

## Cubic Structures in Z-Algebras



In 2012, Jun et al. [36] have introduced a remarkable theory, namely, the theory of cubic sets. This structure is comprised of an interval-valued fuzzy set and a fuzzy set. This chapter is divided into two sections. In the first section, we introduce the notion of cubic Z-Subalgebras in Z-algebras. In the second section, we initiate the notion of cubic Z-Ideals in Z-algebras. Also we prove some interesting results.

### 9.1 Cubic Z-Subalgebras in Z-algebras

In this section, we define the notion of Cubic Z-Subalgebra of a Z-algebra and proved some interesting results.

**Definition 9.1.1:** Let  $(X, *, 0)$  be a Z-algebra. A cubic set  $A = (\tilde{\mu}_A, \omega_A)$  in X is called a **cubic Z-Subalgebra** of X if it satisfies the following conditions:

- (i)  $\tilde{\mu}_A(x * y) \geq r \min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$
- (ii)  $\omega_A(x * y) \leq \max\{\omega_A(x), \omega_A(y)\}$ , for all  $x, y \in X$ .

**Example 9.1.2:** Consider a Z-algebra  $X = \{0, 1, 2, 3\}$  be a set with the following Cayley table :

*	0	1	2	3
0	0	1	2	3
1	0	1	2	1
2	0	2	2	3
3	0	1	3	3

Define a cubic set A in X by  $\tilde{\mu}_A(x) = \begin{cases} [0.6, 0.7] & \text{if } x \in \{0, 2\} \\ [0.1, 0.2] & \text{otherwise} \end{cases}$  and  $\omega_A(x) = \begin{cases} 0.2 & \text{if } x \in \{0, 2\} \\ 0.6 & \text{otherwise} \end{cases}$

Then A is a cubic Z-Subalgebra of a Z-algebra X.

**Theorem 9.1.3:** The intersection of any set of cubic Z-Subalgebras of a Z-algebra X is also a cubic Z-Subalgebra of X.

**Proof:** Let  $A_i = \langle x, \tilde{\mu}_{A_i}(x), \omega_{A_i}(x) \rangle | x \in X$  where  $i \in \Omega$  an index set, be a set of cubic Z-Subalgebras of a Z-algebra X. For  $x, y \in X$ ,

$$\begin{aligned} \tilde{\mu}_{\cap A_i}(x * y) &= r \inf \tilde{\mu}_{A_i}(x * y) \geq r \inf \{r \min \{ \tilde{\mu}_{A_i}(x), \tilde{\mu}_{A_i}(y) \} \} = r \min \{ r \inf \tilde{\mu}_{A_i}(x), r \inf \tilde{\mu}_{A_i}(y) \} \\ &= r \min \{ \tilde{\mu}_{\cap A_i}(x), \tilde{\mu}_{\cap A_i}(y) \} \end{aligned}$$

$$\begin{aligned} \text{and } \omega_{\cup A_i}(x * y) &= \sup \omega_{A_i}(x * y) \leq \sup \{ \max \{ \omega_{A_i}(x), \omega_{A_i}(y) \} \} = \max \{ \sup \omega_{A_i}(x), \sup \omega_{A_i}(y) \} \\ &= \max \{ \omega_{\cup A_i}(x), \omega_{\cup A_i}(y) \} \end{aligned}$$

Hence  $\bigcap_{i \in \Omega} A_i = \langle x, \tilde{\mu}_{\cap A_i}(x), \omega_{\cup A_i}(x) \rangle | x \in X$  is a cubic Z-Subalgebra of X.

We remark here that the union, P-intersection and P-union of any sets of cubic Z-Subalgebras in Z-algebras is need not be a cubic Z-subalgebra. We now provide some conditions that the union, P-intersection and P-union of any cubic Z-Subalgebras in Z-algebras are still a cubic Z-Subalgebra in the following three theorems.

**Theorem 9.1.4:** Let  $A_i = \langle x, \tilde{\mu}_{A_i}(x), \omega_{A_i}(x) \rangle | x \in X$  where  $i \in \Omega$  an index set, be cubic Z-Subalgebras of a Z-algebra X. If

$$r \sup \{ r \min \{ \tilde{\mu}_{A_i}(x), \tilde{\mu}_{A_i}(y) \} \} = r \min \{ r \sup \tilde{\mu}_{A_i}(x), r \sup \tilde{\mu}_{A_i}(y) \} \quad \text{and}$$

$\inf \{ \max \{ \omega_{A_i}(x), \omega_{A_i}(y) \} \} = \max \{ \inf \omega_{A_i}(x), \inf \omega_{A_i}(y) \}$ , for all  $x, y \in X$ , then the union of  $A_i$  is again a cubic Z-Subalgebra of X.

**Proof:** For any  $x, y \in X$ ,

$$\begin{aligned} \tilde{\mu}_{\cup A_i}(x * y) &= r \sup \tilde{\mu}_{A_i}(x * y) \geq r \sup \{ r \min \{ \tilde{\mu}_{A_i}(x), \tilde{\mu}_{A_i}(y) \} \} = r \min \{ r \sup \tilde{\mu}_{A_i}(x), r \sup \tilde{\mu}_{A_i}(y) \} \\ &= r \min \{ \tilde{\mu}_{\cup A_i}(x), \tilde{\mu}_{\cup A_i}(y) \} \end{aligned}$$

$$\begin{aligned} \text{and } \omega_{\cap A_i}(x * y) &= \inf \omega_{A_i}(x * y) \leq \inf \{ \max \{ \omega_{A_i}(x), \omega_{A_i}(y) \} \} = \max \{ \inf \omega_{A_i}(x), \inf \omega_{A_i}(y) \} \\ &= \max \{ \omega_{\cap A_i}(x), \omega_{\cap A_i}(y) \} \end{aligned}$$

Hence  $\bigcup_{i \in \Omega} A_i = \langle x, \tilde{\mu}_{\cup A_i}(x), \omega_{\cap A_i}(x) \rangle | x \in X$  is a cubic Z-Subalgebra of a Z-algebra X.

**Theorem 9.1.5:** Let  $A_i = \left\langle x, \tilde{\mu}_{A_i}(x), \omega_{A_i}(x) \mid x \in X \right\rangle$  where  $i \in \Omega$  an index set, be cubic Z-Subalgebras of a Z-algebra X. If  $\inf\{\max\{\omega_{A_i}(x), \omega_{A_i}(y)\}\} = \max\{\inf \omega_{A_i}(x), \inf \omega_{A_i}(y)\}$ , for all  $x, y \in X$ , then the P-intersection of  $A_i$  is again a cubic Z-Subalgebra of X.

**Proof:** For any  $x, y \in X$ ,

$$\begin{aligned} \tilde{\mu}_{\cap A_i}(x * y) &= r \inf \tilde{\mu}_{A_i}(x * y) \geq r \inf\{r \min\{\tilde{\mu}_{A_i}(x), \tilde{\mu}_{A_i}(y)\}\} = r \min\{r \inf \tilde{\mu}_{A_i}(x), r \inf \tilde{\mu}_{A_i}(y)\} \\ &= r \min\{\tilde{\mu}_{\cap A_i}(x), \tilde{\mu}_{\cap A_i}(y)\} \end{aligned}$$

$$\begin{aligned} \text{and } \omega_{\cap A_i}(x * y) &= \inf \omega_{A_i}(x * y) \leq \inf\{\max\{\omega_{A_i}(x), \omega_{A_i}(y)\}\} = \max\{\inf \omega_{A_i}(x), \inf \omega_{A_i}(y)\} \\ &= \max\{\omega_{\cap A_i}(x), \omega_{\cap A_i}(y)\} \end{aligned}$$

which shows that  $P\left(\bigcap_{i \in \Omega} A_i\right) = (\tilde{\mu}_{\cap A_i}, \omega_{\cap A_i})$  a cubic Z-Subalgebra of a Z-algebra X.

**Theorem 9.1.6:** Let  $A_i = \left\langle x, \tilde{\mu}_{A_i}(x), \omega_{A_i}(x) \mid x \in X \right\rangle$  where  $i \in \Omega$  an index set, be cubic Z-Subalgebras of a Z-algebra X. If  $r \sup\{r \min\{\tilde{\mu}_{A_i}(x), \tilde{\mu}_{A_i}(y)\}\} = r \min\{r \sup \tilde{\mu}_{A_i}(x), r \sup \tilde{\mu}_{A_i}(y)\}$ , for all  $x, y \in X$ , then the P-union of  $A_i$  is again a cubic Z-Subalgebra of X.

**Proof:** For any  $x, y \in X$ ,

$$\begin{aligned} \tilde{\mu}_{\cup A_i}(x * y) &= r \sup \tilde{\mu}_{A_i}(x * y) \geq r \sup\{r \min\{\tilde{\mu}_{A_i}(x), \tilde{\mu}_{A_i}(y)\}\} = r \min\{r \sup \tilde{\mu}_{A_i}(x), r \sup \tilde{\mu}_{A_i}(y)\} \\ &= r \min\{\tilde{\mu}_{\cup A_i}(x), \tilde{\mu}_{\cup A_i}(y)\} \end{aligned}$$

$$\begin{aligned} \text{and } \omega_{\cup A_i}(x * y) &= \sup \omega_{A_i}(x * y) \leq \sup\{\max\{\omega_{A_i}(x), \omega_{A_i}(y)\}\} = \max\{\sup \omega_{A_i}(x), \sup \omega_{A_i}(y)\} \\ &= \max\{\omega_{\cup A_i}(x), \omega_{\cup A_i}(y)\} \end{aligned}$$

which shows that  $P\left(\bigcup_{i \in \Omega} A_i\right) = (\tilde{\mu}_{\cup A_i}, \omega_{\cup A_i})$  is a cubic Z-Subalgebra of X.

**Theorem 9.1.7:** Cubic set  $A = (\tilde{\mu}_A, \omega_A)$  of a Z-algebra X is a cubic Z-Subalgebra of X where  $\tilde{\mu}_A = [\mu_A^L, \mu_A^U]$  if and only if  $\mu_A^L, \mu_A^U$  and  $(\omega_A)^c$  are fuzzy Z-Subalgebras of X.

**Proof:** Let  $\mu_A^L, \mu_A^U$  and  $(\omega_A)^c$  are fuzzy Z-Subalgebras of a Z-algebra X and  $x, y \in X$ .

Then,  $\tilde{\mu}_A(x * y) \geq r \min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

Also  $(\omega_A)^c(x * y) \geq \min\{(\omega_A)^c(x), (\omega_A)^c(y)\}$

$\Rightarrow 1 - \omega_A(x * y) \geq \min\{1 - \omega_A(x), 1 - \omega_A(y)\} \geq 1 - \max\{\omega_A(x), \omega_A(y)\}$

implies  $\omega_A(x * y) \leq \max\{\omega_A(x), \omega_A(y)\}$

Thus A is cubic Z-Subalgebra of a Z-algebra X.

Conversely, assume that A is a cubic Z-Subalgebra of a Z-algebra X.

For any  $x, y \in X$ , we have  $\mu_A^L(x * y) \geq \min\{\mu_A^L(x), \mu_A^L(y)\}$ ,  $\mu_A^U(x * y) \geq \min\{\mu_A^U(x), \mu_A^U(y)\}$

Also,  $\omega_A(x * y) \leq \max\{\omega_A(x), \omega_A(y)\}$

$\Rightarrow 1 - \omega_A(x * y) \geq 1 - \max\{\omega_A(x), \omega_A(y)\}$

$\Rightarrow 1 - \omega_A(x * y) \geq \min\{1 - \omega_A(x), 1 - \omega_A(y)\}$

$\Rightarrow (\omega_A)^c(x * y) \geq \min\{(\omega_A)^c(x), (\omega_A)^c(y)\}$

Thus  $\mu_A^L$ ,  $\mu_A^U$  and  $(\omega_A)^c$  are fuzzy Z-Subalgebras of a Z-algebra X.

**Theorem 9.1.8:** If a cubic set  $A = (\tilde{\mu}_A, \omega_A)$  in a Z-algebra X is a cubic Z-Subalgebra of X then for  $[s_1, s_2] \in D[0,1]$  and  $t \in [0,1]$  the interval-valued upper  $[s_1, s_2]$ -level subset  $U(\tilde{\mu}_A; [s_1, s_2])$  and lower t-level subset  $L(\omega_A; t)$  of A are Z-Subalgebras of X.

**Proof:** Assume that A is a cubic Z-Subalgebra of a Z-algebra X and let  $[s_1, s_2] \in D[0,1]$  be such that  $x, y \in U(\tilde{\mu}_A; [s_1, s_2])$ . Then clearly,  $\tilde{\mu}_A(x * y) \geq [s_1, s_2]$ .

Thus  $U(\tilde{\mu}_A; [s_1, s_2])$  is a Z-Subalgebra of a Z-algebra X.

Let  $t \in [0,1]$  be such that  $x, y \in L(\omega_A; t)$ . Then,  $\omega_A(x) \leq t$  and  $\omega_A(y) \leq t$ .

It follows that,  $\omega_A(x * y) \leq \max\{\omega_A(x), \omega_A(y)\} \leq \max\{t, t\} = t$  so that  $x * y \in L(\omega_A; t)$ .

Hence,  $L(\omega_A; t)$  is a Z-Subalgebra of a Z-algebra X.

This completes the proof.

Converse of the above theorem stated as follows can be easily proved.

**Theorem 9.1.9:** Let  $A = (\tilde{\mu}_A, \omega_A)$  be a cubic set of a Z-algebra X, such that the sets  $U(\tilde{\mu}_A; [s_1, s_2])$  and  $L(\omega_A; t)$  are Z-Subalgebras of X for every  $[s_1, s_2] \in D[0,1]$  and  $t \in [0,1]$ . Then  $A = (\tilde{\mu}_A, \omega_A)$  is a cubic Z-Subalgebra of X.

**Theorem 9.1.10:** Any Z-Subalgebra of a Z-algebra X can be realized as both the interval-valued upper  $[s_1, s_2]$ - level and lower t-level subsets of some cubic Z-Subalgebra of X.

**Proof:** Let Y be a Z-Subalgebra of a Z-algebra X and A be a cubic set on X defined by

$$\tilde{\mu}_A(x) = \begin{cases} [s_1, s_2] & \text{if } x \in Y \\ [0, 0] & \text{otherwise} \end{cases} \quad \text{and} \quad \omega_A(x) = \begin{cases} t & \text{if } x \in Y \\ 1 & \text{otherwise} \end{cases}$$

where  $[s_1, s_2] \in D[0, 1]$  and  $t \in [0, 1]$  with  $[s_1, s_2] > [0, 0]$  and  $t < 1$ .

**To prove:** A is a cubic Z-Subalgebra of a Z-algebra X.

Let  $x, y \in X$ . We consider the following cases.

**Case (1):** If  $x, y \in Y$  then  $x * y \in Y$ .

$$\text{Therefore, } \tilde{\mu}_A(x * y) = [s_1, s_2] = r \min \{[s_1, s_2], [s_1, s_2]\} = r \min \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$$

$$\text{and } \omega_A(x * y) = t = \max\{t, t\} = \max\{\omega_A(x), \omega_A(y)\}$$

**Case (2):** If  $x, y \notin Y$  then  $\tilde{\mu}_A(x) = [0, 0] = \tilde{\mu}_A(y)$  and  $\omega_A(x) = 1 = \omega_A(y)$ .

$$\text{Now, } \tilde{\mu}_A(x * y) = [0, 0] = r \min \{[0, 0], [0, 0]\} = r \min \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$$

$$\text{and } \omega_A(x * y) \leq 1 = \max\{1, 1\} = \max\{\omega_A(x), \omega_A(y)\}$$

**Case (3):** If  $x \in Y$  and  $y \notin Y$  then  $\tilde{\mu}_A(x) = [s_1, s_2]$ ,  $\tilde{\mu}_A(y) = [0, 0]$ ,  $\omega_A(x) = t$  and  $\omega_A(y) = 1$ .

$$\text{Thus, } \tilde{\mu}_A(x * y) \geq [0, 0] = r \min \{[s_1, s_2], [0, 0]\} = r \min \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$$

$$\text{and } \omega_A(x * y) \leq 1 = \max\{t, 1\} = \max\{\omega_A(x), \omega_A(y)\}$$

**Case (4):** If  $y \in Y$  and  $x \notin Y$  then  $\tilde{\mu}_A(x) = [0, 0]$ ,  $\tilde{\mu}_A(y) = [s_1, s_2]$ ,  $\omega_A(x) = 1$  and

$$\omega_A(y) = t. \text{ Thus, } \tilde{\mu}_A(x * y) \geq [0, 0] = r \min \{[0, 0], [s_1, s_2]\} = r \min \{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$$

$$\text{and } \omega_A(x * y) \leq 1 = \max\{1, t\} = \max\{\omega_A(x), \omega_A(y)\}$$

Therefore, A is a cubic Z-Subalgebra of a Z-algebra X.

Obviously,  $U(\tilde{\mu}_A; [s_1, s_2]) = Y = L(\omega_A; t)$ .

**Theorem 9.1.11:** Let Y be a subset of a Z-algebra X and A be a cubic set on X which is given in the proof of Theorem 9.1.10. If Y be realized as interval-valued upper  $[s_1, s_2]$ -level

Z-Subalgebra and lower t-level Z-Subalgebra of cubic Z-Subalgebra A of X, then Y is a Z-Subalgebra of a Z-algebra X.

**Proof:** Let Y be a subset of a Z-algebra X and Let A be a cubic Z-Subalgebra of a Z-algebra X such that  $U(\tilde{\mu}_A; [s_1, s_2]) = Y = L(\omega_A; t)$ .

**To prove:** Y is a Z-Subalgebra of a Z-algebra X.

Let  $x, y \in Y$ .

Then,  $\tilde{\mu}_A(x) = [s_1, s_2] = \tilde{\mu}_A(y)$  and  $\omega_A(x) = t = \omega_A(y)$ .

Thus, 
$$\tilde{\mu}_A(x * y) \geq r \min \{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \} = r \min \{ [s_1, s_2], [s_1, s_2] \} = [\min \{ s_1, s_1 \}, \{ s_2, s_2 \}] = [s_1, s_2]$$

$\Rightarrow \tilde{\mu}_A(x * y) \geq [s_1, s_2]$ .

and  $\omega_A(x * y) \leq \max \{ \omega_A(x), \omega_A(y) \} = \max \{ t, t \} = t$ .

Thus  $x * y \in Y$ .

Hence Y is a Z-Subalgebra of a Z-algebra X.

**Theorem 9.1.12:** Let h be a Z-homomorphism from a Z-algebra  $(X, *, 0)$  onto a Z-algebra  $(Y, *, 0')$  and A be a cubic Z-Subalgebra of X with rsup-inf property. Then the image of A,  $h(A) = \{ \langle y, \tilde{\mu}_{h(A)}(y), \omega_{h(A)}(y) \rangle \mid y \in Y \}$  is a cubic Z-Subalgebra of Y.

**Proof:** Let  $a, b \in Y$  with  $x_0 \in h^{-1}(a)$  and  $y_0 \in h^{-1}(b)$  such that

$$\tilde{\mu}_A(x_0) = r \sup_{t \in h^{-1}(a)} \tilde{\mu}_A(t); \quad \tilde{\mu}_A(y_0) = r \sup_{t \in h^{-1}(b)} \tilde{\mu}_A(t) \quad \text{and}$$

$$\omega_A(x_0) = \inf_{t \in h^{-1}(a)} \omega_A(t); \quad \omega_A(y_0) = \inf_{t \in h^{-1}(b)} \omega_A(t)$$

Now,

$$\begin{aligned} \tilde{\mu}_{h(A)}(a * b) &= r \sup_{t \in h^{-1}(a * b)} \tilde{\mu}_A(t) \geq \tilde{\mu}_A(x_0 * y_0) \geq r \min \{ \tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0) \} \\ &= r \min \left\{ r \sup_{t \in h^{-1}(a)} \tilde{\mu}_A(t), r \sup_{t \in h^{-1}(b)} \tilde{\mu}_A(t) \right\} \\ &= r \min \{ \tilde{\mu}_{h(A)}(a), \tilde{\mu}_{h(A)}(b) \} \end{aligned}$$

That is,  $\tilde{\mu}_{h(A)}(a * b) \geq \min \{ \tilde{\mu}_{h(A)}(a), \tilde{\mu}_{h(A)}(b) \}$

$$\begin{aligned}
 \text{Also } \omega_{h(A)}(a *' b) &= \inf_{t \in h^{-1}(a *' b)} \omega_A(t) \leq \omega_A(x_0 * y_0) \leq \max\{\omega_A(x_0), \omega_A(y_0)\} \\
 &= \max\left\{ \inf_{t \in h^{-1}(a)} \omega_A(t), \inf_{t \in h^{-1}(b)} \omega_A(t) \right\} \\
 &= \max\{\omega_{h(A)}(a), \omega_{h(A)}(b)\}
 \end{aligned}$$

That is,  $\omega_{h(A)}(a *' b) \leq \max\{\omega_{h(A)}(a), \omega_{h(A)}(b)\}$

Therefore  $h(A)$  is a cubic Z-Subalgebra of a Z-algebra Y.

**Theorem 9.1.13:** Assume that  $h: (X, *, 0) \rightarrow (Y, *', 0')$  is a Z-homomorphism of Z-algebras and  $A_i = (\tilde{\mu}_{A_i}, \omega_{A_i})$  is a cubic Z-Subalgebras of X, where  $i \in \Omega$ . If

$$\inf\{\max\{\omega_{A_i}(x), \omega_{A_i}(y)\}\} = \max\{\inf \omega_{A_i}(x), \inf \omega_{A_i}(y)\}, \text{ for all } x, y \in X, \text{ then } h\left(\mathcal{P}\left(\bigcap_{i \in \Omega} A_i\right)\right) \text{ is}$$

a cubic Z-Subalgebra of Y.

**Proof:** Let  $A_i = (\tilde{\mu}_{A_i}, \omega_{A_i})$  be cubic Z-Subalgebras of a Z-algebra X, where  $i \in \Omega$  satisfying  $\inf\{\max\{\omega_{A_i}(x), \omega_{A_i}(y)\}\} = \max\{\inf \omega_{A_i}(x), \inf \omega_{A_i}(y)\}$ , for all  $x, y \in X$ . Then by Theorem

9.1.5,  $\mathcal{P}\left(\bigcap_{i \in \Omega} A_i\right)$  is a cubic Z-Subalgebra of X. Hence  $h\left(\mathcal{P}\left(\bigcap_{i \in \Omega} A_i\right)\right)$  is also a cubic Z-Subalgebra

of Y.

**Theorem 9.1.14:** Consider  $h: (X, *, 0) \rightarrow (Y, *', 0')$  is a Z-homomorphism of Z-algebras and  $A_i = (\tilde{\mu}_{A_i}, \omega_{A_i})$  is a cubic Z-Subalgebras of X, where  $i \in \Omega$ . If

$$r \sup\{r \min\{\tilde{\mu}_{A_i}(x), \tilde{\mu}_{A_i}(y)\}\} = r \min\{r \sup \tilde{\mu}_{A_i}(x), r \sup \tilde{\mu}_{A_i}(y)\}, \text{ for all } x, y \in X, \text{ then}$$

$h\left(\mathcal{P}\left(\bigcup_{i \in \Omega} A_i\right)\right)$  is also a cubic Z-Subalgebra of Y.

**Proof:** Let  $A_i = (\tilde{\mu}_{A_i}, \omega_{A_i})$  be cubic Z-Subalgebras of a Z-algebra X, where  $i \in \Omega$  satisfying

$$r \sup\{r \min\{\tilde{\mu}_{A_i}(x), \tilde{\mu}_{A_i}(y)\}\} = r \min\{r \sup \tilde{\mu}_{A_i}(x), r \sup \tilde{\mu}_{A_i}(y)\}, \text{ for all } x, y \in X. \text{ Then by}$$

Theorem 9.1.6,  $\mathcal{P}\left(\bigcup_{i \in \Omega} A_i\right)$  is a cubic Z-Subalgebra of X. Hence  $h\left(\mathcal{P}\left(\bigcup_{i \in \Omega} A_i\right)\right)$  is also a cubic

Z-Subalgebra of Y.

**Theorem 9.1.15:** Let  $h:(X,*,0) \rightarrow (Y,*,0')$  be a Z-homomorphism of Z-algebras and  $B = (\tilde{\mu}_B, \omega_B)$  is a cubic Z-Subalgebra of Y then the inverse image  $h^{-1}(B) = \left\{ \langle x, \tilde{\mu}_{h^{-1}(B)}(x), \omega_{h^{-1}(B)}(x) \rangle \mid x \in X \right\}$  of B under h is a cubic Z-Subalgebra of X.

**Proof:** Let  $B = (\tilde{\mu}_B, \omega_B)$  be a cubic Z-Subalgebra of Y.

**To prove:**  $h^{-1}(B)$  is a cubic Z-Subalgebra of a Z-algebra X.

Let  $x, y \in X$ ,

$$\begin{aligned} \text{Then } \tilde{\mu}_{h^{-1}(B)}(x * y) &= \tilde{\mu}_B(h(x * y)) = \tilde{\mu}_B(h(x) *' h(y)) \geq r \min \{ \tilde{\mu}_B(h(x)), \tilde{\mu}_B(h(y)) \} \\ &= r \min \{ \tilde{\mu}_{h^{-1}(B)}(x), \tilde{\mu}_{h^{-1}(B)}(y) \} \end{aligned}$$

$$\Rightarrow \tilde{\mu}_{h^{-1}(B)}(x * y) \geq r \min \{ \tilde{\mu}_{h^{-1}(B)}(x), \tilde{\mu}_{h^{-1}(B)}(y) \}$$

$$\begin{aligned} \text{and } \omega_{h^{-1}(B)}(x * y) &= \omega_B(h(x * y)) = \omega_B(h(x) *' h(y)) \leq \max \{ \omega_B(h(x)), \omega_B(h(y)) \} \\ &= \max \{ \omega_{h^{-1}(B)}(x), \omega_{h^{-1}(B)}(y) \} \end{aligned}$$

$$\Rightarrow \omega_{h^{-1}(B)}(x * y) \leq \max \{ \omega_{h^{-1}(B)}(x), \omega_{h^{-1}(B)}(y) \}$$

Thus  $h^{-1}(B)$  is a cubic Z-Subalgebra of a Z-algebra X.

**Theorem 9.1.16:** Let  $h:(X,*,0) \rightarrow (Y,*,0')$  be a Z-homomorphism of Z-algebras. Let  $B_j = (\tilde{\mu}_{B_j}, \omega_{B_j})$  be cubic Z-Subalgebras of Y, where  $j \in \Omega$ . If

$$\inf \{ \max \{ \omega_{B_j}(x), \omega_{B_j}(y) \} \} = \max \{ \inf \omega_{B_j}(x), \inf \omega_{B_j}(y) \}, \text{ for all } x, y \in Y, \text{ then } h^{-1} \left( P \left( \bigcap_{j \in \Omega} B_j \right) \right)$$

is also a cubic Z-Subalgebra of X.

**Proof:** Let  $B_j = (\tilde{\mu}_{B_j}, \omega_{B_j})$  be cubic Z-Subalgebras of Y, where  $j \in \Omega$  satisfying  $\inf \{ \max \{ \omega_{B_j}(x), \omega_{B_j}(y) \} \} = \max \{ \inf \omega_{B_j}(x), \inf \omega_{B_j}(y) \}$ , for all  $x, y \in Y$ . Then by Theorem

9.1.5,  $P \left( \bigcap_{j \in \Omega} B_j \right)$  is a cubic Z-Subalgebra of Y. Hence  $h^{-1} \left( P \left( \bigcap_{j \in \Omega} B_j \right) \right)$  is also a cubic

Z-Subalgebra of X.

**Theorem 9.1.17:** Let  $h : (X, *, 0) \rightarrow (Y, *, 0')$  be a Z-homomorphism of Z-algebras. Let  $B_j = (\tilde{\mu}_{B_j}, \omega_{B_j})$  be cubic Z-Subalgebras of Y, where  $j \in \Omega$ . If  $r \sup \{r \min \{ \tilde{\mu}_{B_j}(x), \tilde{\mu}_{B_j}(y) \} \} = r \min \{ r \sup \tilde{\mu}_{B_j}(x), r \sup \tilde{\mu}_{B_j}(y) \}$ , for all  $x, y \in Y$ , then  $h^{-1} \left( P \left( \bigcup_{j \in \Omega} B_j \right) \right)$  is also a cubic Z-Subalgebra of X.

**Proof:** Let  $B_j = (\tilde{\mu}_{B_j}, \omega_{B_j})$  be cubic Z-Subalgebras of Y, where  $j \in \Omega$  satisfying  $r \sup \{r \min \{ \tilde{\mu}_{B_j}(x), \tilde{\mu}_{B_j}(y) \} \} = r \min \{ r \sup \tilde{\mu}_{B_j}(x), r \sup \tilde{\mu}_{B_j}(y) \}$ , for all  $x, y \in Y$ . Then by Theorem 9.1.6,  $P \left( \bigcup_{j \in \Omega} B_j \right)$  is a cubic Z-Subalgebra of Y. Hence  $h^{-1} \left( P \left( \bigcup_{j \in \Omega} B_j \right) \right)$  is also a cubic Z-Subalgebra of X.

**Theorem 9.1.18:** If A and B be cubic Z-subalgebras of a Z-algebra X then  $A \times B$  is also a cubic Z-Subalgebra of  $X \times X$ .

**Proof:** Let A and B be cubic Z-subalgebras of a Z-algebra X.

**To prove:**  $A \times B$  is also a cubic Z-Subalgebra of  $X \times X$ .

Take  $(x_1, y_1)$  and  $(x_2, y_2) \in X \times X$ . Then clearly,

$$\tilde{\mu}_{A \times B}((x_1, y_1) * (x_2, y_2)) \geq r \min \{ \tilde{\mu}_{A \times B}(x_1, y_1), \tilde{\mu}_{A \times B}(x_2, y_2) \}$$

$$\begin{aligned} \text{and } \omega_{A \times B}[(x_1, y_1) * (x_2, y_2)] &= \omega_{A \times B}[(x_1 * x_2), (y_1 * y_2)] \\ &= \max \{ \omega_A(x_1 * x_2), \omega_B(y_1 * y_2) \} \\ &\leq \max \{ \max \{ \omega_A(x_1), \omega_A(x_2) \}, \max \{ \omega_B(y_1), \omega_B(y_2) \} \} \\ &= \max \{ \max \{ \omega_A(x_1), \omega_B(y_1) \}, \max \{ \omega_A(x_2), \omega_B(y_2) \} \} \\ &= \max \{ \omega_{A \times B}(x_1, y_1), \omega_{A \times B}(x_2, y_2) \} \end{aligned}$$

This proves that the Cartesian product of two cubic Z-Subalgebras is again a cubic Z-Subalgebra of X.

## 9.2 Cubic Z-Ideals in Z-algebras

In this section, the notion of Cubic Z-ideals in Z-algebras is defined and corresponding results are proved.

**Definition 9.2.1 :** Let  $(X, *, 0)$  be a Z-algebra. A cubic set  $A = (\tilde{\mu}_A, \omega_A)$  in  $X$  is called a **cubic Z-ideal** of  $X$  if it satisfies the following conditions:

- (i)  $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$  and  $\omega_A(0) \leq \omega_A(x)$
- (ii)  $\tilde{\mu}_A(x) \geq r \min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$
- (iii)  $\omega_A(x) \leq \max\{\omega_A(x * y), \omega_A(y)\}$  , for all  $x, y \in X$  .

**Example 9.2.2:** Consider a Z-algebra  $X = \{0, 1, 2, 3\}$  with the following Cayley table :

*	0	1	2	3
0	0	1	2	3
1	0	1	3	1
2	0	3	2	1
3	0	1	1	3

Define a cubic set  $A$  in  $X$  by  $\tilde{\mu}_A(x) = [0.6, 0.8]$  and  $\omega_A(x) = 0.2$  , for all  $x \in X$  .Then,  $A$  is a cubic Z-ideal of a Z-algebra  $X$ .

**Theorem 9.2.3:** The intersection of any set of cubic Z-ideals of a Z-algebra  $X$  is also a cubic Z-ideal of  $X$ .

**Proof:** Let  $A_i = \langle\langle x, \tilde{\mu}_{A_i}(x), \omega_{A_i}(x) \rangle\rangle | x \in X$  where  $i \in \Omega$  an index set, be a set of cubic Z-ideals of a Z-algebra  $X$  . Then for any  $x, y \in X$  ,

$$\tilde{\mu}_{\cap A_i}(0) = r \inf \tilde{\mu}_{A_i}(0) \geq r \inf \tilde{\mu}_{A_i}(x) = \tilde{\mu}_{\cap A_i}(x)$$

$$\omega_{\cup A_i}(0) = \sup \omega_{A_i}(0) \leq \sup \omega_{A_i}(x) = \omega_{\cup A_i}(x)$$

$$\begin{aligned} \tilde{\mu}_{\cap A_i}(x) &= r \inf \tilde{\mu}_{A_i}(x) \geq r \inf \{r \min\{\tilde{\mu}_{A_i}(x * y), \tilde{\mu}_{A_i}(y)\}\} = r \min\{r \inf \tilde{\mu}_{A_i}(x * y), r \inf \tilde{\mu}_{A_i}(y)\} \\ &= r \min\{\tilde{\mu}_{\cap A_i}(x * y), \tilde{\mu}_{\cap A_i}(y)\} \end{aligned}$$

$$\begin{aligned} \text{and } \omega_{\cup A_i}(x) &= \sup \omega_{A_i}(x) \leq \sup \{\max\{\omega_{A_i}(x * y), \omega_{A_i}(y)\}\} = \max\{\sup \omega_{A_i}(x * y), \sup \omega_{A_i}(y)\} \\ &= \max\{\omega_{\cup A_i}(x * y), \omega_{\cup A_i}(y)\} \end{aligned}$$

Hence  $\bigcap_{i \in \Omega} A_i = (\tilde{\mu}_{\cap A_i}, \omega_{\cap A_i})$  is a cubic Z-ideal of a Z-algebra X.

We remark here that the union, P-intersection and P-union of any sets of cubic Z-ideals in Z-algebras is need not be a cubic Z-ideal. We now provide some conditions that the union, P-intersection and P-union of any cubic Z-ideals in Z-algebras are still a cubic Z-ideal in the following three theorems.

**Theorem 9.2.4:** Let  $A_i = (\tilde{\mu}_{A_i}, \omega_{A_i})$  be a set of cubic Z-ideals of a Z-algebra X, where  $i \in \Omega$  an index set. If  $r \sup \{r \min \{\tilde{\mu}_{A_i}(x * y), \tilde{\mu}_{A_i}(y)\}\} = r \min \{r \sup \tilde{\mu}_{A_i}(x * y), r \sup \tilde{\mu}_{A_i}(y)\}$  and  $\inf \{\max \{\omega_{A_i}(x * y), \omega_{A_i}(y)\}\} = \max \{\inf \omega_{A_i}(x * y), \inf \omega_{A_i}(y)\}$ , for all  $x, y \in X$ , then the union of  $A_i$  is again a cubic Z-ideal of X.

**Proof:** For any  $x, y \in X$ ,

$$\tilde{\mu}_{\cup A_i}(0) = r \sup \tilde{\mu}_{A_i}(0) \geq r \sup \tilde{\mu}_{A_i}(x) = \tilde{\mu}_{\cup A_i}(x)$$

$$\omega_{\cap A_i}(0) = \inf \omega_{A_i}(0) \leq \inf \omega_{A_i}(x) = \omega_{\cap A_i}(x)$$

$$\begin{aligned} \tilde{\mu}_{\cup A_i}(x) &= r \sup \tilde{\mu}_{A_i}(x) \geq r \sup \{r \min \{\tilde{\mu}_{A_i}(x * y), \tilde{\mu}_{A_i}(y)\}\} \\ &= r \min \{r \sup \tilde{\mu}_{A_i}(x * y), r \sup \tilde{\mu}_{A_i}(y)\} \\ &= r \min \{\tilde{\mu}_{\cup A_i}(x * y), \tilde{\mu}_{\cup A_i}(y)\} \end{aligned}$$

$$\begin{aligned} \text{and } \omega_{\cap A_i}(x) &= \inf \omega_{A_i}(x) \leq \inf \{\max \{\omega_{A_i}(x * y), \omega_{A_i}(y)\}\} \\ &= \max \{\inf \omega_{A_i}(x * y), \inf \omega_{A_i}(y)\} \\ &= \max \{\omega_{\cap A_i}(x * y), \omega_{\cap A_i}(y)\} \end{aligned}$$

Hence  $\bigcup_{i \in \Omega} A_i = (\tilde{\mu}_{\cup A_i}, \omega_{\cap A_i})$  is a cubic Z-ideal of a Z-algebra X.

**Theorem 9.2.5:** Let  $A_i = (\tilde{\mu}_{A_i}, \omega_{A_i})$  be a set of cubic Z-ideals of a Z-algebra X, where  $i \in \Omega$  an index set. If  $\inf \{\max \{\omega_{A_i}(x * y), \omega_{A_i}(y)\}\} = \max \{\inf \omega_{A_i}(x * y), \inf \omega_{A_i}(y)\}$ , for all  $x, y \in X$ , then the P-intersection of  $A_i$  is again a cubic Z-ideal of X.

**Proof:** For any  $x, y \in X$ ,

$$\tilde{\mu}_{\cap A_i}(0) = r \inf \tilde{\mu}_{A_i}(0) \geq r \inf \tilde{\mu}_{A_i}(x) = \tilde{\mu}_{\cap A_i}(x)$$

$$\omega_{\cap A_i}(0) = \inf \omega_{A_i}(0) \leq \inf \omega_{A_i}(x) = \omega_{\cap A_i}(x)$$

$$\begin{aligned} \tilde{\mu}_{\cap A_i}(x) &= r \inf \tilde{\mu}_{A_i}(x) \geq r \inf \{r \min \{ \tilde{\mu}_{A_i}(x * y), \tilde{\mu}_{A_i}(y) \} \} = r \min \{ r \inf \tilde{\mu}_{A_i}(x * y), r \inf \tilde{\mu}_{A_i}(y) \} \\ &= r \min \{ \tilde{\mu}_{\cap A_i}(x * y), \tilde{\mu}_{\cap A_i}(y) \} \end{aligned}$$

$$\begin{aligned} \text{and } \omega_{\cap A_i}(x) &= \inf \omega_{A_i}(x) \leq \inf \{ \max \{ \omega_{A_i}(x * y), \omega_{A_i}(y) \} \} = \max \{ \inf \omega_{A_i}(x * y), \inf \omega_{A_i}(y) \} \\ &= \max \{ \omega_{\cap A_i}(x * y), \omega_{\cap A_i}(y) \} \end{aligned}$$

Hence  $P\left(\bigcap_{i \in \Omega} A_i\right) = (\tilde{\mu}_{\cap A_i}, \omega_{\cap A_i})$  is a cubic Z-ideal of a Z-algebra X.

**Theorem 9.2.6:** Let  $A_i = (\tilde{\mu}_{A_i}, \omega_{A_i})$  be a set of cubic Z-ideals of a Z-algebra X, where  $i \in \Omega$  an index set. If  $r \sup \{r \min \{ \tilde{\mu}_{A_i}(x * y), \tilde{\mu}_{A_i}(y) \} \} = r \min \{ r \sup \tilde{\mu}_{A_i}(x * y), r \sup \tilde{\mu}_{A_i}(y) \}$ , for all  $x, y \in X$ , then the P-union of  $A_i$  is again a cubic Z-Subalgebra of X.

**Proof:** For any  $x, y \in X$ ,

$$\tilde{\mu}_{\cup A_i}(0) = r \sup \tilde{\mu}_{A_i}(0) \geq r \sup \tilde{\mu}_{A_i}(x) = \tilde{\mu}_{\cup A_i}(x)$$

$$\omega_{\cup A_i}(0) = \sup \omega_{A_i}(0) \leq \sup \omega_{A_i}(x) = \omega_{\cup A_i}(x)$$

$$\begin{aligned} \tilde{\mu}_{\cup A_i}(x) &= r \sup \tilde{\mu}_{A_i}(x) \geq r \sup \{r \min \{ \tilde{\mu}_{A_i}(x * y), \tilde{\mu}_{A_i}(y) \} \} = r \min \{ r \sup \tilde{\mu}_{A_i}(x * y), r \sup \tilde{\mu}_{A_i}(y) \} \\ &= r \min \{ \tilde{\mu}_{\cup A_i}(x * y), \tilde{\mu}_{\cup A_i}(y) \} \end{aligned}$$

$$\begin{aligned} \text{and } \omega_{\cup A_i}(x) &= \sup \omega_{A_i}(x) \leq \sup \{ \max \{ \omega_{A_i}(x * y), \omega_{A_i}(y) \} \} = \max \{ \sup \omega_{A_i}(x * y), \sup \omega_{A_i}(y) \} \\ &= \max \{ \omega_{\cup A_i}(x * y), \omega_{\cup A_i}(y) \} \end{aligned}$$

Hence  $P\left(\bigcup_{i \in \Omega} A_i\right) = (\tilde{\mu}_{\cup A_i}, \omega_{\cup A_i})$  is a cubic Z-ideal of a Z-algebra X.

**Theorem 9.2.7:** Cubic set  $A = (\tilde{\mu}_A, \omega_A)$  of a Z-algebra X is a cubic Z-ideal of X where

$\tilde{\mu}_A = [\mu_A^L, \mu_A^U]$  if and only if  $\mu_A^L, \mu_A^U$  and  $(\omega_A)^c$  are fuzzy Z-ideals of X.

**Proof:** Let  $\mu_A^L, \mu_A^U$  and  $(\omega_A)^c$  are fuzzy Z-ideals of a Z-algebra X and  $x, y \in X$ .

$$\text{Consider } \tilde{\mu}_A(0) = [\mu_A^L(0), \mu_A^U(0)] \geq [\mu_A^L(x), \mu_A^U(x)] = \tilde{\mu}_A(x) \tag{1}$$

$$\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$$

$$\begin{aligned}
 &\geq [\min\{\mu_A^L(x * y), \mu_A^L(y)\}, \min\{\mu_A^U(x * y), \mu_A^U(y)\}] \\
 &= r \min\{[\mu_A^L(x * y), \mu_A^U(x * y)], [\mu_A^L(y), \mu_A^U(y)]\} \\
 &= r \min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}
 \end{aligned} \tag{2}$$

Also,  $1 - \omega_A(0) = (\omega_A)^c(0) \geq (\omega_A)^c(x) = 1 - \omega_A(x)$

$$\Rightarrow \omega_A(0) \leq \omega_A(x) \tag{3}$$

$$\begin{aligned}
 1 - \omega_A(x) = (\omega_A)^c(x) &\geq \min\{(\omega_A)^c(x * y), (\omega_A)^c(y)\} \\
 &= \min\{1 - \omega_A(x * y), 1 - \omega_A(y)\} \\
 &= 1 - \max\{\omega_A(x * y), \omega_A(y)\} \\
 \Rightarrow \omega_A(x) &\leq \max\{\omega_A(x * y), \omega_A(y)\}
 \end{aligned} \tag{4}$$

From (1), (2), (3) and (4) we get,  $A = (\tilde{\mu}_A, \omega_A)$  is a cubic Z-ideal of a Z-algebra X.

Conversely, for any  $x, y \in X$ ,

$$\begin{aligned}
 [\mu_A^L(0), \mu_A^U(0)] = \tilde{\mu}_A(0) &\geq \tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)] \\
 \Rightarrow \mu_A^L(0) \geq \mu_A^L(x) \quad \text{and} \quad \mu_A^U(0) &\geq \mu_A^U(x)
 \end{aligned} \tag{5}$$

$$\text{and } \mu_A^L(x) \geq \min\{\mu_A^L(x * y), \mu_A^L(y)\} \quad \text{and} \quad \mu_A^U(x) \geq \min\{\mu_A^U(x * y), \mu_A^U(y)\} \tag{6}$$

$$\text{Also, } (\omega_A)^c(0) = 1 - \omega_A(0) \geq 1 - \omega_A(x) = (\omega_A)^c(x) \tag{7}$$

$$\begin{aligned}
 (\omega_A)^c(x) = 1 - \omega_A(x) &\geq 1 - \max\{\omega_A(x * y), \omega_A(y)\} \\
 &= \min\{1 - \omega_A(x * y), 1 - \omega_A(y)\} \\
 &= \min\{(\omega_A)^c(x * y), (\omega_A)^c(y)\} \\
 \Rightarrow (\omega_A)^c(x) &\geq \min\{(\omega_A)^c(x * y), (\omega_A)^c(y)\}
 \end{aligned} \tag{8}$$

From (5),(6),(7) and (8), we get  $\mu_A^L, \mu_A^U$  and  $(\omega_A)^c$  are fuzzy Z-ideals of a Z-algebra X.

Analogously, the following theorems can be proved.

**Theorem 9.2.8:** Let  $A = (\tilde{\mu}_A, \omega_A)$  be a cubic set in a Z-algebra X. Then A is a cubic Z-ideal of X if and only if for all  $[s_1, s_2] \in D[0,1]$  and  $t \in [0,1]$ , the sets  $U(\tilde{\mu}_A; [s_1, s_2])$  and  $L(\omega_A; t)$  of A are either empty or Z-ideals of X.

**Theorem 9.2.9:** Let h be a Z-homomorphism from a Z-algebra  $(X, *, 0)$  onto a Z-algebra  $(Y, *, 0')$  and A be a cubic Z-ideal of X with rsup-inf property. Then image of A denoted by  $h(A)$  is a cubic Z-ideal of Y.

**Theorem 9.2.10:** Let  $h : (X, *, 0) \rightarrow (Y, *, 0')$  be a Z-homomorphism of Z-algebras. If B is a cubic Z-ideal of Y, then  $h^{-1}(B)$  is a cubic Z-ideal of X.

**Theorem 9.2.11:** Let  $h : (X, *, 0) \rightarrow (Y, *, 0')$  be an Z-epimorphism of Z-algebras. Let B be a cubic set of Y. If  $h^{-1}(B)$  is a cubic Z-ideal of X then B is a cubic Z-ideal of Y.

**Proof:** Assume  $h^{-1}(B)$  is a cubic Z-ideal of a Z-algebra X.

**To prove:** B is a cubic Z-ideal of a Z-algebra Y.

Let  $y \in Y$ , there exists  $x \in X$  such that  $h(x) = y$ . Then

$$(i) \quad \tilde{\mu}_B(y) = \tilde{\mu}_B(h(x)) = \tilde{\mu}_{h^{-1}(B)}(x) \leq \tilde{\mu}_{h^{-1}(B)}(0) = \tilde{\mu}_B(h(0)) = \tilde{\mu}_B(0')$$

$$\text{This implies,} \quad \tilde{\mu}_B(0') \geq \tilde{\mu}_B(y)$$

$$(ii) \quad \omega_B(y) = \omega_B(h(x)) = \omega_{h^{-1}(B)}(x) \geq \omega_{h^{-1}(B)}(0) = \omega_B(h(0)) = \omega_B(0')$$

$$\text{This implies,} \quad \omega_B(0') \leq \omega_B(y)$$

Let  $x, y \in Y$ . Then there exists  $a, b \in X$  such that  $h(a) = x$  and  $h(b) = y$ . It follows that

$$\begin{aligned} (iii) \quad \tilde{\mu}_B(x) &= \tilde{\mu}_B(h(a)) = \tilde{\mu}_{h^{-1}(B)}(a) \geq r \min \{ \tilde{\mu}_{h^{-1}(B)}(a * b), \tilde{\mu}_{h^{-1}(B)}(b) \} \\ &= r \min \{ \tilde{\mu}_B(h(a * b)), \tilde{\mu}_B(h(b)) \} \\ &= r \min \{ \tilde{\mu}_B(h(a) *' h(b)), \tilde{\mu}_B(h(b)) \} \\ &= r \min \{ \tilde{\mu}_B(x *' y), \tilde{\mu}_B(y) \} \end{aligned}$$

$$\text{This implies,} \quad \tilde{\mu}_B(x) \geq r \min \{ \tilde{\mu}_B(x *' y), \tilde{\mu}_B(y) \}$$

$$(iv) \quad \omega_B(x) = \omega_B(h(a)) = \omega_{h^{-1}(B)}(a) \leq \max \{ \omega_{h^{-1}(B)}(a * b), \omega_{h^{-1}(B)}(b) \}$$

$$\begin{aligned}
 &= \max \{ \omega_B(h(a * b)), \omega_B(h(b)) \} \\
 &= \max \{ \omega_B(h(a) *' h(b)), \omega_B(h(b)) \} \\
 &= \max \{ \omega_B(x *' y), \omega_B(y) \}
 \end{aligned}$$

This implies,  $\omega_B(x) \leq \max \{ \omega_B(x *' y), \omega_B(y) \}$

From (i) ,(ii), (iii) and (iv) we get, B is a cubic Z-ideal of a Z-algebra Y.

**Theorem 9.2.12:** If A and B be cubic Z-ideals of Z-algebra X then  $A \times B$  is a cubic Z-ideal in  $X \times X$  .

**Proof:** Let A and B be cubic Z-ideals in a Z-algebra X.

**To prove:**  $A \times B$  is a cubic Z-ideal in  $X \times X$  .

Let  $(x_1, x_2) \in X \times X$  ,

$$\tilde{\mu}_{A \times B}(0,0) = r \min \{ \tilde{\mu}_A(0), \tilde{\mu}_B(0) \} \geq r \min \{ \tilde{\mu}_A(x_1), \tilde{\mu}_B(x_2) \} = \tilde{\mu}_{A \times B}(x_1, x_2)$$

$$\text{Hence } \tilde{\mu}_{A \times B}(0,0) \geq \tilde{\mu}_{A \times B}(x_1, x_2) \tag{1}$$

$$\omega_{A \times B}(0,0) = \max \{ \omega_A(0), \omega_B(0) \} \leq \max \{ \omega_A(x_1), \omega_B(x_2) \} = \omega_{A \times B}(x_1, x_2)$$

$$\text{Hence } \omega_{A \times B}(0,0) \leq \omega_{A \times B}(x_1, x_2) \tag{2}$$

Let  $(x_1, x_2), (y_1, y_2) \in X \times X$ . Then,

$$\begin{aligned}
 \tilde{\mu}_{A \times B}(x_1, x_2) &= r \min \{ \tilde{\mu}_A(x_1), \tilde{\mu}_B(x_2) \} \\
 &\geq r \min \{ r \min \{ \tilde{\mu}_A(x_1 * y_1), \tilde{\mu}_A(y_1) \}, r \min \{ \tilde{\mu}_B(x_2 * y_2), \tilde{\mu}_B(y_2) \} \} \\
 &= r \min \{ r \min \{ \tilde{\mu}_A(x_1 * y_1), \tilde{\mu}_B(x_2 * y_2) \}, r \min \{ \tilde{\mu}_A(y_1), \tilde{\mu}_B(y_2) \} \} \\
 &= r \min \{ \tilde{\mu}_{A \times B}((x_1 * y_1), (x_2 * y_2)), \tilde{\mu}_{A \times B}(y_1, y_2) \} \\
 &= r \min \{ \tilde{\mu}_{A \times B}((x_1, x_2) * (y_1, y_2)), \tilde{\mu}_{A \times B}(y_1, y_2) \}
 \end{aligned} \tag{3}$$

$$\begin{aligned}
 \text{and } \omega_{A \times B}(x_1, x_2) &= \max \{ \omega_A(x_1), \omega_B(x_2) \} \\
 &\leq \max \{ \max \{ \omega_A(x_1 * y_1), \omega_A(y_1) \}, \max \{ \omega_B(x_2 * y_2), \omega_B(y_2) \} \} \\
 &= \max \{ \max \{ \omega_A(x_1 * y_1), \omega_B(x_2 * y_2) \}, \max \{ \omega_A(y_1), \omega_B(y_2) \} \} \\
 &= \max \{ \omega_{A \times B}((x_1 * y_1), (x_2 * y_2)), \omega_{A \times B}(y_1, y_2) \} \\
 &= \max \{ \omega_{A \times B}((x_1, x_2) * (y_1, y_2)), \omega_{A \times B}(y_1, y_2) \}
 \end{aligned} \tag{4}$$

Hence  $A \times B$  is a cubic Z-ideal in  $X \times X$  .

**Theorem 9.2.13:** Let A and B be two cubic sets of a Z-algebra X. If  $A \times B$  is a cubic Z-ideal of  $X \times X$ , the following are true.

(i)  $\tilde{\mu}_A(0) \geq \tilde{\mu}_B(y)$  and  $\tilde{\mu}_B(0) \geq \tilde{\mu}_A(x)$  for all  $x, y \in X$ .

(ii)  $\omega_A(0) \leq \omega_B(y)$  and  $\omega_B(0) \leq \omega_A(x)$  for all  $x, y \in X$ .

**Proof :** Assume that  $\tilde{\mu}_B(y) > \tilde{\mu}_A(0)$  and  $\tilde{\mu}_A(x) > \tilde{\mu}_B(0)$  for some  $x, y \in X$ .

$$\begin{aligned} \text{Then } \tilde{\mu}_{A \times B}(x, y) &= r \min\{\tilde{\mu}_A(x), \tilde{\mu}_B(y)\} > r \min\{\tilde{\mu}_B(0), \tilde{\mu}_A(0)\} \\ &= \tilde{\mu}_{A \times B}(0, 0) \text{ which is a contradiction.} \end{aligned}$$

Similarly, assume that  $\omega_A(x) < \omega_B(0)$  and  $\omega_B(y) < \omega_A(0)$  for some  $x, y \in X$ .

Then  $\omega_{A \times B}(x, y) = \max\{\omega_A(x), \omega_B(y)\} < \max\{\omega_B(0), \omega_A(0)\} = \omega_{A \times B}(0, 0)$  which is also a contradiction. Thus proving the result.

**Theorem 9.2.14:** Let A and B be two cubic sets of a Z-algebra X such that  $A \times B$  is a cubic Z-ideal of  $X \times X$ . Then either A or B is a cubic Z-ideal of X.

**Proof :** Now by above Theorem 9.2.13, if we take  $\tilde{\mu}_A(0) \geq \tilde{\mu}_B(y)$  and  $\omega_A(0) \leq \omega_B(y)$  for all  $y \in X$ ,

$$\tilde{\mu}_{A \times B}(0, y) = r \min\{\tilde{\mu}_A(0), \tilde{\mu}_B(y)\} = \tilde{\mu}_B(y) \quad \text{and} \quad \omega_{A \times B}(0, y) = \max\{\omega_A(0), \omega_B(y)\} = \omega_B(y) \quad (1)$$

Take  $(x_1, y_1)$  and  $(x_2, y_2) \in X \times X$ .

Since  $A \times B$  is a cubic Z-ideal of  $X \times X$ .

$$\begin{aligned} \tilde{\mu}_{A \times B}(x_1, y_1) &\geq r \min\{\tilde{\mu}_{A \times B}((x_1, y_1) * (x_2, y_2)), \tilde{\mu}_{A \times B}(x_2, y_2)\} \\ &= r \min\{\tilde{\mu}_{A \times B}(x_1 * x_2, y_1 * y_2), \tilde{\mu}_{A \times B}(x_2, y_2)\} \end{aligned} \quad (2)$$

Putting  $x_1 = x_2 = 0$  in (2) we get,

$$\tilde{\mu}_{A \times B}(0, y_1) \geq r \min\{\tilde{\mu}_{A \times B}(0, y_1 * y_2), \tilde{\mu}_{A \times B}(0, y_2)\} \quad \text{and by (1),}$$

$$\tilde{\mu}_B(y_1) \geq r \min\{\tilde{\mu}_B(y_1 * y_2), \tilde{\mu}_B(y_2)\}$$

Analogously, we can prove  $\omega_B(y_1) \leq \max\{\omega_B(y_1 * y_2), \omega_B(y_2)\}$

Hence B is a cubic Z-ideal of X.

By Theorem 9.2.13, assume that  $\tilde{\mu}_B(0) \geq \tilde{\mu}_A(x)$  and  $\omega_B(0) \leq \omega_A(x)$  then A is a cubic Z-ideal of a Z-algebra X.

Therefore, either  $A$  or  $B$  is a cubic  $Z$ -ideal of a  $Z$ -algebra  $X$ .

This completes the proof.