3

$\psi^*\alpha$-Continuous Maps in Topological Spaces

3.1 Introduction

The notion of continuity is one of the most important concept in the study of topological spaces. The study of characterizations and generalizations of continuity has been done by various generalized closed sets. Several researchers working in the field of general topology have shown interest in studying the properties of generalizations of continuous maps. Levine (1960), Arya and Gupta (1974), Jain (1980) and Noiri (1984) introduced respectively strongly continuous maps, completely continuous maps, totally continuous maps and perfectly continuous maps which are some of stronger forms of continuous maps. Mashhour et al.(1983), Balachandran et al.(1991), Devi et al.(1997) and Ramya and Parvathi (2013) introduced α-continuous maps, g-continuous maps, αg-continuous maps(gα-continuous maps) and ψg-continuous maps respectively which are some of the weaker forms of continuous maps. Dontchev (1996) and Jafari and Noiri (2001) defined respectively contra continuous maps and contra α-continuous maps which are independent of their respective continuous maps.

In this chapter, the concepts of $\psi^*\alpha$-continuous maps, quasi $\psi^*\alpha$-continuous maps, perfectly $\psi^*\alpha$-continuous maps, totally $\psi^*\alpha$-continuous maps and strongly $\psi^*\alpha$-continuous maps and contra $\psi^*\alpha$-continuous maps are introduced and some of their properties and their interrelations are obtained.

3.2 $\psi^*\alpha$-continuous maps

In this section, $\psi^*\alpha$-continuous maps in topological spaces are introduced and their properties are derived. It is shown that the composition of two $\psi^*\alpha$-continuous maps need not be $\psi^*\alpha$-continuous.

Definition 3.2.1 A map $f : (X, \tau) \to (Y, \sigma)$ is called **$\psi^*\alpha$-continuous** if $f^{-1}(V)$ is $\psi^*\alpha$-closed in (X, τ) for every closed set V in (Y, σ).

Theorem 3.2.2 A map $f : (X, \tau) \to (Y, \sigma)$ is $\psi^*\alpha$-continuous if and only if $f^{-1}(V)$ is $\psi^*\alpha$-open for each open set V in (Y, σ).

Proposition 3.2.3

(i) Every continuous map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-continuous map.

(ii) Every completely continuous map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-continuous map.

(iii) Every α-continuous map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-continuous map.

Proof: Since every closed, regular closed and α-closed set is $\psi^*\alpha$-closed, the statements of the proposition hold good.

The converse of the statements in the above proposition need not be true as seen from the following examples.

Example 3.2.4 Let X = Y = {a, b, c}, τ = {ϕ, {a, b}, X} and σ = {ϕ, {a}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = b, f(b) = a, f(c) = c. Then f is $\psi^*\alpha$-continuous but not continuous, not completely continuous and not α-continuous, since for the closed set {b, c} in (Y, σ), $f^{-1}(\{b, c\}) = \{a, c\}$ is not closed, not regular closed and not α-closed in (X, τ).

Proposition 3.2.5 Every $\psi^*\alpha$-continuous map f : (X, τ) → (Y, σ) is gα-continuous, αg-continuou, $\tilde{g}_\alpha$ -continuous, ψ-continuous, $\psi\hat{g}$ -continuous, ψg-continuous and $\alpha\hat{g}$ -continuous.

Proof: Since every $\psi^*\alpha$-closed set is gα-closed, αg-closed, $\tilde{g}_\alpha$-closed, ψ-closed, $\psi\hat{g}$-closed, ψg-closed and $\alpha\hat{g}$-closed, the result follows.

The converse of the above proposition is not true as seen from the following examples.

Example 3.2.6 Let X = Y = {a, b, c, d}, τ = {ϕ, {d}, {a, b}, {a, b, d}, X} and σ = {ϕ, {a}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = b, f(b) = a, f(c) = c, f(d)=d. Then f is gα-continuous, αg-continuous, $\tilde{g}_\alpha$ -continuous, $\psi\hat{g}$ -continuous, ψg-continuous and $\alpha\hat{g}$ -continuous but not $\psi^*\alpha$-continuous, since for the closed set {b, c, d} in (Y, σ), $f^{-1}(\{b, c, d\}) = \{a, c, d\}$ is not $\psi^*\alpha$-closed in (X, τ).

Example 3.2.7 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = {ϕ, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = b, f(c) = a. Then f is ψ- continuous but not $\psi^*\alpha$-continuous, since for the closed set {c} in (Y, σ), f^{-1}({c}) = {a} is not $\psi^*\alpha$-closed in (X, τ).

Remark 3.2.8 Let f : (X, τ) → (Y, σ) be a map. When (X, τ) is ${}_{\psi*\alpha}T_c$-space (resp. ${}_{\psi*\alpha}T_\alpha$ -space, ${}_{g\alpha}T_{\psi*\alpha}$ -space, ${}_{\alpha g}T_{\psi*\alpha}$-space and ${}_{\psi g}T_{\psi*\alpha}$-space), the concept of continuity (resp. α-continuity, gα-continuity, αg-continuity and ψg-continuity) and $\psi^*\alpha$-continuity coincides.

Proposition 3.2.9 Every strongly continuous (resp. totally continuous) map f : (X, τ) → (Y, σ) is a $\psi^*\alpha$-continuous map but not conversely.

Proof: Let V be a closed set in (Y, σ). Since f : (X, τ) → (Y, σ) is strongly continuous (resp. totally continuous), f^{-1}(V) is clopen in (X, τ). Since every closed set is $\psi^*\alpha$-closed, f^{-1}(V) is $\psi^*\alpha$-closed in (X, τ). Hence f is a $\psi^*\alpha$-continuous map.

Example 3.2.10 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X} and σ = {ϕ,{a}, {a, b}, {a, c}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = a, f(b) = c, f(c) = b. Then f is $\psi^*\alpha$-continuous but not strongly continuous and not totally continuous, since {b} is closed in (Y, σ) but f^{-1}({b}) = {c} is not open in (X, τ).

Remark 3.2.11 The following examples show that semi continuous maps and $\psi^*\alpha$- continuous maps are independent.

Example 3.2.12 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = {ϕ, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = a, f(c) = b. Then f is semi continuous but not $\psi^*\alpha$-continuous, since for the closed set {c} in (Y, σ), f^{-1}({ c}) = {a} is semi closed but not $\psi^*\alpha$-closed in (X, τ).

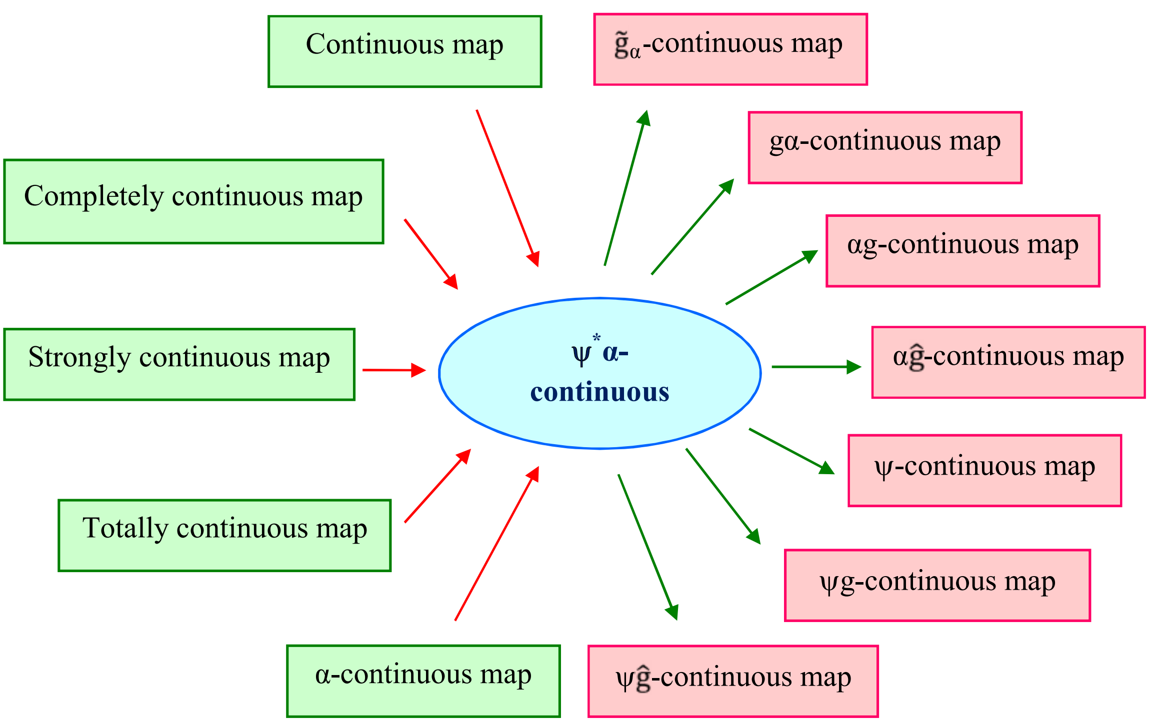
Example 3.2.13 Let X = Y = {a, b, c}, τ = {ϕ, {a, b}, X} and σ = {ϕ, {a}, Y}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is $\psi^*\alpha$-continuous but not semi continuous, since for the closed set {b, c} in (Y, σ) f^{-1}({b, c}) = {b, c} is $\psi^*\alpha$-closed but not semi closed in (X, τ).

Remark 3.2.14 The following examples show that the notion of g-(resp. g^*, $\hat{g}$, $g^\#$, ${}^*g$, $\tilde{g}$, $g^\# p^\#$) continuity and $\psi^*\alpha$-continuity are independent.

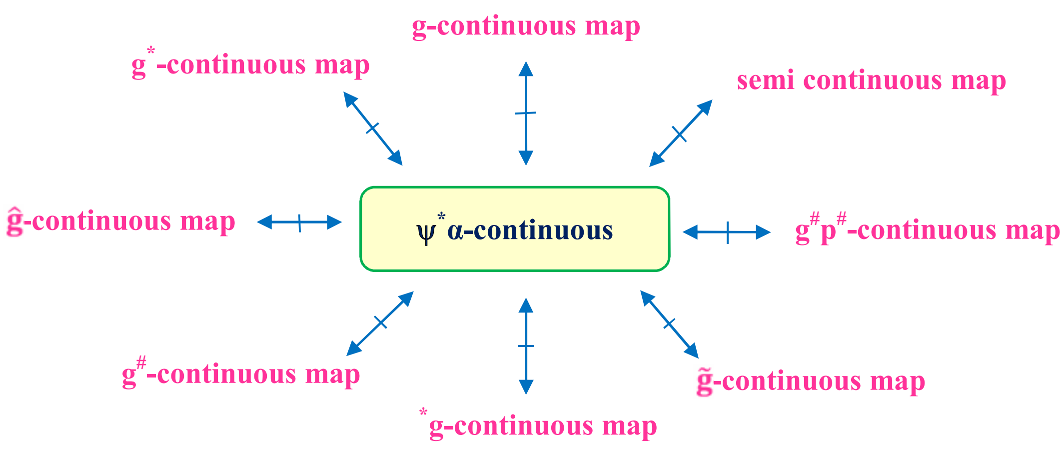
Example 3.2.15 Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{d\}, \{a, b\}, \{a, b, d\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map defined by $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(d)=d$. Then f is g-(resp. g^*, $\hat{g}$, $g^\#$, ${}^*g$, $\tilde{g}$, $g^\# p^\#$) continuous but not $\psi^*\alpha$-continuous, since for the closed set $\{b, c, d\}$ in (Y, σ), $f^{-1}(\{b, c, d\})=\{a, c, d\}$ is g-(resp. g^*, $\hat{g}$, $g^\#$, ${}^*g$, $\tilde{g}$, $g^\# p^\#$) closed but not $\psi^*\alpha$-closed in (X, τ).

Example 3.2.16 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map defined by $f(a) = b$, $f(b) = c$, $f(c) = a$. Then f is $\psi^*\alpha$-continuous but not g-(resp. g^*, $\hat{g}$, $g^\#$, ${}^*g$, $\tilde{g}$, $g^\# p^\#$) continuous, since for the closed set $\{c\}$ in (Y, σ) $f^{-1}(\{c\}) = \{b\}$ is $\psi^*\alpha$-closed but not g-(resp. g^*, $\hat{g}$, $g^\#$, ${}^*g$, $\tilde{g}$, $g^\# p^\#$) closed in (X, τ).

Remark 3.2.17 The dependency relations between various types of continuous maps with $\psi^*\alpha$-continuous maps are given in the following diagram.



Remark 3.2.18 The independency relations between various types of continuous maps with $\psi^*\alpha$-continuous maps are given in the following diagram.



Theorem 3.2.19 If $f : (X, \tau) \to (Y, \sigma)$ is a $\psi^*\alpha$-continuous map, then $f(\psi^*\alpha cl(U)) \subseteq cl(f(U))$ for every subset U of (X, τ).

Proof: Let U be any subset of (X, τ). Then $cl(f(U))$ is closed in (Y, σ). Since f is $\psi^*\alpha$-continuous, $f^{-1}(cl(f(U)))$ is $\psi^*\alpha$-closed in (X, τ). Since $f(U) \subseteq cl(f(U))$, $U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(cl(f(U)))$ and hence $f^{-1}(cl(f(U)))$ is a $\psi^*\alpha$-closed set containing U. By definition of $\psi^*\alpha$-closure, we have $\psi^*\alpha cl(U) \subseteq f^{-1}(cl(f(U)))$ which implies that $f(\psi^*\alpha cl(U)) \subseteq cl(f(U))$.

Corollary 3.2.20 Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous map. Then for every subset U of (X, τ), $f(\psi^*\alpha cl(U)) \subseteq cl(f(U))$.

Proof: Follows from **Proposition 3.2.3 (i)** and by **Theorem 3.2.19.**

Theorem 3.2.21 Let $f : (X, \tau) \to (Y, \sigma)$ be a map from a topological space (X, τ) into a topological space (Y, σ). Then the following statements are equivalent.

(a) For each point x in (X, τ) and each open set V in (Y, σ) containing f(x), there exists a $\psi^*\alpha$-open set U in (X, τ) containing x such that $f(U) \subseteq V$.

(b) For every subset A of (X, τ), $f(\psi^*\alpha cl(A)) \subseteq cl(f(A))$.

(c) For every subset B of (Y, σ), $\psi^*\alpha cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

Proof: (a) $\Leftrightarrow$ (b) Suppose that (a) holds and let $y \in f(\psi^*\alpha cl(A))$. Then $y = f(x)$ for some $x \in \psi^*\alpha cl(A) \subseteq X$. Let V be any open set in (Y, σ) containing f(x).Then by hypothesis, there exists a $\psi^*\alpha$-open set U in (X, τ) containing x such that $f(U) \subseteq V$. By **Theorem 2.5.15** we get $U \cap A \neq \phi$. Then $f(U \cap A) \neq \phi$, which implies that $V \cap f(A) \neq \phi$. Hence $y = f(x) \in cl(f(A))$. Therefore $f(\psi^*\alpha cl(A)) \subseteq cl(f(A))$.

Conversely, suppose that (b) holds and let $x \in X$ and let V be any open set in (Y, σ) containing f(x). Let $A = f^{-1}(V^c)$ then $x \notin A$. By (b), $f(\psi^*\alpha cl(A)) \subseteq cl(f(A)) \subseteq cl(f(f^{-1}(V^c))) \subseteq cl(V^c) = V^c$. Therefore $f^{-1}(f(\psi^*\alpha cl(A))) \subseteq f^{-1}(V^c)$ which implies $\psi^*\alpha cl(A) \subseteq f^{-1}(V^c) = A$. Hence $A = \psi^*\alpha cl(A)$. Since $x \notin A$, $x \notin \psi^*\alpha cl(A)$. Then there exists a $\psi^*\alpha$-open set U containing x such that $U \cap A = \phi$ and hence $f(U) \subseteq f(A^c) \subseteq V$.

(b) $\Leftrightarrow$ (c) Suppose that (b) holds and let B be any subset of Y. Replacing A by $f^{-1}(B)$ from (b), $f(\psi^*\alpha cl\ (f^{-1}(B))) \subseteq cl(f(f^{-1}(B))) \subseteq cl(B)$. Hence $\psi^*\alpha cl\ (f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

Conversely, suppose that (c) holds and let $B = f(A)$ where A is a subset of X. Then $\psi^*\alpha cl\ (A) \subseteq \psi^*\alpha cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$. Therefore $f(\psi^*\alpha cl(A)) \subseteq cl(B) = cl(f(A))$.

Proposition 3.2.22 If $f : (X, \tau) \to (Y, \sigma)$ is a $\psi^*\alpha$-continuous map and $g : (Y, \sigma) \to (Z, \eta)$ is a continuous (resp. strongly continuous) map, then $g \circ f : (X, \tau) \to (Z, \eta)$ is a $\psi^*\alpha$-continuous map

Proof: Let V be a closed set in (Z, η). Since g is continuous (resp. strongly continuous), $g^{-1}(V)$ is closed (resp. clopen) in (Y, σ). Since f is $\psi^*\alpha$-continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $\psi^*\alpha$-closed in (X, τ). Therefore $g \circ f$ is a $\psi^*\alpha$-continuous map

Remark 3.2.23 The **Proposition 3.2.22** is also true if g is a completely continuous map, since every regular closed set is closed.

Remark 3.2.24 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a continuous (resp. an α-continuous) map and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is a continuous map, then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a $\psi^*\alpha$-continuous map.

Remark 3.2.25 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an α-irresolute map and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is a continuous map , then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a $\psi^*\alpha$-continuous map.

Proposition 3.2.26 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an α-irresolute map and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be a $\psi^*\alpha$-continuous map. If (Y, σ) is a ${}_{\psi^*\alpha}T_\alpha$-space, then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is an α-continuous map.

Proof: Let V be a closed set in (Z, η). Since g is $\psi^*\alpha$-continuous, $g^{-1}(V)$ is $\psi^*\alpha$-closed in (Y, σ). Since (Y, σ) is a ${}_{\psi^*\alpha}T_\alpha$-space, $g^{-1}(V)$ is α-closed in (Y, σ). Since f is α-irresolute, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is α-closed in (X, τ). Hence $g \circ f$ is an α-continuous map.

Remark 3.2.27 The composition of two $\psi^*\alpha$-continuous maps need not be a $\psi^*\alpha$-continuous map as seen from the following example.

Example 3.2.28 Let X = Y = Z = {a, b, c}, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$, $\sigma = \{\phi, \{a, b\}, Y\}$ and $\eta = \{\phi, \{a\}, Z\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map and $g : (Y, \sigma) \rightarrow (Z,\eta)$ be a map defined by g(a) = b, g(b) = a, g(c) = c. Then the maps f and g are $\psi^*\alpha$-continuous but their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is not a $\psi^*\alpha$-continuous map, since for the closed set {b, c} in (Z,η), $(g \circ f)^{-1}(\{b, c\}) = \{a, c\}$ is not $\psi^*\alpha$-closed in (X, τ).

Proposition 3.2.29 Let f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η) be $\psi^*\alpha$-continuous maps. Then g∘f : (X, τ) → (Z, η) is also a $\psi^*\alpha$-continuous map, if (Y, σ) is a ${}_{\psi*\alpha}T_c$-space.

Proof: Let V be a closed set in (Z, η). Since g is $\psi^*\alpha$-continuous, $g^{-1}(V)$ is $\psi^*\alpha$-closed in (Y, σ). Since (Y, σ) is a ${}_{\psi*\alpha}T_c$-space, $g^{-1}(V)$ is closed in (Y, σ). Since f is $\psi^*\alpha$-continuous, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $\psi^*\alpha$-closed in (X, τ). Hence g∘f is a $\psi^*\alpha$-continuous map.

Proposition 3.2.30 Let f : (X, τ) → (Y, σ) be a gα-continuous map and g : (Y, σ) → (Z, η) be a continuous. Then g∘f : (X, τ) → (Z, η) is a $\psi^*\alpha$-continuous map, if (X, τ) is a ${}_{g\alpha}T_{\psi*\alpha}$-space.

Proof: Let V be a closed set in (Z, η). Since g is continuous, $g^{-1}(V)$ is closed in (Y, σ). Since f is gα-continuous, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is gα-closed in (X, τ). Since (X, τ) is a ${}_{g\alpha}T_{\psi*\alpha}$-space, $(g\circ f)^{-1}(V)$ is $\psi^*\alpha$-closed in (X, τ). Hence g∘f is a $\psi^*\alpha$-continuous map.

Proposition 3.2.31 Let f : (X, τ) → (Y, σ) be a αg-continuous map and g : (Y, σ) → (Z,η) be a continuous map. Then g∘f : (X, τ) → (Z, η) is a $\psi^*\alpha$-continuous map, if (X, τ) is a ${}_{\alpha g}T_{\psi*\alpha}$-space.

Proof: Let V be a closed set in (Z, η). Since g is continuous, $g^{-1}(V)$ is closed in (Y, σ). Since f is αg-continuous, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is αg-closed in (X, τ). Since (X, τ) is a ${}_{\alpha g}T_{\psi*\alpha}$-space, $(g\circ f)^{-1}(V)$ is $\psi^*\alpha$-closed in (X, τ). Hence g∘f is a $\psi^*\alpha$-continuous map.

Proposition 3.2.32 Let f : (X, τ) → (Y, σ) be a ψg-continuous map and g : (Y, σ) → (Z,η) be a continuous map. Then g∘f : (X, τ) → (Z, η) is a $\psi^*\alpha$-continuous map, if (X, τ) is a ${}_{\psi g}T_{\psi*\alpha}$-space

Proof: Let V be a closed set in (Z, η). Since g is continuous, $g^{-1}(V)$ is closed in (Y, σ). Since f is ψg-continuous, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is ψg-closed in (X, τ). Since (X, τ) is a ${}_{\psi g}T_{\psi*\alpha}$-space, $(g\circ f)^{-1}(V)$ is $\psi^*\alpha$-closed in (X, τ). Hence g∘f is a $\psi^*\alpha$-continuous map.

Proposition 3.2.33 Let f : (X, τ) → (Y, σ) be a $\psi^*\alpha$-continuous map and g : (Y, σ) → (Z, η) be an αg-continuous map. Then g∘f : (X, τ) →(Z, η) is a $\psi^*\alpha$-continuous map, if (Y, σ) is a $_{\alpha}T_b$-space

Proof: Let V be a closed set in (Z, η). Since g is αg-continuous, $g^{-1}(V)$ is αg-closed in (Y, σ). Since (Y, σ) is a $_{\alpha}T_b$-space, $g^{-1}(V)$ is closed in (Y, σ). Since f is $\psi^*\alpha$-continuous, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $\psi^*\alpha$-closed in (X, τ). Hence g∘f is a $\psi^*\alpha$-continuous map.

Definition 3.2.34 A map f : (X, τ) → (Y, σ) is said to be pre ψg-closed (resp. pre ψg-open) if f(U) is ψg-closed (resp. ψg-open) in (Y, σ) for every ψg-closed (resp. ψg-open) set U in (X, τ).

Theorem 3.2.35 Let f : (X, τ) → (Y, σ) be a bijective pre ψg-closed and pre α-open map and if (X, τ) is a $_{\psi*\alpha}T_{\alpha}$-space, then (Y, σ) is also a $_{\psi*\alpha}T_{\alpha}$-space.

Proof: Let y ∈ Y. Since f is a bijection map, y = f(x) for some x ∈ X. By hypothesis (X, τ) is a $_{\psi*\alpha}T_{\alpha}$ -space and so {x} is either ψg-closed or α-open by **Theorem 2.6.7.** If {x} is ψg-closed then {y} = f({x}) is ψg-closed, since f is pre ψg-closed. Also {y} = f({x}) is α-open, as f is pre α-open. Hence by **Theorem 2.6.7,** (Y, σ) is a $_{\psi*\alpha}T_{\alpha}$-space.

Theorem 3.2.36 Let f : (X, τ) → (Y, σ) be a bijective, pre ψg-closed and $\psi^*\alpha$-continuous map from (X, τ) to an α-space (Y, σ). If A is $\psi^*\alpha$-open subset of (Y, σ), then $f^{-1}(A)$ is a $\psi^*\alpha$-open set in (X, τ).

Proof: Let A be a $\psi^*\alpha$-open set in (Y, σ) and F be any ψg-closed set in (X, τ) such that $F \subseteq f^{-1}(A)$. Then f(F) ⊆ A. By hypothesis f(F) is ψg-closed in (Y, σ) and A is $\psi^*\alpha$-open in (Y, σ). Hence f(F) ⊆ αint(A) by **Theroem 2.5.11** and so $F \subseteq f^{-1}(\alpha int(A))$. Since f is $\psi^*\alpha$-continuous and (Y, σ) is an α-space, $f^{-1}(\alpha int(A))$ is $\psi^*\alpha$-open in (X, τ). Therefore $F \subseteq \alpha int(f^{-1}(\alpha int(A))) \subseteq \alpha int(f^{-1}(A))$ and so $f^{-1}(A)$ is $\psi^*\alpha$-open in (X, τ).

Theorem 3.2.37 Let f : (X, τ) → (Y, σ) be a ψg-irresolute and pre α-closed map. If A is a $\psi^*\alpha$-closed set in (X, τ), then f(A) is a $\psi^*\alpha$-closed set in (Y, σ).

Proof: Let A be a $\psi^*\alpha$-closed set in (X, τ) and U be any ψg-open set in (Y, σ) such that f(A) ⊆ U. Then A ⊆ $f^{-1}(U)$. By hypothesis αcl(A) ⊆ $f^{-1}(U)$. Thus f(αcl(A)) ⊆ U and f(αcl(A)) is α-closed. Now αcl(f (A)) ⊆ αcl(f (αcl(A))) = f (αcl(A)) ⊆ U. That is αcl(f(A)) ⊆ U and so f(A) is a $\psi^*\alpha$-closed set in (Y, σ).

3.3 Quasi $\psi^*\alpha$-continuous maps and perfectly $\psi^*\alpha$-continuous maps

In this section, quasi $\psi^*\alpha$-continuous and perfectly $\psi^*\alpha$-continuous maps in topological spaces are introduced and their properties are studied.

Definition 3.3.1 A map f : (X, τ) → (Y, σ) is called **quasi $\psi^*\alpha$-continuous** if $f^{-1}(V)$ is closed in (X, τ) for every $\psi^*\alpha$-closed set V in (Y, σ).

Example 3.3.2 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X} and σ = {ϕ, {a}, {b, c}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = b, f(b) = a, f(c) = c. Then f is quasi $\psi^*\alpha$-continuous.

Theorem 3.3.3 A map f : (X, τ) → (Y, σ) is quasi $\psi^*\alpha$-continuous if and only if the inverse image of every $\psi^*\alpha$-open set in (Y, σ) is open in (X, τ).

Proof: (Necessity) Let f : (X, τ) → (Y, σ) be quasi $\psi^*\alpha$-continuous and V be any $\psi^*\alpha$-open set in (Y, σ). Then Y – V is $\psi^*\alpha$-closed in (Y, σ). Since f is quasi $\psi^*\alpha$-continuous, $f^{-1}(Y - V) = X - f^{-1}(V)$ is closed in (X, τ). Hence $f^{-1}(V)$ is open in (X, τ).

(Sufficiency): Let F be any $\psi^*\alpha$-closed set in (Y, σ). Then Y – F is $\psi^*\alpha$ -open in (Y, σ). By assumption, $f^{-1}(Y - F) = X - f^{-1}(F)$ is open in (X, τ) which implies that $f^{-1}(F)$ is closed in (X, τ). Hence f is quasi $\psi^*\alpha$-continuous.

Definition 3.3.4 A map f : (X, τ) → (Y, σ) is called **perfectly $\psi^*\alpha$-continuous** if $f^{-1}(V)$ is clopen in (X, τ) for every $\psi^*\alpha$-closed set V in (Y, σ).

Example 3.3.5 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X}, and σ = {ϕ, {a}, {b, c}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = a, f(c) = b. Then f is perfectly $\psi^*\alpha$-continuous.

Theorem 3.3.6 A map $f : (X, \tau) \to (Y, \sigma)$ is perfectly $\psi^*\alpha$-continuous if and only if the inverse image of every $\psi^*\alpha$-open set in (Y, σ) is clopen in (X, τ).

Proof: (Necessity) Let V be any $\psi^*\alpha$-open set in (Y, σ). Then $Y - V$ is $\psi^*\alpha$-closed in (Y, σ). Since f is perfectly $\psi^*\alpha$-continuous, $f^{-1}(Y - V) = X - f^{-1}(V)$ is clopen in (X, τ). Hence $f^{-1}(V)$ is clopen in (X, τ).

(Sufficiency): Let F be any $\psi^*\alpha$-closed set in (Y, σ). Then $Y - F$ is $\psi^*\alpha$-open in (Y, σ). By assumption, $f^{-1}(Y - F) = X - f^{-1}(F)$ is clopen in (X, τ) which implies that $f^{-1}(F)$ is clopen in (X, τ). Hence f is a perfectly $\psi^*\alpha$-continuous map

Proposition 3.3.7 If $f : (X, \tau) \to (Y, \sigma)$ is a perfectly $\psi^*\alpha$-continuous map, then it is a quasi $\psi^*\alpha$-continuous map but not conversely.

Proof: Let V be a $\psi^*\alpha$-closed set in (Y, σ). Since $f : (X, \tau) \to (Y, \sigma)$ is a perfectly $\psi^*\alpha$-continuous map, $f^{-1}(V)$ is clopen in (X, τ). Hence f is a quasi $\psi^*\alpha$-continuous map.

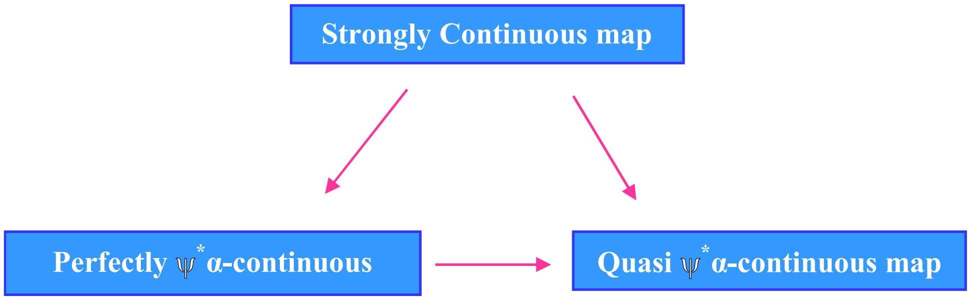
Example 3.3.8 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$, and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map defined by $f(a) = c$, $f(b) = a$, $f(c) = b$. Then f is quasi $\psi^*\alpha$-continuous but not perfectly $\psi^*\alpha$-continuous, since for the $\psi^*\alpha$-closed sets $\{b\}$, $\{c\}$, $\{b, c\}$ in (Y, σ), $f^{-1}(\{b\}) = \{c\}$, $f^{-1}(\{c\}) = \{a\}$, $f^{-1}(\{b, c\}) = \{a, c\}$ are closed but not open in (X, τ).

Proposition 3.3.9 Every strongly continuous map $f : (X, \tau) \to (Y, \sigma)$ is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map but not conversely.

Proof: Let V be a $\psi^*\alpha$-closed set in (Y, σ). Since $f : (X, \tau) \to (Y, \sigma)$ is strongly continuous, for any subset V, $f^{-1}(V)$ is both open and closed in (X, τ). Hence f is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map.

Example 3.3.10 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) but not strongly continuous, since for the subset $\{a, b\}$ in (Y, σ), $f^{-1}(\{a, b\}) = \{a, b\}$ is not clopen in (X, τ).

Remark 3.3.11 From the above results we have the following diagram.



Theorem 3.3.12 Let f : (X, τ) → (Y, σ) be a map. If (X, τ) is a discrete topological space and (Y, σ) be any topological space then the following statements are equivalent

(i) f is perfectly $\psi^*\alpha$-continuous

(ii) f is quasi $\psi^*\alpha$-continuous.

Proof: (i) ⇒ (ii) follows from **Proposition 3.3.7**

(ii) ⇒ (i) Let V be any $\psi^*\alpha$-closed set in (Y, σ). By hypothesis, $f^{-1}(V)$ is closed in (X, τ). Since (X, τ) is a discrete space, $f^{-1}(V)$ is open in (X, τ). Hence $f^{-1}(V)$ is clopen in (X, τ). Hence f is a perfectly $\psi^*\alpha$-continuous map.

Proposition 3.3.13 Let f : (X, τ) → (Y, σ) be a continuous map. If (Y, σ) is a $_{\psi^*\alpha}T_c$-space then f is a quasi $\psi^*\alpha$-continuous map.

Proof: Let V be a $\psi^*\alpha$-closed set in (Y, σ). Since (Y, σ) is a $_{\psi^*\alpha}T_c$-space, V is closed in (Y, σ). Since f is continuous, $f^{-1}(V)$ is closed in (X, τ). Hence f is a quasi $\psi^*\alpha$-continuous map.

Proposition 3.3.14 Let f : (X, τ) → (Y, σ) be a continuous map. If (Y, σ) is a $_{\psi^*\alpha}T_c$-space and a discrete space. Then f is a perfectly $\psi^*\alpha$-continuous map.

Proof: Let V be a $\psi^*\alpha$-closed set in (Y, σ). Since (Y, σ) is a $_{\psi^*\alpha}T_c$-space, V is closed in (Y, σ). Since (Y, σ) is a discrete space, V is open in (Y, σ). Since f is continuous, $f^{-1}(V)$ is clopen in (X, τ). Hence f is a perfectly $\psi^*\alpha$-continuous map.

Proposition 3.3.15 Every totally continuous map f : (X, τ) → (Y, σ) is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map, if (Y, σ) is a $_{\psi^*\alpha}T_c$-space.

Proof: Let V be a $\psi^*\alpha$-closed set in (Y, σ). Since (Y, σ) is a $_{\psi^*\alpha}T_c$-space, V is closed in (Y, σ). Since f is totally continuous, $f^{-1}(V)$ is both open and closed in (X, τ). Hence f is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map.

Proposition 3.3.16 Every quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map is an α-irresolute map but not conversely.

Proof: Let f : (X, τ) → (Y, σ) be a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map. Let V be an α-closed set in (Y, σ). Since every α-closed set is $\psi^*\alpha$-closed, V is $\psi^*\alpha$-closed in (Y, σ). Since f is quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous), $f^{-1}(V)$ is closed (resp. clopen) in (X τ). Since every closed set is α-closed, $f^{-1}(V)$ is α-closed in (X, τ). Hence f is an α-irresolute map

Example 3.3.17 Let X = Y = {a, b, c}, τ = {ϕ, {a}, X} and σ = {ϕ, {a}, {a, b}, {a, c}, Y} Let f : (X, τ) → (Y, σ) be a map defined by f(a) = a, f(b) = c, f(c) = b. Then f is α-irresolute but not quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous), since {c} is $\psi^*\alpha$-closed in (Y, σ) but $f^{-1}(\{c\})$ = {b} is not closed in (X, τ).

Proposition 3.3.18 Every quasi $\psi^*\alpha$-continuous map is a $\psi^*\alpha$-continuous map but not conversely.

Proof: Let f : (X, τ) → (Y, σ) be a quasi $\psi^*\alpha$-continuous map. Let V be a closed set in (Y, σ). Since every closed set is $\psi^*\alpha$-closed, V is $\psi^*\alpha$-closed in (Y, σ). Since f is quasi $\psi^*\alpha$-continuous, $f^{-1}(V)$ is closed in (X τ). Since every closed set is $\psi^*\alpha$-closed, $f^{-1}(V)$ is $\psi^*\alpha$-closed in (X, τ). Hence f is a $\psi^*\alpha$-continuous map.

Example 3.3.19 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = {ϕ, {a, b}, Y}. Let f : (X, τ) $\rightarrow$ (Y, σ) be the identity map. Then f is $\psi^*\alpha$-continuous but not quasi $\psi^*\alpha$-continuous, since {a, c} is $\psi^*\alpha$-closed in (Y, σ) but f^{-1}({a, c}) = {a, c} is not closed in (X, τ).

Proposition 3.3.20 Let f : (X, τ) $\rightarrow$ (Y, σ) and g : (Y, σ) $\rightarrow$ (Z, η) be any two maps. Then g∘f : (X, τ) $\rightarrow$ (Z, η) is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map.

(i) if g is a strongly continuous map and f is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map.

(ii) if g is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous)map and f is a continuous map.

Proof: (i) Let V be any $\psi^*\alpha$-closed set in (Z, η). Since g is strongly continuous, g^{-1}(V) is both open and closed in (Y, σ). Since every closed set is $\psi^*\alpha$-closed, g^{-1}(V) is $\psi^*\alpha$-closed in (Y, σ). Since f is quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous), $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is closed (resp. clopen) in (X, τ). Hence g∘f is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map.

(ii) Let V be any $\psi^*\alpha$-closed set in (Z, η). Since g is quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous), g^{-1}(V) is closed (resp. clopen) in (Y, σ). Since f is continuous, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is closed (resp. clopen) in (X, τ). Hence g∘f is a quasi $\psi^*\alpha$-continuous (resp. a perfectly $\psi^*\alpha$-continuous) map.

Proposition 3.3.21 If f : (X, τ) $\rightarrow$ (Y, σ) is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map and g : (Y, σ) $\rightarrow$ (Z, η) is a perfectly $\psi^*\alpha$-continuous (resp. quasi $\psi^*\alpha$-continuous) map, then g∘f : (X, τ) $\rightarrow$ (Z, η) is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map.

Proof: Since every closed set is $\psi^*\alpha$-closed, the result follows.

Proposition 3.3.22 If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are quasi $\psi^*\alpha$- continuous (resp. perfectly $\psi^*\alpha$-continuous) maps, then $g \circ f : (X, \tau) \to (Z, \eta)$ is also a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map.

Proof : Let V be any $\psi^*\alpha$-closed set in (Z, η). Then $g^{-1}(V)$ is closed (resp. clopen) in (Y, σ) as g is quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous). Since every closed set is $\psi^*\alpha$-closed, $g^{-1}(V)$ is $\psi^*\alpha$-closed in (Y, σ). Since f is quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous), $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is closed (resp. clopen) in (X, τ) and hence $g \circ f$ is a quasi $\psi^*\alpha$-continuous (resp. perfectly $\psi^*\alpha$-continuous) map.

3.4 Totally $\psi^*\alpha$-continuous maps and strongly $\psi^*\alpha$-continuous maps

In this section, some new classes of maps called totally $\psi^*\alpha$-continuous maps and strongly $\psi^*\alpha$-continuous maps are introduced and their properties are discussed.

Definition 3.4.1 A subset A of (X, τ) is called **$\psi^*\alpha$-clopen** if it is both $\psi^*\alpha$-open and $\psi^*\alpha$-closed in (X, τ).

Definition 3.4.2 A map $f : (X, \tau) \to (Y, \sigma)$ is called **totally $\psi^*\alpha$-continuous** if $f^{-1}(V)$ is a $\psi^*\alpha$-clopen set in (X, τ) for every open set V in (Y, σ).

Example 3.4.3 Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map defined by $f(a) = a$, $f(b) = c$, $f(c) = b$. Then f is totally $\psi^*\alpha$-continuous.

Proposition 3.4.4 A map $f : (X, \tau) \to (Y, \sigma)$ is totally $\psi^*\alpha$-continuous if and only if the inverse image of every closed subset of (Y, σ) is a $\psi^*\alpha$-clopen subset of (X, τ).

Proof : (Necessity) Let V be any closed set in (Y, σ).Then $Y - V$ is open in (Y, σ). Since f is totally $\psi^*\alpha$-continuous, $f^{-1}(Y - V) = X - f^{-1}(V)$ is $\psi^*\alpha$-clopen in (X, τ) which implies that $f^{-1}(V)$ is $\psi^*\alpha$-clopen in (X, τ).

(Sufficiency): Let U be any open set in (Y, σ). Then Y – U is closed in (Y, σ). By assumption, $f^{-1}(Y - U) = X - f^{-1}(U)$ is $\psi^*\alpha$-clopen in (X, τ) which implies that $f^{-1}(U)$ is $\psi^*\alpha$-clopen in (X, τ). Hence f is totally $\psi^*\alpha$-continuous.

Proposition 3.4.5 Every totally $\psi^*\alpha$-continuous map is a $\psi^*\alpha$-continuous map but not conversely.

Proof : Follows from the definitions of totally $\psi^*\alpha$-continuous and $\psi^*\alpha$-continuous maps.

Example 3.4.6 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, {a, c}, X} and σ = {ϕ, {a}, Y}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is $\psi^*\alpha$-continuous but not totally $\psi^*\alpha$-continuous, since for the open set {a} in (Y, σ), $f^{-1}(\{a\}) = \{a\}$ is $\psi^*\alpha$-open but not $\psi^*\alpha$ -closed in (X, τ).

Definition 3.4.7 A map f : (X, τ) → (Y, σ) is called **strongly $\psi^*\alpha$-continuous** if $f^{-1}(V)$ is a $\psi^*\alpha$-clopen set in (X, τ) for every subset V in (Y, σ).

Example 3.4.8 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, X} and σ = {ϕ, {c}, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = a, f(b) = c, f(c) = b. Then f is strongly $\psi^*\alpha$-continuous.

Remark 3.4.9 Every strongly $\psi^*\alpha$-continuous map is a totally $\psi^*\alpha$-continuous map but not conversely.

Proof : Follows from the definitions of totally $\psi^*\alpha$-continuous and strongly $\psi^*\alpha$-continuous maps.

Example 3.4.10 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b, c}, X} and σ = {ϕ, {a}, Y}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is totally $\psi^*\alpha$-continuous but not strongly $\psi^*\alpha$-continuous, since for the subset {b} in (Y, σ), $f^{-1}(\{b\}) = \{b\}$ is not $\psi^*\alpha$-clopen in (X, τ).

Proposition 3.4.11 Let f : (X, τ) → (Y, σ) be a totally $\psi^*\alpha$-continuous map where (Y, σ) is a discrete topological space. Then f is a strongly $\psi^*\alpha$-continuous map

Proof: Let V be any subset of (Y, σ). Since (Y, σ) is a discrete topological space, V is open in (Y, σ). Since f is totally $\psi^*\alpha$-continuous, $f^{-1}(V)$ is $\psi^*\alpha$-clopen in (X, τ). Hence f is a strongly $\psi^*\alpha$-continuous map

Theorem 3.4.12 Every perfectly $\psi^*\alpha$-continuous map is a totally $\psi^*\alpha$-continuous map but not conversely. .

Proof: Let V be any open set in (Y, σ). Then V is $\psi^*\alpha$ -open in (Y, σ). Since $f : (X, \tau) \rightarrow (Y, \sigma)$ is perfectly $\psi^*\alpha$-continuous, $f^{-1}(V)$ is clopen in (X, τ). Since every open set is $\psi^*\alpha$-open and every closed set is $\psi^*\alpha$-closed, $f^{-1}(V)$ is $\psi^*\alpha$-clopen in (X, τ). Hence f is a totally $\psi^*\alpha$-continuous map

Example 3.4.13 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X} and σ = {ϕ, {b}, Y}. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is totally $\psi^*\alpha$-continuous but not perfectly $\psi^*\alpha$-continuous, since for the $\psi^*\alpha$-open set {b, c} in (Y, σ), $f^{-1}(\{b, c\}) = \{b, c\}$ is not clopen in (X, τ).

3.5 Contra $\psi^*\alpha$-continuous maps

In this section, contra $\psi^*\alpha$-continuous maps in topological spaces are introduced and some of their properties are obtained.

Definition 3.5.1 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called **contra $\psi^*\alpha$-continuous** if $f^{-1}(V)$ is $\psi^*\alpha$-open in (X, τ) for every closed set V of (Y, σ).

Example 3.5.2 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = {ϕ, {a}, Y}. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by f(a) = b, f(b) = c, f(c) = a. Then f is contra $\psi^*\alpha$-continuous.

Proposition 3.5.3 If a map $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra α-continuous, then it is a contra $\psi^*\alpha$-continuous map but not conversely.

Proof: Let V be any closed set in (Y, σ). Since f is contra α-continuous, $f^{-1}(V)$ is α-open in (X, τ). Since every α-open set is $\psi^*\alpha$-open and so $f^{-1}(V)$ is $\psi^*\alpha$-open. Hence f is contra $\psi^*\alpha$-continuous.

Example 3.5.4 Let X = Y = {a, b, c}, τ = {ϕ, {a, b}, X} and σ = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = b, f(b) = c, f(c) = a. Then f is contra $\psi^*\alpha$-continuous but not contra α-continuous, since for the closed set {c} in (Y, σ), f^{-1}({c}) = {b} is not α-open in (X, τ).

Proposition 3.5.5 Every totally $\psi^*\alpha$-continuous map is a contra $\psi^*\alpha$-continuous map but not conversely.

Proof : Let f : (X, τ) → (Y, σ) be a totally $\psi^*\alpha$-continuous map. Let V be any closed subset of (Y, σ). Then f^{-1}(V) is $\psi^*\alpha$-clopen in (X, τ). Hence f is contra $\psi^*\alpha$-continuous.

Example 3.5.6 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = { ϕ, {a}, {b},{a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = a, f(c) = b. Then f is contra $\psi^*\alpha$-continuous but not totally $\psi^*\alpha$-continuous, since for the closed set {b, c} in (Y, σ), f^{-1}({b, c}) = {a, c} is $\psi^*\alpha$-open but not $\psi^*\alpha$-closed in (X, τ).

Proposition 3.5.7 Every strongly $\psi^*\alpha$-continuous map is contra $\psi^*\alpha$-continuous but not conversely.

Proof: Follows from the definitions of strongly $\psi^*\alpha$-continuous and contra $\psi^*\alpha$-continuous maps.

Example 3.5.8 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = { ϕ, {a}, {b}, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = a, f(c) = b. Then f is contra $\psi^*\alpha$-continuous but not strongly $\psi^*\alpha$-continuous, since for the subset {b} in (Y, σ), f^{-1}({b}) = {c} is $\psi^*\alpha$-closed but not $\psi^*\alpha$-open in (X, τ).

Proposition 3.5.9 Every perfectly $\psi^*\alpha$-continuous map is contra $\psi^*\alpha$-continuous but not conversely.

Proof: Let f : (X, τ) → (Y, σ) be a perfectly $\psi^*\alpha$-continuous map. Let V be a closed set in (Y, σ). Since every closed set is $\psi^*\alpha$-closed, V is $\psi^*\alpha$-closed in (Y, σ). Since f is perfectly $\psi^*\alpha$-continuous, f^{-1}(V) is clopen in (X, τ). Since every open set is $\psi^*\alpha$-open, f^{-1}(V) is $\psi^*\alpha$ -open in (X, τ). Hence f is contra $\psi^*\alpha$-continuous.

Example 3.5.10 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = {ϕ, {a}, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = b, f(b) = c, f(c) = a. Then f is contra $\psi^*\alpha$-continuous but not perfectly $\psi^*\alpha$-continuous, since for the $\psi^*\alpha$-closed set {c} in (Y, σ), f^{-1}({c}) = {b} is open but not closed in (X, τ).

Remark 3.5.11 The following examples show that contra $\psi^*\alpha$-continuous map and $\psi^*\alpha$- continuous map are independent.

Example 3.5.12 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = {ϕ, {a}, {b}, {a, b}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = a, f(c) = b. Then f is contra $\psi^*\alpha$-continuous but not $\psi^*\alpha$-continuous, since {c} is closed in (Y, σ), but f^{-1}({c}) = {a} is not $\psi^*\alpha$-closed in (X, τ).

Example 3.5.13 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = {ϕ, {a, c}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = a, f(c) = b. Then f is $\psi^*\alpha$-continuous but not contra $\psi^*\alpha$-continuous, since {b} is closed in (Y, σ), but f^{-1}({b}) = {c} is not $\psi^*\alpha$-open in (X, τ).

Theorem 3.5.14 A map f : (X, τ) → (Y, σ) is contra $\psi^*\alpha$-continuous if and only if $f^{-1}(U)$ is $\psi^*\alpha$-closed in (X, τ) for every open set U in (Y, σ).

Proof : (Necessity) Let f : (X, τ) → (Y, σ) be contra $\psi^*\alpha$-continuous and U be any open set in (Y, σ). Then Y – U is closed in (Y, σ). Since f is contra $\psi^*\alpha$-continuous, $f^{-1}(Y - U) = X - f^{-1}(U)$ is $\psi^*\alpha$-open in (X, τ) which implies that $f^{-1}(U)$ is $\psi^*\alpha$-closed in (X, τ).

(Sufficiency): Let V be any closed set in (Y, σ). Then Y – V is open in (Y, σ). By assumption, $f^{-1}(Y - V) = X - f^{-1}(V)$ is $\psi^*\alpha$-closed in (X, τ) which implies that $f^{-1}(V)$ is $\psi^*\alpha$-open in (X, τ). Hence f is contra $\psi^*\alpha$-continuous.

Proposition 3.5.15 If f : (X, τ) → (Y, σ) is a contra $\psi^*\alpha$-continuous map and g : (Y, σ) → (Z, η) is a continuous map, then g∘f : (X, τ) → (Z, η) is a contra $\psi^*\alpha$-continuous map

Proof: Let V be any closed set in (Z, η). Then $g^{-1}(V)$ is closed in (Y, σ) as g is continuous. Since f is contra $\psi^*\alpha$-continuous, $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $\psi^*\alpha$-open in (X, τ). Hence g∘f is contra $\psi^*\alpha$-continuous.

Proposition 3.5.16 Let f : (X, τ) → (Y, σ) be perfectly $\psi^*\alpha$-continuous and g : (Y, σ) → (Z, η) be contra $\psi^*\alpha$-continuous then g∘f : (X, τ) → (Z, η) is totally $\psi^*\alpha$-continuous.

Proof: Since every closed set is $\psi^*\alpha$-closed, the result follows.

Remark 3.5.17 The composition of two contra $\psi^*\alpha$-continuous maps need not be a contra $\psi^*\alpha$-continuous map as seen from the following example.

Example 3.5.18 Let X = Y = Z = {a, b, c}, τ = {ϕ, {a}, {a, b}, X}, σ = {ϕ, {a}, {b}, {a, b}, Y} and η= {ϕ, {a}, Z}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = a, f(c) = b and g : (Y, σ) → (Z, η) be a map defined by g(a) = c, g(b) = b, g(c) = a. Then the maps f and g are contra $\psi^*\alpha$-continuous but their composition g∘f : (X, τ) → (Z, η) is not contra $\psi^*\alpha$-continuous, since {b, c} is closed in (Z, η), whereas $(g\circ f)^{-1}(\{b, c\})$ = {b, c} is not $\psi^*\alpha$-open in (X, τ).

Remark 3.5.19 The interrelations between various forms of $\psi^*\alpha$-continuities are given in the following diagram.