Chapter 6

λ_g^α-Homeomorphisms in Topological Spaces

6.1 Introduction

A mapping that is one-to-one between sets such that both the map and its inverse are continuous i.e., bi-continuous is a homeomorphism and that in topology it exists for geometric figures which can be transformed from one into the other by an elastic deformation. More formally, in topology two spaces are said to be equivalent if there exists a homeomorphism between them. Several researchers have formulated the generalization of homeomorphisms on the initiation of the g-homemorphisms and gc-homeomorphisms in topological spaces by Maki et al. [1991]. Later, on the introduction of $\tilde{g}$-closed sets, the notion $\tilde{g}$-homeomorphisms were defined and characterized by Caldas et al. [2009]. Quite recently, Gilbert Rani and Pious Missier [2011], Padma et al. [2015], Delcia and Punitha Tharani [2021] have studied Λ^λ-homeomorphisms, Q^*-homeomorphisms, $g^*\alpha$-homeomorphisms respectively. Moreover, they all have also established important theorems and fundamental properties with the respective mappings.

In this chapter, a very peculiar form of homeomorphisms called as λ_g^α-homeomorphisms and $\lambda_g^\alpha r$-homeomorphisms are defined using λ_g^α-continuity and λ_g^α-irresoluteness. Later, it was found that the developed $\lambda_g^\alpha r$-homeomorphism forms a group under the relation – composition of mappings and it also induces an isomorphism between two groups formed by the topology.

6.2. λ_g^α-Homeomorphisms

The forthcoming section defines, analyzes and investigates the importance of λ_g^α-homeomorphisms and $\lambda_g^\alpha \mathcal{r}$-homeomorphisms in topological spaces.

Definition 6.2.1 A bijective map $u: (M, \mu) \to (N, \nu)$ is called **λ_g^α-homeomorphism** if it is both λ_g^α-open and λ_g^α-continuous. In other words, a bijective map u is a λ_g^α-homeomorphism if both u and $u^{-1}$are λ_g^α-continuous.

Example 6.2.2 Let $M = N = \{i, j, k\}, \mu = \{\phi, \{i\}, \{i, j\}, \{i, k\}, M\}$ and $\nu = \{\phi, \{i\}, \{j, k\}, N\}$. Let $u: (M, \mu) \to (N, \nu)$ be the map defined by $u(i) = j, u(j) = k$ and $u(k) = i$. Then u is a λ_g^α-homeomorphism.

Proposition 6.2.3 Every homeomorphism $u: (M, \mu) \to (N, \nu)$ is a λ_g^α-homeomorphism.

Proof: By definition u is bijective, open and continuous. Using Proposition 3.2.4 and Proposition 5.3.3, u is both λ_g^α-continuous and λ_g^α-open. Hence by Definition 6.2.1 u is λ_g^α-homeomorphism.

The subsequent example shows that the converse of Proposition 6.2.3 does not hold good.

Example 6.2.4 Let $M = N = \{i, j, k\}$, $\mu = \{\phi, \{i\}, \{j, k\}, M\}$ and $\nu = \{\phi, \{i\}, \{j\}, \{i, j\}, N\}$. Let $u: (M, \mu) \to (N, \nu)$ be a map defined by $u(i) = j, u(j) = i$ and $u(k) = k$. Then u is a λ_g^α-homeomorphism but not a homeomorphism, since for the open set $\{i, j\}$ in (N, ν) $u^{-1}(\{i, j\}) = \{i, j\}$ is not open in (M, μ).

Theorem 6.2.5 Let $u: (M, \mu) \to (N, \nu)$ be a bijective λ_g^α-continuous map, then the following statements are equivalent. $u: (M, \mu) \to (N, \nu)$ is a

(i) λ_g^α-open map

(ii) λ_g^α-homeomorphism

(iii) λ_g^α-closed map

Proof:

(i) ⇒ (ii) Assume (i), then u is bijective and λ_g^α-continuous, which proves (ii)

(ii) ⇒ (iii) Assume (ii) and using Proposition 5.3.9

(iii) ⇒ (i) Obvious from Proposition 5.3.9

Theorem 6.2.6 If a bijective map $u: (M, \mu) \rightarrow (N, \nu)$ is both open and λ_g^α-irresolute, then it is a λ_g^α-homeomorphism.

Proof: From the Proposition 5.3.3 and Proposition 3.3.4. we have u is both λ_g^α-open and λ_g^α-continuous. Hence u is a λ_g^α-homeomorphism.

The subsequent example shows that the converse of Proposition 6.2.6 does not hold good.

Example 6.2.7 Let $M = N = \{i, j, k\}$, $\mu = \{\phi, \{i\}, \{i, j\}, M\}$ and $\nu = \{\phi, \{i\}, \{j, k\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be the identity map. Then the map u is a λ_g^α-homeomorphism but it is neither open nor λ_g^α-irresolute, since for the open set $\{i, j\}$ in (M, μ), $u(\{i, j\}) = \{i, j\}$ is not open in (N, ν) and also for the λ_g^α-closed set $\{i, k\}$ in (N, ν), $u^{-1}(\{i, k\}) = \{i, k\}$ is not λ_g^α-closed in (M, μ).

Theorem 6.2.8 If a bijective map $u: (M, \mu) \rightarrow (N, \nu)$ is both λ-open and λ-continuous then it is a λ_g^α-homeomorphism.

Proof: Let S be an open set in (M, μ). As u is λ-open, $u(S)$ is λ-open in (N, ν). By Proposition 2.4.3, $u(S)$ is λ_g^α-open. Therefore u is a λ_g^α-open map.

Let T be an open set in (N,ν). As u is λ-continuous, $u^{-1}(T)$ is λ-open in (M,μ). By Proposition 2.4.3, $u^{-1}(T)$ is λ_g^α-open. Therefore u is a λ_g^α-continuous map. Hence u is a λ_g^α-homeomorphism.

The subsequent example shows that the converse of Proposition 6.2.8 does not hold good.

Example 6.2.9 Let $M=N=\{i,j,k\}$, $\mu=\{\phi,\{i,j\},M\}$ and $\nu=\{\phi,\{i\},\{j,k\},N\}$. Let $u:(M,\mu)\to(N,\nu)$ be a map defined by $u(i)=i, u(j)=k$ and $u(k)=j$. Then u is a λ_g^α-homeomorphism but it is neither λ-open nor λ-continuous, since for the open set $\{i,j\}$ in (M,μ), $u(\{i,j\})=\{i,k\}$ is not λ-open in (N,ν) and also for the closed set $\{i\}$ in (N,ν), $u^{-1}(\{i\})=\{i\}$ is not λ-closed in (M,μ).

Theorem 6.2.10 If a bijective map $u:(M,\mu)\to(N,\nu)$ is both λ-open and λ-irresolute then it is a λ_g^α-homeomorphism.

Proof: Follows from the Proposition 5.3.4. and Proposition 3.2.11.

The subsequent example shows that the converse of Theorem 6.2.10 does not hold good.

Example 6.2.11 Consider M,N,μ,ν and u as in Example 6.2.9, then u is a λ_g^α-homeomorphism but it is neither λ-open nor λ-irresolute, since for the λ-closed set $\{j,k\}$ in (N,ν), $u^{-1}(\{j,k\})=\{j,k\}$ is not λ-closed in (M,μ).

Theorem 6.2.12 If a bijective map $u:(M,\mu)\to(N,\nu)$ and its inverse $u^{-1}:(N,\nu)\to(M,\mu)$ are both λ_g^α-irresolute then it is a λ_g^α-homeomorphism.

Proof: Let S be an open set in (M,μ). Using the Proposition 2.4.3, S is a λ_g^α-open set in (M,μ). Since u^{-1} is λ_g^α-irresolute, $(u^{-1})^{-1}(S)=u(S)$ is λ_g^α-open in (N,ν). Thus u is a λ_g^α-open map.

Let T be an open set in (N,ν). Then T is also a λ_g^{α}-open set in (N,ν). Since u is λ_g^{α}-irresolute, $u^{-1}(T)$ is λ_g^{α}-open in (M,μ). Thus u is a λ_g^{α}-continuous map. Hence u is a λ_g^{α}-homeomorphism.

The subsequent example shows that the converse of Proposition 6.2.12 does not hold good.

Example 6.2.13 Let $M=N=\{i,j,k,l,m\}$, $\mu=\{\phi,\{i\},\{j\},\{i,j\},\{j,k\},\{i,j,k\},\{j,k,l\},\{i,j,k,l\},\{j,k,l,m\},M\}$ and $\nu=\{\phi,\{i\},\{j\},\{i,j\},\{i,j,m\},\{i,j,k,m\},\{i,j,l,m\},N\}$. Let $u:(M,\mu)\to(N,\nu)$ be the identity map. Then u is a λ_g^{α}-homeomorphism but neither u nor u^{-1} is λ_g^{α}-irresolute, since for the λ_g^{α}-closed set $\{i,k\}$ in (M,μ), $(u^{-1})^{-1}(\{i,k\})=u(\{i,k\})=\{i,k\}$ is not λ_g^{α}-closed in (N,ν) and also for the λ_g^{α}-closed set $\{i,j,k,m\}$ in (N,ν), $u^{-1}(\{i,j,k,m\})=\{i,j,k,m\}$ is not λ_g^{α}-closed in (M,μ).

Definition 6.2.14 A bijective map $u:(M,\mu)\to(N,\nu)$ is called **$\lambda_g^{\alpha}r$-homeomorphism** if u is both strongly λ_g^{α}-open and λ_g^{α}-irresolute. In other words, a bijective map $u:(M,\mu)\to(N,\nu)$ is a $\lambda_g^{\alpha}r$-homeomorphism if both u and u^{-1} are λ_g^{α}-irresolute.

The family of all $\lambda_g^{\alpha}r$-homeomorphisms onto itself is denoted by $\lambda_g^{\alpha}r\text{-}H(M,\mu)$.

Example 6.2.15 Let $M=N=\{i,j,k\}$, $\mu=\{\phi,\{i\},\{j\},\{i,j\},\{i,k\},M\}$ and $\nu=\{\phi,\{i\},\{i,j\},\{i,k\},N\}$. Let $u:(M,\mu)\to(N,\nu)$ be a map defined by $u(i)=j, u(j)=k$ and $u(k)=i$. Then u is a $\lambda_g^{\alpha}r$-homeomorphism.

Proposition 6.2.16 Every $\lambda_g^{\alpha}r$-homeomorphism $u:(M,\mu)\to(N,\nu)$ is a λ_g^{α}-homeomorphism.

Proof: By assumption, u is bijective, λ_g^{α}-irresolute and u^{-1} is λ_g^{α}-irresolute. Using Proposition 3.3.4, u and u^{-1} are λ_g^{α}-continuous and hence by definition, the proof follows.

The subsequent example shows that the converse of Proposition 6.2.16 does not hold good.

Example 6.2.17 Consider M, N, μ and ν as in Example 6.2.13. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = i, u(j) = j, u(k) = l, u(l) = k$ and $u(m) = m$. Then u is a λ_g^α-homeomorphism but neither u nor u^{-1} is λ_g^α-irresolute, since for the λ_g^α-closed set $\{i, k\}$ in (M, μ), $(u^{-1})^{-1}(\{i, k\}) = u(\{i, k\}) = \{i, l\}$ is not λ_g^α-closed in (N, ν) and also for the λ_g^α-closed set $\{i, j, k, m\}$ in (N, ν), $u^{-1}(\{i, j, k, m\}) = \{i, j, l, m\}$ is not λ_g^α-closed in (M, μ).

Remark 6.2.18 $\lambda_g^\alpha \mathscr{r}$-homeomorphism and homeomorphism are independent of each other as continuity and λ_g^α-irresoluteness are independent concepts, as the following examples prove.

Example 6.2.19 Let $M = N = \{i, j, k, l\}$, $\mu = \{\phi, \{i\}, \{i, j\}, M\}$ and $\nu = \{\phi, \{i, j\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = k, u(j) = j, u(k) = i$ and $u(l) = l$. Then u is continuous but not λ_g^α-irresolute, since for the λ_g^α-closed set $\{i, k, l\}$ in (N, ν), $u^{-1}(\{i, k, l\}) = \{i, k, l\}$ is not λ_g^α-closed in (M, μ).

Example 6.2.20 Let $M = N = \{i, j, k, l\}$, $\mu = \{\phi, \{i, j, k\}, M\}$ and $\nu = \{\phi, \{i\}, \{i, j\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = i, u(j) = k, u(k) = j$ and $u(l) = l$. Then u is λ_g^α-irresolute but not continuous, since for the closed set $\{k, l\}$ in (N, ν), $u^{-1}(\{k, l\}) = \{j, l\}$ is not closed in (M, μ).

Theorem 6.2.21 For a $\lambda_g^\alpha \mathscr{r}$-homeomorphism $u: (M, \mu) \rightarrow (N, \nu)$, the equality $\lambda_g^\alpha cl\big(u^{-1}(T)\big) = u^{-1}(\lambda_g^\alpha cl(T))$ holds for all λ_g^α-closed sets T in (N, ν).

Proof: Assume that u is a $\lambda_g^\alpha \mathscr{r}$-homeomorphism, which implies it is also a λ_g^α-irresolute map. When $\lambda_g^\alpha cl(T)$ is a λ_g^α-closed set in (N, ν), for $T \subseteq N$, then $u^{-1}(\lambda_g^\alpha cl(T))$ is λ_g^α-closed in (M, μ). Since $T \subseteq \lambda_g^\alpha cl(T)$, $u^{-1}(T) \subseteq u^{-1}(\lambda_g^\alpha cl(T))$. Therefore, $\lambda_g^\alpha cl\big(u^{-1}(T)\big) \subseteq u^{-1}(\lambda_g^\alpha cl(T))$.

Conversely, by hypothesis u^{-1} is a λ_g^α-irresolute map. When $\lambda_g^\alpha cl\big(u^{-1}(T)\big)$ is a λ_g^α-closed set in (M, μ), $(u^{-1})^{-1}(\lambda_g^\alpha cl\big(u^{-1}(T)\big)) = u(\lambda_g^\alpha cl(u^{-1}(T)))$ is λ_g^α-closed in (N, ν).

Since $T \subseteq (u^{-1})^{-1}\big(u^{-1}(T)\big) \subseteq (u^{-1})^{-1}(\lambda_g^{\alpha}cl\big(u^{-1}(T)\big)) = u(\lambda_g^{\alpha}cl\big(u^{-1}(T)\big))$. Therefore $\lambda_g^{\alpha}cl(T) \subseteq u(\lambda_g^{\alpha}cl\big(u^{-1}(T)\big)) \Rightarrow u^{-1}(\lambda_g^{\alpha}cl(T)) \subseteq \lambda_g^{\alpha}cl\big(u^{-1}(T)\big)$. Thus, the equality holds.

Corollary 6.2.22 For a $\lambda_g^{\alpha}r$-homeomorphism $u: (M, \mu) \to (N, \nu)$, the equality $\lambda_g^{\alpha}cl\big(u(T)\big) = u(\lambda_g^{\alpha}cl(T))$ holds for all λ_g^{α}-closed sets T in (N, ν).

Proof: Since u is a $\lambda_g^{\alpha}r$-homeomorphism, u^{-1} is also a $\lambda_g^{\alpha}r$-homeomorphism. Hence using Theorem 6.2.21, $\lambda_g^{\alpha}cl\big((u^{-1})^{-1}(T)\big) = (u^{-1})^{-1}(\lambda_g^{\alpha}cl(T))$, for all λ_g^{α}-closed sets T in (N, ν). Hence the proof follows.

Corollary 6.2.23 For a $\lambda_g^{\alpha}r$-homeomorphism $u: (M, \mu) \to (N, \nu)$, the equality $\lambda_g^{\alpha}int\big(u(S)\big) = u(\lambda_g^{\alpha}int(S))$ holds for all λ_g^{α}-open sets S in (M, μ).

Proof: Using the fact that "$\lambda_g^{\alpha}int(S) = M \setminus [\lambda_g^{\alpha}cl(M \setminus S)]$" and using the Corollary 6.2.22 the proof follows.

Corollary 6.2.24 For a $\lambda_g^{\alpha}r$-homeomorphism $u: (M, \mu) \to (N, \nu)$, the equality $\lambda_g^{\alpha}int\big(u^{-1}(T)\big) = u^{-1}(\lambda_g^{\alpha}int(T))$ holds for all λ_g^{α}-open sets T in (N, ν).

Proof: Since u is $\lambda_g^{\alpha}r$-homeomorphism, u^{-1} is also a $\lambda_g^{\alpha}r$-homeomorphism and by Corollary 6.2.23 the proof follows.

Remark 6.2.25 The composition of two λ_g^{α}-homeomorphisms need not be a λ_g^{α}-homeomorphism as described in the next example.

Example 6.2.26 Let $M = N = K = \{i, j, k\}$, $\mu = \{\phi, \{i\}, \{i, j\}, M\}$, $\nu = \{\phi, \{i\}, \{i, j\}, \{i, k\}, N\}$ and $\kappa = \{\phi, \{i\}, \{j, k\}, K\}$. Let $u: (M, \mu) \to (N, \nu)$ be the identity map, $w: (N, \nu) \to (K, \kappa)$ be a map defined by $w(i) = j, w(j) = i$ and $w(k) = k$. Then both u and w are λ_g^{α}-homeomorphisms but their composition $(w \circ u): (M, \mu) \to (K, \kappa)$ is not a λ_g^{α}-homeomorphism, since for the open set $\{i\}$ in (K, κ), $(w \circ u)(\{i\}) = w\big(u(\{i\})\big) = \{j\}$ is not λ_g^{α}-open in (M, μ). Therefore $(w \circ u)$ is not a λ_g^{α}-open map.

Theorem 6.2.27 The composition of two homeomorphisms is a λ_g^{α}-homeomorphism.

Proof: Let $u: (M, \mu) \to (N, \nu)$ and $w: (N, \nu) \to (K, \kappa)$ be two homeomorphisms. Let Q be the open set in (K, κ). By assumption, w is continuous and so $w^{-1}(Q)$ is an open set in (N, ν). Also u is continuous and therefore $u^{-1}\big(w^{-1}(Q)\big) = (w \circ u)^{-1}(Q)$ is an open set in (M, μ). By Proposition 2.4.3, $(w \circ u)^{-1}(Q)$ is a λ_g^{α}-open set in (M, μ). Hence $(w \circ u): (M, \mu) \to (K, \kappa)$ is λ_g^{α}-continuous.

Let S be an open set S in (M, μ). By assumption, u is an open map and so $u(S)$ is an open set in (N, ν). Also w is an open map and therefore $w(u(S)) = (w \circ u)(S)$ is an open set in (K, κ). By Proposition 2.4.3, $(w \circ u)(S)$ is also a λ_g^{α}-open set in (K, κ) and hence $(w \circ u): (M, \mu) \to (K, \kappa)$ is a λ_g^{α}-open map.

Since u and w are bijective maps, $(w \circ u)$ is also a bijective map. Hence $(w \circ u)$ is a bijective, λ_g^{α}-open and λ_g^{α}-continuous map. Therefore $(w \circ u)$ is a λ_g^{α}-homeomorphism.

Theorem 6.2.28 The composition of two $\lambda_g^{\alpha}\mathscr{r}$-homeomorphisms is a $\lambda_g^{\alpha}\mathscr{r}$-homeomorphism.

Proof: Let $u: (M, \mu) \to (N, \nu)$ and $w: (N, \nu) \to (K, \kappa)$ be two $\lambda_g^{\alpha}\mathscr{r}$-homeomorphisms. Let T be the λ_g^{α}-closed set in (K, κ). Consider $(w \circ u)^{-1}(T) = u^{-1}(w^{-1}(T)) = u^{-1}(S)$, where $S = w^{-1}(T)$. By assumption, S is a λ_g^{α}-closed set in (N, ν) and hence $u^{-1}(S)$ is a λ_g^{α}-closed set in (M, μ). Thus $(w \circ u): (M, \mu) \to (K, \kappa)$ is λ_g^{α}-irresolute.

Let E be a λ_g^{α}-closed set in (M, μ). Consider $[(w \circ u)^{-1}]^{-1}(E) = (w \circ u)(E) = w(u(E)) = w(F)$ where $F = u(E)$. By assumption $w(F)$ is a λ_g^{α}-closed set in (N, ν) and $w(u(E))$ is a λ_g^{α}-closed set in (K, κ). Therefore $(w \circ u)^{-1}: (K, \kappa) \to (M, \mu)$ is λ_g^{α}-irresolute. Hence $(w \circ u)$ is $\lambda_g^{\alpha}\mathscr{r}$-homeomorphism.

Theorem 6.2.29 If $u: (M,\mu) \to (N,\nu)$ and $w: (N,\nu) \to (K,\kappa)$ are bijective maps such that u is open, λ_g^{α}-continuous and w is a homeomorphism then their composition $(w \circ u): (M,\mu) \to (K,\kappa)$ is a λ_g^{α}-homeomorphism.

Proof: Let S be the open set in (K,κ). From assumption, $w^{-1}(S)$ is open in (N,ν) and $(w \circ u)^{-1}(S) = u^{-1}(w^{-1}(S))$ is open. By Proposition 2.4.3, $(w \circ u)^{-1}(S)$ is a λ_g^{α}-open set in (M,μ). Therefore, $(w \circ u)$ is λ_g^{α}-continuous.

Let T be the open set in (M,μ). From assumption, $u(T)$ is open in (N,ν) and $(w \circ u)(T) = w(u(T))$ is open in (K,κ). By Proposition 2.4.3, $(w \circ u)(T) = w(u(T))$ is λ_g^{α}-open in (K,κ). Therefore $(w \circ u)$ is λ_g^{α}-open. Also, since u and w are bijective maps, $(w \circ u)$ is also a bijective map. Hence $(w \circ u)$ is a λ_g^{α}-homeomorphism.

Remark 6.2.30 The above theorem holds good even if u is open and λ_g^{α}-irresolute.

Theorem 6.2.31 The set $\lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$ forms a group under the operation composition of mappings.

Proof: Consider the binary operation $\star: [\lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu) \times \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)] \to \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$ defined by $(p \star q) = (p \circ q)\ \forall\, p, q \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$, where $\circ$ is the usual operation of composition of mappings. Using Theorem 6.2.28, $(p \circ q) \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$, this satisfies the closure property. Associative property holds, since the composition of mappings is always associative. The identity map $I: (M,\mu) \to (M,\mu)$ is a $\lambda_g^{\alpha}\mathscr{r}$-homeomorphism and $I \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$. Also, $p \circ I = I \circ p = p$, $\forall\, p \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$. For any $p \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$, $p^{-1} \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$ and $p \circ p^{-1} = p^{-1} \circ p = I$. Hence the set $\lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$ forms a group under composition of mappings.

Theorem 6.2.32 Let $u: (M,\mu) \to (N,\nu)$ be a $\lambda_g^{\alpha}\mathscr{r}$-homeomorphism, then u induces an isomorphism from the group $\lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$ onto the group $\lambda_g^{\alpha}\mathscr{r}\text{-}H(N,\nu)$.

Proof: Let $\vartheta_u : \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu) \to \lambda_g^{\alpha}\mathscr{r}\text{-}H(N,\nu)$ be a map defined by $\vartheta_u(p) = (u^{-1} \circ p \circ u)$, $\forall\, p \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$.

For $p_1 \neq p_2$, if $\vartheta_u(p_1) = \vartheta_u(p_2)$ then $(u^{-1} \circ p_1 \circ u) = (u^{-1} \circ p_2 \circ u)$. Hence $p_1 = p_2$, which contradicts the assumption. Therefore ϑ_u is one-to-one.

For $q \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(N,\nu)$, choose $p \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$, such that $\vartheta_u(p) = (u^{-1} \circ p \circ u) = q$. That is $p = (u \circ p \circ u^{-1}) \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$. Therefore ϑ_u is onto and hence ϑ_u is a bijective map.

For all $p_1, p_2 \in \lambda_g^{\alpha}\mathscr{r}\text{-}H(M,\mu)$, $\vartheta_u(p_1 \circ p_2) = (u^{-1} \circ (p_1 \circ p_2) \circ u) = (u^{-1} \circ p_1 \circ u) \circ (u^{-1} \circ p_2 \circ u) = \vartheta_u(p_1) \circ \vartheta_u(p_2)$. Hence ϑ_u is a homomorphism. Thus ϑ_u is an isomorphism induced by u.

Theorem 6.2.33 $\lambda_g^{\alpha}\mathscr{r}$-homeomorphism is an equivalence relation in the collection of all topological spaces.

Proof: Let $\mathfrak{D}$ be the collection of all topological spaces. For $(M,\mu),(N,\nu) \in \mathfrak{D}$, define a relation $\mathcal{R}$ on $\mathfrak{D}$ such that $(M,\mu)\,\mathcal{R}\,(N,\nu)$ if and only if (M,μ) and (N,ν) are $\lambda_g^{\alpha}\mathscr{r}$-homeomorphism to each other.

Reflexivity: $(M,\mu)\,\mathcal{R}\,(M,\mu)$ is obvious from the relation defined.

Symmetricity: $(M,\mu)\,\mathcal{R}\,(N,\nu) \Rightarrow (N,\nu)\,\mathcal{R}\,(M,\mu)$

Transitivity: follows from Theorem 6.2.28.

Hence the proof.