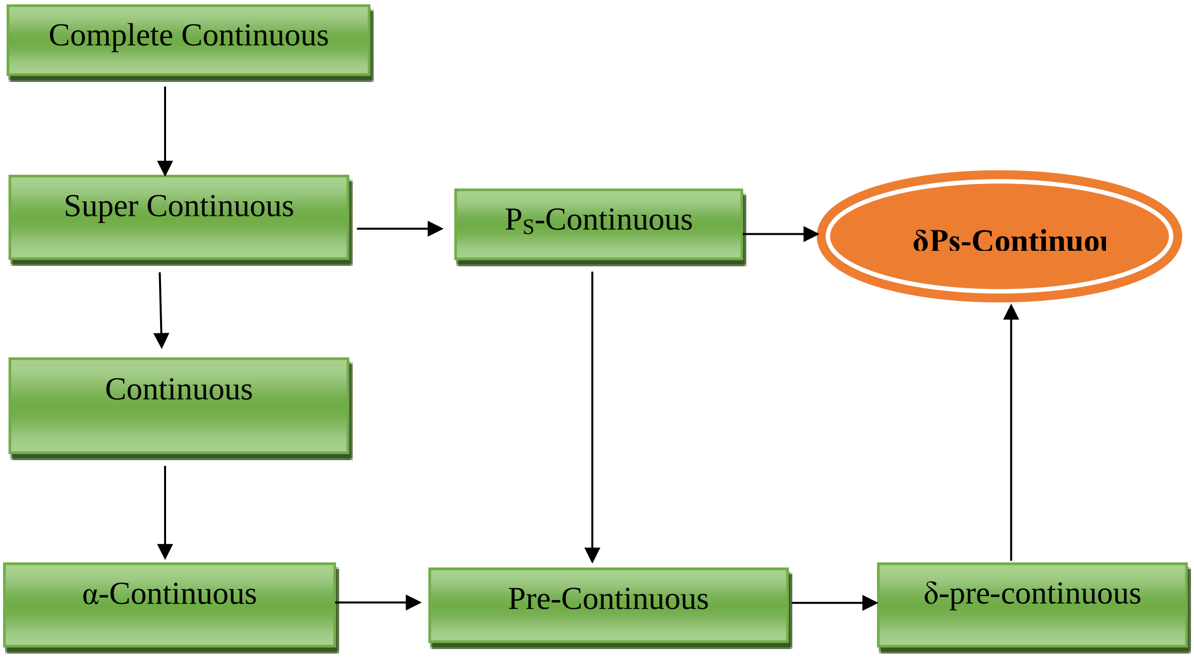

Figure 5.1

Theorem 5.2.7. For a function $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent:

a) f is δP_S-continuous.

b) $f^{-1}(V)$ is a δP_S-open set in X, for each open set V in Y.

c) $f^{-1}(F)$ is a δP_S-closed set in X, for each closed set F in Y.

d) $f(\delta P_S Cl(A))\subseteq Cl(f(A))$, for each $A\subseteq X$.

e) $\delta P_S Cl(f^{-1}(B))\subseteq f^{-1}(Cl(B))$, for each $B\subseteq Y$.

f) $f^{-1}(B)\subseteq \delta P_S Int(f^{-1}(B))$ for each $B\subseteq Y$.

g) $Int(f(A))\subseteq f(\delta P_S Int(A))$, for each $A\subseteq$ X.

Proof. $(a)\Rightarrow(b)$ Let $f:(\mathrm{X},\tau)\to(\mathrm{Y},\sigma)$ be δP_S-continuous function.

To Prove. f^{-1}(V) is δP_Sopen in X for every open set V in Y

Let V be open in Y. Let $x\in f^{-1}$(V). Then f(x) $\in$ V

By Definition 5.2.1, there exists a δP_S-open set U of X containing x such that f(U) $\subseteq$ V $\Rightarrow$ $x\in$ U $\subseteq f^{-1}$(V).

By Definition 2.5.7 x is an δP_S Int point of f^{-1}(V). Since x is arbitrary, f^{-1}(V) is a δP_S-open.

$(b)\Rightarrow(c)$ Let F be any closed set of Y, then $Y\backslash F$ is an open set of Y. By (b), we have $f^{-1}(Y\backslash F)=X\backslash f^{-1}(F)$ is a δP_S-open set in X. Hence $f^{-1}(F)$ is a δP_S-closed set in X.

$(c) \Rightarrow$ (d) Consider A ⊆ X. Then f(A) ⊆ Y. Then Clf(A) = F say, is closed in Y. Then by (c), $f^{-1}(F)$ is δP_S-closed in X.

$\therefore\ \delta P_S Cl(f^{-1}(F) = f^{-1}(F)$ ⟶ (1)

Now, $A \subseteq \delta P_S$Cl(A) ⊆ δP_SClf(F)

Then by (c), $f^{-1}(\delta P_S ClA) \subseteq Cl\big(f(A)\big)$.

$f(\delta P_S Cl(A) \subseteq f(\delta P_S\ Cl(f^{-1}(F))$ ⟶ (2)

(Since, $f(A) \subseteq Cl\big(f(A)\big) = F$ & $A \subseteq f^{-1}(F)$

From (1) & (2),

$$f(\delta P_S\ Cl(A) \subseteq f(f^{-1}(F)) \subseteq F = Cl(f(A))$$

$(d) \Rightarrow (e)$ Let B ⊆ Y and $f^{-1}(B) = A \Rightarrow B = f(A)$

Then by (d) $f(\delta P_S Cl\big(f^{-1}(B)\big) = f(\delta P_S ClA) \subseteq Clf(A) = Cl(B)$

$$= Clf(f^{-1}(B)) \subseteq ClB$$

$$\therefore \delta P_S Cl(f^{-1}(B) \subseteq f^{-1}(ClB)$$

$(e) \Rightarrow (f)$ From lemma prove $\delta P_S Int\ A = (\delta P_S Cl\ A)^C$

$\therefore$ From (e) $(\delta P_S Cl(f^{-1}(B))^C \supseteq (f^{-1}(ClB))^C$

$\Rightarrow \delta P_S IntB \supseteq f^{-1}(Cl\ B)^C = f^{-1}(Int\ B)$

$\therefore f^{-1}(Int\ B) \subseteq \delta P_S IntB$

$(f) \Rightarrow$ (g)Let A ⊆ X. Let B ⊆ Y such that $A = f^{-1}(B)$

$$\Rightarrow f(A) \subseteq B$$

$$\Rightarrow Int\ f(A) \subseteq Int(B)$$

$$\Rightarrow f^{-1}\big(Intf(A)\big) \subseteq f^{-1}(Int\ B) \subseteq \delta P_S Intf^{-1}(B) = \delta P_S IntA$$

$\therefore Int\ f(A) \subseteq f(\delta P_S Int\ A)$

$(g) \Rightarrow (a)$: Let $x \in X$ and let V be any open set of Y containing $f(x)$, then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X. Hence, by (g), we have $Int(f(f^{-1}(V)) \subseteq f(\ \delta P_S Int(f^{-1}(V)))$. So, $Int(V) \subseteq f(\delta P_S Int(f^{-1}(V)))$. Since V is an open set in Y, then $V \subseteq f(\delta P_S Int(f^{-1}(V)))$ which implies that $f^{-1}(V)) \subseteq \delta P_S Int(f^{-1}(V)))$. Therefore, $f^{-1}(V)$ is δP_S-open set in X contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence f is δP_S-continuous.

Proposition 5.2.8. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is δP_S-continuous if and only if f is δ-precontinuous and for each $x \in$ X and each open set V of Y containing $f(x)$, there exists a semi-closed set F in X containing x such that $f(F) \subseteq$ V.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be δP_S-continuous, since every δP_S-open set is δ-preopen, f is δ-precontinuous also. Let $x \in$ X and V be any set containing f(x). From the definition of δP_S-

continuity, there exists a δP_S-open set U of X containing x, such that f(U) ⊆ V. From the definition of δP_S-open set, for all x ∈ U, there exists a semi-closed set F of X such that x ∈ F ⊆ U.

∴ f(F) ⊆ f(U) ⊆ V

⇔ Let the criteria be true. To prove f is δP_S-continuous. From the criteria, f is δ-precontinuous which implies f^{-1}(V) is δ-preopen set.

Let x ∈ f^{-1}(V) then f(x) ∈ V and by hypothesis there exists a semi-closed set F in X such that x ∈ F and f(F) ⊆ V ⇒ x ∈ F ⊆ f^{-1}(V).

∴ f^{-1}(V) is δP_S-open ⇒ F is δP_S-continuous.

Proposition 5.2.9. If $f:(X,\tau) \to (Y,\sigma)$ is a continuous and an open function and V is a δP_S-open set of Y, then f^{-1}(V) is a δP_S-open set of X.

Proof. Let V be a δP_S-open set of Y, then by Proposition 2.2.2, V is a δ-preopen set of Y and V = $\cup F_\alpha$, where F_α is a semi- closed set of Y for each α. Then f^{-1}(V) = f^{-1}($\cup F_\alpha$) = ∪ f^{-1}(F_α), where F_α is semi-closed set of Y for each α. Since f is a continuous and an open function. Then by Proposition 1.3.5, f^{-1}(V) is a preopen set of X and thereby f^{-1}(V) is δ-preopen and by Proposition 1.3.6, f^{-1}(F_α) is a semi-closed set of X for each α. Hence by 2.2.2, f^{-1} (V) is a δP_S-open set of X.

Corollary 5.2.10. If $f:(X,\tau) \to (Y,\sigma)$ is a continuous and open function and F is a δP_S-closed set of Y, then $f^{-1}(F)$ is a δP_S-closed set of X.

The following theorem gives the decomposition of perfect continuity:

Theorem 5.2.11. The following statements are equivalent for a function $f:(X,\tau) \to (Y,\sigma)$:

a) f is perfectly continuous.

b) f is super continuous and contra δ- continuous

c) f is P_S-continuous and contra δ-continuous.

d) f is α-continuous and contra δ-continuous.

e) f is δP_S-continuous and contra δ- continuous.

f) f is δ-pre-continuous and contra δ-continuous.

Proof. This is an immediate consequence of Corollary 2.2.38 and from the definitions of perfectly continuous [1.3.1(f)], super continuous [1.3.1(h)], contra δ-continuous [1.3.2(f)], P_S-continuous [1.3.4], α-continuous [1.3.1(d)], δP_S-continuous [5.2.1] and δ-precontinuous [1.3.2(g)].

The following theorem gives the decomposition of complete continuity in a semi-regular space:

Theorem 5.2.12. The following statements are equivalent for a function $f:(X,\tau)\to(Y,\sigma)$, in a semi-regular space:

a) f is complete continuous.

b) f is P_S-continuous and contra-semicontinuous.

c) f is continuous and contra-semicontinuous.

d) f is α-continuous and contra-semicontinuous.

e) f is δP_S-continuous and contra semicontinuous.

f) f is δ-precontinuous and contra-semicontinuous.

Proof. This is an immediate consequence of Corollary 2.2.39 and the definitions of complete continuous [1.3.1(g)], P_S-continuous [1.3.4], continuous [1.3.1(a)], α-continuous [1.3.1(d)], δP_S-continuous [5.2.1], δ-precontinuous [1.3.2(g)] and contra semi-continuous [1.3.2(e)].

Theorem 5.2.13. The following statements are equivalent for a function $f:(X,\tau)\to(Y,\sigma)$:

a) f is complete continuous

b) f is super continuous and δ-semicomplete continuous

c) f is super continuous and contra- δ -semi θ continuous

d) f is super continuous and contra -δ semicontinuous

e) f is α-continuous and contra- δ-semicontinuous

f) f is δP_S-continuous and contra - δ-semicontinuous

g) f is δP-continuous and contra - δ-semicontinuous

h) A is α-precontinuous and contra-e^* continuous

Proof: This is an immediate consequence of Corollary 2.2.40 and the definitions of complete continuous [1.3.1(g)], super continuous [1.3.1(h)], α-continuous, δP_S-continuous [5.2.1], δ-precontinuous [1.3.2(g)], δ-semicomplete continuous, contra δ-semi θ-continuous, contra δ-semi-continuous and contra e^*-continuous [1.3.2(h)].

Proposition 5.2.14. If $f:(X,\tau)\to(Y,\sigma)$ be θ_S-continuous and precontinuous, then f is δP_S-continuous.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be θ_S-continuous and precontinuous. Let V be an open set in Y. Since f is θ_S continuous and precontinuous function $f^{-1}(V)$ is θ-semiopen and preopen in X. This implies $f^{-1}(V)$is θ-semiopen and δ-preopen. By Corollary 2.2.45, $f^{-1}(V)$ is $\delta P_S O(X)$. Then by Theorem 5.2.7(b), f is δP_S-continuous.

Corollary 5.2.15. If a function $f: (X, \tau) \to (Y, \sigma)$ is almost θ_S-continuous and almost precontinuous, then $f: (X, \tau) \to (Y, \sigma_S)$ is δP_S-continuous.

Proof. Let $f: (X, \tau) \to (Y, \sigma)$ be almost θ_S-continuous and almost precontinuous function. Let V be open in Y, then by definition almost θ_S-continuous and almost precontinuous function, $f^{-1}(V)$ is θ-semiopen and preopen. From Corollary 1.3.24, $f^{-1}(V)$ is P_S-open which implies

$f^{-1}(V)$ is δP_S-open [by Proposition 2.2.17]. Hence by Theorem 5.2.7(b), f is δP_S-continuous.

Corollary 5.2.16. Let X be a locally indiscrete space. Then the function $f: (X, \tau) \to (Y, \sigma)$ is almost continuous if and only if $f: (X, \tau) \to (Y, \sigma_S)$ is δP_S-continuous.

Proof. Follows from Proposition 2.2.31 in which it is proved that in a locally indiscrete space $\delta P_S O(X) = \tau$

Corollary 5.2.17. Let $f: (X, \tau) \to (Y, \sigma)$ be a function and X is locally indiscrete space. Then f is δP_S- continuous if and only if f is continuous.

Proof. Follows from Proposition 2.2.31.

Corollary 5.2.18. Let $f: (X, \tau) \to (Y, \sigma)$ be a function and X is s-regular space. If f is continuous, then f is δP_S-continuous.

Proof. Follows from Proposition 2.2.35 in which it is proved that in a s-regular space $\tau \subseteq \delta P_S O(X)$.

Corollary 5.2.19. Let $f: (X, \tau) \to (Y, \sigma)$ be a function and X is semi-T_1 space. Then f is δP_S-continuous if and only if f is δ-precontinuous.

Proof. Follows from Proposition 2.2.23 in which it is proved that in a semi-T_1 space $\delta P_S O(X) = \delta PO(X)$.

Proposition 5.2.20. Let X be an extremally disconnected space. If the function $f: (X, \tau) \to (Y, \sigma)$ is almost θs-continuous, then $f: (X, \tau) \to (Y, \sigma_S)$ is δP_S-continuous.

Proof. Let $H \in \sigma_S$, then H is a regular open set in (Y, σ). Since $f: (X, \tau) \to (Y, \sigma)$ is almost θs- continuous. Then $f^{-1}(H)$ is θ-semi-open set in X. Since X is extremally disconnected space. Then by Proposition 2.2.46, $f^{-1}(H)$ isδP_S-open set in X. Therefore, by Theorem 5.2.7, $f: (X, \tau) \to (Y, \sigma_S)$ is δP_S-continuous.

Proposition 5.2.21. Let $f: (X, \tau) \to (Y, \sigma)$ be a function. Let B be any basis for τ in Y. Then f is δP_S-continuous if and only if for each $B \in \mathcal{B}$, $f^{-1}(B)$ is a δP_S-open subset of X.

Proof. Necessity. Suppose that f is δP_S-continuous. Then since each $B \in \mathcal{B}$ is an open subset of Y and f is δP_S-continuous. $f^{-1}(B)$ is a δP_S-open subset of X by Theorem 5.2.7.

Sufficiency. Let V be any open subset of Y. Then V = ∪{B_i : i ∈ I} where every B_i is a member of $\mathcal{B}$ for a suitable index set I, it follows that $f^{-1}(V) = f^{-1}(\cup\{B_i : i \in I\}) = \cup f^{-1}(\{B_i : i \in I\})$. Since $f^{-1}(B_i)$ is a δP_S-open subset of X for each i ∈ I, by hypothesis $f^{-1}(V)$ is the union of a family of δP_S-open sets of X and hence is δP_S-open set of X, by Proposition 2.2.7. Therefore, by Theorem 5.2.7(b), f is δP_S-continuous.

Proposition 5.2.22. Let $f: (X, \tau) \to (Y, \sigma)$ be a δP_S-continuous function. If A is either open or regular semi-open subset of X, then f|A:A→Y is δP_S-continuous in the subspace A.

Proof. Let V be any open set of Y. Since f is δP_S- continuous, $f^{-1}(V)$ is δP_S- open set in X, then by Theorem 5.2.7(b). Since A is either open or regular semi-open subset of X. $(f|A)^{-1}(V) = f^{-1}(V) \cap A$ is a δP_S-open subspace of A by Proposition 2.3.6. This shows that f|A:A→Y is δP_S-continuous.

From Proposition 5.2.22, we obtain the following corollary

Corollary 5.2.23. Let $f: (X, \tau) \to (Y, \sigma)$ be a δP_S-continuous function. If A is a regular open subset of X, then f|A:A→Y is δP_S-continuous in the subspace A.

Proof: If A is regular open then it is open. Hence from Proposition 5.2.22, the proof follows.

Proposition 5.2.24. A function $f: (X, \tau) \to (Y, \sigma)$ is δP_S- continuous. If for each $x \in X$, there exists a regular open set A of X containing x such that f|A:A→Y is δP_S-continuous.

Proof. Let $x \in$ X, then by hypothesis, there exists a regular open set A containing x such that f|A:A→Y is δP_S-continuous. Let V be any open set of Y containing f (x), there exists a δP_S-open set U in A containing x such that (f|A)(U) ⊆ V. Since A is regular open set. By Proposition 2.3.2, U is δP_S-open set in X and hence f (U) ⊆ V. This shows that f is δP_S-continuous.

Corollary 5.2.25. Let $\{U_\alpha : \alpha \in \Delta\}$ be a regular open cover of a topological space X. A function $f: (X, \tau) \to (Y, \sigma)$ is δP_S-continuous if and only if $f|U_\alpha : U_\alpha \to (Y, \sigma)$ is δP_S-continuous for each α ∈ Δ.

Proof. Let $\{U_\alpha / \alpha \in \Delta\}$ be a regular open cover of a topological space X. Let $f: (X, \tau) \to (Y, \sigma)$ be a regular open cover of a topological space X. Let $F: X \to Y$ be δP_S-continuous. By Corollary 5.2.23 $f/U_\alpha : U_\alpha \to Y$ is δP_S-continuous ∀ $\alpha \in \Delta$.

Conversely, let $f/U_\alpha : U_\alpha \to Y$ be δP_S-continuous, ∀ $\alpha \in \Delta$

Now, ∀ $x \in X$, there exists $U_\alpha \in \{U_\alpha / \alpha \in \Delta\}$ containing x, since $\{U_\alpha\}$ is a cover for X. Then by Proposition 5.2.24 $f: (X, \tau) \to (Y, \sigma)$ is δP_S-continuous.

Proposition 5.2.26. If $X = R \cup S$, where R and S are regular open sets and $f: (X, \tau) \to (Y, \sigma)$ is a function such that both f|R and f|S are δP_S-continuous, then f is δP_S-continuous.

Proof. Let V be any open set of Y. Then $f^{-1}(V) = (f|R)^{-1}(V) \cup (f|S)^{-1}(V)$. Since f|R and f|S are δP_S- continuous. Then by Theorem 5.2.7(b), $(f|R)^{-1}(V)$ and $(f|S)^{-1}(V)$ are δP_S-open sets in R and S, respectively. Since R and S are regular open sets in X, then by Proposition 2.3.2, $(f|R)^{-1}(V)$ and $(f|S)^{-1}(V)$ are δP_S-open sets in X. Since union of two δP_S-open sets is δP_S-open. Hence $f^{-1}(V)$ is δP_S- open set in X. Therefore, by Theorem 5.2.7(b), f is δP_S-continuous.

The following is the pasting lemma for δP_S-continuity:

Lemma 5.2.27. Let $X = R_1 \cup R_2$, where R_1 and R_2 are regular open sets in X. Let $f:R_1 \to Y$ and $g:R_2 \to Y$ be P_S-continuous. If f (x) = g (x) for each $x \in R_1 \cap R_2$. Then $h:R_1 \cup R_2 \in Y$ such that

$$h(x) = \begin{cases} f(x) \ if \ x \in R_1 \\ g(x) \ if \ x \in R_2 \end{cases}$$

is δP_S-continuous.

Proof. Let O be an open set of Y. Now $h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O)$. Since f is δP_S-continuous, then by Theorem 5.2.7(b), $f^{-1}(O)$ is δP_S-open set in R_1. But R_1 is regular open set in X. Then by Proposition 2.3.2, $f^{-1}(O)$ is δP_S-open set in X. Similarly, $g^{-1}(O)$ is δP_S-open set in R_2 and hence, δP_S -open set in X. Since union of two δP_S-open sets is δP_S-open. Therefore, $h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O)$ is a δP_S -open set in X. Hence by Theorem 5.2.7(b), h is δP_S -continuous**.**

Proposition 5.2.28. Let $f:(X,\tau) \to (Y,\sigma)$be δP_S-continuous surjection and A be either δ-open or regular semi- open subset of X. If f is an δ-open function, then the function $g:A \to f(A)$, defined by g (x) = f (x) for each $x \in A$, is δP_S-continuous.

Proof. Putting H = f (A). Let $x \in A$ and let V be any open set in H containing $g(x)$. Since H is open in Y and V is open in H, then V is open in Y. Since f is δP_S-continuous, there exists a δP_S- open set U in X containing x such that $f(U) \subseteq V$. Taking $W = U \cap A$, since A is either δ-open or regular semi-open subset of X. Then by Corollary 2.3.7, W is a δP_S-open set in A containing x and $g(W) \subseteq V \cap H = V_H$. Then $g(W) \subseteq V_H$. This shows that g is δP_S-continuous.

Proposition 5.2.29. Let $f:(X,\tau) \to (Y,\sigma)$ be a δP_S-continuous. If Y is an open subset of a topological space Z, then $f:(X,\tau) \to (Z,\eta)$ is δP_S-continuous.

Proof. Let V be an open set in Z. Then $V \cap Y$ is open in Y. Since f is δP_S-continuous, by Theorem 5.2.7(b), $f^{-1}(V \cap Y)$ is δP_S-open set in X. But $f(x) \in Y$ for each $x \in X$, and thus $f^{-1}(V) = f^{-1}(V \cap Y)$ is a δP_S-open subset of X. Therefore, by Theorem 5.2.7(b), $f:(X,\tau) \to (Z,\eta)$ is δP_S-continuous.

Remark 5.2.30. In general, composition of two δP_S-continuous functions need not be δP_S-continuous.

Example 5.2.31. Let

$X = Y = Z = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}, \sigma = \{X, \emptyset, \{a, b\}\}$ *and* $\eta = \{X, \emptyset, \{c\}\}$.

Define $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ as identity functions. Then f and g are δP_S-continuous functions but not $g \circ f$ is not δP_S-continuous as $f^{-1}(g^{-1}\{a, b\}) = \{a, b\}$ is not δP_S-closed in (X, τ).

Theorem 5.2.32. Let $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ be functions. Then the composition function $g \circ f: X \to Z$ is δP_S-continuous if f and g satisfy one of the following conditions:

a) f is δP_S-continuous and g is continuous.

b) f is continuous and open and g is δP_S- continuous.

Proof. a). Let W be any open subset of Z. Since g is continuous, $g^{-1}(W)$ is open subset of Y. Since f is δP_S-continuous, then by Theorem 5.2.7(b), $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is δP_S-open subset in X. Therefore, by Theorem 5.2.7(b), $g \circ f$ is δP_S-continuous.

b). Let W be any open subset of Z. Since g is δP_S-continuous, by Theorem 5.2.7(b), $g^{-1}(W)$ is δP_S- open subset of Y. Since f is continuous and open, then by Theorem 5.2.7(b), $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is a δP_S-open set in X. Hence by Theorem 5.2.7(b), $g \circ f$ is δP_S-continuous.

Proposition 5.2.33: Let $f: (X, \tau) \to (Y, \sigma)$ be an injective, continuous and open function. If Y is $\delta P_S T_0$, then X is $\delta P_S T_0$.

Proof: Let $x, y \in X$ with $x \neq y$. Since f is injective and Y is $\delta P_S T_0$, there exists a δP_S-open set U_x in Y such that f is $f(x) \in U_x$ and $f(y) \notin U_x$ or there exists a δP_S-open set U_y in Y such that $f(y) \in U_y$ and $f(x) \notin U_y$ with $f(x) \neq f(y)$. Since f is continuous and open function, then by Proposition 5.2.9, $f^{-1}(U_x)$ is δP_S-open set in X such that $x \in f^{-1}(U_x)$ and $y \notin f^{-1}(U_x)$ or $f^{-1}(U_y)$ δP_S-open set in X such that $y \in f^{-1}(U_y)$ and $x \notin f^{-1}(U_y)$. This shows that X is $\delta P_S T_0$.

Proposition 5.2.34: Let $f: (X, \tau) \to (Y, \sigma)$ be an injective, continuous and open function. If Y is $\delta P_S T_1$, then X is $\delta P_S T_1$-space.

Proof: Proof follows from Proposition 5.2.33.

Proposition 5.2.35: Let $f: (X, \tau) \to (Y, \sigma)$ be an injective, continuous and open function. If Y is $\delta P_S T_2$, then X is $\delta P_S T_2$-space.

Proof: Proof follows from Proposition 4.2.1.10.

Proposition 5.2.36: If $f:(X,\tau)\to(Y,\sigma)$ isa□scontinuous surjection function and X is extremally disconnected and δP_S-compact space, then Y is compact.

Proof: Follows from Proposition 4.6.1 and Proposition 5.2.20

Proposition 5.2.37: If $f:(X,\tau)\to(Y,\sigma)$is a continuous and open function. If A is δP_S-set, then f (A) is δP_S-set.

Proof: Let $\{V_\alpha: \alpha \in\Delta\}$ be any cover of f (A) by δP_S-open sets of Y. Since f is continuous and open function. By Proposition 5.2.7(b), $\{f^{-1}$ (V_α): $\alpha \in\Delta\}$ is a cover of A by δP_S-open sets of X. Since A is δP_S-set, there exists a finite subset Δ_0 of Δ such that $A \subseteq\cup\{f^{-1}(V_\alpha):\ \alpha\in\Delta_0\}$. Thus, we have f (A) $\subseteq\cup\{V_\alpha: \alpha \in\Delta_0\}$. This shows that $f(A)$ is δP_S-set.

Corollary 5.2.38: If X is a δP_S-compact space and $f:(X,\tau)\to(Y,\sigma)$ is a continuous and open surjection function, then Y is δP_S-compact.

5.3 Perfectly δP_S-Continuity and Strongly δP_S-Continuity

Definition 5.3.1. A map $f:(X,\tau)\to(Y,\sigma)$ is called **perfectly δP_S-continuous** if the inverse image of every δP_S-open set in (Y, σ) is a clopen set in (X, τ).

Proposition 5.3.2. A function $f:(X,\tau)\to(Y,\sigma)$ is perfectly δP_S-continuous if and only if the inverse image of every δP_S-closed set in Y is clopen in X.

Proof: (Necessary): Let $f:(X,\tau)\to(Y,\sigma)$ be perfectly δP_S-continuous and V be any δP_S-closed set in (Y, σ). Then Y\V is δP_S-open set in (Y, σ). Since f is perfectly δP_S-continuous, $f^{-1}(Y-V) = X \setminus f^{-1}(V)$ is clopen in (X, τ) which implies that f^{-1} (V) is clopen in (X, τ).

(Sufficiency): Let the inverse image of every δP_S-closed set in (Y,σ) is clopen in (X, τ). Let U be any δP_S-open set in (Y,σ). Then (Y \ U) is a δP_S-closed set in (Y, σ). By our assumption, $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is clopen in (X,τ) which implies that f^{-1} (U) is clopen in (X,τ) and hence f is perfectly δP_S -continuous.

Proposition 5.3.3. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be any two functions. Then their composition $g\circ f:(X,\tau)\to(Z,\eta)$ is perfectly δP_S-continuous if g is perfectly δP_S-continuous and f is continuous.

Proof. Let $g:(Y,\sigma)\to(Z,\eta)$ be perfectly δP_S-continuous and $f:(X,\tau)\to(Y,\sigma)$ be continuous. Let V be a δP_S-closed set in Z. Then g^{-1} (V) is open as well as closed in Y. Since f is continuous, $f^{-1}(g^{-1}(V)) = (g\circ f)^{-1}$ (V) is clopen in X. Hence $g\circ f$ is perfectly δP_S-continuous.

Proposition 5.3.4. Every strongly continuous function is a perfectly δP_S-continuous function.

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ be a strongly continuous function and B be a δP_S-closed set in Y. By hypothesis, f^{-1}(B) is clopen in X. This proves f is a perfectly δP_S-continuous function.

Proposition 5.3.5. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be any two functions. Then their composition $g\circ f:(X,\tau)\to(Z,\eta)$ is continuous if g is δP_S-continuous and f is perfectly δP_S-continuous.

Proof: Let $g:(Y,\sigma)\to(Z,\eta)$ be δP_S-continuous and $f:(X,\tau)\to(Y,\sigma)$ be perfectly δP_S-continuous. Let V be a closed set in Z. Then g^{-1}(V) is a δP_S-closed in Y. Further, since f is perfectly δP_S-continuous, $(f^{-1}\big(g^{-1}(V)\big)=(g\circ f)^{-1}(V)$ is clopen in X and thereby closed in X. Hence $g\circ f$ is continuous.

Theorem 5.3.6. For topological spaces $(X,\tau),(Y,\sigma),(Z,\eta),f:(X,\tau)\to(Y,\sigma),g:(Y,\sigma)\to(Z,\eta)$ and $g\circ f:(X,\tau)\to(Z,\eta)$ the following results are true.

a) If f is perfectly δP_S-continuous and g is super continuous then $g\circ f$ is perfectly continuous.
b) If f is continuous and g is perfectly δP_S-continuous then $g\circ f$ is perfectly δP_S-continuous.
c) If f is super continuous and g is perfectly δP_S-continuous then $g\circ f$ is perfectly δP_S-continuous.

Proof: a) Let $f:(X,\tau)\to(Y,\sigma)$, be perfectly δP_S-continuous function and $g:(Y,\sigma)\to(Z,\eta)$ be a super continuous function. Let W be a open subset of Z. Then g^{-1}(W) is δ-open in Y and hence δP_S-open in Y. Further, $f^{-1}(g^{-1}(W))=(g\circ f)^{-1}(W)$ is clopen in X. Hence $g\circ f$ is a perfectly continuous function.

b) Let $f:(X,\tau)\to(Y,\sigma)$, be a continuous function and $g:(Y,\sigma)\to(Z,\eta)$be a perfectly δP_S-continuous function. Let W be a δP_S-open subset of Z. Then g^{-1}(W) is clopen in Y and hence $f^{-1}(g^{-1}(W))=(g\circ f)^{-1}(W)$ is clopen in X. Hence $g\circ f$ is a perfectly δP_S-continuous function.

c) Let $f:(X,\tau)\to(Y,\sigma)$, be a super continuous function and $g:(Y,\sigma)\to(Z,\eta)$be a perfectly δP_S-continuous function. Let W be a δP_S-open subset of Z. Then g^{-1}(W) is clopen in Y. Then $f^{-1}(g^{-1}(W))=(g\circ f)^{-1}(W)$ is δ-closed and δ-open in X. Thereby it is δP_S-closed as well as δP_S-open in X. Hence $g\circ f$ is a perfectly δP_S-continuous function.

Definition 5.3.7. A function $f:(X,\tau)\to(Y,\sigma)$ is called **strongly $\boldsymbol{\delta P_S}$-continuous** if the inverse image of every subset of (Y, σ) is δP_S-clopen in (X, τ).

Proposition 5.3.8. If f is a strongly δP_S-continuous and g is any function then $g \circ f$ is a strongly δP_S- continuous.

Proof: Let $f: (X, \tau) \to (Y, \sigma)$, be a strongly δP_S-continuous function and $g: (Y, \sigma) \to (Z, \eta)$ be a quasi δP_S-continuous function. Let W be any subset of Z. Then $g^{-1}(W)$ is a subset in Y. Further, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is δP_S-clopen in X. Hence $g \circ f$ is a strongly δP_S-continuous function.

5.4 Quasi δP_S-Continuity and Totally δP_S-Continuity

Definition 5.4.1. A function $f: (X, \tau) \to (Y, \sigma)$ is called **quasi δP_S-continuous** if the inverse image of every δP_S-open set in (Y, σ) is open in (X, τ).

Example 5.4.2 Let X = Y = {a,b,c} with τ = {X, ∅, {a}, {b}, {a,b}, {a,c}} and σ = {Y, ∅, {a,b}}. Let $f: (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is quasi δP_S-continuous.

Proposition 5.4.3. A function $f: (X, \tau) \to (Y, \sigma)$is quasi δP_S-continuous if and only if the inverse image of every δP_S-closed set in (Y, σ) is closed in (X, τ).

Proof (Necessary): Let $f: (X, \tau) \to (Y, \sigma)$ be quasi δP_S-continuous function. Let V be a δP_S-closed in (Y, σ) which implies that $(Y \backslash V)$ is δP_S-open in (Y, σ). Since f is quasi δP_S-continuous, $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is open in (X, τ) and hence $f^{-1}(V)$ is closed in (X, τ).

(Sufficiency): Let U be δP_S-open set in (Y, σ) which implies $(Y \backslash U)$ is δP_S-closed set in (Y, σ). By assumption $f^{-1}(Y \backslash U) = X \backslash f^{-1}(U)$ is open in (X, τ). Hence f is quasi δP_S-continuous.

Theorem 5.4.4. For a function $f: (X, \tau) \to (Y, \sigma)$, the following are equivalent:

(a) quasi δP_S-continuous;

(b) $f^{-1}(B)$ is closed in X, for every δP_S-closed B in Y.

(c) For each $x \in X$ and each δP_S-open set B containing f(x), there exists an open set A containing x such that $f(A) \subseteq B$.

Proof. (a) ⇔ (b) is obvious.

(b) ⇒ (c) Let $x \in X$ and let B be an open set containing f(x) then by hypothesis $f^{-1}(B)$ is a δP_S-open set containing x. Let $A = f^{-1}(B)$ then $f(A) = f(f^{-1}(B)) \subseteq B$.

(c) ⇒ (a) Let B be δP_S-open in Y with $x \in f^{-1}(B) \Rightarrow f(x) \Rightarrow B$ then by hypothesis there exists an open set A containing x such that $f(A) \subseteq B \Rightarrow A \subseteq f^{-1}(B)$. The result follows as $f^{-1}(B)$ can be written as the union of open sets.

Proposition 5.4.5. Let (X, τ) be a partition space, (Y, σ) be a topological space and $f: (X, \tau) \to (Y, \sigma)$ be any function. Then f is quasi δP_S-continuous.

Proof. Follows from the definition of a partition space in which every open set is closed.

Proposition 5.4.6. Let $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ be any two functions. Then their composition $g \circ f: (X, \tau) \to (Z, \eta)$ is continuous if g is δP_S-continuous and f is quasi δP_S-continuous.

Proof. Let $g: (Y, \sigma) \to (Z, \eta)$ be δP_S-continuous and $f: (X, \tau) \to (Y, \sigma)$ be quasi δP_S-continuous. Let V be a closed set in Z. Then $g^{-1}(V)$ is δP_S-closed in Y. Further, $f^{-1}\left(g^{-1}(V)\right) = (g \circ f)^{-1}(V)$ is closed in X. Hence $g \circ f$ is continuous.

Theorem 5.4.7. For topological spaces $(X, \tau), (Y, \sigma), (Z, \eta), f: (X, \tau) \to (Y, \sigma)$, g: $(Y, \sigma) \to (Z, \eta)$ and $g \circ f: (X, \tau) \to (Z, \eta)$ the following results are true.

(a) If f is quasi δP_S-continuous and g is super continuous then $g \circ f$ is continuous.

(b) If f is super continuous and g is quasi δP_S-continuous then $g \circ f$ is quasi δP_S-continuous.

Proof: (a) Let $f: (X, \tau) \to (Y, \sigma)$ be quasi δP_S-continuous function and $g: (Y, \sigma) \to (Z, \eta)$ be a super continuous function. Let W be a closed subset of Z. Then $g^{-1}(W)$ is δ-closed in Y. Thus $g^{-1}(W)$ is δP_S-closed in Y. Further, $f^{-1}\left(g^{-1}(W)\right) = (g \circ f)^{-1}(W)$ is closed in X. Hence $g \circ f$ is a continuous function.

(b) Let $f: (X, \tau) \to (Y, \sigma)$ be a continuous function and $g: (Y, \sigma) \to (Z, \eta)$be a quasi δP_S-continuous function. Let W be a δP_S-open subset of Z. Then $g^{-1}(W)$ is open in Y and hence $f^{-1}\left(g^{-1}(W)\right) = (g \circ f)^{-1}(W)$ is open in X. Hence $g \circ f$ is a quasi δP_S-continuous function.

Totally Continuous

Definition 5.4.8. A function $f: (X, \tau) \to (Y, \sigma)$ is called **totally δP_S-continuous** if the inverse image of every open subset of (Y, σ) is δP_S- clopen in (X, τ).

Proposition 5.4.9. A function $f: (X, \tau) \to (Y, \sigma)$ is totally δP_S-continuous iff f^{-1}(V) is δP_S-clopen in (X, τ) for every closed set V in (Y, σ).

Proof: It is obvious.

Proposition 5.4.10. Every totally δP_S-continuous function is δP_S-continuous but not conversely.

Proof: Follows from the definitions of totally δP_S-continuous and δP_S-continuous functions.

Example 5.4.11. Let X = Y = {a, b, c} with τ = {X, ∅,{a}, {a, b}, {a, c}} and σ = {Y, ∅, {a}, {a, b}, {a, c}}. Let $f: (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is δP_S-continuous but not totally δP_S-continuous since for the open subset {a} in (Y, σ), f^{-1}({a}) = {a} is δP_S-open set but not δP_S-closed set in (X, τ).

Proposition 5.4.12. For topological spaces (X, τ), (Y, σ), (Z, η), $f:(X,\tau)\to(Y,\sigma)$, $g:(Y,\sigma)\to(Z,\eta)$ and $g\circ f:(X,\tau)\to(Z,\eta)$ then if f is a totally δP_S-continuous and g is a continuous (resp. super continuous) then $g\circ f$ is a totally δP_S-continuous.

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ be a totally δP_S-continuous function and $g:(Y,\sigma)\to(Z,\eta)$ be a continuous (resp. super continuous) function. Let W be any open subset of Z. Then $g^{-1}(W)$ is open (resp. δ-open) in Y. Thereby it is open in Y. Further, $f^{-1}\left(g^{-1}(W)\right)=(g\circ f)^{-1}(W)$ is δP_S-clopen in X. Hence $g\circ f$ is a totally δP_S-continuous function.

Proposition 5.4.13. Every strongly δP_S-continuous function is a totally δP_S-continuous function but not conversely.

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ be a strongly δP_S-continuous function and B be an open subset of Y. By hypothesis, $f^{-1}(B)$ is δP_S-clopen in X. This proves f is a totally δP_S-continuous function.

Example 5.4.14. Consider $X=\{a,b,c,d\},\tau=\{X,\emptyset,\{a,b\},\{c,d\}\}$ and $f:(X,\tau)\to(X,\sigma)$ is an identity function. Then f is totally δP_S-continuous function but not strongly δP_S-continuous.

5.5 Contra δP_S-Continuous Functions

Definition 5.5.1: A function $f:(X,\tau)\to(Y,\sigma)$ is said to be **contra δP_S-continuous**, for every open subset V of Y, $f^{-1}(V)$ is δP_S-closed.

Theorem 5.5.2: The following results supervene from their definitions directly:

a) Every contra complete continuous function is contra δP_S-continuous.
b) Every contra super continuous function is contra δP_S-continuous.
c) Every contra P_S-continuous function is contra δP_S-continuous.
d) Every contra δP_S-continuous function is contra δ-precontinuous.

Proof: (a) Since every regular open is a δP_S-open set [Proposition 2.2.15].

(b) Since every δ-open set is a δP_S-open set [Proposition 2.2.14].

(c) Since every P_S-open set is a δP_S-open set [Proposition 2.2.13].

(d) Since every δP_S-open set is a δP-open set [Definition 2.2.1].

Remark 5.5.3: From the above Proposition we have the following figure:

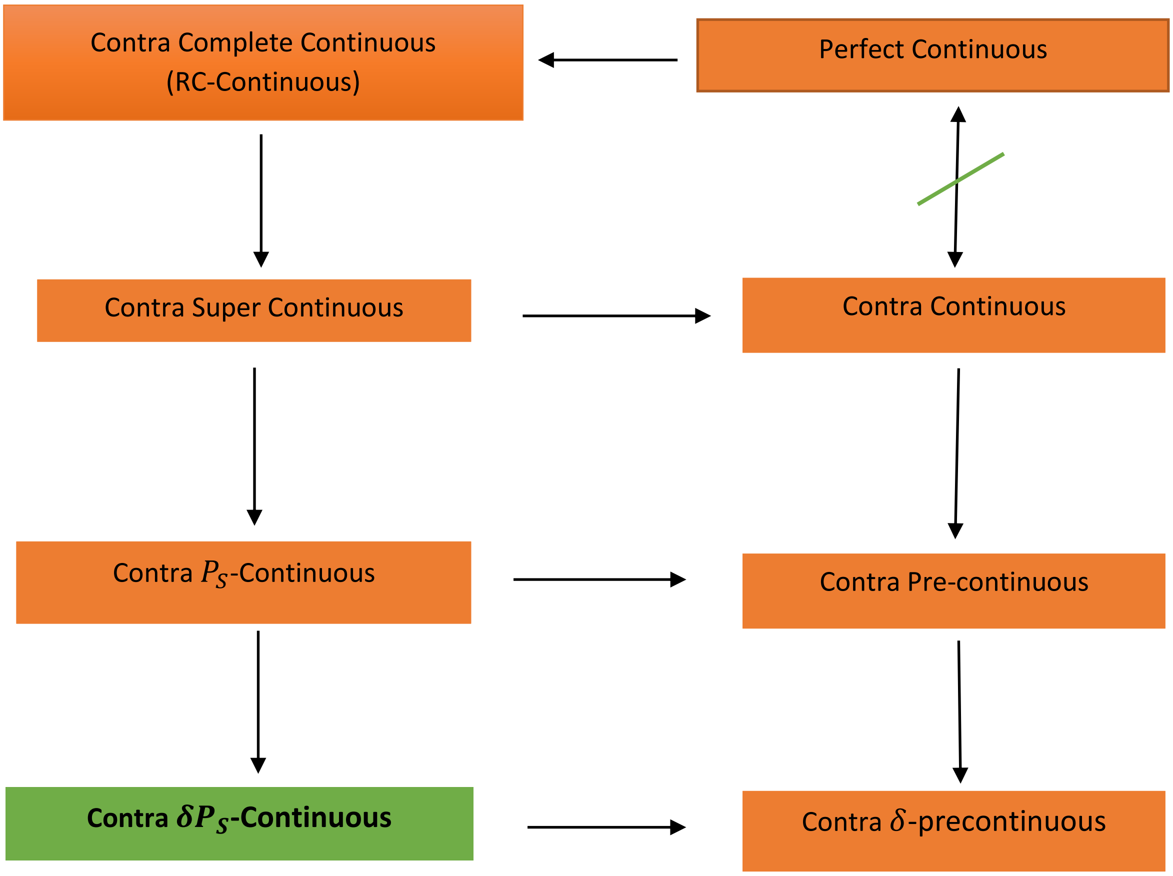

Figure 5.2

Example 5.5.4: Let $X = \{a, b, c, d\}$ with the two topologies τ = {∅, X, {c}, {a, d}, {a, c, d}} and σ = {∅, X, {a}, {b, c}, {a, b, c}}. Let $f: (X, \tau) \to (X, \sigma)$ be a function defined as follows: $f(a) = f(b) = f(c) = a$ and $f(d) = b$. Then f is contra-precontinuous but not contra- δP_S-continuous, since $\{b, c\} \in \sigma$ but $f^{-1}(\{b, c\}) = \{d\}$ is not δP_S-closed in (X, τ).

Example 5.5.5: Let $X = \{a, b, c\}$ with the two topologies

$\tau = \{X, \emptyset, \{a\}\}, \sigma = \{X, \emptyset, \{a\}, \{a, b\}. \{a, c\}\}$. Then $\delta P_S O(\tau) = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be identity function. Since every closed set in σ has inverse image in $\delta P_S O(\tau)$. $\therefore f$ is contra δP_S-continuous function. But not contra super continuous and contra complete continuous. Since δ-open sets in (X, τ) are only X and $\emptyset$.

Remark 5.5.6: The following two examples will show that the concepts of contra δP_S-continuous and δP_S-continuous are independent from each other.

Example 5.5.7: Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$; then $\delta P_S O(X, \tau) = \tau$. Let $f: (X, \tau) \to (X, \tau)$ be a function defined as follows: $f(a) = f(b) = c$ and $f(c) = b$. Then f is contra δP_S-continuous, but not δP_S-continuous, since $\{b\}$ is an open set of (X, τ) but $f^{-1}(\{b\}) = \{c\}$ is not δP_S-open in (X, τ).

Example 5.5.8: Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$; then $\delta P_S O(X, \tau) = \tau$. Let $f: (X, \tau) \to (X, \tau)$ be a function defined as follows: $f(a) = f(b) = c$ and $f(c) = b$. Then f is δP_S-continuous, but not contra δP_S-continuous, since $\{c\}$ is an closed set of (X, τ) but $\{c\}$ is not δP_S-open in (X, τ).

Theorem 5.5.9: For a function $f: (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(a) f is contra δP_S-continuous.

(b) For every closed subset F of $Y, f^{-1}(F)$ is δP_S-open.

(c) For every $x \in X$ and every closed subset F of Y containing $f(x)$, there exists a δP_S-open subset U of X containing x such that $f(U) \subseteq F$.

(d) $f(\delta P_S Cl(A)) \subseteq \ker(f(A))$ for every subset A of X.

(e) $\delta P_S Cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$ for every subset B of Y.

Proof: $(a) \Rightarrow (b)$ It follows from the fact that closedness (resp. δP_S-closedness) is the complement of openness (resp. δP_S-openness).

$(b) \Rightarrow (c)$ It follows from the definition of contra δP_S-continuity.

$(c) \Rightarrow (d)$ Let $A \subseteq X$ and let $y \in f(\delta P_S Cl(A))$. Suppose $y \notin \ker(f(A)) = \cap \{V / V \in \sigma$ and $f(A) \subseteq V\}$. Then there exists an open subset V of Y such that $f(A) \subseteq V$ and $y \notin V$. Let $x \in \delta P_S Cl(A)$ such that $y = f(x)$. Then $f(x) \in Y \setminus V$, which is closed in Y. By (c) there exists a δP_S-open subset U of X for which $x \in U$ and $f(U) \subseteq Y \setminus V$. Since $f(A) \subseteq V, A \cap U = \emptyset$. Since U is δP_S-open, it follows that $x \notin \delta P_S Cl(A)$.

$(d) \Rightarrow (e)$ Let $B \subseteq Y$. It follows from (d) that

$$f(\delta P_S Cl(f^{-1}(B))) \subseteq \ker\left(f\left(f^{-1}(B)\right)\right) \subseteq \ker(B)$$

and thus $\delta P_S Cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$.

$(e) \Rightarrow (a)$ Let V be an open subset of Y. Using (e) we obtain $\delta P_S Cl\left(f^{-1}(V)\right) \subseteq f^{-1}(\ker(V)) = f^{-1}(V)$ and, since $\delta P_S Cl(f^{-1}(A))$ is δP_S-closed, it follows that $f^{-1}(V)$ is δP_S-closed.

Definition 5.5.10: A function $f: (X, \tau) \to (Y, \sigma)$ is said to be **usc-continuous** if, for every closed subset F of $Y, f^{-1}(F)$ is a union of semi-closed sets

The implications below follow from the definition of a δP_S-open set and the fact that regular open sets are both δ-preopen and semi-closed.

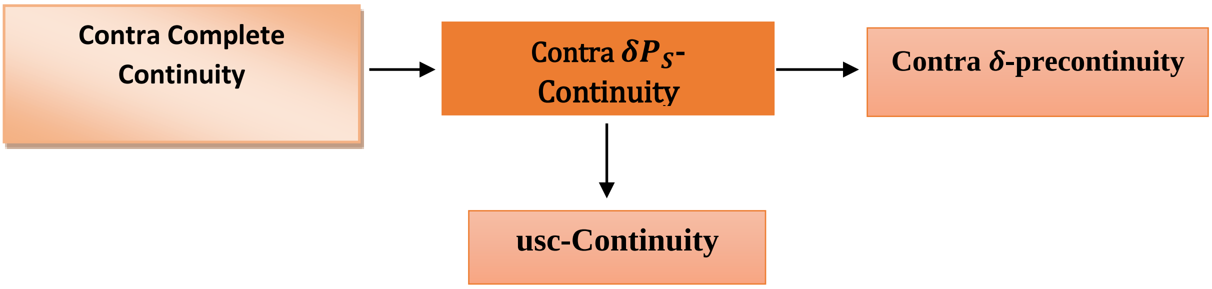

Figure 5.3

Proposition 5.5.11: Every totally δP_S-continuous function is a contra δP_S-continuous function but not conversely.

Proof: Let $f: (X, \tau) \to (Y, \sigma)$ be a totally δP_S-continuous function and B be a closed subset of Y. By hypothesis, $f^{-1}(B)$ is δP_S-clopen in X. This proves f is a contra δP_S-continuous function.

Example 5.5.12: Consider $X = \{a, b, c\}, \tau = \sigma = \{X, \emptyset, \{a\}\}$ and $f: (X, \tau) \to (X, \sigma)$ is an identity function. Then f is contra δP_S-continuous function but not totally δP_S-continuous.

Note 5.5.13: δP_S-continuity and contra δP_S-continuity are independent.

Example 5.5.14: Let $X = \{a, b, c\}$ with two topologies $\tau = \sigma = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $f: (X, \tau) \to (X, \sigma)$ be identity functions is δP_S-continuous, but not contra δP_S-continuous.

Example 5.5.15: Let $X = \{a, b, c\}$ with the two topologies $\tau = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}, \sigma = \{X, \emptyset, \{a\}, \{a, b\}. \{a, c\}\}$. Then $\delta P_S O(\tau) = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be identity function. Since every closed set in σ has inverse image in $\delta P_S O(\tau)$. $\therefore f$ is contra δP_S-continuous function, but not δP_S-continuous.

Proposition 5.5.16: If $f: (X, \tau) \to (Y, \sigma)$ is contra δP_S-continuous and Y is a C-space, then f is δP_S-continuous.

Proof: Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Since Y is a C-space, there exists a closed subset F of Y such that $x \in F \subseteq V$. Then, since f is contra δP_S-continuous, From Theorem 5.5.9(c), implies that there exists a δP_S-open subset U of X containing x such that $f(U) \subseteq F$. Hence $f(U) \subseteq V$, which proves that f is δP_S-continuous.

Proposition 5.5.17: If $f: (X, \tau) \to (Y, \sigma)$ is contra δP_S-continuous and Y is either regular or T_1, then f is δP_S-continuous.

Proof: Let x be any arbitrary point of X and V be an open set of Y containing $f(x)$ such that $Cl(G) \subseteq Y$. Since f is contra δP_S-continuous, by Theorem 5.5.9(c), there exists a δP_S-open set U of X containing x such that $f(U) \subseteq Cl(G)$. Then $f(U) \subseteq Cl(G) \subseteq V$. Hence f is δP_S-continuous.

Proposition 5.5.18: A function $f:(X,\tau) \to (Y,\sigma)$ is contra δP_S-continuous if and only if f is contra-precontinuous and for each $x \in X$ and each closed set F of Y containing $f(x)$, there exists a semi-closed set E in X containing x such that $f(E) \subseteq F$.

Proof. Necessity. Let $x \in X$ and let F be any closed set of Y containing $f(x)$. Since f is contra δP_S-continuous, then by Theorem 5.5.9(b), there exists a δP_S-open set U of X containing x such that $f(U) \subseteq F$. Since U is δP_S-open set. Then for each $x \in U$, there exists a semi-closed set E of X such that $x \in E \subseteq U$. Therefore, we have $f(E) \subseteq F$. And also, since f is contra δP_S-continuous. Then f is contra precontinuous.

Sufficiency. Let F be any closed set of Y. We have to show that $f^{-1}(F)$ is δP_S-open set in X. Since f is contra precontinuous, then $f^{-1}(F)$ is preopen set which is δ-preopen in X. Let $x \in f^{-1}(F)$. Then $f(x) \in F$. By hypothesis, there exists a semi-closed set E of X containing x such that $f(E) \subseteq F$. Which implies that $x \in E \subseteq f^{-1}(F)$. Therefore $f^{-1}(F)$ is δP_S-open set in X. Hence by Theorem 5.5.9(b), f is contra δP_S-continuous.

Proposition 5.5.19: If a function $f:(X,\tau) \to (Y,\sigma)$ is contra δP_S-continuous and Y is regular, then f is δP_S-continuous.

Proof. Let x be any arbitrary point of X and V be an open set of Y containing $f(x)$. Since Y is regular, then there exists an open set G in Y containing $f(x)$ such that $Cl(G) \subseteq V$. Since f is contra δP_S-continuous, so by Theorem 5.5.9(b), there exists a δP_S-open set U of X containing $x$such that $f(U) \subseteq Cl(G) \subseteq V$. Hence, f is δP_S-continuous.

Theorem 5.5.20: The following statements are equivalent for a function $f:(X,\tau) \to (Y,\sigma)$:

a) f is perfectly continuous
b) f is contra δ- continuous and super continuous
c) f is contra P_S-continuous and super continuous
d) f is contra α-continuous and super continuous
e) f is contra δP_S-continuous and super continuous
f) f is contra δ-precontinuous and super continuous

Proof. The proof follows from Corollary 2.2.38.

Theorem 5.5.21: For any topological space the following statements are equivalent for a function $f:(X,\tau) \to (Y,\sigma)$:

a) f is RC continuous
b) f is contra super continuous and δ-SR-continuous
c) f is contra super continuous and δ-Sθ-continuous

d) f is contra α-continuous and δ-semi-continuous

e) f is contra δP_S continuous and δ-semi-continuous

f) f is contra δ-pre-continuous and δ-semi-continuous

g) f is contra α continuous and e^*-semi-continuous

Proof: Follows from Corollary 2.2.40.

Theorem 5.5.22: In a semi-regular space the following conditions are equivalent for $f:(X,\tau) \rightarrow (Y,\sigma)$:

a) a) f is RC continuous

b) f is contra P_S-continuous and semi-continuous

c) f is contra continuous and semi-continuous

d) f is contra α-continuous and semi-continuous

e) f is contra δP_S continuous and semi-continuous

f) f is contra δ-pre-continuous and semi-continuous

Proof: Follows from Corollary 2.2.39.

Corollary 5.5.23: Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function and X is semi- T_1 space. Then f is contra δP_S-continuous if and only if f is contra δ-precontinuous.

Proof: Follows from Proposition 2.2.23, Since in a semi-T_1-space $\delta P_S O(X) = \delta PO(X)$.

Corollary 5.5.24: Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function and X is locally indiscrete space. Then f is contra δP_S-continuous if and only if f is contra continuous.

Proof: Follows from Proposition 2.2.31.

Corollary 5.5.25: If X is both semi- T_1 and locally indiscrete space, the following statements are equivalent for a function $f:(X,\tau) \rightarrow (Y,\sigma)$:

a) f is contra δP_S-continuous

b) f is contra δ-precontinuous

c) f is contra precontinuous

d) f is contra continuous

Proof: Follows from Corollary 5.5.23, Corollary 5.5.24 and from Proposition 2.2.23.

Corollary 5.5.26: Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function and Y is locally indiscrete space. Then f is contra δP_S-continuous if and only if f is δP_S-continuous.

Proof: In a locally indiscrete space, $\delta P_S O(X,\tau) = \tau$. Moreover, in a locally indiscrete space every open set is closed. Hence every δP_S-open set is δP_S-closed. Thus, the concepts δP_S-continuity and contra δP_S-continuity coincide.

Corollary 5.5.27: Let $f:(X,\tau)\to(Y,\sigma)$ be a function and X is s-regular space. If f is contra continuous, then f is contra δP_S-continuous.

Proof: Follows from Proposition 2.2.35.

Proposition 5.5.28: If the function $f:(X,\tau)\to(Y,\sigma)$ is almost contra δ-precontinuous and either S-continuous or θ-irresolute, then $f:(X,\tau)\to(Y,\sigma_S)$ is contra δP_S-continuous.

Proof: Let $H\in\sigma_S$, then H is regular open set in (Y,σ). Since $f:(X,\tau)\to(Y,\sigma)$ is almost contra δ-precontinuous. Then $f^{-1}(H)$ is δ-preclosed set in X. Again since $f:(X,\tau)\to(Y,\sigma)$ is either S-continuous or θ-irrseolute, then $f^{-1}(H)$ is the intersection of regular open sets of X and hence is the intersection of semi-open sets of X. By Proposition 2.4.2, $f^{-1}(H)$ is a δP_S-closed set of X. Hence $f:(X,\tau)\to(Y,\sigma_S)$ is contra δP_S-continuous.

Proposition 5.5.29: Let X be a semi- T_1 space. A function $f:(X,\tau)\to(Y,\sigma)$ is almost contra δ-precontinuous if and only if $f:(X,\tau)\to(Y,\sigma_S)$ is contra- δP_S-continuous.

Proof. Let $H\in\sigma_S$, then H is regular open set in (Y,σ). Since $f:(X,\tau)\to(Y,\sigma)$ is almost contra δ-precontinuous. Then $f^{-1}(H)$ is preclosed set in X. Since X is semi-T_1 space. Then by Proposition 2.2.23, $f^{-1}(H)$ is δP_S-closed set in X. Therefore, $f:(X,\tau)\to(Y,\sigma_S)$ is contra-δP_S-continuous.

Conversely, Let H be any regular open set in (Y,σ), then $H\in\sigma_S$. Since $f:(X,\tau)\to(Y,\sigma_S)$ is contra δP_S-continuous. Then $f^{-1}(H)$ is δP_S-closed set in X and hence it is δ-preclosed set in X. Therefore, $f:(X,\tau)\to(Y,\sigma)$ is almost contra δ-precontinuous.

Proposition 5.5.30. Let X be a locally indiscrete space. A function $f:(X,\tau)\to(Y,\sigma)$ is (θ,s)-continuous if and only if $f:(X,\tau)\to(Y,\sigma_S)$ is contra δP_S-continuous.

Proof. Let $H\in\sigma_S$, then H is regular open set in (Y,σ). Since $f:(X,\tau)\to(Y,\sigma)$ is (θ,s)-continuous. Then $f^{-1}(H)$ is closed set in X. Since X is locally indiscrete space. Then by Proposition 2.2.31, $f^{-1}(H)$ is δP_S-closed set in X. Therefore, $f:(X,\tau)\to(Y,\sigma_S)$ is contra δP_S-continuous.

Conversely, Let H be any regular open set in (Y,σ), then $H\in\sigma_S$. Since $f:(X,\tau)\to(Y,\sigma_S)$ is contra δP_S-continuous. Then $f^{-1}(H)$ is δP_S-closed set in X. Since X is locally indiscrete space. Then by Proposition 2.2.31, $f^{-1}(H)$ is closed set in X. Therefore, $f:(X,\tau)\to(Y,\sigma)$ is (θ,s)-continuous.

5.5.1 Properties

In this section, we give some properties of contra δP_S-continuous functions.

Proposition 5.5.1.1: Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be contra- δP_S-continuous function. If A is either open or regular semi-open subset of X, then $f|A: A \rightarrow Y$ is contra δP_S-continuous in the subspace A.

Proof. Let F be any closed set of Y. Since f is contra δP_S-continuous. Then by Theorem 5.5.9(b), $f^{-1}(F)$ is δP_S-open set in X. Since A is either open or regular semiopen subset of X. Then by Proposition 2.3.5, $(f|A)^{-1}(F) = f^{-1}(F) \cap A$ is a δP_S-open subspace of A. Therefore, by Theorem 5.5.9(b), $f|A: A \rightarrow Y$ is contra δP_S-continuous.

From Theorem 5.5.1.1, we obtain the following corollary

Corollary 5.5.1.2. Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a contra- δP_S-continuous function. If A is regular open subset of X, then $f|A: A \rightarrow Y$ is contra- δP_S-continuous in the subspace A.

Proposition 5.5.1.3. A function $f:(X,\tau) \rightarrow (Y,\sigma)$ is contra- δP_S-continuous, if for each $x \in X$, there exists a regular open set A of X containing x such that $f|A: A \rightarrow Y$ is contra- δP_S-continuous in the subspace A.

Proof. Let $x \in X$, then by hypothesis, there exists a regular open set A containing x such that $f|A: A \rightarrow Y$ is contra- δP_S-continuous. Let F be any closed set of Y containing $f(x)$. By Theorem 5.5.9(c), there exists a δP_S-open set U in A containing x such that $(f|A)(U) \subseteq F$. Since A is regular open set. By Proposition 5.5.11(c), U is δP_S-open set in X and hence $f(U) \subseteq F$. Therefore, by Theorem 5.5.9(c), f is contra- δP_S-continuous

Corollary 5.5.1.4: Let $\{U_\alpha : \alpha \in \Delta\}$ be a regular open cover of a topological space X. A function $f:(X,\tau) \rightarrow (Y,\sigma)$ is contra- δP_S-continuous if and only if $f|U_\alpha: U_\alpha \rightarrow Y$ is contra- δP_S-continuous for each $\alpha \in \Delta$.

Proof. Follows from Corollary 5.5.1.2 and Proposition 5.5.1.3.

Proposition 5.5.1.5. If $X = R \cup S$, where R and S are regular open sets and $f:(X,\tau) \rightarrow (Y,\sigma)$ is a function such that both f | R and f | S are contra- δP_S-continuous, then f is contra- δP_S-continuous.

Proof. Let F be any closed set of Y. Then $f^{-1}(F) = (f|R)^{-1}(F) \cup (f|S)^{-1}(F)$. Since $f|R$ and $f|S$ are contra δP_S-continuous. Then by Theorem 5.5.9(b), $(f|R)^{-1}(F)$ and $(f|S)^{-1}(F)$ are δP_S-open sets in R and S, respectively. Since R and S are regular open sets in X, then by Proposition 2.3.2, $(f|R)^{-1}(F)$ and $(f|S)^{-1}(F)$. are δP_S-open sets in X. Since union of two δP_S-open sets is δP_S-open, $f^{-1}(F)$ is δP_S-open set in X. Therefore, by Theorem 5.5.9(b), f is contra- δP_Scontinuous.

In general, if $X = \cup \{K_\alpha : \alpha \in \Delta\}$, where each K_α is a regular open set and $f: (X, \tau) \to (Y, \sigma)$ is a function that $f|K_\alpha$ is contra δP_S-continuous for each α, then f is contra δP_S-continuous.

The following is the pasting lemma for contra δP_S-continuous functions:

Lemma 5.5.1.6: Let $X = R_1 \cup R_2$, where R_1 and R_2 are regular open sets in X. Let $f: R_1 \to Y$ and $g: R_2 \to Y$ be contra δP_S-continuous. If $f(x) = g(x)$ for each $x \in R_1 \cap R_2$. Then $h: R_1 \cup R_2 \to Y$ such that

$$h(x) = \begin{cases} f(x) & if\ x \in R_1 \\ g(x) & if\ \ x \in R_2 \end{cases} \text{ is contra } \delta P_S\text{-continuous.}$$

Proof: Let E be a closed set of Y. Now $h^{-1}(E) = f^{-1}(E) \cup g^{-1}(E)$. Since f is contra δP_S-continuous, then by Theorem 5.5.9(b), $f^{-1}(E)$ is δP_S-open set in R_1. But R_1 is regular open set in X. Then by Proposition 2.3.2, $f^{-1}(E)$ is δP_S-open set in X. Similarly, $g^{-1}(E)$ is δP_S-open set in R_2 and hence, δP_S-open set in X. Since union of two δP_S-open sets is δP_S-open. Therefore, $h^{-1}(E) = f^{-1}(E) \cup g^{-1}(E)$ is a δP_S-open set in X. Hence by Theorem 5.5.9(b), h is contra δP_S-continuous.

Proposition 5.5.1.7: Let $f: (X, \tau) \to (Y, \sigma)$ be contra δP_S-continuous surjection and A be either open or regular semi-open subset of X. If f is a closed function, then the function $g: A \to f(A)$, defined by $g(x) = f(x)$ for each $x \in A$, is contra δP_S-continuous.

Proof: Putting $H = f(A)$. Let $x \in A$ and let F be any closed set in H containing $g(x)$. Since H is closed in Y. Since f is contra δP_S-continuous, by Theorem 5.5.9(c), there exists a δP_S-open set U in X containing x such that $f(U) \subseteq F$. Taking $W = U \cap A$, since A is either open or regular semi-open subset of X. Then by Proposition 2.3.5, W is a δP_S-open set in A containing x and $g(W) \subseteq F_Y \cap H = F_H$. Then $g(W) \subseteq F_H$. Therefore, by Theorem 5.5.9(c), g is contra δP_S-continuous.

Remark 5.5.1.8: Composition of contra δP_S-continuous need not be contra δP_S-continuous function.

Example 5.5.1.9: Let $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ such that $X = Y = Z = \{a, b, c\}, \tau = \{X, \emptyset, \{c\}, \{a, b\}\}, \sigma = \{X, \emptyset, \{a\}\}, \eta = \{X, \emptyset, \{a, b\}\}$; $\delta P_S(\tau) = \tau, \delta P_S(\sigma) = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then f and g are contra δP_S-continuous but $g \circ f$ is not contra δP_S-continuous, since $(g \circ f)^{-1}(\{a, b\}) = \{b, c\} \notin \delta P_S$-closed sets of τ.

We shall obtain some conditions for the composition of two functions to be contra δP_S-continuous.

Theorem 5.5.1.10: Let $f:(X,\tau) \to (Y,\sigma)$ and $g:(Y,\sigma) \to (Z,\eta)$ be functions. Then the composition $g \circ f:(X,\tau) \to (Z,\eta)$ is contra δP_S-continuous if f and g satisfy one of the following conditions:

a) f is contra δP_S-continuous and g is continuous.

b) f is continuous and open and g is contra δP_S-continuous

c) f is δP_S-continuous and g is contra continuous.

Proof: a) Let W be any open subset of Z. Since g is continuous, $g^{-1}(W)$ is open subset of Y. Since f is contra δP_S-continuous, then $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is δP_S-closed subset in X. Therefore $g \circ f$ is contra δP_S-continuous.

b) Let F be any closed subset of Z. Since g is contra δP_S-continuous, by Theorem 5.5.9(b), $g^{-1}(F)$ is δP_S-open subset of Y. Since f is continuous and open, then by Theorem 5.2.9, $f^{-1}\left(g^{-1}(F)\right) = (g \circ f)^{-1}(F)$ is a δP_S-open set in X. Hence by Theorem 5.5.9(b), $g \circ f$ is contra δP_S-continuous.

c) Let W be any open subset of Z. Since g is contra continuous, then $g^{-1}(W)$ is closed subset of Y. Since f is δP_S-continuous, by Theorem 5.2.7(c), $(g \circ f)^{-1}(W) = f^{-1}\left(g^{-1}(W)\right)$ is δP_S-closed subset in X. Therefore, $g \circ f$ is contra δP_S-continuous.

Proposition 5.5.1.11: If $f:(X,\tau) \to (Y,\sigma)$ is a function and $g:(Y,\sigma) \to (Z,\eta)$ is injecticve and closed, and $g \circ f:(X,\tau) \to (Z,\eta)$ is contra δP_S-continuous, then f is contra δP_S-continuous.

Proof: Let F be a closed subset of Y. Since g is closed, $g(F)$ is closed in Z. Then, since $g \circ f$ is contra δP_S-continuous and g is injective, we see that $f^{-1}(F) = f^{-1}\left(g^{-1}\left(g(F)\right)\right) = (g \circ f)^{-1}(g(F))$ is δP_S-open in X, which proves that f is contra δP_S-continuous.

Proposition 5.5.1.12: The set of all points of x of X at which $f:(X,\tau) \to (Y,\sigma)$ is not contra δP_S-continuous is identical with the union of the δP_S-boundaries of the inverse images of closed subsets of Y containing $f(x)$.

Proof: If f is not contra δP_S-continuous at $x \in X$, then there exists a closed set F of Y containing $f(x)$ such that $f(U) \cap Y \backslash F \neq \emptyset$ for every δP_S-open set U of containing x. This means tthat for every δP_S-open set $U$of X containing x, we must have have $U \cap X \backslash f^{-1}(F) \neq \emptyset$. Therefore, we have $x \in \delta P_S(Cl(X \backslash f^{-1}(F))$. But $x \in f^{-1}(F)$ and hence $x \in \delta P_S(Cl(f^{-1}(F))$. This means that $x \in \delta P_S Bd(f^{-1}(F))$.

On the other hand, suppose that $x \in \delta P_S Bd(f^{-1}(F))$ for some closed subset F of Y such that $f(x) \in F$. Supoose that f is contra δP_S-continuous at x. Then by Theorem 5.5.9(c), there exists a δP_S-open set U of X containing x such that $f(U) \subseteq F$. Then we have $U \subseteq f^{-1}(F)$. This shows that $x \in \delta P_S Int(f^{-1}(F))$. Therefore, we have $x \notin \delta P_S(Cl(X \backslash f^{-1}(F))$ and $x \notin \delta P_S Bd(f^{-1}(F))$. But this is a contradiction. This means that f is not contra δP_S-continuous.

Definition 5.5.1.13: For a subset A of a topological space (X, τ), $\boldsymbol{\delta P_S}$**-frontier** of A is denoted by $\delta P_S F(A)$ and defined as $\delta P_S F(A) = \delta P_S Cl(A) \backslash \delta P_S Int(A)$.

Proposition 5.5.1.14: Let $f: (X, \tau) \to (Y, \sigma)$ be a function and let $x \in X$. Then f is not contra δP_S-continuous at x if and only if x is a member of the δP_S-frontier of the inverse image of a closed subset of Y containing $f(x)$.

Proof: (⇒) Assume f is not contra δP_S-continuous at x. Then by Theorem 5.5.9(c), there exists a closed subset F of Y such that $f(x) \in F$ and, for every δP_S-open subset U of X containing $x, f(U) \nsubseteq F$. Hence $U \cap f^{-1}(Y \backslash F) \neq \emptyset$ for every δP_S-open subset U of X containing x, which implies that $x \in \delta P_S Cl(f^{-1}(Y \backslash F))$ and hence $x \in \delta P_S Cl\left(f^{-1}(F)\right) \cap \delta P_S Cl\left(f^{-1}(Y \backslash F)\right) = \delta P_S Fr\left(f^{-1}(F)\right)$.

(⇐) Let $x \in X$ and assume that $x \in \delta P_S Fr(f^{-1}(F))$ for some closed subset F of Y containing $f(x)$. Suppose f is contra δP_S-continuous at x. Then there exists a δP_S-open subset U of X containing x such that $f(U) \subseteq F$. Then we see that $x \in U \subseteq f^{-1}(F)$ and hence that $x \notin \delta P_S Cl(Y \backslash f^{-1}(F))$ and thus $x \notin \delta P_S Fr(f^{-1}(F))$. Therefore f is not contra δP_S-continuous at x.

Definition 5.5.1.15: A function $f: (X, \tau) \to (Y, \sigma)$ is said to have $\boldsymbol{\delta P_S}$**-closed graph** if for every $(x, y) \in X \times Y \backslash G(f)$ there exists a δP_S-open set U of X such that $x \in U \subseteq X$ and an open set V such that $y \in V \subseteq Y$ for which $(x, y) \in U \times V \subseteq X \times Y \backslash G(f)$.

Proposition 5.5.1.16: If $f: (X, \tau) \to (Y, \sigma)$ is contra δP_S-continuous and Y is T_2, then $G(f)$ is δP_S-closed.

Proof: Assume $(x, y) \in X \times Y \backslash G(f)$, then since $y \neq f(x)$, there exist disjoint open subsets V and W of X and Y, respectively, such that $f(x) \in V$ and $y \in W$. Since f is contra δP_S-continuous, there exists a δP_S-open subset U of X such that $x \in U$ and $f(U) \subseteq V$. Because $V \cap W = \emptyset, f(U) \cap W = \emptyset$ and thus $(x, y) \in U \times W \subseteq X \times Y \backslash G(f)$. Therefore $G(f)$ is δP_S-closed.

Definition 5.5.1.17: A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is said to have a **contra δP_S-closed graph** if, for every $(x,y)\in X\times Y\backslash G(f)$, there exists a δP_S-open set U of X such that $x\in U\subseteq X$ and a closed set F such that $y\in F\subseteq Y$ for which $(x,y)\in U\times F\subseteq X\times Y\backslash G(f)$.

Proposition 5.5.1.18: A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is contra δP_S-continuous and Y is Urysohn, then $G(f)$ is contra δP_S-closed.

Proof: Assume $(x,y)\in X\times Y\backslash G(f)$. Then, since $y\neq f(x)$, there exist open subsets V and W of X and Y, respectively, such that $f(x)\in V$ and $y\in W$ and $Cl(V)\cap Cl(W)=\emptyset$. Since f is contra δP_S-continuous, there exists a δP_S-open subset U of X containing x such that $f(U)\subseteq Cl(V)$. Then we have $(x,y)\in U\times Cl(W)\subseteq X\times Y\backslash G(f)$, which proves that $G(f)$ is contra δP_S-closed.

Proposition 5.5.1.19: Assume Y is an Urysohn space and that $f_i: X_i\rightarrow Y$ for $i=1,2$ is contra δP_S-continuous for each i. The set $A=\{(x_1,x_2): f_1(x_1)=f_2(x_2)\}$ is δP_S-closed in the product space $X_1\times X_2$.

Proof: Suppose $(x_1,x_2)\in(X_1\times X_2)\backslash A$. Then $f_1(x_1)\neq f_2(x_2)$ and, since Y is Urysohn, there exists open sets V_1 and V_2 containing $f_1(x_1)$ and $f_2(x_2)$, respectively, such that $Cl(V_1)\cap Cl(V_2)=\emptyset$. Since products of δP_S-open sets are δP_S-open, ${f_1}^{-1}(Cl(V_1))\times {f_2}^{-1}(Cl(V_2))$ is δP_S-open in $X_1\times X_2$. Then we see that $(x_1,x_2)\in {f_1}^{-1}\big(Cl(V_1)\big)\times {f_2}^{-1}\big(Cl(V_2)\big)\subseteq(X_1\times X_2)\backslash A$. Since the union of δP_S-open sets are δP_S-open, it follows that A is δP_S-closed in $X\times Y$.

Proposition 5.5.1.20: Let $f_\alpha:(X,\tau)\rightarrow(Y_\alpha,\sigma)$ be a function for every $\alpha\in\mathcal{A}$ and let $f: X\rightarrow\prod_{\alpha\in\mathcal{A}}Y_\alpha$ be the product function given by $f(x)=(f_\alpha(x))_{\alpha\in\mathcal{A}}$ for every $x\in X$. If f is contra δP_S-continuous, then f_α is contra δP_S-continuous for every $\alpha\in A$.

Proof: Let $\beta\in\mathcal{A}$ and $p_\beta:\prod_{\alpha\in\mathcal{A}}Y_\alpha\rightarrow Y_\beta$ be the βth projection function. Since $f_\beta=p_\beta\circ f$, it follows from Theorem 5.5.1.10(b) that f_β is contra δP_S-continuous.