3. Batch Arrival Retrial G - Queue with Multistage and Multi-Optional Services, Priority and Collision

A single server batch arrival two phase retrial G-queue with multistages in second phase is considered. Upon the arrival of a batch of positive customers, if the server is busy, the arriving batch either joins the orbit or collides with the customer in service resulting in all being shifted to the orbit or one of the customers in the batch pushes out the customer in service to the orbit to get his own service. The arrival of negative customer brings the server down and makes the interrupted customer to leave the system. Analytical expressions for joint distribution of the server state and orbit length are derived. Useful measures are calculated. Stochastic decomposition law is verified and special cases are discussed. Numerical results are presented to explore the effect of several parameters on the system measures.

3.1 Mathematical Description of the Model

Consider a single server batch arrival two phase retrial G-queue under pre-emptive priority service. Two phase service includes first phase of essential service and the second phase of multistage services. The first phase service is needed to all the arriving customers. The second phase consists of M stages of sequential services. In each stage, there are multi-optional heterogeneous services. In particular, stage i (i=1,2,....,M) consists of k_i optional services.

Arrival Process

Positive customers arrive in batches according to the Poisson process with rate λ^+. At every arrival epoch, a batch of k customers arrives with probability C_k, k=1,2,3,..... The generating function of the sequence $\{C_k\}$ is C(z) with first two moments m_1 and m_2. Negative customers arrive in single according to the Poisson process with rate λ^-.

Upon the arrival of a batch of positive customers, if the server is free, then one of the customers from the batch enters the first phase essential service and others join the orbit. If the server is busy, then the arriving batch proceeds to the server with probability δ or join the orbit with complementary probability $\bar{\delta}=1-\delta$.

Retrial Process

If the batch proceeds to the server then with probability α one of the customers in the arriving batch enters the service and the remaining customers along with the interrupted customer join the orbit or with probability $\bar{\alpha}=1-\alpha$ the batch collides with the customer in service and all being shifted to the orbit. Service of the interrupted customer starts from the beginning. The inter-retrial time is generally distributed with distribution function A(x) and Laplace-Stieltjes transform $A^*(s)$.

Service Process

All the customers join the system for essential service. After completion of essential service, the customer moves to first stage of second phase and opts j_1^{th} (j_1=1,2,....,k_1) option with probability p_{j_1} or leaves the system with probability q_0. In general, after the completion of i^{th} stage (i=1,2,….,M) service, the customer may opt j_{i+1}^{th} (j_{i+1}=1,2,....,k_{i+1}) option in $(i+1)^{th}$ stage with probability $p_{j_{i+1}}$ or leave the system with probability q_i. Priority and collision occur when the server is busy. The essential service time is arbitrarily distributed with distribution function $B_0(x)$, Laplace-Stieltjes transform $B_0^*(s)$ and n^{th} factorial moment $\mu_0^{(n)}$. The service time of i^{th} stage (i=1,2,….,M), j_i^{th} (j_i=1,2,....,k_i) optional service is arbitrarily distributed with distribution function $B_{i,j_i}(x)$, Laplace-Stieltjes transform $B_{i,j_i}^*(s)$ and n^{th} factorial moment $\mu_{i,j_i}^{(n)}$.

Breakdown and Repair Process

Arrival of negative customer removes the customer being in service and makes the server down. The failed server is sent for repair immediately. The repair time of the server failed during the essential service is generally distributed with distribution function $R_0(x)$, Laplace-Stieltjes transform $R_0^*(s)$, n^{th} factorial moment $\beta_0^{(n)}$. The repair time of the server failed during i^{th} (i=1,2,.....,M) stage j_i^{th} (j_i =1,2,......,k_i)

optional service follows general distribution with distribution function $R_{i,j_i}(x)$, Laplace-Stieltjes transform $R^*_{i,j_i}(s)$ and n^{th} factorial moment $\beta^{(n)}_{i,j_i}$.

The schematic representation of the model under consideration is shown in Fig. 3.1.

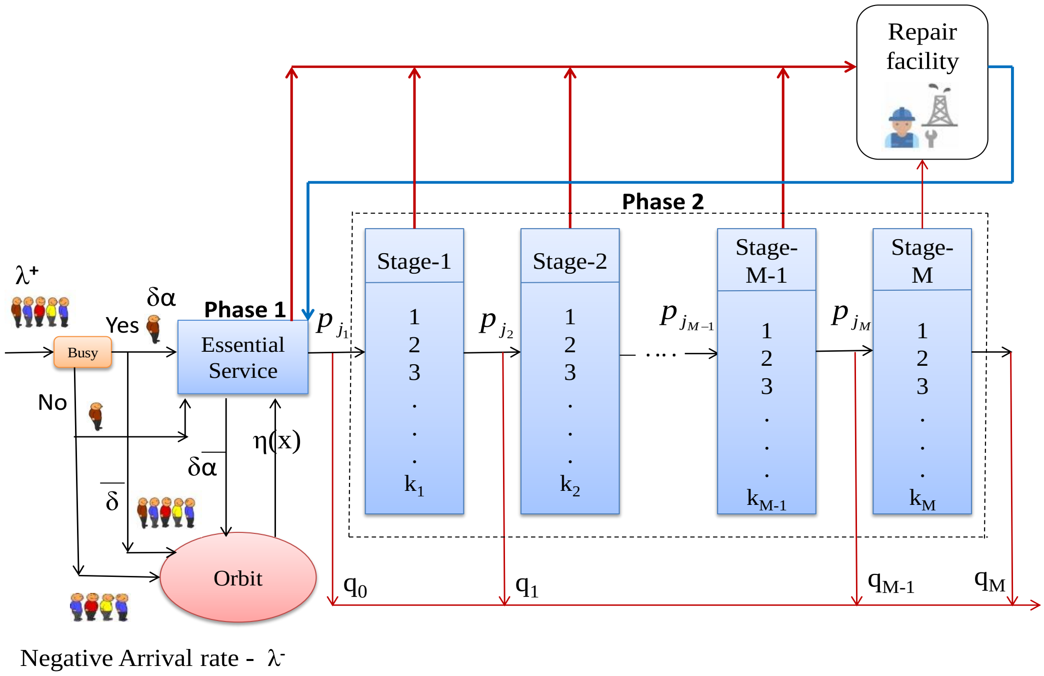

Fig. 3.1 Batch Arrival Retrial G-Queue with Multistage and Multi-Optional Services, Priority and Collision

The state of the system at time t can be described by the Markov process $\{X(t), t \geq 0\} = \{C(t), N(t), \xi_1(t), \xi_2(t), \xi_3(t)\}$, where C(t) denotes the server state 0, 1, 2, 3 or 4 according as the server being idle, busy in essential service, busy in i^{th} stage j_i^{th} optional service, repair during essential service or repair during i^{th} stage j_i^{th} optional service. N(t) corresponds to the customers in the orbit at time t. Define the supplementary variables $\xi_1(t)$, $\xi_2(t)$ and $\xi_3(t)$ as follows

$\xi_1(t)$ = elapsed retrial time, if C(t) = 0

$\xi_2(t)$ = elapsed service time, if C(t) = 1 and 2

$\xi_3(t)$ = elapsed repair time, if C(t) = 3 and 4

The functions $\eta(x)$, $\mu_0(x)$, $\mu_{i,j_i}(x)$, $\beta_0(x)$ and $\beta_{i,j_i}(x)$ are the conditional completion rates for repeated attempts, essential service, i^{th} stage j_i^{th} optional service, repair during essential service and repair during i^{th} stage j_i^{th} optional service respectively. Then

$$\eta(x) = \frac{dA(x)}{1-A(x)}, \quad \mu_0(x) = \frac{dB_0(x)}{1-B_0(x)}, \quad \mu_{i,j_i}(x) = \frac{dB_{i,j_i}(x)}{1-B_{i,j_i}(x)}, \quad i = 1,2,3,\ldots\ldots,M, \quad j_i = 1,2,3,\ldots\ldots,k_i,$$

$$\beta_0(x) = \frac{dR_0(x)}{1-R_0(x)}, \quad \beta_{i,j_i}(x) = \frac{dR_{i,j_i}(x)}{1-R_{i,j_i}(x)}, \quad i = 1,2,3,\ldots\ldots,M, \quad j_i = 1,2,3,\ldots\ldots,k_i.$$

3.2 Orbit Size Distribution at Random Epoch

For the process, $\{N(t)\,;\, t \geq 0\}$, the probability densities are defined as follows.

$$I_0(t) = P\{C(t) = 0, \quad N(t) = 0\}$$

$$I_n(x,t) = P\{C(t) = 0, \quad N(t) = n, \quad x < \xi_1(t) \leq x + dx\}, \quad n \geq 1$$

$$P_{0,n}(x,t)dx = P\{C(t) = 1, \quad N(t) = n, \quad x < \xi_2(t) \leq x + dx\}, \quad n \geq 0$$

$$P_{i,j_i,n}(x,t)dx = P\{C(t) = 2, \quad N(t) = n, \quad x < \xi_2(t) \leq x + dx\}, \; n \geq 0, \; i = 1,2,\ldots.,M, \; j_i = 1,2,\ldots.,k_i$$

$$R_{0,n}(x,t)dx = P\{C(t) = 3, \quad N(t) = n, \quad x < \xi_3(t) \leq x + dx\}, \quad n \geq 0$$

$$R_{i,j_i,n}(x,t)dx = P\{C(t) = 4, \; N(t) = n, \; x < \xi_3(t) \leq x + dx\}, \quad n \geq 0, \; i = 1,2,\ldots.,M, \; j_i = 1,2,\ldots.,k_i$$

The steady state governing equations of the model under consideration are given below

$$\lambda^+ I_0 = q_0 \int_0^\infty P_{0,0}(x)\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i} \int_0^\infty P_{i,j_i,0}(x)\mu_{i,j_i}(x)dx$$
$$+ \int_0^\infty R_{0,0}(x)\beta_0(x)dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,0}(x)\beta_{i,j_i}(x)dx \quad (3.1)$$

$$\frac{d}{dx} I_n(x) = -(\lambda^+ + \eta(x))\, I_n(x), \quad n \geq 1 \quad (3.2)$$

$$\frac{d}{dx} P_{0,n}(x) = -(\lambda^+ + \lambda^- + \mu_0(x))\, P_{0,n}(x) + \lambda^+ \bar{\delta}\, (1-\delta_{0n}) \sum_{k=1}^{n} C_k P_{0,n-k}(x), \; n \geq 0 \quad (3.3)$$

$$\frac{d}{dx} P_{i,j_i,n}(x) = -(\lambda^+ + \lambda^- + \mu_{i,j_i}(x))\, P_{i,j_i,n}(x) + \lambda^+ \bar{\delta}\, (1-\delta_{0n}) \sum_{k=1}^{n} C_k P_{i,j_i,n-k}(x),$$
$$n \geq 0, \; i = 1,2,\ldots.,M, \; j_i = 1,2,\ldots.,k_i \quad (3.4)$$

$$\frac{d}{dx}R_{0,n}(x) = -(\lambda^+ + \beta_0(x))\,R_{0,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n}C_k R_{0,n-k}(x)\ , n \ge 0 \tag{3.5}$$

$$\frac{d}{dx}R_{i,j_i,n}(x) = -(\lambda^+ + \beta_{i,j_i}(x))\,R_{i,j_i,n}(x) + \lambda^+(1-\delta_{0n})\sum_{k=1}^{n}C_k R_{i,j_i,n-k}(x)\ ,$$
$$n \ge 0,\ i = 1,2,\ldots.,M,\ \ j_i = 1,2,\ldots.,k_i \tag{3.6}$$

with boundary conditions

$$I_1(0) = q_0\int_0^\infty P_{0,1}(x)\mu_0(x)dx + \sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,1}(x)\mu_{i,j_i}(x)dx + \int_0^\infty R_{0,1}(x)\beta_0(x)dx$$
$$+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,1}(x)\beta_{i,j_i}(x)dx \tag{3.7}$$

$$I_n(0) = q_0\int_0^\infty P_{0,n}(x)\mu_0(x)dx + \sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n}(x)\mu_{i,j_i}(x)dx + \int_0^\infty R_{0,n}(x)\beta_0(x)dx$$
$$+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i,n}(x)\beta_{i,j_i}(x)dx + \lambda^+\delta\bar{\alpha}\sum_{k=1}^{n-1}C_k\int_0^\infty P_{0,n-(k+1)}(x)\,dx$$
$$+\lambda^+\delta\bar{\alpha}\sum_{k=1}^{n-1}C_k\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n-(k+1)}(x)\,dx,\ n \ge 2 \tag{3.8}$$

$$P_{0,0}(0) = \lambda^+ C_1 I_0 + \int_0^\infty I_1(x)\eta(x)dx \tag{3.9}$$

$$P_{0,n}(0) = \lambda^+ C_{n+1}I_0 + \lambda^+\sum_{k=1}^{n}C_k\int_0^\infty I_{n-k+1}(x)\,dx + \int_0^\infty I_{n+1}(x)\eta(x)dx$$
$$+\lambda^+\delta\alpha\sum_{k=1}^{n}C_k\int_0^\infty P_{0,n-k}(x)\,dx + \lambda^+\delta\alpha\sum_{k=1}^{n}C_k\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i,n-k}(x)\,dx\ , n \ge 1 \tag{3.10}$$

$$P_{1,j_1,n}(0) = p_{j_1}\int_0^\infty P_{0,n}(x)\mu_0(x)dx,\ \ n \ge 0,\ \ j_1 = 1,2,\ldots.,k_1 \tag{3.11}$$

$$P_{i,j_i,n}(0) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^\infty P_{i-1,j_{i-1},n}(x)\mu_{i,j_i}(x)dx,\ n \ge 0,\ i = 2,3,\ldots\ldots,M,\ \ j_i = 1,2,\ldots.,k_i \tag{3.12}$$

$$R_{0,n}(0) = \lambda^-\int_0^\infty P_{0,n}(x)dx\ ,\ \ n \ge 0 \tag{3.13}$$

$$R_{i,j_i,n}(0) = \lambda^-\int_0^\infty P_{i,j_i,n}(x)dx\ ,\ \ n \ge 0,\ i = 2,3,\ldots\ldots,M,\ \ j_i = 1,2,\ldots.,k_i \tag{3.14}$$

By defining the probability generating functions I(x,z), P_0(x,z), $P_{i,j_i}(x,z)$, R_0(x,z) and $R_{i,j_i}(x,z)$ as

$$\left.\begin{array}{ll} I(x,z)=\sum_{n=1}^{\infty} I_n(x)z^n \quad ; & P_0(x,z)=\sum_{n=0}^{\infty} P_{0,n}(x)z^n \\ P_{i,j_i}(x,z)=\sum_{n=0}^{\infty} P_{i,j_i,n}(x)z^n\,; & R_0(x,z)=\sum_{n=0}^{\infty} R_{0,n}(x)z^n \\ \text{and}\ \ R_{i,j_i}(x,z)=\sum_{n=0}^{\infty} R_{i,j_i,n}(x)z^n & \end{array}\right\} \tag{3.15}$$

Using the definition in (3.15), equations (3.1) to (3.6) give

$$\left(\frac{d}{dx}+\lambda^{+}+\eta(x)\right)I(x,z)=0 \tag{3.16}$$

$$\left(\frac{d}{dx}+\lambda^{+}(1-\bar{\delta}C(z))+\lambda^{-}+\mu_0(x)\right)P_0(x,z)=0 \tag{3.17}$$

$$\left(\frac{d}{dx}+\lambda^{+}(1-\bar{\delta}C(z))+\lambda^{-}+\mu_{i,j_i}(x)\right)P_{i,j_i}(x,z)=0,\quad i=1,2,.....,M,\ \ j_i=1,2,.......,k_i \tag{3.18}$$

$$\left(\frac{d}{dx}+\lambda^{+}(1-C(z))+\beta_0(x)\right)R_0(x,z)=0 \tag{3.19}$$

$$\left(\frac{d}{dx}+\lambda^{+}(1-C(z))+\beta_{i,j_i}(x)\right)R_{i,j_i}(x,z)=0\,,\quad i=1,2,.....,M,\ \ j_i=1,2,........,k_i \tag{3.20}$$

Solutions of the partial differential equations (3.16) to (3.20) are respectively

$$I(x,z)\ =\ I(0,z)\,e^{-\lambda^{+}x}(1-A(x)) \tag{3.21}$$

$$P_0(x,z)\ =\ P_0(0,z)e^{-(\lambda^{+}+\lambda^{-}-\lambda^{+}\bar{\delta}\,C(z))x}(1-B_0(x)) \tag{3.22}$$

$$P_{i,j_i}(x,z)=\ P_{i,j_i}(0,z)e^{-(\lambda^{+}+\lambda^{-}-\lambda^{+}\bar{\delta}\,C(z))x}(1-B_{i,j_i}(x)),\ i=1,2,......,M,\ \ j_i=1,2,.....,k_i \tag{3.23}$$

$$R_0(x,z)\ =\ R_0(0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))x}(1-R_0(x)) \tag{3.24}$$

$$R_{i,j_i}(x,z)=R_{i,j_i}(0,z)e^{-(\lambda^{+}-\lambda^{+}C(z))x}(1-R_{i,j_i}(x)),\ i=1,2,......,M,\ \ j_i=1,2,.....,k_i \tag{3.25}$$

Equations (3.7) to (3.14) yield

$$\begin{aligned} I(0,z)\ =\ & q_0\int_0^{\infty}P_0(x,z)\,\mu_0(x)dx+\sum_{i=1}^{M}q_i\sum_{j_i=1}^{k_i}\int_0^{\infty}P_{i,j_i}(x,z)\,\mu_{i,j_i}(x)dx \\ & +\int_0^{\infty}R_0(x,z)\beta_0(x)dx+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^{\infty}R_{i,j_i}(x,z)\beta_{i,j_i}(x)dx \\ & +\lambda^{+}\delta\bar{\alpha}\,zC(z)\int_0^{\infty}P_0(x,z)\,dx+\lambda^{+}\delta\bar{\alpha}\,zC(z)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^{\infty}P_{i,j_i}(x,z)\,dx-\lambda^{+}I_0 \end{aligned} \tag{3.26}$$

$$P_0(0,z) = \frac{1}{z}[\lambda^+ C(z) I_0 + \int_0^\infty I(x,z)\eta(x)dx + \lambda^+ C(z)\int_0^\infty I(x,z)dx]$$

$$+\lambda^+ \delta \alpha C(z)\int_0^\infty P_0(x,z)\,dx + \lambda^+ \delta\alpha C(z)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i}(x,z)\,dx \tag{3.27}$$

$$P_{1,j_1}(0,z) = p_{j_1}\int_0^\infty P_0(x,z)\mu_0(x)dx,\quad j_1 = 1,2,......,k_1 \tag{3.28}$$

$$P_{i,j_i}(0,z) = p_{j_i}\sum_{j_{i-1}=1}^{k_{i-1}}\int_0^\infty P_{i-1,j_{i-1}}(x,z)\mu_{i-1,j_{i-1}}(x)dx,\quad i = 2,3,......,M,\quad j_i = 1,2,......,k_i \tag{3.29}$$

$$R_0(0,z) = \lambda^- \int_0^\infty P_0(x,z)dx \tag{3.30}$$

$$R_{i,j_i}(0,z) = \lambda^- \int_0^\infty P_{i,j_i}(x,z)dx,\quad i = 1,2,3,......,M,\quad j_i = 1,2,......,k_i \tag{3.31}$$

Using the expressions in equations (3.21) to (3.25), equation (3.26) yields

$$I(0,z) = q_0\int_0^\infty P_0(0,z)e^{-k(z)x}(1-B_0(x))\,\mu_0(x)dx + \sum_{i=1}^{M} q_i \sum_{j_i=1}^{k_i}\int_0^\infty P_{i,j_i}(0,z)e^{-k(z)x}$$

$$(1-B_{i,j_i}(x))\mu_{i,j_i}(x)dx + \int_0^\infty R_0(0,z)e^{-h(z)x}(1-R_0(x))\,\beta_0(x)\,dx$$

$$+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^\infty R_{i,j_i}(0,z)e^{-h(z)x}(1-R_{i,j_i}(x))\beta_{i,j_i}(x)\,dx + \lambda^+\delta\bar{\alpha}\,z\,C(z)$$

$$[\int_0^\infty P_0(0,z)e^{-k(z)x}(1-B_0(x))\,dx + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(0,z)e^{-k(z)x}(1-B_{i,j_i}(x))] - \lambda^+ I_0$$

$$= q_0\, P_0(0,z)\,B_0^*(k(z)) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} q_i\, P_{i,j_i}(0,z)\,B_{i,j_i}^*(k(z)) + R_0(0,z)\,R_0^*(h(z))$$

$$+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(0,z)\,R_{i,j_i}^*(h(z)) + \lambda^+\delta\bar{\alpha}\,z\,C(z)[P_0(0,z)(1-B_0^*(k(z)))/k(z))$$

$$+\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} q_i\, P_{i,j_i}(0,z)((1-B_{i,j_i}^*(k(z)))/k(z)) - \lambda^+ I_0 \tag{3.32}$$

where

$$k(z) = \lambda^+(1-\bar{\delta}C(z)) + \lambda^-$$

$$h(z) = \lambda^+(1-C(z))$$

Substituting the expressions (3.21) to (3.23) in equations (3.27) to (3.31) and solving, we get

$$P_0(0,z) = \frac{1}{z}[\lambda^+ C(z) I_0 + \left(A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))\right) I(0,z)]$$
$$+ \lambda^+ \delta \alpha C(z) [P_0(0,z)((1 - B_0^*(k(z))/k(z))$$
$$+ \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} P_{i,j_i}(0,z)((1 - B_{i,j_i}^*(k(z)))/k(z))] \quad (3.33)$$

$$P_{1,j_1}(0,z) = p_{j_1} P_0(0,z) B_0^*(k(z)), \; j_1 = 1,2,......,k_1 \quad (3.34)$$

$$P_{i,j_i}(0,z) = p_{j_i} \sum_{j_{i-1}=1}^{k_{i-1}} P_{i-1,j_{i-1}}(0,z) B_{i-1,j_{i-1}}^*(k(z)), \quad i = 2,3,......,M, \; j_i = 1,2,......,k_i \quad (3.35)$$

$$R_0(0,z) = \lambda^- P_0(0,z)(1 - B_0^*(k(z)))\big/k(z) \quad (3.36)$$

$$R_{i,j_i}(0,z) = \lambda^- P_{i,j_i}(0,z)(1 - B_{i,j_i}^*(k(z)))\big/k(z), \; i = 2,3,......,M, \; j_i = 1,2,......,k_i \quad (3.37)$$

From the equation (3.35), we have

$$P_{i,j_i}(0,z) = p_{j_i} \sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}} \sum_{j_{i-2}=1}^{k_{i-2}} P_{i-2,j_{i-2}}(0,z) B_{i-2,j_{i-2}}^*(k(z)) B_{i-1,j_{i-1}}^*(k(z))$$

$$= p_{j_i} \sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}} \sum_{j_{i-2}=1}^{k_{i-2}} p_{j_{i-2}} \sum_{j_{i-3}=1}^{k_{i-3}} P_{i-3,j_{i-3}}(0,z) B_{i-3,j_{i-3}}^*(k(z)) B_{i-2,j_{i-2}}^*(k(z)) B_{i-1,j_{i-1}}^*(k(z))$$

$$= p_{j_i} \sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}} \sum_{j_{i-2}=1}^{k_{i-2}} p_{j_{i-2}} \cdots \sum_{j_2=1}^{k_2} p_{j_2} \sum_{j_1=1}^{k_1} p_{j_1} P_0(0,z) B_0^*(k(z)) B_{1,j_1}^*(k(z))$$
$$B_{2,j_2}^*(k(z)).....B_{i-2,j_{i-2}}^*(k(z)) B_{i-1,j_{i-1}}^*(k(z))$$

$$= p_{j_i}\left[\sum_{j_1=1}^{k_1} p_{j_1} B_{1,j_1}^*(k(z)) \sum_{j_2=1}^{k_2} p_{j_2} B_{2,j_2}^*(k(z)).......\sum_{j_{i-1}=1}^{k_{i-1}} p_{j_{i-1}} B_{i-1,j_{i-1}}^*(k(z))\right] B_0^*(k(z)) P_0(0,z)$$

$$= p_{j_i}\left[\prod_{l=1}^{i-1} \sum_{j_l=1}^{k_l} p_{j_l} B_{l,j_l}^*(k(z))\right] B_0^*(k(z)) P_0(0,z)$$

$$= p_{j_i} \Lambda_{i-1}^*(k(z)) B_0^*(k(z)) P_0(0,z), \; i = 2,3,......,M, \; j_i = 1,2,......,k_i \quad (3.38)$$

where

$$\Lambda_0^*(k(z)) = 1 , \qquad \Lambda_i^*(k(z)) = \prod_{l=1}^{i} \sum_{j_l=1}^{k_l} p_{j_l} B_{l,j_l}^*(k(z))$$

Substituting the expressions (3.34) and (3.38) in the equation (3.37), we obtain

$$R_{i,j_i}(0,z) \;=\; \lambda^- \; p_{j_i} \Lambda_{i-1}^*(k(z)) B_0^*(k(z)) \; P_0(0,z) \; (1 - B_{i,j_i}^*(k(z)))\big/ k(z),$$

$$i = 1,2,3,......,M, \quad j_i = 1,2,......,k_i \qquad (3.39)$$

Using the equations (3.34), (3.36), (3.38) and (3.39), the equation (3.32) becomes

$$I(0,z) = T_1(z) \; P_0(0,z) - \lambda^+ I_0 \qquad (3.40)$$

where

$$\begin{aligned} T_1(z) \;=\; & \sum_{i=0}^{M} q_i \; \Lambda_i^*(k(z)) \; B_0^*(k(z)) + \lambda^- ((1 - B_0^*(k(z)))/k(z)) R_0^*(h(z)) \\ & + \lambda^- \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i\text{-}1}^*(k(z))((1 - B_{i,j_i}^*(k(z)))/k(z)) R_{i,j_i}^*(h(z)) B_0^*(k(z)) \\ & + \lambda^+ \delta \, \bar{\alpha} \; z C(z) [(1 - B_0^*(k(z)))/k(z)) \\ & + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i\text{-}1}^*(k(z))((1 - B_{i,j_i}^*(k(z)))/k(z)) B_0^*(k(z))] \end{aligned}$$

Using the expression (3.38), equation (3.33) becomes

$$\begin{aligned} z \, P_0(0,z) \;=\; & \lambda^+ C(z) I_0 + (A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))) \; I(0,z) \\ & + \lambda^+ \delta \, \alpha z \; C(z) \; P_0(0,z) [((1 - B_0^*(k(z))/k(z)) \\ & + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i\text{-}1}^*(k(z)) \; ((1 - B_{i,j_i}^*(k(z)))/k(z)) B_0^*(k(z))] \end{aligned} \qquad (3.41)$$

Substituting the expression of I(0,z) in terms of $P_0(0,z)$ in equation (3.41) and simplifying, we have

$$P_0(0,z) \;=\; \lambda^+ I_0 A^*(\lambda^+)(C(z) - 1) \; \big/ \; D(z) \qquad (3.42)$$

where

$$D(z) = z - T_1(z) \; (A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))) - \lambda^+ \delta \, \alpha z \; C(z) T_2(z)$$

$$T_2(z) = ((1 - B_0^*(k(z))/k(z)) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i\text{-}1}^*(k(z))((1 - B_{i,j_i}^*(k(z)))/k(z)) B_0^*(k(z))$$

Inserting the equation (3.42) in equations (3.34), (3.36), (3.38), (3.39) and (3.40), we obtain

$$I(0,z) = \lambda^{+} I_0 \{ C(z) [T_1(z) + \lambda^{+} \delta \alpha z T_2(z)] - z] \} \,/\, D(z) \qquad (3.43)$$

$$P_{1,j_1}(0,z) = p_{j_1} \lambda^{+} I_0 A^{*}(\lambda^{+})(C(z)-1) B_0^{*}(k(z)) \,/\, D(z), \qquad j_1 = 1,2,......,k_1 \qquad (3.44)$$

$$P_{i,j_i}(0,z) = p_{j_i} \lambda^{+} I_0 A^{*}(\lambda^{+})(C(z)-1) \Lambda_{i-1}^{*}(k(z)) B_0^{*}(k(z)) \big/ D(z),$$

$$i = 2,3,......,M, \; j_i = 1,2,......,k_i \qquad (3.45)$$

$$R_0(0,z) = \lambda^{-} \lambda^{+} I_0 A^{*}(\lambda^{+})(C(z)-1)((1 - B_0^{*}(k(z)))/k(z)) \big/ D(z) \qquad (3.46)$$

$$R_{i,j_i}(0,z) = p_{j_i} \lambda^{-} \lambda^{+} I_0 A^{*}(\lambda^{+})(C(z)-1) \Lambda_{i-1}^{*}(k(z))((1 - B_{i,j_i}^{*}(k(z)))/k(z)) B_0^{*}(k(z)) \big/ D(z),$$

$$i = 1,2,3,......,M, \; j_i = 1,2,......,k_i \qquad (3.47)$$

Now, we derive the probability generating functions of the server state and orbit size.

- The probability generating function of the orbit size when the server is idle in the non-empty system is given by

$$I(z) = \int_0^{\infty} I(x,z)dx$$

$$= I_0 (1 - A^{*}(\lambda^{+})) \{ C(z)[T_1(z) + \lambda^{+} \delta \alpha z T_2(z)] - z \} \,/\, D(z) \qquad (3.48)$$

- The probability generating function of the orbit size when the server is busy in providing first phase service is given by

$$P_0(z) = \int_0^{\infty} P_0(x,z)dx$$

$$= \lambda^{+} I_0 A^{*}(\lambda^{+})(C(z)-1)((1 - B_0^{*}(k(z)))/k(z)) \big/ D(z) \qquad (3.49)$$

- The probability generating function of the orbit size when the server is busy in providing second phase services is given by

$$P(z) = \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} P_{i,j_i}(z)$$

$$= \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} \int_0^{\infty} P_{i,j_i}(x,z)dx$$

$$= \lambda^{+} I_0 A^{*}(\lambda^{+})(C(z)-1)B_0^{*}(k(z))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^{*}(k(z))((1-B_{i,j_i}^{*}(k(z)))/k(z)) \Big/ D(z) \tag{3.50}$$

- The probability generating function of the orbit size when the server in first phase is under repair is given by

$$R_0(z) = \int_0^{\infty} R_0(x,z)dx$$

$$= \lambda^{-} I_0 A^{*}(\lambda^{+})((1-B_0^{*}(k(z)))/k(z))(R_0^{*}(h(z))-1) \big/ D(z) \tag{3.51}$$

- The probability generating function of the orbit size when the server in second phase is under repair is given by

$$R(z) = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \sum_{i=1}^{M}\sum_{j_i=1}^{k_i}\int_0^{\infty} R_{i,j_i}(x,z)dx$$

$$= \lambda^{-} I_0 A^{*}(\lambda^{+})\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^{*}(k(z))\Lambda_{i-1}^{*}(k(z))((1-B_{i,j_i}^{*}(k(z)))/k(z))(R_{i,j_i}^{*}(h(z))-1) \Big/ D(z) \tag{3.52}$$

Using the normalizing condition

$$I_0 + \lim_{z\to 1} I(z) + \lim_{z\to 1} P_0(z) + \lim_{z\to 1}\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + \lim_{z\to 1} R_0(z) + \lim_{z\to 1}\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z) = 1 ,$$

we obtain

$$I_0 = \frac{\begin{array}{l}\{1 - q_0 f_0^{(1)} - \sum_{i=1}^{M} q_i [M_i^{(1)} B_0^{*}(\lambda^{-}+\lambda^{+}\delta) + \Lambda_i^{*}(\lambda^{-}+\lambda^{+}\delta) f_0^{(1)}] - m_1(1-A^{*}(\lambda^{+}))[(\lambda^{-}+\lambda^{+}\delta)T_3 \\ + \sum_{i=0}^{M} q_i \Lambda_i^{*}(\lambda^{-}+\lambda^{+}\delta) B_0^{*}(\lambda^{-}+\lambda^{+}\delta)] - (\lambda^{-}\lambda^{+} m_1/(\lambda^{-}+\lambda^{+}\delta))T_5 - \lambda^{+}\delta[(1+m_1)T_3 + T_4]\}\end{array}}{A^{*}(\lambda^{+})\, T_6} \tag{3.53}$$

where

$$T_3 = (1/(\lambda^{-}+\lambda^{+}\delta))\,[1 - B_0^{*}(\lambda^{-}+\lambda^{+}\delta)(1-h_1)]$$

$$T_4 = (\lambda^+ \bar{\delta} m_1/(\lambda^- + \lambda^+\delta))T_3 - (1/(\lambda^- + \lambda^+\delta))\{ f_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^- + \lambda^+\delta) B_0^*(\lambda^- + \lambda^+\delta) f_{i,j_i}^{(1)}$$

$$- h_1 f_0^{(1)} - h_2 B_0^*(\lambda^- + \lambda^+\delta)\}$$

$$T_5 = (1/(\lambda^- + \lambda^+\delta))[(1 - B_0^*(\lambda^- + \lambda^+\delta))\, \beta_0^{(1)} + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^- + \lambda^+\delta)$$

$$(1 - B_{i,j_i}^*(\lambda^- + \lambda^+\delta))\, B_0^*(\lambda^- + \lambda^+\delta)\, \beta_{i,j_i}^{(1)}]$$

$$T_6 = 1 - \sum_{i=1}^{M} q_i [\Lambda_i^*(\lambda^- + \lambda^+\delta) f_0^{(1)} + M_i^{(1)} B_0^*(\lambda^- + \lambda^+\delta)] + \sum_{j_1=1}^{k_1} p_{j_1} f_0^{(1)} - \lambda^+\delta T_3$$

$$- h_1 f_0^{(1)} - h_2 B_0^*(\lambda^- + \lambda^+\delta) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^- + \lambda^+\delta) f_{i,j_i}^{(1)} B_0^*(\lambda^- + \lambda^+\delta)$$

$$h_1 = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^- + \lambda^+\delta)\,(1 - B_{i,j_i}^*(\lambda^- + \lambda^+\delta))$$

$$h_2 = \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} M_{i-1}^{(1)}\,(1 - B_{i,j_i}^*(\lambda^- + \lambda^+\delta))$$

$$f_0^{(1)} = \lambda^+ m_1 \bar{\delta} \int_0^\infty x e^{-\lambda^- x} b_0(x)dx\,, \qquad f_{i,j_i}^{(1)} = \lambda^+ m_1 \bar{\delta} \int_0^\infty x e^{-\lambda^- x} b_{i,j_i}(x)dx$$

$$M_i^{(1)} = \lim_{z\to 1} \Lambda_i^{*\prime}(k(z)) = \sum_{m=1}^{i} \left[\sum_{j_m=1}^{k_m} p_{j_m} B_{m,j_m}^{*\prime}(\lambda^- + \lambda^-\delta) \left(\prod_{\substack{n=1\\ n\neq m}}^{i} \sum_{j_n=1}^{k_n} p_{j_n} B_{n,j_n}^*(\lambda^- + \lambda^-\delta) \right) \right],$$

$$M_i^{(2)} = \lim_{z\to 1} \Lambda_i^{*\prime\prime}(k(z))$$

$$= \sum_{m=1}^{i} \left[\sum_{j_m=1}^{k_m} p_{j_m} B_{m,j_m}^{*\prime\prime}(\lambda^- + \lambda^-\delta) \left(\prod_{\substack{n=1\\ n\neq m}}^{i} \sum_{j_n=1}^{k_n} p_{j_n} B_{n,j_n}^*(\lambda^- + \lambda^-\delta) \right) \right]$$

$$+ 2 \sum_{m=1}^{i-1} \left[\left(\sum_{j_m=1}^{k_m} p_{j_m} B_{m,j_m}^{*\prime}(\lambda^- + \lambda^-\delta) \right) \left(\sum_{j_{m+1}=1}^{k_{m+1}} p_{j_{m+1}} B_{m+1,j_{m+1}}^{*\prime}(\lambda^- + \lambda^-\delta) \right) \left(\prod_{\substack{n=1\\ n\neq m\\ n\neq m+1}}^{i} \sum_{j_n=1}^{k_n} p_{j_n} B_{n,j_n}^*(\lambda^- + \lambda^-\delta) \right) \right]$$

The probability generating function of the orbit size is given by

$$P_q(z) = I_0 + I(z) + P_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \frac{I_0 A^*(\lambda^+)\{z - \sum_{i=0}^{M} q_i \Lambda_i^*(k(z)) B_0^*(k(z)) + [\lambda^+(C(z)-1) - \lambda^+ \delta\, z\, C(z) - \lambda^-]\, T_2(z)\}}{D(z)} \tag{3.54}$$

The probability generating function of the system size is given by

$$P_s(z) = I_0 + I(z) + zP_0(z) + z\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} P_{i,j_i}(z) + R_0(z) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \frac{I_0\ A^*(\lambda^+)\{z - \sum_{i=0}^{M} q_i\ \Lambda_i^*(k(z))B_0^*(k(z)) + [\lambda^+ z(C(z)-1) - \lambda^+\ \delta\ z\ C(z) - \lambda^-]\ T_2(z)\}}{D(z)} \tag{3.55}$$

3.3 Stability Condition

The necessary and sufficient condition for the system to be stable is

$$q_0 f_0^{(1)} + \sum_{i=0}^{M} q_i \Lambda_i^*(\lambda^- + \lambda^+\delta)\, f_0^{(1)} + \sum_{i=1}^{M} q_i M_i^{(1)} B_0^*(\lambda^- + \lambda^+\delta) + \lambda^+\delta\,[(1+m_1)T_3 + T_4]$$
$$-(\lambda^- + \lambda^+\delta\bar{\alpha})\ T_3 + m_1(1 - A^*(\lambda^+))\sum_{i=0}^{M} q_i\ \Lambda_i^*(\lambda^- + \lambda^+\delta)B_0^*(\lambda^- + \lambda^+\delta) - \lambda^-\lambda^+ m_1 T_5 < 1$$

3.4 Performance Measures

Let $N_I(z), N_{P_0}(z), N_P(z), N_{R_0}(z)$ and $N_R(z)$ denotes the numerators of $I(z)$, $P_0(z), P(z), R_0(z)$ and $R(z)$ respectively.

- The probability that the server is idle in the non-empty system is

$$I = \lim_{z\to 1} I(z)$$

$$= \frac{I_0(1 - A^*(\lambda^+))\{m_1 + q_0 f_0^{(1)} + \sum_{i=1}^{M} q_i [M_i^{(1)} B_0^*(\lambda^- + \lambda^+\delta) + \Lambda_i^*(\lambda^- + \lambda^+\delta) f_0^{(1)}] + \lambda^-\lambda^+ m_1 T_5 + (\lambda^+\delta(1+m_1) + \lambda^+ m_1)T_3 - 1\}}{D'} \tag{3.56}$$

where

$$D' = 1 - q_0 f_0^{(1)} - \sum_{i=1}^{M} q_i [M_i^{(1)} B_0^*(\lambda^- + \lambda^+\delta) + \Lambda_i^*(\lambda^- + \lambda^+\delta) f_0^{(1)}] - m_1(1 - A^*(\lambda^+))$$
$$[\sum_{i=0}^{M} q_i\ \Lambda_i^*(\lambda^-)B_0^*(\lambda^- + \lambda^+\delta) + (\lambda^- + \lambda^+\delta)T_3] - \lambda^-\lambda^+ m_1 T_5 - \lambda^+\delta\,[(1+m_1)T_3 + T_4]$$

- Mean number of customers in the orbit when the server is idle in the non-empty system is

$$L_I = \lim_{z\to 1}\frac{d}{dz} I(z)$$

$$= \frac{D' N_I'' - N_I' D''}{2 D'^2} \tag{3.57}$$

where

$$N_I' = I_0 (1 - A^*(\lambda^+)) \{ m_1 + q_0 f_0^{(1)} + \sum_{i=1}^{M} q_i [M_i^{(1)} B_0^*(\lambda^- + \lambda^+ \delta) + \Lambda_i^*(\lambda^- + \lambda^+ \delta) f_0^{(1)}]$$
$$+ \lambda^- \lambda^+ m_1 T_5 + (\lambda^+ \delta \, (1 + m_1) + \lambda^+ m_1) T_3 - 1 \}$$

$$N_I'' = I_0 (1 - A^*(\lambda^+)) \{ m_2 + 2 m_1 [q_0 f_0^{(1)} + \sum_{i=1}^{M} q_i [M_i^{(1)} B_0^*(\lambda^- + \lambda^+ \delta) + \Lambda_i^*(\lambda^- + \lambda^+ \delta) f_0^{(1)}]$$
$$+ \lambda^- \lambda^+ m_1 T_5 + \lambda^- T_4 + \lambda^+ \delta \, (T_3 + T_4) + \lambda^+ \delta \overline{\alpha} m_1 T_3] + 2 \lambda^+ \delta \alpha T_4 + q_0 f_0^{(2)}$$
$$+ \sum_{i=1}^{M} q_i [M_i^{(2)} B_0^*(\lambda^- + \lambda^+ \delta) + 2 M_i^{(1)} f_0^{(1)} + \Lambda_i^*(\lambda^- + \lambda^+ \delta) f_0^{(2)}] - \lambda^- {\lambda^+}^2 m_1 T_7$$
$$+ 2 \lambda^- T_8 (\lambda^- \lambda^+ m_1 / (\lambda^- + \lambda^+ \delta)) [f_0^{(1)} \beta_0^{(1)} - \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} (M_{i-1}^{(1)} B_0^*(\lambda^- + \lambda^+ \delta)$$
$$+ \Lambda_{i-1}^*(\lambda^- + \lambda^+ \delta) f_0^{(1)}) (1 - B_{i,j_i}^*(\lambda^- + \lambda^+ \delta)) \beta_{i,j_i}^{(1)} + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^- + \lambda^+ \delta)$$
$$B_0^*(\lambda^- + \lambda^+ \delta) f_{i,j_i}^{(1)} \beta_{i,j_i}^{(1)}] + \lambda^+ \delta \overline{\alpha} (2 m_1 + m_2) T_3 + 2 \lambda^+ \delta \overline{\alpha} (1 + m_1) T_4 + \lambda^+ \delta T_8 \}$$

$$D'' = - q_0 f_0^{(2)} - \sum_{i=1}^{M} q_i [\Lambda_i^*(\lambda^- + \lambda^+ \delta) f_0^{(2)} + 2 M_i^{(1)} f_0^{(1)} + M_i^{(2)} B_0^*(\lambda^- + \lambda^+ \delta)]$$
$$- \lambda^- ((\lambda^+)^2 m_1^2 T_7 + \lambda^+ m_2 T_5) + 2 (\lambda^- \lambda^+ m_1 / (\lambda^- + \lambda^+ \delta)) \{ f_0^{(1)} \beta_0^{(1)}$$
$$+ \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^- + \lambda^+ \delta) B_0^*(\lambda^- + \lambda^+ \delta) f_{i,j_i}^{(1)} - \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} [M_{i-1}^{(1)} B_0^*(\lambda^- + \lambda^+ \delta)$$
$$+ \Lambda_{i-1}^*(\lambda^- + \lambda^+ \delta) f_0^{(1)}] \beta_{i,j_i}^{(1)} \} - \lambda^- T_8 - \lambda^+ \delta \; [(2 m_1 + m_2) T_3 + 2 \, (1 + m_1) T_4 + T_8]$$
$$- 2 m_1 (1 - A^*(\lambda^+)) [q_0 f_0^{(1)} + \lambda^- \lambda^+ m_1 T_5 + \sum_{i=1}^{M} q_i \, (\Lambda_i^*(\lambda^- + \lambda^+ \delta) f_0^{(1)}$$
$$+ M_i^{(1)} B_0^*(\lambda^- + \lambda^+ \delta)) + \lambda^- T_4 + \lambda^+ \delta \overline{\alpha} ((1 + m_1) T_3 + T_4)]$$

$$T_7 = (1 / (\lambda^- + \lambda^+ \delta)) [(1 - B_0^*(\lambda^-)) \beta_0^{(2)} + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^- + \lambda^+ \delta)$$
$$(1 - B_{i,j_i}^*(\lambda^- + \lambda^+ \delta)) B_0^*(\lambda^- + \lambda^+ \delta) \beta_{i,j_i}^{(2)}]$$

$$T_8 = (\lambda^+ \overline{\delta} m_2 / (\lambda^- + \lambda^+ \delta)) T_3 - 2 (\lambda^+ \overline{\delta} m_1 / (\lambda^- + \lambda^+ \delta)^2) [f_0^{(1)} + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^- + \lambda^+ \delta)$$
$$B_0^*(\lambda^- + \lambda^+ \delta) f_{i,j_i}^{(1)}] \} + 2 (\lambda^+ \overline{\delta} m_1 / (\lambda^- + \lambda^+ \delta))^2 T_3 - (1 / (\lambda^- + \lambda^+ \delta)) \{ f_0^{(2)}$$
$$+ \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^- + \lambda^+ \delta) B_0^*(\lambda^- + \lambda^+ \delta) f_{i,j_i}^{(2)} + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} p_{j_i} [M_{i-1}^{(2)} B_0^*(\lambda^- + \lambda^+ \delta)$$
$$+ 2 M_{i-1}^{(1)} f_0^{(1)} + \Lambda_{i-1}^*(\lambda^- + \lambda^+ \delta) f_0^{(2)}] (1 - B_{i,j_i}^*(\lambda^- + \lambda^+ \delta)) \}$$

- The probability that the server is busy in essential service is

$$P_0 = \lim_{z\to 1} P_0(z)$$

$$= \frac{I_0(\lambda^+ m_1/(\lambda^- + \lambda^+\delta))A^*(\lambda^+)(1 - B_0^*(\lambda^- + \lambda^+\delta))}{D'} \quad (3.58)$$

- Mean number of customers in the orbit when the server is busy in essential service is

$$L_{P_0} = \lim_{z\to 1} \frac{d}{dz} P_0(z)$$

$$= \frac{D' N''_{P_0} - N'_{P_0} D''}{2D'^2} \quad (3.59)$$

where

$$N'_{P_0} = I_0(\lambda^+ m_1/(\lambda^- + \lambda^+\delta))A^*(\lambda^+)(1 - B_0^*(\lambda^- + \lambda^+\delta))$$

$$N''_{P_0} = I_0 A^*(\lambda^+)\,(1/(\lambda^- + \lambda^+\delta))\,[\,\lambda^+ m_2(1 - B_0^*(\lambda^- + \lambda^+\delta)) - 2\lambda^+ m_1 f_0^{(1)}$$
$$+ 2((\lambda^+ m_1)^2 \bar{\delta}/(\lambda^- + \lambda^+\delta))(1 - B_0^*(\lambda^- + \lambda^+\delta))]$$

- The probability that the server is busy in optional services is

$$P = \lim_{z\to 1} \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} P_{i,j_i}(z)$$

$$= \frac{I_0(\lambda^+ m_1/(\lambda^- + \lambda^+\delta))A^*(\lambda^+)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\,\Lambda_{i-1}^*(\lambda^- + \lambda^+\delta)B_0^*(\lambda^- + \lambda^+\delta)(1 - B_{i,j_i}^*(\lambda^- + \lambda^+\delta))}{D'}$$

(3.60)

- Mean number of customers in the orbit when the server is busy in optional services is

$$L_P = \lim_{z\to 1} \frac{d}{dz} P(z)$$

$$= \frac{D' N''_P - N'_P D''}{2D'^2} \quad (3.61)$$

where

$$N'_P = I_0 A^*(\lambda^+)(\lambda^+ m_1/(\lambda^- + \lambda^+\delta))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda^*_{i-1}(\lambda^- + \lambda^+\delta) B^*_0(\lambda^- + \lambda^+\delta)(1 - B^*_{i,j_i}(\lambda^- + \lambda^+\delta))$$

$$\begin{aligned} N''_P = {} & I_0 A^*(\lambda^+)\,(1/(\lambda^- + \lambda^+\delta))\{\lambda^+ m_2 \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B^*_0(\lambda^- + \lambda^+\delta)\Lambda^*_{i-1}(\lambda^- + \lambda^+\delta)(1 - B^*_{i,j_i}(\lambda^- + \lambda^+\delta)) \\ & + 2\,\lambda^+ m_1 [h_1 f_0^{(1)} + h_2 B^*_0(\lambda^- + \lambda^+\delta) - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda^*_{i-1}(\lambda^- + \lambda^+\delta) B^*_0(\lambda^- + \lambda^+\delta) f^{(1)}_{i,j_i}] \\ & + 2((\lambda^+ m_1)^2 \bar{\delta}/(\lambda^- + \lambda^+\delta))\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda^*_{i-1}(\lambda^- + \lambda^+\delta)\,(1 - B^*_{i,j_i}(\lambda^- + \lambda^+\delta)) B^*_0(\lambda^- + \lambda^+\delta)\} \end{aligned}$$

- The probability that the server is under repair during essential service is

$$\begin{aligned} R_0 &= \lim_{z\to 1} R_0(z) \\ &= \frac{I_0(\lambda^- \lambda^+ m_1/(\lambda^- + \lambda^+\delta)) A^*(\lambda^+)(1 - B^*_0(\lambda^-))\beta_0^{(1)}}{D'} \end{aligned} \tag{3.62}$$

- Mean number of customers in the orbit when the server is under repair during essential service is

$$\begin{aligned} L_{R_0} &= \lim_{z\to 1} \frac{d}{dz} R_0(z) \\ &= \frac{D' N''_{R_0} - N'_{R_0} D''}{2D'^2} \end{aligned} \tag{3.63}$$

where

$$N'_{R_0} = I_0(\lambda^- \lambda^+ m_1/(\lambda^- + \lambda^+\delta)) A^*(\lambda^+)(1 - B^*_0(\lambda^- + \lambda^+\delta))\beta_0^{(1)}$$

$$\begin{aligned} N''_{R_0} = {} & (\lambda^-/\lambda^- + \lambda^+\delta) I_0 A^*(\lambda^+)[(\lambda^+)^2 m_1^2 (1 - B^*_0(\lambda^- + \lambda^+\delta))\beta_0^{(2)} \\ & + \lambda^+ m_2 (1 - B^*_0(\lambda^- + \lambda^+\delta))\beta_0^{(1)} - 2\lambda^+ m_1 f_0^{(1)}\beta_0^{(1)} \\ & + 2((\lambda^+ m_1)^2 \bar{\delta}/(\lambda^- + \lambda^+\delta))(1 - B^*_0(\lambda^-))\beta_0^{(1)}] \end{aligned}$$

- The probability that the server is under repair during optional services is

$$R = \lim_{z\to 1} \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} R_{i,j_i}(z)$$

$$= \frac{I_0(\lambda^-\lambda^+ m_1/(\lambda^-+\lambda^+\delta))A^*(\lambda^+)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-+\lambda^+\delta)(1-B_{i,j_i}^*(\lambda^-+\lambda^+\delta))\Lambda_{i-1}^*(\lambda^-+\lambda^+\delta)\beta_{i,j_i}^{(1)}}{D'}$$

(3.64)

- Mean number of customers in the orbit when the server is under repair during optional services is

$$L_R = \lim_{z\to 1}\frac{d}{dz}R(z)$$

$$= \frac{D' N_R'' - N_R' D''}{2D'^2} \qquad (3.65)$$

$$N_R' = I_0(\lambda^-\lambda^+ m_1/(\lambda^-+\lambda^+\delta))A^*(\lambda^+)\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} B_0^*(\lambda^-+\lambda^+\delta)(1-B_{i,j_i}^*(\lambda^-+\lambda^+\delta))\Lambda_{i-1}^*(\lambda^-+\lambda^+\delta)\beta_{i,j_i}^{(1)}$$

$$\begin{aligned}N_R'' = \ & I_0 A^*(\lambda^+)(\lambda^-/(\lambda^-+\lambda^+\delta))\{2((\lambda^+ m_1)^2\bar{\delta}/(\lambda^-+\lambda^+\delta))B_0^*(\lambda^-+\lambda^+\delta) \\ & \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-+\lambda^+\delta)(1-B_{i,j_i}^*(\lambda^-+\lambda^+\delta)\beta_{i,j_i}^{(1)} + 2\lambda^+ m_1[h_1 f_0^{(1)} \\ & + h_2 B_0^*(\lambda^-+\lambda^+\delta) - \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-+\lambda^+\delta)B_0^*(\lambda^-+\lambda^+\delta))f_{i,j_i}^{(1)}\beta_{i,j_i}^{(1)}] \\ & + (\lambda^+)^2 m_1^2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-+\lambda^+\delta)(1-B_{i,j_i}^*(\lambda^-+\lambda^+\delta))B_0^*(\lambda^-+\lambda^+\delta)\beta_{i,j_i}^{(2)} \\ & + \lambda^+ m_2\sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i}\Lambda_{i-1}^*(\lambda^-+\lambda^+\delta)(1-B_{i,j_i}^*(\lambda^-+\lambda^+\delta))B_0^*(\lambda^-+\lambda^+\delta)\beta_{i,j_i}^{(1)}\}\end{aligned}$$

Let $N_q(z)$ be the numeraror of $P_q(z)$.

- Average orbit size is given by

$$L_q = \lim_{z\to 1}\frac{d}{dz}P_q(z)$$

$$= \frac{D' N_q'' - N_q' D''}{2D'^2} \qquad (3.66)$$

where

$$\begin{aligned}N_q' \ = \ & I_0 A^*(\lambda^+)\{1-q_0 f_0^{(1)} + \sum_{i=1}^{M} q_i(\Lambda_i^*(\lambda^-+\lambda^+\delta)f_0^{(1)} + M_i^{(1)}B_0^*(\lambda^-+\lambda^+\delta)) \\ & + (\lambda^+ m_1\bar{\delta} - \lambda^+\delta)T_3 - (\lambda^-+\lambda^+\delta)T_4\}\end{aligned}$$

$$N_q'' = I_0 A^*(\lambda^+)\{-q_0 f_0^{(2)} - \sum_{i=1}^{M} q_i \Lambda_i^*(\lambda^- + \lambda^+\delta) f_0^{(2)} - 2\sum_{i=1}^{M} q_i M_i^{(1)} f_0^{(1)} B_0^*(\lambda^- + \lambda^+\delta)$$
$$- \sum_{i=1}^{M} q_i M_i^{(2)} B_0^*(\lambda^- + \lambda^+\delta) + (\lambda^+ m_2 \bar{\delta} - 2\lambda^+\delta m_1) T_3 + 2(\lambda^+ m_1 \bar{\delta} - \lambda^+\delta) T_4$$
$$- (\lambda^- + \lambda^+ \delta) T_8 \}$$

- Expected system size is given by

$$L_s = \lim_{z\to 1} \frac{d}{dz} P_s(z)$$
$$= L_q + P_0 + P \tag{3.67}$$

3.5 Reliability Indices

Let $\mathcal{A}$ (t) be the availability of the server at time t, and let $\mathcal{F}$ (t) be the failure frequency

Theorem 3.1

The steady state availability of the server is

$$\mathcal{A} = \frac{\begin{array}{l} I_0 A^*(\lambda^+)\{1 + \sum_{j_1=1}^{k_1} p_{j_1} f_0^{(1)} - \sum_{i=1}^{M} q_i [\Lambda_i^*(\lambda^- + \lambda^+\delta) f_0^{(1)} + M_i^{(1)} B_0^*(\lambda^- + \lambda^+\delta)] - \lambda^+\delta T_3 \\ - h_1 f_0^{(1)} - h_2 B_0^*(\lambda^- + \lambda^+\delta) + \sum_{i=1}^{M}\sum_{j_i=1}^{k_i} p_{j_i} \Lambda_{i-1}^*(\lambda^- + \lambda^+\delta) f_{i,j_i}^{(1)} B_0^*(\lambda^- + \lambda^+\delta) - \lambda^- \lambda^+ m_1 T_5 \} \end{array}}{D'} \tag{3.68}$$

Proof.

$$\mathcal{A} = I_0 + \lim_{z\to 1} [\int_0^\infty I(x,z)\, dx + \int_0^\infty P_0(x,z) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} \int_0^\infty P_{i,j_i}(x,z)\, dx]$$
$$= I_0 + \lim_{z\to 1} [I(z) + P_0(z) + P(z)]$$
$$= I_0 + I + P_0 + P \tag{3.69}$$

Equation (3.68) is obtained by substituting the expressions of I, P_0 and P in equation (3.69).

Theorem 3.2

The steady state failure frequency of the server is

$$\mathcal{F} = \frac{I_0 A^*(\lambda^+) \lambda^- \lambda^+ m_1 T_3}{D'} \tag{3.70}$$

Proof:

$$\mathcal{F} = \lambda^- \lim_{z\to 1} [\int_0^\infty P_0(x,z) + \sum_{i=1}^{M} \sum_{j_i=1}^{k_i} \int_0^\infty P_{i,j_i}(x,z)\, dx]$$

$$= \lambda^- \lim_{z\to 1} [P_0(z) + P(z)]$$

$$= \lambda^- (P_0 + P) \tag{3.71}$$

Equation (3.70) is obtained by substituting the expression of P_0 and P in equation (3.71).

3.6 Stochastic Decomposition

Theorem 3.3

The mean number of customers in the system (L_s) is the sum of two independent random variables, one of which is the mean number of customers in the batch arrival G-queue with multistage and multi-optional services, priority and collision (L_C) and the mean number of customers in the orbit when the server is idle (L_{Id}).

$$L_s = L_C + L_{Id} \tag{3.72}$$

Proof:

The probability generating function of the system size $\phi(z)$ of batch arrival G-queue with multistage and multi-optional services, priority and collision is given by

$$\phi(z) = \lim_{A^*(\lambda^+)\to 1} P_s(z)$$

$$= \frac{D'\{z - q_0 B_0^*(k(z)) - \sum_{i=0}^{M} q_i \Lambda_i^*(k(z)) B_0^*(k(z)) + [\lambda^+ z (C(z)-1) - \lambda^+ \delta\, z\, C(z) - \lambda^-]\, T_2(z)\}}{T_6\, [z - T_1(z) - \lambda^+ \delta \alpha z C(z)\, T_2(z)]} \tag{3.73}$$

The probability generating function of the number of customers in the orbit when the system is idle is given by

$$\psi(z) = \frac{I_0 + I(z)}{I_0 + I(1)}$$

$$= \frac{[z - T_1(z) - \lambda^+ \delta \alpha z C(z)\, T_2(z)]\, D'}{D(z)\, T_6} \tag{3.74}$$

From the equations (3.55), (3.73) and (3.74), we see that

$$P_s(z) = \phi(z)\ \psi(z)$$

$$L_s = \lim_{z \to 1} \frac{d}{dz} P_s(z)$$

$$= \phi'(1) + \psi'(1)$$

$$= L_C + L_{Id}$$

3.7 Special Cases

Case (i) : If $\lambda^- = 0$, M = 1, k_1 = 1, C(z) = z, α = 1 and q_1 = 1, then the model becomes an M/G/1 retrial queueing system with two-phase service and preemptive resume. In this case, the probability generating functions of the orbit size at different states of the server are given by

$$I(z) = \frac{I_0 (1 - A^*(\lambda^+))\, z (1 - z)\, [q_0 B_0^*(k(z)) + p_1 B_1^*(k(z)) B_0^*(k(z)) - 1]}{D(z)}$$

$$P_0(z) = \frac{I_0 A^*(\lambda^+)\, (z - 1)\, (1 - B_0^*(k(z)))}{D(z)}$$

$$P_1(z) = \frac{I_0 A^*(\lambda^+)\, (z - 1)\, p_1 (1 - B_1^*(k(z)))\, B_0^*(k(z))}{D(z)}$$

where

$$D(z) = z(1 - z) - B_0^*(k(z))(q_0 + p_1 B_1^*(k(z)))\, [(1 - z + \delta z)(z + (1 - z) A^*(\lambda^+)) - \delta z^2]$$

$$k(z) = \lambda^+ (1 - z) + \delta \lambda^+ z$$

The above results coincide with the results of Krishna Kumar et al. (2002).

Case (ii) : If M = 0 and q_0 = 1, then the system reduces to batch arrival retrial G-queue with priority and collision. In this case, the probability generating functions of the orbit size when the server is idle, busy and under repair are expressed as

$$I(z) = \frac{I_0 (1 - A^*(\lambda^+)) \{ \left(C(z) B_0^*(k(z)) - z\right)\left(\lambda^+ (1 - \bar{\delta} C(z)) + \lambda^-\right) + C(z)(1 - B_0^*(k(z)))\left(\lambda^- R_0^*(h(z)) + \lambda^+ \delta \bar{\alpha} z C(z) + \lambda^+ \delta \alpha z\right)}{D(z)}$$

$$P_0(z) = \frac{\lambda^+ I_0 A^*(\lambda^+) (C(z) - 1) (1 - B_0^*(k(z)))}{D(z)}$$

$$R_0(z) = \frac{\lambda^- I_0 A^*(\lambda^+) (C(z) - 1) (1 - B_0^*(k(z))) (R_0^*(h(z)) - 1)}{D(z)}$$

where

$$D(z) = [z - \left(A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))\right) B_0^*(k(z))]\left(\lambda^+ (1 - \bar{\delta} C(z)) + \lambda^-\right) - (1 - B_0^*(k(z))) [\left(A^*(\lambda^+) + C(z)(1 - A^*(\lambda^+))\right)\left(\lambda^- R_0^*(h(z)) + \lambda^+ \delta \bar{\alpha} z C(z)\right) - \lambda^+ \delta \alpha z C(z)]$$

$$k(z) = \lambda^+ (1 - \bar{\delta} C(z)) + \lambda^- , \quad h(z) = \lambda^+ (1 - C(z))$$

The above results agree with the results of Sumitha and Udaya Chandrika (2015b) without delay in repair and orbital search.

3.8 Practical Justification of the Model

In Optical Burst Switching (OBS) network, multiple incoming packets are assembled into a burst. The bursts are categorized as positive bursts and negative bursts.

When a positive burst enter into the source node of the network and finds the server free, burst transmission takes place normally. If the server is busy, the burst would enter into the buffer (orbit) at the source node and the burst retransmission (retrial) takes place after sometime.

Arrival of high priority positive burst moves the low priority positive burst being in the service to the buffer. Thus the bursts with highest priority are served first. The burst transmission collision also occurs while a node is transmitting a burst. If another burst passes through the node on the same wavelength.

All possible paths for the selected source and destination pair are divided into two phases, first phase of essential path and the second phase of multi-optional paths. In second phase, the multi-optional paths are subdivided into stages consisting of all possible node to node transmission. Here the selection of nodes in each stage is collectively known as the multistage and multi-optional services. After the completion of the first essential path, the burst may reach the destination directly. Otherwise, the burst is allowed to pass through the stage by stage multi-optional paths.

Arrival of negative burst removes the positive burst in transmission from the network and makes the server breakdown. The repair of the failed server starts immediately. Negative customer can be considered as some kind of work cancelling signal.

3.9 Numerical Results

In this section, numerical results of the effects of various system parameters on the performance measures are presented. It is assumed that the retrial time, first phase service time, second phase service time and repair time during first and second phase are exponentially distributed with respective parameters η, μ_0, μ_{i,j_i}, β_0 and β_{i,j_i}, where i=1,2,...,M and j_i = 1,2,....,k_i . The arbitrary values for the parameters are chosen as $\lambda^+ = 1.5$, $\lambda^- = 0.2$, $\eta = 30$, M = 3, $k_1 = 2$, $k_2 = 3$, $k_3 = 2$, p_1 = [0.5, 0.3], p_2 = [0.2, 0.3, 0.1], p_3 = [0.4, 0.2], $\delta = 0.4$, $q_0 = 0.2$, q = [0.4 0.4 1], $\alpha = 0.7$, $\mu_0 = 15$, $\beta_0 = 2$, μ_1 = [25 14], μ_2 = [22 12 15], μ_3 = [20 25], β_1 = [1 3], β_2 = [5 7 10], β_3 = [12 14], $c_1 = 0.5$, $c_2 = 0.5$.

System performance measures I_0 – the probability that the system is empty, I – the probability that the server is idle in the non-empty system, P_0 – the probability that the server is busy in first phase, P – the probability that the server is busy in second phase, R_0 – the probability that the server is under repair during first phase service, R – the probability that the server is under repair during second phase service and L_s – the mean system size are evaluated for different values of δ, α, p_{11} and p_{12} and presented respectively in Tables 3.1, 3.2, 3.3 and 3.4.

Table 3.1 reveals that increase in δ, decreases I_0 and increases I , P_0 , P , R_0 , R and L_s . From the Table 3.2, it is observed that increase in α decreases I and increases

I_0 and L_s and measures P_0 , P , R_0 and R are independent of α. Table 3.3 and 3.4 depict the effect of p_{11} and p_{12} on the system measures. It is noted that increase in p_{11} and p_{12} increases I, P, R and L_s and decreases I_0 , P_0 and R_0 .

Influence of the parameters λ^- , μ_{11} and β_{11} on the mean values L_I , L_P and L_R are displayed in Fig. 3.2 to 3.4.

These figures indicate that

- increase in λ^-, increases L_I , L_P and L_R
- increase in μ_{11} and β_{11} , decreases L_I , L_P and L_R

Table 3.1 Performance Measures versus δ

δ	I_0	I	P_0	P	R_0	R	L_s
0.1	0.4169	0.0390	0.1245	0.3685	0.0124	0.0361	0.5409
0.3	0.3606	0.0458	0.1379	0.3977	0.0138	0.0392	0.9403
0.5	0.2977	0.0532	0.1530	0.4301	0.0153	0.0427	1.5825
0.7	0.2270	0.0616	0.1701	0.4663	0.0170	0.0465	2.9897
0.9	0.1473	0.0709	0.1895	0.5069	0.0190	0.0509	6.6251

Table 3.2 Performance Measures versus α

α	I_0	I	P_0	P	R_0	R	L_s
0.1	0.3233	0.0561	0.1453	0.4135	0.0145	0.0409	1.3472
0.3	0.3255	0.0539	0.1453	0.4135	0.0145	0.0409	1.6093
0.5	0.3278	0.0516	0.1453	0.4135	0.0145	0.0409	1.8676
0.7	0.3300	0.0494	0.1453	0.4135	0.0145	0.0409	2.1223
0.9	0.3323	0.0472	0.1453	0.4135	0.0145	0.0409	2.3735

Table 3.3 Performance Measures versus p_{11}

p_{11}	I_0	I	P_0	P	R_0	R	L_s
0.2	0.4610	0.0451	0.1460	0.3031	0.0146	0.0255	0.8433
0.3	0.3954	0.0472	0.1456	0.3584	0.0146	0.0332	1.0061
0.4	0.3300	0.0494	0.1453	0.4135	0.0145	0.0409	1.2036
0.5	0.2650	0.0515	0.1449	0.4683	0.0145	0.0485	1.4614
0.6	0.2003	0.0537	0.1445	0.5228	0.0145	0.0561	1.8376

Table 3.4 Performance Measures versus p_{12}

p_{12}	I_0	I	P_0	P	R_0	R	L_s
0.2	0.5201	0.0430	0.1464	0.2631	0.0146	0.0086	0.7310
0.3	0.4513	0.0452	0.1460	0.3259	0.0146	0.0120	0.8880
0.4	0.383	0.0474	0.1456	0.3883	0.0146	0.0152	1.0703
0.5	0.315	0.0495	0.1451	0.4503	0.0145	0.0185	1.2943
0.6	0.2475	0.0517	0.1447	0.5119	0.0145	0.0218	1.5940

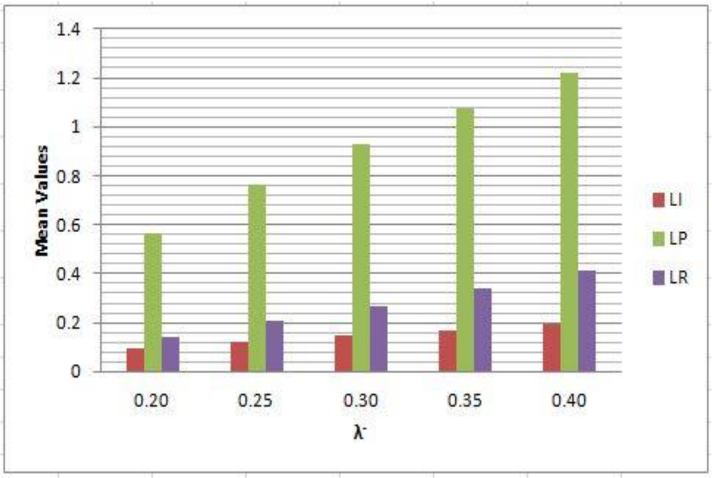

Fig. 3.2 Mean Values L_I , L_P and L_R versus λ^-

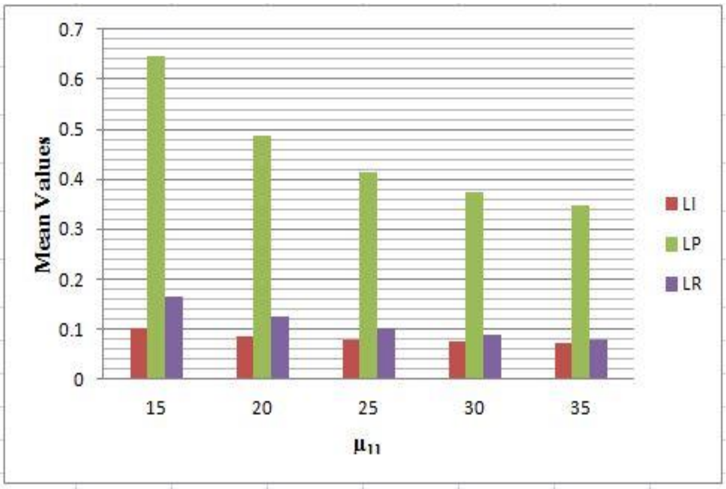

Fig. 3.3 Mean Values L_I , L_P and L_R versus μ_{11}

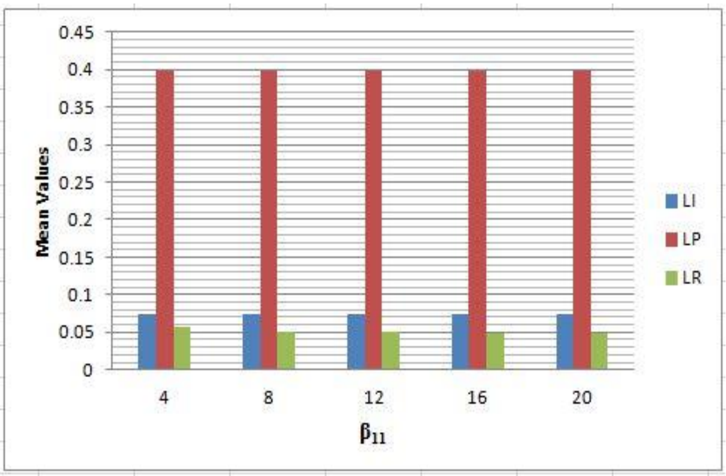

Fig. 3.4 Mean Values L_I , L_P and L_R versus β_{11}