Markovian Queues With
General Bulk Service
And Balking

BY

K. Devasena

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE
AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE - 641 043
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN MATHEMATICS

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CONTENTS

CONTENTS

INTRODUCTION

INTRODUCTION

> The essence of Mathematics is its freedom, said Cantor, freedom of construct, freedom to make assumptions. These aspects of Mathematics are recognised in construct vision and formalism yet cantor was a platonist, beleiver in a Mathematical reality that transcends human mind.
>
> PHILIPS J. DAVIS AND
> REUBEN HERSH.

The Queueing theory had its origin in 1909. When A.K. Erlang (1878-1929) published his fundamental paper relating to the study of congestion in telephone traffic.

A queue or waiting line is formed when customers (or Units) needing some kind of service arrive at a service channel which offers such facility. A queueing system can be described as the flow of units for service, forming or joining the queue. If the service is not immediatedly available, and leaving the system after being served (or sometimes without being served).

The basic four characteristics of queueing systems are

(i) Arrival Pattern

(ii) Service Mechanism.
(iii) Queue Discipline
(iv) Service Channels.

ARRIVAL PATTERN

The arrival pattern is measured in terms of the mean arrival rate or mean inter arrival time. Generally the arrival of customers is deterministic but in most of the cases the arrival pattern is probabilistic or stochastic. Arrival of customers may occur either singly or in batches. Most of the arrival pattern is assumed to follow poisson process. In some cases arriving units may go through several exponential phases before entering for service. This type of arrival pattern is known as Erlang Process.

SERVICE MECHANISM

This means the arrangement of service facility to serve the customers. If there are infinite number of servers then all the customers are served instantaneously on arrival and these will be no queue.

If the number of servers is finite then the customers are served according to specific order. Further, the customers may be served in batches of fixed size or variable size rather than individually by the same server such as a computer with parallel processing or people boarding a bus. The service system in this case is called bulk service system.

QUEUE DISCIPLINE

The queue discipline is a rule by which the customers are selected for service when a queue has been formed. The most common discipline is the "First come, first serve" (FCFS) or "first in, first out" (FIFO) rule under which the customers are served in strict order of their arrivals. Sometimes the customers are served on the basis "Last in First out" (LIFO). Another queue descipline is "Service in Random Order" (SIRO) rule according to which the customers are served randomly irrespective of their arrivals in the system. "Priority" is another queue discipline, where the customers are given priority upon entering the system.

SERVICE CHANNELS

The number of services or service channels in a queueing model may be finite or infinite. Depending on the model, if the number of servers is more than one, the customers may form a single queue waiting for service or seperate queue in front of each server.

NOTATION

Kendall (4) designed a convenient notation to denote queueing systems. A queueing process is described by a service of symbols and slashes such as A/B/X/Y/Z.

Where

A : indicates the interarrival distribution

B : the service pattern as described by probability distribution for service time.

X : the number of parallel service channels.

Y : the restriction on system capacity.

Z : the queue discipline.

The following letters are used for signifying distributions.

M : for negative exponential inter arrival or service time distribution.

D : for constant inter arrival or service times

E_k : for the K - Erlang distribution.

G : for general or arbitary inter arrival on service time distribution.

Bulk service queueing structure occupy an important place in our daily life, mainly at taxi stand, unscheduled car ferry, a ground floor station of an elevator etc. Arora (1) Ghare (3) have studied bulk service queue.

Neuts (10) has introduced the general bulk service for single channel system with capacity range (a, b). Under this rule, the service starts only when a minimum number of customers 'a' is present in the queue and maximum service capacity is 'b'. If the server is free and queue length is 'a' or more but less than or equal to b, then entire queue is taken up for service. If the queue size is greater than 'b', only 'b' customers are taken up for service while others wait in the queue.

Borthakur (2) has discussed the M/M (a,b)/ 1 queue and obtained the steady state solution. Medhi (6) has obtained distribution of waiting time in the queue. Medhi (5,7) has also discussed a method for direct steady state solution and the busy period distribution.

Medhi and Borthakur (9) have considered the system of two channel homogenous server case under the general bulk service rule. Medhi (8) have discussed the steady state waiting time distribution of M/M (a, b)/ C system.

Neuts and Natarajan (11) have considered the same model as a Markov process in a matrix geometric approach and they have expressed the stationary waiting time distribution in a usable algorithmic form.

In this dissertation the queueing system M/M a,b/2 with two types of balking is analysed. (i) An arriving customer will definitely join the system whatever may be queue length if he finds atleast one server is busy otherwise he may balk. (ii) An arriving customer will definitely join the system if the queue length is 'a' and also if atleast one server is free. If not either he joins the system with probability β or leave the system with probability $1-\beta$.

The queueing model M/Ma,b/2 with balking (i) and (ii) are studied in chapter I. For both the models the waiting time distribution and the expected queue lengths are obtained. Also it is proved that the little's formula is not true for system M/Ma,b/2 with balking.

In the second chapter, attempt is made to verify Little's formula for M/Ma,b/1 with balking. By deriving the expressions for E(T) and Lq for both the types of balking, it is proved, that for the single server Markovian queue with bulk service and balking the formula is not true if the balking probability $\beta \neq 1$.

TWO SERVER BULK SERVICE
WITH BALKING

CHAPTER - I

TWO - SERVER - BULK SERVICE WITH BALKING

In this chapter the two server Markovian queue with general bulk service and balking is considered. In the first section it is assumed that an arriving customer may balk if he finds both the servers are busy. In the second section the system with the assumption that an arriving customer would not balk if he gets immediate service is analysed. For both types the steady state probabilities, waiting time distribution and the expected number in the queue are derived.

Consider a queueing process with two servers, that is completely specified by the following.

(i) Customers arrive to the service station in accordance with poisson process with parameter λ .

(ii) Both the servers serve the system according to a general bulk service rule with same service rate μ.

(iii) Whenever a server becomes free and if the queue length is greater than or equal to a, then the server takes into service immediately, a batch containing k $(a \leqslant k \leqslant b)$ customers waiting at the head of the line.

The model leads to the state space $[(i,n)/i=0,1,\ 0 \leqslant n \leqslant a-1] \cup [(2,n)/n \geqslant 0]$. In the state space the first index denote the number of busy servers. The index n represents the number of waiting customers in the queue.

Let $P\,in\,(t)$ denote the probability that the system is in state (i,n) at time t.

SECTION (I)

Assume that if an arriving customer finds both the servers are busy, then he joins the system with probability β and wait in the line.

The differential difference equations are.

$$
\begin{aligned}
P0,0^{1}(t) &= -\lambda P\,00\,(t) + \mu P1,0\,(t) \\
P0,n^{1}(t) &= -\lambda P0,n\,(t) + \lambda P0,n-1\,(t) + \mu P1,n\,(t) \\
P1,0^{1}(t) &= -(\lambda+\mu)\,P1,0(t) + 2\mu\,P2,0(t) + \lambda P0a-1(t) \\
P1,n^{1}(t) &= -(\lambda+\mu)\,P1,n(t) + \lambda P1,n-1(t) + 2\mu P2n(t) \\
P2,0^{1}(t) &= -(\lambda\beta+2\mu)\,P2n(t) + \lambda P1,a-1(t) \\
&\qquad + 2\mu \sum_{S=a}^{b} P2,s
\end{aligned}
$$

$$P2,n^{1}(t) = -(\lambda\beta + 2\mu)\, P2,n\,(t) + \lambda\beta\, P2,n-1(t) + 2\mu\, P2,n+b\,(t)$$

The steady state equations are given by

$$\lambda P00 = \mu P10 \qquad (2)$$

$$\lambda P0n = \lambda\, P0n-1 + \mu P1n \quad (1 \leq n \leq a-1) \qquad (3)$$

$$(\lambda + \mu)\; P10 = 2\mu P20 + \lambda\, P0a-1 \qquad (4)$$

$$(\lambda + \mu)\; P1n = \lambda P1n-1 + 2\mu P2n \qquad (1 \leq n \leq a-1) \qquad (5)$$

$$(\lambda\beta + 2\mu)\; P20 = \lambda P1a-1 + \mu 2 \sum_{s=a}^{b} P2S \qquad (6)$$

$$(\lambda\beta + 2\mu)\; P2n = \lambda\beta\; P2n-1 + 2\mu P2n+b \qquad (7)$$

The characteristic equation of (7) can be written as $h\,(Z) = 2\mu Z^{b+1} - (\lambda\beta + 2\mu)\, Z + \lambda\beta = 0$ (8)

Suppose $f\,(Z) = -(\lambda\beta + 2\mu)\, Z$ and

$g\,(Z) = 2\mu Z^{b+1} + \lambda\beta.$

Then for $|Z| = 1$ we have.

$$/g\,(Z)/ = /2\mu Z^{b+1} + \lambda\beta\,/ < / -(\lambda\beta + 2\mu)\, Z/ = /f(Z)/$$

Since has only one zero inside $|z| = 1$ by Rouche's theorem only one zero of $h\,(Z)$ falls inside 1z1 = 1. Denote this root of $h\,(Z)$ by $|r|$, $o < r < 1$ and the other b roots as $r_1, r_2, \ldots, r_b$ ($|r_i| \geq 1$) i = 1 to

b. The system is then stable if $h'(1) > 1$.

[i.e.] if $\rho = \frac{\lambda\beta}{2b\mu} < 1$ (9)

Thus equation (7) has the solution,

$$P2n = Ar^n + \sum_{i=1}^{b} Ai\ ri^n \quad n \geq 0.$$

Since $\sum_{n=0}^{\infty} P2n < 1$, we must have $Ai = 0$. for $1 \leq i \leq b$.

Hence $P2n = Ar^n \quad (n \geq 0)$

Eliminating A we get

$$P2n = P20\ r^n \quad (n \geq 0) \tag{10}$$

Define

$$\theta = \frac{\lambda}{\lambda+\mu} \tag{11}$$

from equation (5) we get

$$P1n = A\left(\frac{\lambda}{\lambda+\mu}\right)^n + 2\mu \frac{r^{n+1}}{(\lambda+\mu)r-\lambda} P20$$

taking n = 0 and using (4).

$$\left(\frac{2\mu}{\lambda+\mu}\right) P20 + \left(\frac{\lambda}{\lambda+\mu}\right) P0a-1 = A + \frac{2\mu r}{(\lambda+\mu)r-\lambda} P20$$

$$A = \theta P0a-1 - \frac{2\mu\lambda}{(\lambda+\mu)[(\lambda+\mu)r-\lambda]} P20$$

$$A = \theta Poa-1 - \frac{2\mu\theta}{(\lambda+\mu)r-\lambda} P20$$

Hence

$$P1n = \theta^{n+1} \quad P0a-1 - \frac{2\mu\theta^{n+1}}{(\lambda + \mu)\ r-\lambda} P20 + \frac{2\mu r^{n+1}}{(\lambda + \mu)\ r-\lambda} P20$$

$$\mathbf{P1n} = \theta^{n+1} \quad Poa-1 + \frac{2\mu}{(\lambda+\mu)\ r-\lambda} (r^{n+1} - \theta^{n+1})\ P20 \qquad (12)$$

Putting $K = 1,2 \dots\dots\dots n$ in equation (3) and adding we get

$$\lambda P0n = \lambda P00 + \mu \sum_{k=1}^{n} P1k$$

using (2) in the above equation gives.

$$P\ 0n = \frac{\mu}{\lambda} \sum_{k=0}^{n} P1k$$

$$= \frac{\mu}{\lambda} (\sum_{K=0}^{n} \theta^{n+1} \quad P0a-1 + \frac{2\mu\ P20}{(\lambda+\mu)\ r-\lambda} (r^{n+1} - \theta^{n+1})$$

$$= \frac{\mu}{\lambda} [\frac{\theta - \theta^{n+2}}{1-\theta} \quad P0a-1 + \frac{2\mu}{(\lambda+\mu)\ r-\lambda}$$

$$(\frac{r-r^{n+2}}{1 - r} - \frac{\theta - \theta^{n+2}}{1-\theta})$$

$$\mathbf{P0n} = (1-\theta^{n+1}) \quad P0a-1 + (\frac{2\mu}{\lambda(1-r)} - \frac{2\mu^2 r^{n+2}}{\lambda^2 (r/\theta-1)\ (1-r)} + \frac{2\mu\theta^{n+1}}{\lambda(r/\theta-1)} P20 \qquad (13)$$

putting n = a-1 in the above equation we get

$$P0a-1 = (1-\theta^{a})\,P0a-1 + \left(\frac{2\mu}{\lambda(1-r)} - \frac{2\mu^{2}r^{a+1}}{\lambda^{2}(r/\theta-1)\,(1-r)} + \frac{2\mu\theta^{a}}{\lambda(r/\theta-1)}\right) P20$$

$$\theta^{a}P0a-1 = \frac{1}{\lambda}\left(\frac{2\mu}{1-r}\left(1-\frac{\mu r^{a+1}}{\lambda(r/\theta-1)}\right) + \frac{2u\theta^{a}}{(r/\theta-1)}\right) P20$$

$$P0a-1 = \frac{1}{\theta^{a}}\left(\frac{2\mu(1-r^{a})}{(1-r)} - \frac{2\mu(\gamma^{a}-\theta^{a})}{(r/\theta-1)}\right)P20 \qquad (14)$$

we have all the steady state probabilities are in term of P20. We can derive P20 using the normalized equation.

$$\sum_{n=0}^{a-1}[P0n + P1n] + \sum_{n=0}^{\infty} P2n = 1$$

using equations (12), (13) and (14) we get

$$\sum_{n=0}^{a-1}\left[\left(\left(\frac{2\mu}{(\lambda+\mu)r-\lambda}(r^{n+1}-\theta^{n+1}) + \frac{2\mu}{\lambda(1-r)} - \frac{2\mu^{2}r^{n+2}}{\lambda^{2}(r/\theta-1)\,(1-r)} + \frac{2\mu\theta^{n+1}}{\lambda(r/\theta-1)}\right) P20 + Poa-1\right] + \sum_{n=0}^{\infty} P20r^{n} = 1$$

$$\sum_{n=0}^{a-1}\left[\frac{2\mu r^{n+1}}{(\lambda+u)r-\lambda} + \frac{2\mu}{\lambda(1-r)} - \frac{2\mu^{2}r^{n+2}}{\lambda^{2}(r/\theta-1)\,(1-r)} + \frac{2\mu(1-r^{a})}{\lambda\theta^{a}(1-r)} - \frac{2\mu(r^{a}-\theta^{a})}{\lambda\,\theta^{a}(r/\theta-1)}\right]P20 + \frac{1}{1-r}\,P20 = 1$$

$$\left[\frac{2}{(\lambda+\mu)r-\lambda}\,\frac{(r-r^{a+1})}{(1-r)}+\frac{a}{\lambda\theta^{a}}\left(\frac{2\mu(1-r^{a})}{1-r}-\frac{2\mu(r^{a}-\theta^{a})}{(r/\theta-1)}\right)+\frac{2\mu a}{(1-r)\lambda}-\frac{2\mu(r^{2}-r^{a+2})}{\lambda^{2}(1-r)^{2}(r/\theta-1)}+\frac{1}{(1-r)}\right]P20=1.$$

$$\left[\frac{2\mu r(1-r^{a})}{[(\lambda+u)r-\lambda](1-r)}+\frac{a}{\lambda\theta^{a}}\left(\frac{(2\mu(1-r^{a})}{(1-r)}-\frac{2\mu(r^{a}-\theta^{a})}{(r/\theta-1)}\right)+\frac{2\mu a}{\lambda(1-r)}+\frac{2\mu r^{2}(1-r^{a})}{\lambda^{2}(1-r)^{2}(r/\theta-1)}+\frac{1}{(1-r)}\right]P20=1.$$

$$P20=\left[\frac{1}{1-r}+\frac{2\mu a}{\lambda(1-r)}+\frac{a}{\lambda\theta^{a}}\left(\frac{2\mu(1-r^{a})}{1-r}-\frac{2\mu(r^{a}-\theta^{a})}{(r/\theta-1)}\right)-\frac{2\mu r(1-r^{a})}{\lambda(1-r)^{2}}\right]^{-1} \qquad (15)$$

WAITING TIME DISTRIBUTION

Now we find the waiting time distribution in the equilibrium state. Let the random variable T denote the waiting time in the queue for an arriving unit.

Let $\nu(t)$ be the probability density function (p.d.f) of T.

$f(\alpha, k, t)$ be the p.d.f. of gamma distribution with parameter α, k.

$$f(\alpha, k, t) = \frac{\alpha^{k} t^{k-1} e^{-\alpha t}}{\Gamma(k)}, \quad t \geq 0, \ k=1,2,..$$

$\Gamma_x(\alpha, k,)$ be the incomplete gamma function.

$$\Gamma_x(\alpha, k) = \int_0^x f(\alpha, k, t)\, dt$$

$$= 1 - \sum_{s=0}^{k-1} \frac{e^{-x\alpha}(\alpha x)^{S}}{S\,!}$$

An arriving customer finds the system in one of the following state (16)

(i) $(0, a-1), (1, a-1)$

(ii) $(0,n), (1,n)$ $\quad 0 \leq n \leq a - 2.$

(iii) $(2,n)$ $\quad a - 1 \leq m \leq b - 1$ $\Big\}$ $n = kb + m$

(iv) $(2,n)$ $\quad 0 \leq m \leq a - 2$ $\Big\}$ $k = 0, 1, 2....$

In case (i) the arriving unit does not wait and the probability of zero dealy is thus $[P0a-1+P1a-1]$. In all other case, the unit has to wait and the probability of blocking is thus $1 - [P0a - 1 + P1a - 1]$.

In case (ii), the arriving unit has to wait for the arrival of $(a - 1 - n)$ unit and the time needed for $(a - 1 - n)$ arrival has a gamma distribution with parameters λ, $a - 1 - n$.

In case (iii), the arriving unit has to wait for the completion of services of (k + 1) batches and the time required for this has a gamma distribution with parameter $2\mu, k + 1$.

In case (iv) the arriving unit has to wait till either the services of (k + 1) groups are completed or (a - 1 - m) units arrive, whichever occur later, this duration which is given by the maximum of two gamma variaties may be denoted by the random variable Z.

Z = max (gamma variate with parameter $\lambda\beta$,
, a - 1 - m ; gamma variable
with parameters $2\mu, k + 1$)

$$Z = \max (X, Y) \text{ (say)} \tag{17}$$

The distribution function Fz(t) and the p.d.f hz(t) of Z are given by

$$\begin{aligned} Fz(t) &= Pr (Z \leq t) \\ &= Pr (\max (X, Y) \leq t) \\ &= Pr (X \leq t, Y \leq t) \\ &= Pr (X \leq t) \, Pr (Y \leq t) \\ &= \Gamma_t(\lambda\beta, a - 1 - m) \, \Gamma_t (2\mu, k + 1) \end{aligned}$$

$$\text{and } hz(t) = Fz^1(t)$$

$$= f(\lambda\beta, a-1-m)\, \Gamma_t(2\mu, k+1) + f(2\mu; k+1)\, \Gamma_t(\lambda\beta, a-1-m) \qquad (18)$$

It follows that p.d.f $\nu(t)$ of T is given by

$$\nu(t) = \sum_{n=0}^{a-2} [P1n + P0n]\, f(\lambda, a-1-n, t)$$

$$+ \sum_{k=0}^{\infty} \sum_{m=a-1}^{b-1} P2, kb+m\, f(2\mu, k+1, t)$$

$$+ \sum_{k=0}^{\infty} \sum_{m=0}^{a-2} P2\ kb+m\ hz(t) \qquad (19).$$

Denote

$$\sum_{k=0}^{m-1} \frac{z^k}{k!} = e(m,z)\ ;\ e-z\ e(m,z) = E(m,z)\ m \geq 1. \qquad (20).$$

Then by the following arguments at Medhi [6,7], we have

$$\sum_{q=0}^{a-2} f(\lambda, a-1-q, t) = \sum_{q=0}^{a-2} \frac{\lambda^{a-1-q}\, t^{a-2-q}\, e^{-t\lambda}}{\Gamma(a-1-q)}$$

$$= \lambda e^{-\lambda t}\, e(a-1, \lambda t)$$

$$\sum_{q=0}^{a-2} r^q f(\lambda, a-1-q, t) = \lambda e^{-\lambda t}\, r^{a-2}\, e\left(a-1, \frac{\lambda t}{r}\right)$$

$$\int_0^{\infty} t\, E(a-1, \lambda t)\, dt = \frac{a(a-1)}{2\lambda^2}$$

$$\int_0^\infty t\, e^{-\lambda t}\, e\left(a-1, \frac{t\lambda}{r}\right) dt = \frac{(r^a-1) - a(r-1)}{\lambda^2\, r^{a-2}\, (r-1)^2}$$

$$\int_0^\infty t\, e^{-\lambda t\beta}\, e\left(a-1, \frac{t\lambda\beta}{r}\right) dt = \frac{(r^a-1) - a(r-1)}{\lambda^2\, \beta^2\, r^{a-2}\, (r-1)^2}$$

$$\int_0^\infty t\, e^{-\mu(1-r^b)t}\, E(a-1, \lambda t)\, dt = \frac{r^2(1-r^a) - ar^{a+1}(1-r)}{\lambda^2\, (1-r)^2}$$

$$\int_0^\infty t\, (1-r^b) e^{-\mu(1-r^b)t}\, dt = \frac{1}{\mu^2 \quad (1-r^b)} \qquad (21)^S$$

The first term on the righthand side of the equation (19) is

$$\sum_{n=0}^{a-2} [P1n + P0n]\; f(\lambda, a-1-n, t)$$

using equation (12), (13) and $(21)^S$ we get

$$\sum_{n=0}^{a-2} \left[\left(\frac{2\mu}{(1-r)} + \frac{2\mu r^{n+1}}{(\lambda+\mu)r-\lambda} - \frac{2\mu^2 r^{n+2}}{\lambda[(\lambda+\mu)r-\lambda](1-r)}\right) P20\right.$$

$$\left. + P0a-1\right] f(\lambda, a-1-n, t).$$

$$\left[\frac{2\mu}{(1-r)\lambda} P20 + P0a-1\right] \lambda e^{-\lambda t} e\,(a-1, \lambda t)$$

$$- \frac{2\mu}{(1-r)} r^{a-1} \lambda e^{-\lambda t} e\left(a-1, \frac{\lambda t}{r}\right)\Big]\Big]\; P20$$

$$\frac{2\mu}{1-r} E\,(a-1, \lambda t)\ P20 + \lambda E(a-1, \lambda t)\ P0a-1$$

$$- \frac{2\mu r^{a-1}}{(1-r)}\ e(a-1, \frac{t\lambda}{r})\ \ e^{-\lambda t} P20 \qquad (22)$$

Second term on the right hand side of equation (19)

$$\sum_{k=0}^{\infty} \sum_{m=a-1}^{b-1} P2\ kb+m\ f(2\mu, k+1, t)$$

usng equation (10) we get

$$\sum_{k=0}^{\infty} \sum_{m=a-1}^{b-1} r^{kb+m}\ f(2\mu, k+1, t)\ P20$$

$$\sum_{m=a-1}^{b-1} r^{m} \sum_{k=0}^{\infty} r^{kb} \frac{(2\mu)^{k+1}\ e^{-2\mu t}\ t^{k}}{k!}\ P20$$

$$\sum_{m=a-1}^{b-1} r^{m} 2\mu \sum_{k=0}^{\infty} \frac{(2\mu t r^{b})^{k}\ \ e^{-2\mu t}}{k!}\ P20$$

$$\sum_{m=a-1}^{b-1} r^{m}\ 2\mu e^{-2\mu t}\ e^{2\mu r^{b} t}\qquad P20$$

$$\left[\frac{r^{a-1} - r^{b}}{1-r}\right] 2\mu e^{-2\mu t(1-r^{b})}\ P20 \qquad (23)$$

The third term on the right hand side of equation (19) consists of two terms corresponding to those of equation (18) and using equation (10) we get

$$\sum_{k=0}^{\infty}\sum_{m=0}^{a-2} r^{kb+m}[f(\lambda\beta,a-1-m,t)\Gamma_t(2\mu,k+1,)$$

$$+\Gamma_t(\lambda\beta,a-1-m)\,f(2\mu,k+1,t)]P20 \qquad III$$

consider the first term of the above equation

$$\sum_{k=0}^{\infty}\sum_{m=0}^{a-2} r^{kb+m} f(\lambda\beta,a-1-m,t)\Gamma_t(2\mu,K+1)\,P20$$

$$\sum_{m=0}^{a-2} r^{m} f(\lambda\beta,a-1-m,t)\sum_{K=0}^{\infty} r^{kb}\int_0^t f(2\mu,k+1,t)\,dt\ p20$$

using (21) we get.

$$\lambda\beta e^{-\lambda\beta t} r^{a-2} e(a-1,\frac{\lambda\beta t}{r})\int_0^t\sum_{k=0}^{\infty}2\mu e^{-2\mu t}\frac{(2\mu t r^{b})^{k}}{k!}\,dtP20$$

$$\lambda\beta e^{-\lambda\beta t} r^{a-2} e(a-1,\frac{\lambda\beta t}{r})\,2\mu\int_0^t e^{-2\mu t(1-r^{b})}\,dt\ P20$$

$$\frac{\lambda\beta}{(1-r^{b})} e^{-\lambda\beta t} r^{a-2} e(a-1,\frac{\lambda\beta t}{r})\,(1-e^{-2\mu t(1-r^{b})})\ P20$$

$$\frac{2\mu}{1-r} e^{-\lambda\beta t} r^{a-1} e(a-1,\frac{\lambda\beta t}{r})\,(1-e^{-2\mu t(1-r^{b})}))$$

(24)

The second term of the equation

$$\sum_{k=0}^{\infty}\sum_{m=0}^{a-2} r^{kb+m}\Gamma_t(\lambda\beta,a-1-m)\,f(2\mu,k+1,t)\ P20$$

$$\sum_{k=0}^{\infty}\sum_{m=0}^{a-2} r^{kb+m}\left[1-\sum_{q=0}^{a-m-2}\frac{e^{-\lambda\beta t}(\lambda\beta t)^{q}}{q!}\right]\frac{2\mu e^{-2\mu t}(2\mu t)^{k}}{k!}\;P20$$

$$\Bigg[\sum_{k=0}^{\infty}\sum_{m=0}^{a-2} r^{kb+m}\,2\mu\, e^{-2\mu t}\frac{(2\mu t)^{k}}{k!}$$

$$-\sum_{m=0}^{a-2}\sum_{k=0}^{\infty} r^{kb+m}\sum_{q=0}^{a-m-2}\frac{e^{-\lambda\beta t}(\lambda\beta t)^{q}}{q!}\;\frac{2\;e^{-2\mu t}(2\mu t)^{k}}{k!}\Bigg]\;P20$$

$$\Bigg[\sum_{k=0}^{\infty} r^{kb}2\mu\frac{e^{-2\mu t}(2\mu t)^{k}}{k!}-\sum_{k=0}^{\infty} r^{kb}\sum_{q=0}^{a-m-2}\frac{e^{-\lambda\beta t}(\lambda\beta t)^{q}}{q!}$$

$$\frac{2\mu e^{-2\mu t}(2\mu t)^{k}}{k!}\Bigg]\left[\frac{1-r^{a-1}}{1-r}\right]\;P20$$

$$\frac{2\mu e^{-2\mu t}(1-r^{b})}{1-r}\Bigg[1-r^{a-1}-\sum_{q=0}^{a-m-2}\frac{e^{-\lambda\beta t}(\lambda\beta t)^{q}}{q!}$$

$$+\,r^{a-1}\sum_{q=0}^{a-m-2}\frac{e^{-\lambda\beta t}(\lambda\beta t)^{q}}{q!}\Bigg]\;P20$$

$$\frac{2\mu e^{-2\mu t}(1-r^{b})}{1-r}\Bigg[1-r^{a-1}-E(a-1,\lambda\beta t)$$

$$+\,r^{a-1}e^{-\lambda\beta t}\,e\left(a-1,\frac{\lambda\beta t}{r}\right)\Bigg]\;P20 \qquad (25)$$

Adding equations (24) and (25). We get the equation III as follows.

$$\left[\frac{2\mu e^{-2\mu t}(1-r^{b})}{1-r} \left[1-r^{a-1} - E(a-1,\lambda\beta t)\right] + \frac{2\mu e^{-\lambda\beta t}}{1-r} r^{a-1} e(a-1, \frac{\lambda\beta t}{r}) \right] P20 \qquad (26)$$

Hence by the equations (22) (23) and (26) we get the expression for the probability density functon $\gamma(t)$ as.

$$\gamma(t) = \left[\frac{2\mu}{1-r} E(a-1,\lambda t) - \frac{2\mu r^{a-1}}{1-r} e^{-\lambda t} e(a-1, \frac{\lambda t}{r}) - \frac{r^{b} 2\mu e^{-2\mu t(1-r^{b})}}{1-r} + \frac{r^{a-1}\, 2\mu e^{-\lambda\beta t}}{1-r} e(a-1\ \frac{\lambda t\beta}{r}) + \frac{2\mu e^{-2\mu t(1-r^{b})}}{1-r} (1-E(a-1,\lambda\beta t)) \right] P20 + \lambda E(a-1,\lambda t)\ P0a-1 \qquad (27)$$

The Expected waiting time in the queue E (T) is given by

$$E(T) = \int_{0}^{\infty} t\, \gamma(t)\, dt$$

$$= \int_{0}^{\infty} t \left[\frac{2\mu}{1-r} E(a-1,\lambda t) - \frac{2\mu r^{a-1}}{1-r} e^{-\lambda t} e(a-1, \frac{\lambda t}{r}) \right.$$

$$- \frac{r^{b} 2\mu e^{-2\mu t(1-r^{b})}}{1-r} + \frac{r^{a-1} \; 2\mu e^{-\lambda\beta t}}{1-r} e(a-1, \frac{\lambda\beta t}{r})$$

$$+ \frac{2\mu e^{-2\mu t} \; (1-r^{b})}{1-r} (1 - E(a-1, \lambda\beta t)] \; P20 \; dt$$

$$+ \int_{0}^{t} t \lambda E(a-1, \lambda t) \; dt \; P0a\text{-}1$$

$$= [\frac{2\mu}{1-r} \; \frac{a(a-1)}{2\lambda^{2}} \cdot - \frac{2\mu r^{a-1}}{1-r} \; \frac{(r^{a}-1) - a(r-1)}{\lambda^{2} r^{a-2} (r-1)^{2}}$$

$$+ \frac{(1 - r^{b})}{2\mu(1-r) \; (1-r^{b})^{2}} + \frac{2\mu r \, [(r^{a}-1) - a(r-1)]}{(r-1)^{2} \; (1-r) \, \lambda^{2} \beta^{2}}$$

$$- \frac{2\mu}{1-r} \; \frac{r^{2}(1-r^{a}) - ar^{a+1}(1-r)}{\lambda^{2} \beta^{2} \; (1-r)^{2}} \;] \; P20$$

$$+ \frac{\lambda a(a-1)}{2\lambda^{2}} \; P0a\text{-}1$$

$$E(T) = [\frac{2\mu r}{\lambda^{2}(1-r)^{3}} [r^{a} - 1 - a(r-1)] \; [\frac{1}{\beta^{2}} - 1] + \frac{\mu a(a-1)}{\lambda^{2} \; (1-r)}$$

$$+ \frac{r}{(1-r)^{2} \lambda\beta} - \frac{2\mu}{\lambda^{2} \; \beta^{2} (1-r)^{3}} [r^{2}(1-r^{a}) - ar^{a+1}(1-r)]]$$

$$P20 + \frac{a(a-1)}{2\lambda} \; P0a\text{-}1 \qquad (28)$$

The Expected number of customer waiting in the queue is given by

$$Lq = \sum_{n=0}^{a-1} n\,[P0n + P1n] + \sum_{n=0}^{\infty} n\,P2n$$

and using equations (12), (13) and (10) we get

$$Lq = \sum_{n=0}^{a-1} n\,\left[\frac{2\mu}{\lambda(1-r)} - \frac{2\mu^2\,r^{n+2}}{\lambda^2(r/\theta-1)(1-r)} + \frac{2\,\theta^{n+1}\mu}{\lambda(r/\theta-1)}\right]$$

$$P20 + \sum_{n=0}^{a-1} n(1-\theta^{n+1})\,P0a-1 + \sum_{n=0}^{a-1} n\,\theta^{n+1}\,P0a-1$$

$$+ \sum_{n=0}^{a-1} n\,\frac{2\mu}{(\lambda+\mu)r-\lambda}\,(r^{n+1}-\theta^{n+1})\,P20$$

$$+ \sum_{n=0}^{\infty} n r^n\,P20$$

$$= \left[\frac{\mu\,a(a-1)}{\lambda(1-r)} - \frac{2\mu r^2}{\lambda(1-r)}\left[\frac{(1-r^a) - ar^{a-1}\,(1-r)}{(1-r)^2}\right]\right.$$

$$\left. + \frac{r}{(1-r)^2}\right]P20 + \frac{a(a-1)}{2}\,P0a-1$$

$$Lq = \left[\frac{r}{(1-r)^2} + \frac{\mu a(a-1)}{\lambda(1-r)} - \frac{2\mu}{(1-r)^3}\,(r^2(1-r^a) - ar^{a+1}(1-r))\right.$$

$$\left.\vphantom{\frac{r}{(1-r)^2}}\right]P20 + \frac{a(a-1)}{2}\,P0a-1 \qquad (29)$$

The Little's formula is not true.

SECTION - II.

Assume that an arriving customer would not balk if atleast one server is free and the queue length is found to be a-1.

That is, if an arriving customer finds the system in state (0, a-1) or in state (1, a-1) then he would not balk. In all other cases he will join the system with probability β and balk with probability $1-\beta$.

The differential difference equations are.

$$
\begin{aligned}
P00^{1}(t) &= -\lambda\beta P00(t) + \mu P10(t) && (30)\\
P0n^{1}(t) &= -\lambda\beta P0n(t) + \lambda\beta P0n-1(t) + \mu P1n(t) &&\\
P10^{1}(t) &= -(\lambda\beta+\mu) P10(t) + \lambda P0a-1(t) + 2\mu P20(t) &&\\
P1n^{1}(t) &= -(\lambda\beta+\mu) P1n(t) + \lambda\beta P1,n-1t) + 2\mu P2n(t) &&\\
P2n^{1}(t) &= -(\lambda\beta+2\mu) P2n(t) + \lambda\beta P2n-1(t) + 2\mu P2n+b(t) &&\\
P20^{1}(t) &= -(\lambda\beta+2\mu) P20(t) + \lambda P1a-1(t) &&\\
&\qquad + 2\mu \sum_{S=a}^{b} P2,s &&\\
P0a-1^{1}(t) &= -\lambda P0a-1(t) + \lambda\beta P0a-2(t) + \mu P1a-1(t) &&\\
P1a-1^{1}(t) &= -(\lambda+\mu) P1a-1(t) + \lambda\beta P1a-2(t) + 2\mu P2a-1 &&
\end{aligned}
$$

The steady state equations are given by

$$\lambda\beta P00 = \mu P10 \qquad (31)$$

$$\lambda\beta\ P0n = \lambda\beta\ P0n-1 + \mu P\ 1n \qquad (32)$$

$$(\lambda\beta + \mu)\, P10 = \lambda\, P0a-1 + 2\mu\, P20 \qquad (33)$$

$$(\lambda\beta + \mu)\ P1n = \lambda\beta P1n-1 + 2\mu\, P2n \qquad (34)$$

$$(\lambda\beta + 2\mu)\ P2n = \lambda\beta P2n-1 + 2\mu\ P2n+b \qquad (35)$$

$$(\lambda\beta + 2\mu)\ P20 = \lambda P1a-1 + 2\mu \sum_{s=a}^{b} P2S \qquad (36)$$

$$\lambda P0a-1 = \lambda\beta\ P0a-2 + \mu\ P1a-1 \qquad (37)$$

$$(\lambda + \mu)\ P1a-1 = \lambda\beta\ P1a-2 + 2\mu\, P2a-1 \qquad (38)$$

Equation (35) has b+1 roots using Rouche's theorem we get only one root say r lies in side $|Z| = 1$ and all the other roots $r_1\ r_2$ ----- r_b lie outside the unit circle.

Since $\sum_{n=0}^{\infty} P2n < 1$

$P2n = Ar^n$ for $n \geq 0$

taking $n = 0$ we get $A = P20$

$$P2n = P20\ r^n \qquad (39).$$

Define $\theta = \dfrac{\lambda\beta}{\lambda\beta + \mu}$ (40).

from equation (34) we get

$$P1n = A\left(\frac{\lambda\beta}{\lambda\beta+\mu}\right)^{n} + 2\mu\,\frac{r^{n+1}}{(\lambda\beta+\mu)r-\lambda\beta}\,P20$$

taking n = 0 and using (33)

$$\frac{2\mu}{\lambda\beta+\mu}\,P20 + \frac{\lambda}{\lambda\beta+\mu}\,P0a-1 = A + \frac{2\mu r}{(\lambda\beta+\mu)r-\lambda\beta}\,P20$$

$$A = \frac{\theta}{\beta}\,P0a-1 + \left[\frac{2\mu}{\lambda\beta+\mu} - \frac{2\mu r}{(\lambda\beta+\mu)r-\lambda\beta}\right]P20$$

$$= \frac{\theta}{\beta}\,P0a-1 - \frac{2\mu\lambda\beta}{(\lambda\beta+\mu)[(\lambda\beta+\mu)r-\lambda\beta]}\;]\;P20$$

$$= \frac{\theta}{\beta}\,P0a-1 - \frac{2\mu\theta}{(\lambda\beta+\mu)r-\lambda\beta}\,P20$$

$$P1n = \frac{\theta^{n+1}}{\beta}\,P0a-1 - \frac{2\mu\,\theta^{n+1}}{(\lambda\beta+\mu)r-\lambda\beta}\,P20 + \frac{2\mu\,r^{n+1}}{(\lambda\beta+\mu)r-\lambda\beta}\,P20$$

$$\mathbf{P1n} = \frac{\theta^{n+1}}{\beta}\,P0a-1 - \frac{2\mu}{(\lambda\beta+\mu)r-\lambda\beta}\left(r^{n+1}-\theta^{n+1}\right)P20 \qquad (41)$$

putting k = 1, z .... n in equation (32) and adding we get

$$\lambda\beta P0n = \lambda\beta\,P00 + \mu\sum_{k=1}^{n} P1k$$

using (31) in the above equaition gives

$$P0n = \frac{\mu}{\lambda\beta}\sum_{k=0}^{n} P1k$$

$$= \frac{\mu}{\lambda\beta} \left[\sum_{k=0}^{n} \frac{\theta^{n+1}}{\beta} P0a-1 + \frac{2\mu(r^{n+1} - \theta^{n+1})}{(\lambda\beta + \mu)\ r-\lambda\beta} P20 \right]$$

$$= \frac{\mu}{\lambda\beta} \left[\frac{\theta - \theta^{n+2}}{\beta\,(1-\theta)} P0a-1 + \frac{2\mu}{(\lambda\beta + \mu)\ r-\lambda\beta} \right.$$

$$\left. \left[\frac{r-r^{n+2}}{1-r} - \frac{\theta - \theta^{n+2}}{1-\theta} \right] P20 \right.$$

$$= \frac{\mu}{\lambda\beta} \frac{\theta(1-\theta^{n+2})}{\beta\,(1-\theta)} Poa-1 + \frac{2\mu}{(\lambda\beta + \mu)\ r-\lambda\beta} P20$$

$$\left[\left(\frac{r}{1-r} - \frac{\theta}{1-\theta} \right) - \frac{r^{n+2}}{1-r} + \frac{\theta^{n+2}}{1-\theta} \right]$$

$$\mathbf{P0n} = \frac{1}{\beta} (1- \theta^{n+1}) P0a-1 + \left[\frac{2\mu}{(1-r)\lambda\beta} - \frac{2\mu^2\ r^{n+2}}{(\lambda\beta)^2 (r/\theta - 1)(1-r)} + \frac{2\mu\,\theta^{n+1}}{\lambda\beta(r/\theta - 1)} \right] P20. \qquad (42)$$

using equation (38) yield.

$$P1a-1 = \frac{\lambda\beta}{\lambda + \mu} P1a-2 + \frac{2\mu}{\lambda+\mu} P2a-1$$

$$= \frac{\lambda\beta}{\lambda + \mu} \left[\frac{2\mu}{(\lambda\beta + \mu)\ r-\lambda\beta} (r^{a-1} - \theta^{a-1}) P20 + \frac{\theta^{a-1}}{\beta} P0a-1 \right] + \frac{2\mu}{\lambda+\mu} r^{a-1}\ P20$$

$$= \frac{2\mu}{(\lambda + \mu)\ (1-r/\theta)} (\theta^{a-1} - r^{a-1})\ P20 + \left(\frac{\lambda}{\lambda+\mu} \right)$$

$$\theta^{a-1}\ P0a-1 + \frac{2\mu}{\lambda+\mu}\ r^{a-1}\ P20$$

$$P1a-1 = \frac{2\mu\ P20}{(\lambda+\mu)\ (\theta-r)}\ (\theta^{a} - r^{a}) + \frac{\lambda}{\lambda+\mu}\theta^{a-1}\ P0a-1$$

(43)

Substituting equation (43) in (36) we get p_{0a-1}

$$(\lambda\beta+2\mu)P20 = \frac{2\mu\lambda\ (\theta^{a}-r^{a})}{(\lambda+\mu)\ (\theta-r)} + \frac{\lambda^{2}}{\lambda+\mu}\ \theta^{a-1}\ Poa-1 + \frac{2\mu(r^{a}-r^{b+1})}{1-r}\ P20$$

$$\frac{\lambda^{2}}{\lambda+\mu}\ \theta^{a-1}\ Poa-1 = \frac{2\mu}{1-r}\ (1-r^{a})\ P20 - \frac{2\mu\lambda\ (\theta^{a}-r^{a})}{(\lambda+\mu)\ (\theta-r)}\ P20$$

$$Poa-1 = \frac{\lambda+\mu}{\lambda^{2}\theta^{a-1}}\ P20\ \left[\frac{2\mu}{1-r}\ (1-r^{a}) - \frac{2\mu\lambda}{(\lambda+\mu)\ (\theta-r)}\ (\theta^{a} - r^{a})\right]$$

(44)

Now we have all the probabilities in terms of P20 substituting all the expression in equation (37) it is easy to find the expression for P20. The results can be checked by substituting the value of Pin $i=0,1,\ 0 \leq n \leq a-1$ and P2n, $n \geq 0$ in the normalised equation.

$$\sum_{n=0}^{a-1} [Pon + P1n] + \sum_{n=0}^{\infty} P2n = 1$$

$$\sum_{n=0}^{a-2} [P_{0n} + P_{1n}] + P_{0a-1} + P_{1a-1} + \sum_{n=0}^{\infty} P_{2n}=1$$

$$\sum_{n=0}^{a-2} \left[\frac{2\mu}{(1-r)\lambda\beta} + \frac{2\mu\theta^{n+1}}{\lambda\beta(r/\theta-1)} - \frac{2\mu^{2}r^{n+2}}{(\lambda\beta)^{2}\,(1-r)\,(r/\theta-1)}\right] P_{20}+$$

$$\sum_{n=0}^{a-2} \frac{1-\theta^{n+1}}{\beta} P_{0a-1} + \sum_{n=0}^{a-2} \frac{\theta^{n+1}}{\beta} P_{0a-1}$$

$$+ \sum_{n=0}^{a-2} \frac{2\mu}{(\lambda\beta+\mu)\,r-\lambda\beta} (r^{n+1} - \theta^{n+1})\, P_{20} +$$

$$\frac{\lambda+\mu}{\lambda^{2}\theta^{a-1}} \left[\frac{2\mu\,(1-r^{a})}{1-r} - \frac{2\mu\,(\theta^{a} - r^{a})}{(\lambda+\mu)\,(\theta-r)}\right] P_{20}$$

$$+ \frac{2\mu\,(\theta^{a} - r^{a})}{(\lambda+\mu)\,(\theta-r)} P_{20} + P_{20} \frac{2\mu\,(1-r^{a})}{\lambda(1-r)}$$

$$- \frac{2\mu\,(\theta^{a}-r^{a})}{(\lambda+\mu)\,(\theta-r)} P_{20} + \frac{1}{1-r} P_{20} = 1$$

Hence we get,

$$\frac{(a-1)2\mu}{\lambda\beta(1-r)} - \frac{2\mu^{2}\, r^{2}}{(\lambda\beta)^{2}\,(1-r)\,(r/\theta-1)} (1+r+\dots+r^{a-2})$$

$$+ \frac{2\mu\, r}{\lambda\beta\,(r/\theta-1)} (1+r+\dots\; r^{a-2}) + \left[\frac{(a-1)}{\beta} + 1\right]$$

$$[\frac{\lambda+\mu}{\lambda^{2}\theta^{a-1}} [\frac{2\mu}{1-r} (1-r^{a}) - \frac{2\mu}{(\lambda+\mu)(\theta-r)} (\theta^{a} - r^{a})] + \frac{2\mu(1-r^{a})}{\lambda\,(1-r)} + \frac{1}{1-r}]$$

$$P_{20} = 1$$

$$P20 = \left[\frac{(a-1)\, 2\mu}{\lambda\beta\,(1-r)} + \frac{2\mu(1-r^{a})}{\lambda\,(1-r)} - \frac{2\mu r\,(1-r^{a-1})}{\lambda\beta\,(1-r)^{2}} + \frac{\lambda+\mu}{\lambda^{2}\theta^{a-1}} \left[\frac{2\mu}{1-r}(1-r^{a}) - \frac{2\mu\lambda}{(\lambda+\mu)\,(\theta-r)}(\theta^{a}-r^{a}) \right] \left[\frac{a-1}{\beta} + 1 \right] + \frac{1}{1-r} \right]^{-1} \quad (45)$$

WAITING TIME DISTRIBUTION

Now we find the waiting time distribution in the equilibrium state. Let the random variable T denote the waiting time in the queue for an arriving unit.

Let $\nu(t)$ be the probability density function (p.d.f) of T.

$f(\alpha, k, t)$ be the probability density function of gamma distribution with parameter α, k.

$$\text{(i.e)} \quad f(\alpha, k, t) = \frac{\alpha^{k}\, t^{k-1}\, e^{-\alpha t}}{\Gamma(K)} \qquad t \geqslant 0,\ k = 1, 2 \ldots \quad (46)$$

$\Gamma_x(\alpha, K)$ be the incomplete gamma function

$$\text{(i.e)}\ \Gamma_x(\alpha, k) = \int_0^x f(\alpha, k, t)\, dt = 1 - \sum_{s=0}^{k-1} \frac{e^{-\alpha x}(\alpha x)^{s}}{S!}$$

An arriving customer finds the system in one of the following state (47)

(i) $(0, a-1)$, $(1- a-1)$

(ii) $(0, n)$, $(1, n)$ $0 \leq n \leq a-2$

$$\left.\begin{array}{lll} (iii) & (2, n) & a-1 \leq m \leq b-1 \\ (iv) & (2, n) & o \leq m \leq a-1 \end{array}\right\} \begin{array}{l} n = Kb + m \\ k = 0,1,2 \ldots \end{array}$$

In case (i) the arriving unit does not wait and the probability of zero delay is thus $[Poa-1 + P1a-1]$. In all other case, the unit has to wait and the probability of blocking is thus $1 - [Poa-1 + P1a-1]$

In case (ii) the arriving unit has to wait for the arrival of $(a-1-n)$ unit and the time needed for $(a-1-n)$ arrival has a gamma distribution with parameter $\lambda\beta$, $a-1-n$.

In case (iii) the arriving unit has to wait for the completion of services of $(k+1)$ batches and the time required for this has a gamma distribution with parameter 2μ, $K+1$.

In case of (iv) the arriving unit has to wait till either the services of $(K+1)$ groups are completed or $(a-1-m)$ units arrive, whichever occur later, this duration which is given by the maximum of two gamma variates may be denoted by the random variable Z.

Z = max (gamma variate with parameter $\lambda\beta$, $a-1-m$; gamma variate with parameters 2μ, $K+1$)

$$Z = \max(X,Y) \text{ (say)} \tag{48}$$

The distribution function $F_z(t)$ and the p.d.f $h_z(t)$ of Z are given by

$$\begin{aligned} F_z(t) &= \Pr(Z \leq t) \\ &= \Pr(\max(X,Y) \leq t) \\ &= \Pr(X \leq t,\ Y \leq t) \\ &= \Pr(X \leq t)\ \Pr(Y \leq t) \\ &= \Gamma_t(\lambda\beta, a-1-m)\ \Gamma_t(2\mu, K+1) \end{aligned}$$

and

$$\begin{aligned} h_z(t) &= F_z'(t) \\ &= f(\lambda\beta, a-1-m)\ \Gamma_t(2\mu, K+1) \\ &\quad + \Gamma_t(\lambda\beta, a-1-m)\ f(2\mu, K+1) \end{aligned} \tag{49}$$

It follows that probability density function $\nu(t)$ of T is given by

$$\begin{aligned} \nu(t) &= \sum_{n=0}^{a-2} [P_{1n} + P_{on}]\ f(\lambda\beta, a-1-n, t) \\ &\quad + \sum_{K=o}^{\infty} \sum_{m=a-1}^{b-1} P_{2\,Kb+m}\ f(2\mu, K+1, t) \\ &\quad + \sum_{K=o}^{\infty} \sum_{m=o}^{a-2} P_{2\,Kb+m}\ h_z(t) \end{aligned} \tag{50}$$

Denote

$$\sum_{k=0}^{m-1} \frac{z^k}{K\,!} = e\,(m,z); \quad e^{-z}\, e(m,z) = E(m,z) \quad m \geq 1 \qquad (51)$$

Then the following arguments at Medhi [6, 7] we have

$$\sum_{q=0}^{a-2} f(\lambda\beta, a-1-q, t) = \sum_{q=0}^{a-2} \frac{(\lambda\beta)^{a-1-q}\, t^{a-2-q}\, e^{-\lambda\beta t}}{\Gamma(a-1-q)}$$

$$= \lambda\beta\, e^{-\lambda\beta t}\, e(a-1, \lambda\beta t).$$

$$\sum_{q=0}^{a-2} r^q\, f\,(\lambda\beta,\ a-1-q, t) = \lambda\beta e^{-\lambda\beta t}\, r^{a-2}\, e(a-1, \frac{t\lambda\beta}{r})$$

$$\int_0^\infty t\, E\,(a-1, \lambda\beta t)\,dt = \frac{a(a-1)}{2\,\lambda^2\,\beta^2}$$

$$\int_0^\infty t\, e^{-\lambda\beta t}\, e(a-1, \frac{\lambda\beta t}{r})\, dt = \frac{(r^a-1) - a(r-1)}{\lambda^2\,\beta^2\; r^{a-2}\; (r-1)^2}$$

$$\int_0^\infty t\, E\,(a-1, \lambda\beta\, t)\; e^{-\mu(1-r^b)t}\, dt =$$

$$= \frac{r^2\,(1-r^a) - ar^{a+1}\,(1-r)}{\lambda^2\,\beta^2\,(1-r)^2}$$

$$\int_0^\infty t(1-r^b)\; e^{-\mu(1-r^b)t}\, dt = \frac{1}{\mu^2\,(1-r^b)}$$

The first term on the right hand side of equation (50)is

$$\sum_{n=0}^{a-2} (P1n + P0n)\ f\ (\lambda\beta,\ a-1-n,\ t)$$

$$\sum_{n=0}^{a-2} \frac{\theta^{n+1}}{\beta}\ Poa-1 + \frac{2\mu}{(\lambda\beta+u)r-\lambda\beta}\ (r^{n+1} - \theta^{n+1})\ P20 +$$

$$\frac{1-\theta^{n+1}}{\beta}\ Poa-1 + (\frac{2\mu}{(1-r)\lambda\beta} - \frac{2\mu^{2} r^{n+1}}{\lambda^{2}(r/\theta -1)(1-r)} + \frac{2\mu\theta^{n+1}}{\lambda\beta(r/\theta-1)})]$$

$$f(\lambda\beta,\ a-1-n,t)$$

$$\sum_{n=0}^{a-2} (\frac{P0a-1}{\beta} + \frac{2\mu}{(1-r)\lambda\beta}\ P20 - \frac{2\mu r^{n+1}}{\lambda\beta(1-r)}\ P20)$$

$$f\ (\lambda\beta,a-1-n,t)$$

$$(\frac{1}{\beta}\ P0a-1 + \frac{2\mu}{(1-r)\lambda\beta}\ P20)\ \lambda\beta\ e^{-\lambda\beta t}\ e(a-1,\lambda\beta t)$$

$$- \frac{2\mu}{\lambda\beta(1-r)}\ r^{a-1}\ e^{-t\lambda\beta}\lambda\beta\ e(a-1,\lambda\beta t/r)P20$$

$$\lambda\ E\ (a-1,\lambda\beta\ t)\ \ P0a-1 + \frac{2\mu}{(1-r)}\ E\ (a-1,\lambda\beta t)\ P20$$

$$- \frac{2\mu}{1-r}\ r^{a-1}\ e^{-\lambda\beta t} e(a-1,\frac{\lambda\beta t}{r})\ P20 \qquad (53)$$

Second term on the right hand side of equation (50) is

$$\sum_{k=0}^{\infty} \sum_{m=a-1}^{b-1} P2\ kb+m\ f(2\mu, K+1, t)$$

$$\sum_{m=a-1}^{b-1} r^{m} \sum_{k=0}^{\infty} \frac{r^{kb} (2\mu)^{K+1} e^{-2\mu t} t^{k}}{K!} \quad P20$$

$$\sum_{m=a-1}^{b-1} r^{m} \frac{2\mu}{1} \sum_{k=0}^{\infty} \frac{e^{-2\mu t} (2\mu t r^{b})^{K}}{K!} \quad P20$$

$$\sum_{m=a-1}^{b-1} r^{m}\, 2\mu\, e^{-2\mu t}\, e^{2\mu r^{b} t} \quad P20$$

$$\left[\frac{r^{a-1} - r^{b}}{1 - r}\right] 2\mu\ e^{-2\mu t(1-r^{b})} \qquad (54)$$

The third term on the right hand side of equation (50) consists of two terms corresponding to those of (49) and using (39) we get

$$\sum_{m=0}^{a-2} \sum_{k=0}^{\infty} r^{kb+m}\ h_{2}(t)\ P20$$

$$\sum_{k=0}^{\infty} \sum_{m=0}^{a-2} r^{kb+m} (f(\lambda\beta, a-1-m, t)\ \Pi t\ (2\mu, K+1)$$

$$+\Pi t\ (\lambda\beta, a-1-m)\ f(2\mu, K+1, t))\ P20$$

The first term of the above equation is

$$\sum_{K=0}^{\infty}\sum_{m=0}^{a-2} r^{kb+m}\, f(\lambda\beta, a-1-m, t)\, \Pi_t(2\mu, K+1) \quad P20$$

$$\sum_{m=0}^{a-2} r^{m}\, f(\lambda\beta, a-1-m, t) \sum_{k=0}^{\infty} r^{kb} \int_0^t f(2\mu, K+1, t)\, dt \quad P20$$

using (23) we get

$$\lambda\beta\, e^{-\lambda\beta t}\, r^{a-2}\, e\left(a-1, \frac{\lambda t\beta}{r}\right) \sum_{K=0}^{\infty} \int_0^t e^{-2\mu t}\, 2\mu\, \frac{(2\mu t r^{b})^{k}}{K!}\, dt \quad P20$$

$$\lambda\beta\, e^{-\lambda\beta t}\, r^{a-2}\, e\left(a-1, \frac{t\lambda\beta}{r}\right) 2\mu \int_0^t e^{-2\mu t(1-r^{b})}\, dt \quad P20$$

$$\frac{\lambda\beta}{(1-r^{b})}\, e^{-\lambda\beta t}\, r^{a-2}\, e\left(a-1, \frac{t\lambda\beta}{r}\right) \{1 - e^{-2\mu t(1-r^{b})}) \quad P20$$

$$\frac{2\mu}{1-r}\, e^{-\lambda\beta t}\, r^{a-2}\, e\left(a-1, \frac{t\lambda\beta}{r}\right) \left(1-e^{-2\mu t(1-r^{b})}\right) \quad P20 \qquad (55)$$

The second term of equation is

$$\sum_{k=0}^{\infty}\sum_{m=0}^{a-2} r^{bk+m}\, \Pi_t(\lambda\beta, a-1-m)\, f(2\mu, K+1, t) \quad P20$$

$$\sum_{k=0}^{\infty}\sum_{m=0}^{a-2} r^{bk+m} \left[1-\sum_{q=0}^{a-m-2} \frac{e^{-\lambda\beta t}(\lambda\beta t)^{q}}{q!} \right] \frac{2\, e^{-2\mu t}(2\mu t)^{k}}{K!} \quad P20$$

$$\sum_{k=0}^{\infty} \sum_{m=0}^{a-2} \frac{r^{kb+m}\, 2\, e^{-2\mu t} (2\mu t)^k}{K\,!} P20$$

$$- \sum_{k=0}^{\infty} \sum_{m=0}^{a-2} r^{kb+m} P20\ \underline{2\mu e^{-2\mu t} (2\mu t)^k}$$

$$\sum_{q=0}^{a-m-2} \frac{e^{-\lambda\beta t} (\lambda\beta t)^q}{q\,!}$$

$$\left(\sum_{k=0}^{\infty} r^{kb}\, 2\mu e^{-2\mu t} \frac{(2\mu t)^k}{k\,!} - \sum_{k=0}^{\infty} r^{kb} \right.$$

$$\left. \sum_{q=0}^{a-2-m} \frac{e^{-\lambda\beta t} (\lambda\beta t)^q}{q\,!} \frac{2\, e^{-2\mu t} (2\mu t)^k}{k\,!} \right)$$

$$\left(\frac{1-r^{a-1}}{1-r}\right) P20$$

$$\left(2\mu e^{-2\mu t}\, e^{2\mu t r^b} \left(1 - \sum_{q=0}^{a-m-2} \frac{e^{-\lambda\beta t} (\lambda\beta t)^q}{q\,!}\right)\right.$$

$$\left.\left(\frac{1-r^{a-1}}{1-r}\right)\right) P20$$

$$\frac{2\mu\, e^{-2\mu t(1-r^b)}}{(1-r)} \left(1-r^{a-1} - \sum_{q=0}^{a-m-2} \frac{e^{-\lambda\beta t} (\lambda\beta t)^q}{q\,!}\right.$$

$$\left. + r^{a-1} \sum_{q=0}^{a-m-2} \frac{e^{-\lambda\beta t} (\lambda\beta t)^q}{q\,!} \right) P20$$

$$\frac{2\mu e^{-2\mu t(1-r^b)}}{(1-r)}\left(1-r^{a-1} - E(a-1,\lambda\beta t) + r^{a-1} e^{-t\lambda\beta} e(a-1, \frac{\lambda\beta t}{r})\right)P20 \qquad (56)$$

Adding equations (55)and (56) we get the third term of the equation (50) as

$$\left[\frac{2\mu e^{-2\mu t(1-r^b)}}{1-r}\left(1-r^{a-1} - E(a-1,\lambda\beta t)\right) + \frac{2\mu e^{-\lambda\beta t}}{1-r} r^{a-1} e(a-1, \frac{\lambda\beta t}{r})\right]P20 \qquad (57)$$

Hence by the equations (53), (54) and (57) we get the expression for the probability density function $\nu(t)$ as

$$\nu(t) = \left[\frac{2\mu}{1-r}E(a-1,\lambda\beta t) + \frac{(1-r^b)\,2\mu}{1-r}e^{-2\mu t(1-r^b)} - \frac{2\mu e^{-2\mu t(1-r^b)}}{1-r}E(a-1,\lambda\beta t)\right]P20 + \lambda E(a-1,\lambda\beta t) \quad P0a-1 \qquad (58)$$

The expected waiting time in the queue E(T) is given by

$$E(T) = \int_0^{\infty} t \, \nu(t)\, dt$$

$$= \int_0^\infty t \left[\left(\frac{2\mu}{1-r} E(a-1,\lambda\beta t) + \frac{(1-r^b)2\mu}{(1-r)} e^{-2\mu t(1-r^b)} - \frac{2\mu e^{-2\mu t(1-r^b)}}{1-r} E(a-1,\lambda\beta t) \right] P20 \right.$$

$$\left. + Po\ a-1 \quad E(a-1,\lambda\beta t) \right] dt$$

$$= \left(\frac{2\mu}{1-r} \left[\frac{a(a-1)}{2\ \lambda^2\ \beta^2} \right] + \frac{2\mu}{(1-r)\ (2\mu)^2\ (1-r^b)} - \frac{2}{1-r} \left[\frac{r^2\ (1-r^a) - a r^{a+1}\ (1-r)}{\lambda^2\ \beta^2\ (1-r)^2} \right] \right) P20$$

$$+ \frac{a(a-1)\lambda}{2\lambda^2\ \beta^2} P0a-1$$

$$E\ (T) = \left(\frac{\mu}{1-r} \left[\frac{a(a-1)}{\lambda^2\ \beta^2} \right] + \frac{1}{2\mu(1-r)\ (1-r^b)} - \frac{2}{(1-r)^3} \left[\frac{r^2\ (1-r^a) - a r^{a+1}\ (1-r)}{\lambda^2\ \beta^2\ (1-r)^2} \right] \right) P20$$

$$+ \frac{a(a-1)}{2\ \lambda\ \beta^2} P0a-1 \tag{59}$$

The Expected number of customer waiting in the queue is given by

$$Lq = \sum_{n=0}^{a-1} n\ [P0n + P1n] + \sum_{n=0}^{\infty} n\ P2n$$

$$= \sum_{n=0}^{a-2} n \left[\frac{P0a-1}{\beta} + \frac{2\mu\ (1-r^{n+1})}{(1-r)\lambda\beta} P20 \right] + (a-1)\ P0a-1$$

$$+ (a-1)\ P1\ a-1 + \sum_{n=0}^{\infty} nr^{n}\ P20$$

$$= \sum_{n=0}^{a-2} n \left[\frac{P0a-1}{\beta} + \left(\frac{2\mu}{\lambda\beta(1-r)} - \frac{2\ \mu r^{n+1}}{\lambda\beta(1-r)} \right) P20 \right] + (a-1)\ P0a-1$$

$$+ (a-1) \frac{2\mu(1-r^{a})}{\lambda(1-r)} P20 + \sum_{n=0}^{\infty} n\ r^{n}\ P20$$

$$= \frac{P0a-1(a-1)(a-2)}{2\beta} + \frac{2\mu}{\lambda\beta(1-r)} \frac{(a-2)(a-1)}{2} P20$$

$$- \frac{2}{\lambda\beta(1-r)} r^{2} \sum_{n=0}^{a-2} nr^{n-1}\ P20 + (a-1)Poa-1 +$$

$$\frac{(a-1)2\mu(1-r^{a})}{(1-r)\lambda} P20 + r \sum_{n=0}^{\infty} nr^{n-1}\ P20$$

$$= (a-1)\ Po\ a-1 \left(\frac{a-2}{\beta} + 1 \right) + \left(\frac{2\mu}{(1-r)} (a-1) \left(\frac{a-2}{\beta} + 1 \right) \right.$$

$$- \frac{2\mu}{\lambda\beta(1-r)} r^{2} \left(\sum_{n=0}^{a-2} \frac{d}{dr} (r^{n}) \right) - \frac{(a-1)\ 2\mu r^{a}}{\lambda(1-r)}$$

$$\left. + r \sum_{n=0}^{\infty} \frac{d}{dr} (r^{n}) \right) P20$$

$$= (a-1)\ Po\ a-1 \left(\frac{a-2}{\beta} + 1 \right) + \left(\frac{2\mu}{(1-r)} (a-1) \left(\frac{a-2}{\beta} + 1 \right) \right.$$

$$- \frac{2\mu}{\lambda\beta(1-r)} r^{2} \left(\frac{(1-r)\ (-(a-1)r^{a-2}) + (1-r^{a-1})}{(1-r)^{2}} \right)$$

$$\left. + \frac{2\ \mu(a-1)r^{a}}{\lambda(1-r)} + \frac{r}{(1-r)^{2}} \right) P20$$

$$Lq = a\,(a-1)\,Po\ a-1\ \left(\frac{a-2}{\beta} + 1\right) + \left(\frac{2\mu}{(1-r)}(a-1)\left(\frac{a-2}{\beta} + 1\right) + \frac{r}{(1-r)^2} - \frac{2\mu}{\lambda(1-r)}\left(\frac{r^2\,(1-r^{a-1}) - (a-1)\,r^a\,(1-r)}{\beta\,(1-r)^2} + (a-1)r^a\right)P20 \qquad (60)$$

It is clear that the Little's formula is not true in this model also. It can be verified that when $\beta = 1$ we have $E(T) = \lambda Lq$.

SINGLE SERVER MARKOVIAN QUEUE WITH BALKING

CHAPTER -2

SINGLE SERVER MARKOVIAN QUEUE WITH BALKING

Here the steady state results are given for the single channel bulk service Markovian queue with balking. It is proved that the little's formule is not true if $\beta \neq 1$.

Type 1

First assume that if an arriving customer finds the server is free he would not balk. The customer joins the queue with probability β.

The steady state equations are

$$\lambda P00 = \mu P10 \qquad (1)$$

$$\lambda P0n = \lambda P0n + \mu P1n \qquad 1 \leq n \leq a-1 \qquad (2)$$

$$(\lambda\beta + \mu) P10 = \lambda P0a-1 + \mu \sum_{s=a}^{b} P1s \qquad (3)$$

$$(\lambda\beta + \mu) P1n = \lambda\beta P1n-1 + \mu P1n+b \qquad (4)$$

The steady state probabilities are given as

$$P1n = r^n P10 \quad \text{for} \quad n \geq 0 \qquad (5)$$

$$P0n = \frac{\mu}{\lambda} \left[\frac{1-r^{n+1}}{1-r}\right] P10 \qquad 0 \leq n \leq a-1 \qquad (6)$$

$$P\,00 = [\frac{a}{1-r} + \frac{(r^{a+1}) - r^{b+1})}{(1-r)^2}]^{-1} \qquad (7)$$

waiting time distribution

$$\gamma(t) = \sum_{n=0}^{a-2} P\,0n \quad f(\lambda, a-1-n, t) + \sum_{k=0}^{\infty} \sum_{m=a-1}^{b-1} P1\ kb+m$$

$$f(\mu, K+1, t) + \sum_{k=o}^{\infty} \sum_{m=o}^{a-2} P1Kb+m\ h\ z(t) \qquad (8)$$

$$\nu(t) = \frac{Poo}{1-r} \lambda[E(a-1, \lambda t) - r^{a-1} e^{-\lambda t} e(a-1, \frac{\lambda t}{r})$$

$$+ r^{a-1} e^{-\lambda\beta t} e(a-1, \frac{\lambda\beta t}{r}) + e^{-\mu t(1-r^b)} (1-r^b)$$

$$- e^{-\mu t(1-r^b)} E(a-1, \lambda\beta t)] \qquad (9)$$

The Expected waiting time in the queue E(T) is given by

$$E(T) = \int_0^{\infty} t\ \nu(t)\ dt$$

$$E(T) = \frac{\lambda P00}{1-r} [\frac{a(a-1)}{2\lambda^2} + \frac{1}{\mu^2(1-r^b)}$$

$$+ \frac{ar^{a+1}(1-r) - r^2(1-r^a)}{\lambda^2\beta^2(1-r)^2}$$

$$+ \frac{r[(r^a-1) - a(r-1)]}{\lambda^2(1-r)^2} [\frac{1}{\beta^2} - 1] \qquad (10)$$

The Expected number of customers Lq waiting in the queue is given by

$$Lq = \sum_{n=0}^{a-1} n\ P\ 0n + \sum_{n=0}^{\infty} n\ P1n$$

$$Lq = P00\ \left[\frac{a\ (a-1)}{2\ (1-r)} + \frac{ar^{a+1} \quad (1-r)\ -\ r^{2}(1-r^{a})}{(1-r)^{2}} + \frac{\lambda r}{\mu(1-r)^{2}} \right]$$

(11)

It is clear that unless $\beta = 1$ the little's formula is not true.

TYPE 2

Assume that if an arriving customer finds the system in state (0, a-1) then he would not balk. In all other cases he will join the system with probability β and balk with probability $1-\beta$.

The steady state equations are

$$\lambda\beta P00 = \mu\ P10 \tag{12}$$

$$\lambda\beta\ P0n = \lambda\beta\ P0n-1 + \mu\ P\ 1n \qquad 0 \leqslant n \leqslant a-1 \tag{13}$$

$$(\lambda\beta + \mu)\ P10 = \lambda\ P0a-1 + \mu \sum_{s=a}^{b} P\ 1s \tag{14}$$

$$(\lambda\beta + \mu)\ P\ 1n = \lambda\beta\ P1n-1 + \mu\ P1n + b \tag{15}$$

$$.\lambda\ P\ 0a-1 = \lambda\beta\ P0a-2 + \mu\ P1a-1 \tag{16}$$

The steady state probabilities are given as

$$P1n = r^{n} P10 \tag{17}$$

$$P0n = \frac{\mu}{\lambda\beta}\left[\frac{1-r^{n+1}}{1-r}\right] P10 \qquad 0 \leq n \leq a-2 \tag{18}$$

$$P0a-1 = \frac{\mu}{\lambda}\left[\frac{(1-r^{a})}{(1-r)}\right] P10 \tag{19}$$

$$P00 = \left[\frac{a-1}{1-r} + \frac{(r^{a}-r^{b+1})}{(1-r)^{2}} + \frac{(1-r^{a})\beta}{(1-r)}\right]^{-1} \tag{20}$$

waiting time distribution

$$\gamma(t) = \sum_{n=0}^{a-2} P0n \; f(\lambda\beta, a-1-n, t) + \sum_{k=0}^{\infty}\sum_{m=a-1}^{b-1} P1\,kb+m \; f(\mu, k+1, t) + \sum_{k=o}^{\infty}\sum_{m=o}^{a-2} P1\,kb+m \; hz(t) \tag{21}$$

$$\gamma(t) = \frac{\lambda\beta}{1-r}\left[E(a-1, \lambda\beta t)\,(1-e^{-\mu t(1-r^{b})}) + e^{-\mu t\,(1-r^{b})}(1-r^{b})\right] P00 \tag{22}$$

The Expected waiting time in the queue E(T) is given as

$$E(T) = \int_{0}^{\infty} t \; \gamma(t)\, dt$$

$$E(T) = \frac{\lambda\beta}{1-r}\left[\frac{a(a-1)}{2\lambda^{2}} + \frac{1}{\mu^{2}(1-r^{b})} + \frac{ar^{a+1}(1-r) - r^{2}(1-r^{a})}{\lambda^{2}\beta^{2}(1-r)^{2}}\right] P00$$

23)

The Expected number of customers Lq waiting in the queue is given by

$$Lq = \sum_{n=0}^{a-1} n\, P0n + \sum_{n=0}^{\infty} n\, P1n$$

$$= \sum_{n=0}^{a-2} n\, P0n + (a-1)\, P0a-1 + \sum_{n=0}^{\infty} n\, P1n$$

$$Lq = \frac{1}{(1-r)} \left[\frac{(a-1)\,(a-2)}{2} + \frac{(a-1)\,(1-r^{a})}{(1-r)} + \frac{ar^{a+1}}{(1-r)} + \frac{r^{2}(r^{a}-r^{b})}{(1-r)^{2}} \right] P00 \qquad (24)$$

Conclusion : -

For queues with bulk service and balking the Little's formule does not hold.

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