A STUDY ON CERTAIN GRAPHS UNDER SPHERICAL FUZZY ENVIRONMENT

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Degree of Master of Science (M.Sc.)

By

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DECLARATION

DECLARATION

I declare that the thesis entitled **"A Study On Certain Graphs under Spherical Fuzzy Environment "** submitted by me for the degree of **Master of Science (M. Sc.)** is the record of work carried out by me during the period from December 2022 to May 2023 under the guidance of **Dr.K. Akalyadevi, M.Sc., M.Phil., Ph.D.,** Assistant Professor and Head, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, and has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship, Titles in this institute or any other University or other similar institution of Higher Learning.

Signature of the Candidate

Signature of the Supervisor

CERTIFICATE

CERTIFICATE

I certify that the thesis entitled **"A Study On Certain Graphs under Spherical Fuzzy Environment"** submitted for the degree of **Master of Science (M. Sc.)** by Ms. Abirami.V, is the record of research work carried out by her during the period from December 2022 to May 2023 under my guidance and supervision, and that this work has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or other Titles in this institute or any other University or institution of Higher Learning.

Signature of the

Head of the Department

Signature of the Supervisor

Signature of the Director

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SYNOPSIS

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Chapter-I deals with an Introduction of graph theory, Review of literature and discussed some of the important definitions like Fuzzy Set, Intuitionistic Fuzzy Set, Pythagorean Fuzzy Set, Spherical Fuzzy Set, Fuzzy Graph, Intutionistic Fuzzy Graph, Pythagorean Fuzzy Graph and Spherical Fuzzy Graph.

Chapter-II deals with the extended concepts of Pythagorean Fuzzy Environment and also described Regular Maximal Product in Pythagorean Fuzzy Set and Regular Residue Product in Pythagorean Fuzzy Set as well as included some operations such as Connectedness, Completeness and Strongness.

Chapter-III deals with new ideas of Spherical Fuzzy Environment, Which is an extension of Pythagorean Fuzzy Environment. We discussed the combination of two Spherical Fuzzy Graphs using Products namely Maximal Product and Residue Product and also included some operations such as Connectedness, Completeness and strongness.

CHAPTER 1

CHAPTER 1

1.1 INTRODUCTION

Fuzzy sets were introduced by Zadeh in 1965. The notion of fuzzy set theory has caused great interest among both pure and applied mathematics. This day's fuzzy set hypothesis has arisen as a likely space of interdisciplinary exploration. It has fruitful applications in different fields as a phenomenal apparatus for addressing human information and discernment.

Atanassov (1983) introduced the concept of intuitionistic fuzzy set as a generalisation of fuzzy sets. Intuitionistic fuzzy models provide more precision, flexibility and compatibility to the system compared to the fuzzy models and he also added new components that determine the degree of non-membership in the definition of fuzzy set. The fuzzy sets give the degree of membership, while intuitionistic fuzzy sets give both the degree of membership and the degree of non-membership, which are more or less independent from each other; the only requirement is that the sum of these two degrees is not greater than one. Intuitionistic fuzzy sets have been applied in a wide variety of fields, including computer science, engineering, mathematics, medicine, chemistry, and economics.

Atanassov (1989) introduced new results on intuitionistic fuzzy sets and also defined two new operations on intuitionistic fuzzy sets and their basic properties. Young et al. (2005) using the notion intuitionistic fuzzy sets, the concept of intuitionistic fuzzy semi-preopen sets and intuitionistic fuzzy semi-precontinuous mappings are introduced.

Yager, proposed a brand-new extension of fuzzy set called Pythagorean fuzzy set (PFS), which has been successfully applied in many fields for decision making procedures. PFS is characterized by a membership and non-membership function satisfies the condition that the square sum of membership and non-membership is less than or equal to one.

Spherical fuzzy set is a generalization of picture fuzzy set and Pythagorean fuzzy set. There is a need of spherical fuzzy set to tackle an interesting scenario emerge when picture fuzzy sets and Pythagorean fuzzy sets both failed to handle.

We can study the neutral degree in spherical fuzzy set where as in Pythagorean fuzzy sets and picture fuzzy sets it doesn't. In spherical fuzzy set, membership degrees are gratifying the condition $0 \leq P^2(x) + I^2(x) + N^2(x) \leq 1$.

In Mathematics, Graph theory is the investigation of graphs, which are numerical designs used to display the relationship between two or more objects or with set of vertices and edges. Mathematics plays an essential role in our day to day life. The graph theory origin can be traced back to Euler's work on the Konigsberg bridges problem in 1735.

Kaufmann (1973) gave first definition of fuzzy graph based on Zadeh's fuzzy relations. But it was Rosenfeld who laid the foundations for fuzzy graph theory in 1975. Generalization of fundamental ideas of graph theory like paths, cycles, trees, connectedness, and their properties to fuzzy graph theory have been done by Rosenfeld. Fuzzy graph models can address the complex, imprecise and uncertain problems where classical graph models may fail. Thus, a fuzzy graph representation is more appropriate to reality than crisp graph representation. Fuzzy graph theory has applications in the modern science and technology especially in the fields of information theory, neural network, expert systems, cluster analysis, medical diagnosis, control theory, etc. Bhattacharya (1987) introduced the notions of eccentricity and centre on the fuzzy graph.

The idea of intuitionistic fuzzy graph was created by Atanassov in 1994. Intuitionistic fuzzy graphs discover more extensive applications in clinical science, the executive's science, designing, software engineering and so forth. In the whole application field, it helps in demonstrating constant framework where the data acquired changes with various degrees of accuracy. Parvathi and Karunambigai in 2006 gave another meaning of intuitionistic fuzzy graphs and discuss some properties. Further investigated different procedure on min-max intuitionistic fuzzy graphs. Akram and Alshehri in 2014 made a new understanding on bends, scaffolds and cutnodes of an intuitionistic fuzzy graph.

Recently, Naz et al. (2018) initially presented the idea of Pythagorean Fuzzy Graph (PFG). Akram et al. (2018) introduced certain notions, including intuitionistic fuzzy Graphs of 3-Type (IFGs3T), Intuitionistic Fuzzy Graphs of 4-

Type (IFGs4T), and Intuitionistic Fuzzy Graphs of n-Type (IFGsnT), and proved that every IFG (n-1)T is an IFGsnT, and also discussed the application of Pythagorean fuzzy graphs in decision making.

Spherical Fuzzy Graph deals with many real time issues and to modify the bounding constraint. The motto of this research is to merge two Spherical Fuzzy Graphs using products namely Maximal Product and Residue Product. Therefore, we discuss with the standard properties such as connectedness, completeness and strongness with appropriate illustrations.

Chapter-I deals with an introduction of graph theory review of literature and discussed some of the important definitions like Fuzzy set, Intuitionistic Fuzzy Set, Pythagorean Fuzzy Set, Spherical Fuzzy Set, Fuzzy Graph, Intutionistic Fuzzy Graph, Pythagorean Fuzzy Graph and Spherical Fuzzy Graph.

Chapter-II deals with the extended concepts of Pythagorean Fuzzy Environment and also described Regular Maximal Product in Pythagorean Fuzzy Set and Regular Residue Product in Pythagorean Fuzzy Set as well as included some operations such as Connectedness, Completeness and Strongness.

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1.2 REVIEW OF LITERATURE

In 1965, Zadeh was introduced a fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterised by a membership (characteristic) function that assigns to each object a grade of membership ranging between zero and one.

In 1983, Atanassov defined the concept Intuitionistic Fuzzy Set (IFS) as a generalization of the concept fuzzy set. Various properties are proved, which are connected to the operations and relations over sets, and with modal and topological operators, defined over the set of IFS's. In 1989, Atanassov defined new results on Intuitionistic fuzzy sets. Two new operators on intuitionistic fuzzy sets are defined and their basic properties are studied. Krassimir T. Atanassov (1998) was established the intuitionistic fuzzy set (IFS) which is generalization of the fuzzy set. Later he extended various concepts and properties of intuitionistic fuzzy set.

In 2005, Young et al. using the notion of intuitionistic fuzzy sets, the concepts of intuitionistic fuzzy semi-preopen sets and intuitionistic fuzzy semi-precontinuous mappings were introduced. The relation between an intuitionistic fuzzy precontinuous mapping and an intuitionistic semi-precontinuous mapping is given. Characterizations of intuitionistic fuzzy semi-preopen sets and intuitionist fuzzy semi-precontinuous mappings are given.

Yager (2013) introduced a Pythagorean fuzzy set (PFS), whose origin from great philosophers, mathematicians named Pythagoras. Yager suggests that PFS is useful for decision making purpose.

In 2014, Yager introduced a class of nonstandard Pythagorean fuzzy subsets whose membership grades are pairs (a,b) satisfying the requirement $a^2 + b^2 \leq 1$. They also introduce a variety of aggregation operations for these Pythagorean fuzzy subsets. Then look at multicriteria decision making in the case where the criteria satisfaction is expressed using Pythagorean membership grades. The issue of having to choose the best alternative in multicriteria decision making leads us to consider the problem of comparing Pythagorean membership grades.

Garg (2018) explored an idea of Pythagorean fuzzy sets in decision-making. Later he proposed concept of Pythagorean fuzzy set (PFS) has been extended to interval-valued Pythagorean fuzzy set and hesitant Pythagorean fuzzy set.

In 2019, Xindong Peng et al. present an overview of the Pythagorean fuzzy set with the aim of offering a clear perspective on the different concepts, tools, and trends related to their extension. In particular, we provide two novel algorithms for decision-making problems in a Pythagorean fuzzy environment. It may serve as a foundation for developing more algorithms in decision-making.

In 1735, graph theory was first introduced by Swiss mathematician Leonhard Euler by solving a Konigsberg bridge problem.

In 1973, Kaufmann gave first definition of fuzzy graph based on Zadeh's fuzzy relations. In 1975, Rosenfeld developed the theory of fuzzy graphs and also defined generalization of fundamental ideas of graph theory like paths, cycles, trees, connectedness, and their properties to fuzzy graph theory.

In 1987, Bhattacharya introduces the notions of eccentricity and center, and their examples indicate that results from (crisp) graph theory do not always have analogues for fuzzy graphs.

In 2006, Atanassov et al. discuss a new generalization of the Intuitionistic Fuzzy Graphs (IFGs), using as a basis the concepts of the Intuitionistic Fuzzy Sets (IFSs), Intuitionistic Fuzzy Relations (IFRs), and Index Matrices (IMs) and their basic concepts.

In 2006, Parvathi et al. introduced a new definition for intuitionistic fuzzy graphs and some properties of intuitionistic fuzzy graphs are considered and the authors introduced the notions of various concepts. These concepts are analysed through suitable illustrations. In 2007, Karunambigai et al. present a model based on dynamic programming to find the shortest paths in intuitionistic fuzzy graphs.

In 2018, Naz et al. proposed the Pythagorean Fuzzy Graph (PFG). They investigate some properties of our proposed graphs. We determine the degree and total degree of a vertex of PFGs. Moreover, they introduce the concept of Pythagorean Fuzzy Preference Relations (PFPRs). In particular, we solve decision-making problems, including hospital evaluations, partner selection in supply chain management, and electronic learning main factors evaluation, by using PFGs.

In 2018, Garg proposes an improved score function for solving Multi-Criteria Decision-Making (MCDM) problems with partially known weight information. In it, the preferences related to criteria are taken in the form of interval-valued Pythagorean fuzzy sets. Based on these preferences and an improved score function, a score matrix has been formulated and then a linear programming-based method has been proposed to solve MCDM problems with unknown attribute weights. Some generalised properties have also been proven with justification. Illustrative examples have been given to show the superiority of the approach over the other existing functions in the decision-making process.

Muhammad Akram et al. (2019) extended the definition of Pythagorean fuzzy sets (PFS) to Pythagorean fuzzy graphs (PFG) and its properties, and studied the regularity of Pythagorean fuzzy graph (PFG) product. The Pythagorean fuzzy graph is an extension of intuitionistic fuzzy graph and it provides an accurate value for the problem which is vague.

In 2019, Akram et al. defined the concept of Pythagorean fuzzy sets for graphs and then combined two Pythagorean Fuzzy Graphs (PFGs) using two new graph products, namely, the maximal product and the residue product. The author investigates the regularity of these products. Moreover, it discusses some eminent properties such as strongness, connectedness, and completeness. Further, it proposes some necessary and sufficient conditions for G1*G2 and G1.G2 to be regular. Finally, decision-making problems concerning the best company for investment and alliance partner selection of a software company are solved by better understanding PFGs.

In 2019, Xindong Peng et al. present an overview of the Pythagorean fuzzy set with the aim of offering a clear perspective on the different concepts, tools, and trends related to their extension. In particular, we provide two novel algorithms for decision-making problems in a Pythagorean fuzzy environment. It may serve as a foundation for developing more algorithms in decision-making.

In 2020, Akalyadevi et al. introduced spherical fuzzy graph in bipolar environment and discussed the operation on bipolar spherical fuzzy graphs namely, symmetric difference and rejection with brief description on degree and total degree of bipolar spherical fuzzy graphs.

In 2020, Akalyadevi et al. Compared to fuzzy set and all other versions of fuzzy set, neutrosophic sets can handle imprecise information in a more effective way. A Neutrosophic cubic set, which is the generalization of neutrosophic set, are more flexibile as well as compatible to the system compared to other existing fuzzy models. On other hand, graph is a very easy way to understand and handle a problem physically in the form of diagrams. We introduce spherical fuzzy neutrosophic cubic graph and single-valued neutrosophic spherical cubic graphs in bipolar setting and discuss some of their properties such as Cartesian product, composition, m-join, n-join, m-union, n-union. We also present a numerical example of the defined model which depicts the advantage of the same. Finally, we define a score function and minimum spanning tree algorithm of an undirected bipolar single-valued neutrosophic spherical cubic graph with a numerical example.

In 2020, Akalyadevi et al. proposed an algorithm for finding minimum spanning tree of an undirected bipolar graph where the edge lengths are represented by bipolar spherical fuzzy number. To construct the minimum spanning tree of undirected bipolar spherical fuzzy graph, a new algorithm and score function based on matrix approach has been introduced. The proposed method compare with some existing method are also discussed.

NOTATIONS

- ❖ FS = Fuzzy set
- ❖ FR = Fuzzy Relation
- ❖ IFS = Intuitionistic Fuzzy Set
- ❖ IFR = Intuitionistic Fuzzy Relation
- ❖ PFS = Pythagorean Fuzzy Set
- ❖ PFR = Pythagorean Fuzzy Relation
- ❖ SFS = Spherical Fuzzy Set
- ❖ SFR = Spherical Fuzzy Graph
- ❖ FG = Fuzzy Graph
- ❖ IFG = Intuitionistic Fuzzy Graph
- ❖ PFG = Pythagorean Fuzzy Graph
- ❖ SFG = Spherical Fuzzy Graph

1.3 PRELIMINIARIES

Definition 1.3.1

A **Fuzzy Set** (FS) on a universe Y is an object of the form represents the membership

$$m=\{\langle \eta, \alpha_m(\eta)\rangle \mid \eta \in Y\}$$

Where $\alpha_m : Y \to [0,1]$ represents the membership functions of m.

Definition 1.3.2

A Fuzzy set on $Y \times Y$ is said to be a **Fuzzy Relation** (FR) on Y, denoted by

$$n = \{\eta\lambda, \alpha_n(\eta\lambda) \mid uv \in Y \times Y\}$$

Where $\alpha_n : Y \times Y \to [0,1]$ represents the membership function of m.

Definition 1.3.3

An **Intuitionistic Fuzzy Set** (IFS) on a universe X is an object of the form

$$m=\{\langle \eta, \alpha_m(\eta), \beta_m(\eta)\rangle \mid \eta \in Y\},$$

Where $\alpha_m : Y \to [0,1]$ and $\beta_m : Y \to [0,1]$ represents the membership and non-membership functions of m, and α_m, β_m satisfies the condition $0 \le \alpha_m(\eta) + \beta_m(\eta) \le 1$ for all $\eta \in Y$.

Definition 1.3.4

An **Intuitionistic Fuzzy Relation** (IFR) $n = (\alpha_n(\eta,\lambda), \beta_n(\eta,\lambda))$ in an Universe $Y \times Y (m(Y \to Y))$ is an intutionistic fuzzy set of the form

$$n = \{\langle (\eta,\lambda), \alpha_n(\eta,\lambda), \beta_n(\eta,\lambda)\rangle \mid (u,v) \in Y \times Y\}$$

Where $\alpha_n : Y \times Y \to [0,1]$ and $\beta_n : Y \times Y \to [0,1]$. The Intuitionistic fuzzy relation m satisfies $0 \le \alpha_n(\eta,\lambda) + \beta_n(\eta,\lambda) \le 1$ for all $\eta, \lambda \in Y$.

Definition 1.3.5

A **Pythagorean Fuzzy Set** (PFS) on a universe Y is an object of the form

$$m=\{\langle \eta,\ \alpha_m(\eta),\beta_m(\eta)\rangle \mid \eta \in Y\}$$

Where $\alpha_m : Y \to [0,1]$ and $\beta_m : Y \to [0,1]$ represents the membership and non-membership functions of m, and α_m, β_m satisfies the condition $0 \le \alpha_m^2(\eta)+\beta_m^2(\eta) \le 1$ for all $\eta \in Y$.

Definition 1.3.6

A Pythagorean Fuzzy Set n on $Y \times Y$ is said to be a **Pythagorean Fuzzy Relation (**PFR) on X, denoted by

$$n=\{\langle \eta\lambda, \alpha_n(\eta\lambda), \beta_n(\eta\lambda)\rangle \mid \eta\lambda \in Y \times Y\}$$

Where $\alpha_n : Y \times Y \to [0,1]$ and $\beta_n : Y \times Y \to [0,1]$ represents the membership function and non- membership functions of n, α_n, β_n satisfies the condition $0 \le \alpha_m^2(\eta\lambda)+\beta_m^2(\eta\lambda) \le 1$ for all $\eta\ \lambda \in Y \times Y$.

Definition 1.3.7

A **Spherical Fuzzy Set** (SFS) on a universe Y is an object of the form

$$m=\{\langle \eta,\ \alpha_m(\eta),\ \beta_m(\eta), \tau_m(\eta)\rangle \mid \eta \in Y\}$$

where $\alpha_m : Y \to [0,1]$, $\beta_m : Y \to [0,1]$ and $\tau_m : Y \to [0,1]$ represents the membership, non-membership and Indeterminancy functions of m, and α_m, β_m and τ_m satisfies the condition $0 \le \alpha_m^2(\eta)+\beta_m^2(\eta)+\tau_m^2(\eta) \le 1$ for all $\eta \in Y$.

Definition 1.3.8

A Spherical Fuzzy Set n on $Y \times Y$ is said to be **Spherical Fuzzy Relation** (SPR) on Y, denoted by

$$n=\{\langle \eta\lambda, \alpha_n(\eta\lambda), \beta_n(\eta\lambda), \tau_n(\eta\lambda)\rangle \mid \eta\lambda \in Y \times Y\}$$

$$\alpha_n : Y \times Y \to [0,1],\ \beta_n : Y \times Y \to [0,1],\ \tau_n : Y \times Y \to [0,1]$$

represents the membership ,non-membership and Indeterminancy of n and α_n, β_n and τ_n satisfies the condition $0 \le \alpha_n^2(\eta\lambda) + \beta_n^2(\eta\lambda) + \tau_n^2(\eta\lambda) \le 1$ for all $\eta\lambda \in Y \times Y$.

Definition 1.3.9

A **Graph** G is finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G, called edges. The vertex set and the edge set of G are respectively denoted by V (G) and E (G) is denoted by G= (V, E).

Definition 1.3.10

The **Degree** of a vertex v in a graph G is the number of edges of G Incident with v and is denoted by $\deg(v)$ and $d(v)$.

Definition 1.3.11

The cardinality of the vertex $V(G)$ is said to be the **Order** of the graph G.

Definition 1.3.12

A Graph in which each vertex has the same degree is a **Regular Graph**.

Definition 1.3.13

A Simple Graph in which each pair of distinct vertices is adjacent is a **Complete Graph**.

Definition 1.3.14

A graph is said to be **Connected** if there exists at least one path between every pair of vertices.

Definition 1.3.15

A Graph can split into an Independent Components. Then it is called as **Disconnected Graph**

Definition 1.3.16

A **Fuzzy Graph** (FG) on a non-empty set Y is a pair $g=(m,n)$ with m a FS on Y and n a fuzzy relation on Y such that

$$\alpha_n(\eta\lambda) \le \alpha_m(\eta) \wedge \alpha_m(\lambda)$$

For all $\eta, \lambda \in Y$, where $m : Y \rightarrow [0,1]$ and $n : Y \times Y \rightarrow [0,1]$.

Definition 1.3.17

An **Intuitionistic Fuzzy Graph** (IFG) on a non-empty set Y is a pair $g = (m, n)$ with m an IFS on Y and n an IFR on Y such that

$$\alpha_n(\eta\lambda) \leq \alpha_m(\eta) \wedge \alpha_m(\lambda),\ \beta_n(\eta\lambda) \geq \beta_m(\eta) \vee \beta_m(\lambda)$$

And $0 \leq \alpha_n(\eta\lambda) + \beta_n(\eta\lambda) \leq 1$ for all $\eta, \lambda \in Y$, where $\alpha_n : Y \times Y \rightarrow [0,1]$ and $\beta_n : Y \times Y \rightarrow [0,1]$ represents the membership function and non- membership functions of n, respectively.

Definition 1.3.18

A **Pythagorean Fuzzy Graph** (PFG) on a non-empty set Y is a pair $g = (m, n)$ with m a PFS on Y and n a PFR on Y such that

$$\alpha_n(\eta\lambda) \leq \alpha_m(\eta) \wedge \alpha_m(\lambda),\ \beta_n(\eta\lambda) \geq \beta_m(\eta) \vee \beta_m(\lambda)$$

and $0 \leq \alpha_n^2(\eta\lambda) + \beta_n^2(\eta\lambda) \leq 1$ for all $\eta, \lambda \in Y$, where $\alpha_n : Y \times Y \rightarrow [0,1]$ and $\beta_n : Y \times Y \rightarrow [0,1]$ represents the membership function and non- membership functions of n, respectively.

Definition 1.3.19

A **Spherical Fuzzy Graph** (SFG) on a non-empty set Y is a pair $g = (m, n)$ with m an SFS on Y and n an SFR on Y such that

$$\alpha_n(\eta\lambda) \leq \alpha_m(\eta) \wedge \alpha_m(\lambda),$$

$$\beta_n(\eta\lambda) \leq \beta_m(\eta) \wedge \beta_m(\lambda),$$

$$\tau_n(\eta\lambda) \geq \tau_m(\eta\lambda) \vee \tau_m(\eta\lambda)$$

and $0 \leq \alpha_n^2(\eta\lambda) + \beta_n^2(\eta\lambda) + \tau_n^2(\eta\lambda) \leq 1$ for all $\eta, \lambda \in Y$, where, $\alpha_n : Y \times Y \rightarrow [0,1]$ $\beta_n : Y \times Y \rightarrow [0,1]$ and $\tau_n : Y \times Y \rightarrow [0,1]$ represents the membership, non-membership and indeterminacy functions of n, respectively.

CHAPTER 2

CHAPTER – 2

Certain Graphs under Pythagorean Fuzzy Environment

2.1 Regular maximal product in Pythagorean fuzzy graphs

Definition 2.1.1

Let $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ be two PFGS of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. The maximal product of g_1 and g_2 is denoted by $g_1 * g_2 = (A_1 * A_2, B_1 * B_2)$ and defined as:

$$\text{(i)}\begin{cases}(\mu_{A_1} * \mu_{A_2})(u_1, u_2) = \mu_{A_1}(u_1) \vee \mu_{A_2}(u_2)\\ (\nu_{A_1} * \nu_{A_2})(u_1, u_2) = \nu_{A}(u_1) \wedge \nu_{A_2}(u_2)\\ \text{for all } (u_1, u_2) \in V_1 \times V_2\end{cases}$$

$$\text{(ii)}\begin{cases}(\mu_{\beta_1} * \mu_{\beta_2})((u, u_2)(u, v_2)) = \mu_{A_1}(u) \vee \mu_{\beta_2}(u_2 v_2)\\ (\nu_{\beta_1} * \nu_{\beta_2})((u, u_2)(u, v_2)) = \nu_{A_1}(u) \wedge \nu_{\beta_2}(u_2 v_2)\\ \text{for all } u \in V_1 \text{ and } u_2 v_2 \in_2\end{cases}$$

$$\text{(iii)}\begin{cases}(\mu_{\beta_1} * \mu_{\beta_2})((\mu_1, z)(v_1, z)) = \mu_{\beta_1}(u_1 v_1) \vee \mu_{A_2}(z)\\ (\nu_{\beta_1} * \nu_{\beta_2})((u_1, z)(v_1, z)) = \nu_{\beta_1}(u_1, v_1) \wedge \nu_{A_2}(z)\\ \text{for all } z \in V_2 \text{ and } u_1 v_1 \in E_1\end{cases}$$

Definition 2.1.2

A Pythagorean fuzzy graph $g = (A, B)$ is said to be a **Strong Pythagorean fuzzy graph** of underlying crisp graph $G = (V, E)$ if

$$\mu_\beta(uv) = \mu_A(u) \wedge \mu_A(v),$$
$$\nu_\beta(uv) = \nu_A(u) \vee \nu_A(v) \text{ for all } u\,v \in E$$

Example 2.1.3

Consider a graph $G = (V, E)$ where $V = \{a, b, c, d, e, f\}$ and $E = \{af, bf, cf, df, ef\}$. Let A and B be Pythagorean fuzzy vertex set and Pythagorean fuzzy edge set defined on V and $V \times V$ respectively.

$$A=\left\langle\left(\frac{a}{0.9},\frac{b}{0.25},\frac{c}{0.6},\frac{d}{0.5},\frac{e}{0.65},\frac{f}{0.4}\right)\times\left(\frac{a}{0.4},\frac{b}{0.8},\frac{c}{0.5},\frac{d}{0.7},\frac{e}{0.6},\frac{f}{0.7}\right)\right\rangle \text{and}$$

$$B=\left\langle\left(\frac{af}{0.4},\frac{bf}{0.25},\frac{cf}{0.4},\frac{df}{0.4},\frac{ef}{0.4}\right)\times\left(\frac{af}{0.7},\frac{bf}{0.8},\frac{cf}{0.7},\frac{df}{0.7},\frac{ef}{0.7}\right)\right\rangle$$

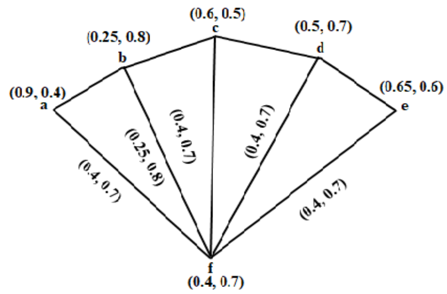

Figure 2.1

By routine calculation, one can see from fig that is a strong Pythagorean fuzzy graph.

Theorem 2.1.4

If $g_1=(A_1,B_1)$ and $g_2=(A_2,B_2)$ are two strong PFGs, then their maximal product is also a strong PFG.

Proof

Let $g_1=(A_1,B_1)$ and $g_2=(A_2,B_2)$ be two strong PFGs of the graph G_1 and G_2 respectively. Then,

$$\mu_{\beta_1}(u_1u_2)=\mu_{A_1}(u_1)\wedge\mu_{A_1}(u_2)$$

$$\mu_{B_2}(v_1v_2)=\mu_{A_2}(v_1)\wedge\mu_{A_2}(v_2)$$

$$\nu_{B_2}(v_1v_2)=\nu_{A_2}(v_1)\vee\nu_{A_2}(v_2) \text{ for all } v_1v_2\in E_2.$$

By definition of maximal product, we have if $u_1=u_2$ and $v_1v_2\in E_2$

$$(\mu_{B_1}*\mu_{B_2})((u_1,v_1)(u_2,v_2))$$

$$=\mu_{A_1}(u_1)\vee\mu_{\beta_2}(v_1v_2)$$

$$=\mu_{A_1}(u_1)\vee\{\mu_{A_2}(v_1)\wedge\mu_{A_2}(v_2)\}$$

$$=\{\mu_{A_1}(u_1)\vee\mu_{A_2}(v_1)\}\wedge\{\mu_{A_1}(u_1)\vee\mu_{A_2}(v_2)\}$$

$$=(\mu_{A_1}*\mu_{A_2})(u_1,v_1)\wedge(\mu_{A_1}*\mu_{A_2})(u_2,v_2)$$

$$(\nu_{B_1}*\nu_{B_2})((u_1,v_1)(u_2,v_2))$$

$$=\nu_{A_1}(u_1)\wedge\nu_{\beta_2}(v_1v_2)$$

$$=\nu_{A_1}(u_1)\wedge\{\nu_{A_2}(v_1)\vee\nu_{A_2}(v_2)\}$$

$$=\{\nu_{A_1}(u_1)\wedge\nu_{A_2}(v_1)\}\vee\{\nu_{A_1}(u_1)\wedge\nu_{A_2}(v_2)\}$$

$$=(\nu_{A_1}*\nu_{A_2})(u_1,v_1)\wedge(\nu_{A_1}*\nu_{A_2})(u_2,v_2).$$

If $v_1=v_2$ and $u_1u_2\in E_1$,

$$(\mu_{B_1}*\mu_{B_2})((u_1,v_1)(u_2,v_2))$$

$$=\mu_{B_1}(u_1u_2)\vee\mu_{A_2}(v_1)$$

$$=\{\mu_{A_1}(u_1)\wedge\mu_{A_1}(u_2)\}\vee\mu_{A_2}(v_1)$$

$$=\{\mu_{A_1}(u_1)\vee\mu_{A_2}(v_1)\}\wedge\{\mu_{A_1}(u_2)\vee\mu_{A_2}(v_1)\}$$

$$=(\mu_{A_1}*\mu_{A_2})(u_1,v_1)\wedge(\mu_{A_1}*\mu_{A_2})(u_2,v_2),$$

$$(\nu_{B_1}*\nu_{B_2})((u_1,v_1)(u_2,v_2))$$

$$=\nu_{B_1}(u_1u_2)\wedge\nu_{A_2}(v_1)$$

$$=\{\nu_{A_1}(u_1)\vee\nu_{A_1}(u_2)\}\wedge\nu_{A_2}(v_1)$$

$$=\{\nu_{A_1}(u_1)\wedge\nu_{A_2}(v_1)\}\vee\{\nu_{A_1}(u_2)\wedge\nu_{A_2}(v_1)\}$$

$$=(\nu_{A_1}*\nu_{A_2})(u_1,v_1)\vee(\nu_{A_1}*\nu_{A_2})(u_2,v_2).$$

Hence the maximal product g_1*g_2 of two strong PFGs is a strong PFG.

Remark 2.1.5

Converse the theorem may not be as it can be seen in the following example. Consider two Pythagorean fuzzy graphs $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ as shows in fig, their maximal product is given in fig,

Here

$\mu_{B_1}(u_1 u_2) \neq \mu_{A_1}(u_1) \wedge \mu_{A_1}(u_2)$

$\nu_{B_1}(u_1 u_2) \neq \nu_{A_1}(u_1) \vee \nu_{A_1}(u_2)$ and

$\mu_{B_1}(v_1 v_2) \neq \mu_{A_2}(v_1) \wedge \mu_{A_2}(v_2)$

$\nu_{B_2}(v_1 v_2) \neq \nu_{A_2}(v_1) \vee \nu_{A_2}(v_2)$

Hence, g_1 and g_2 are not strong PFGs.

But $(\mu_{B_1} * \mu_{B_2})((u,v)(x,y)) = \mu_{A_1}(u,v) \wedge \mu_{A_2}(x,y)$

$(\nu_{B_1} * \nu_{B_2})((u,v)(x,y)) = \nu_{A_1}(u,v) \vee \nu_{A_2}(x,y)$

For all edge $(u,v)\ (x,y)$ in E. Thus, their maximal product $g_1 * g_2$ is a strong PFG.

Definition 2.1.6

Let $g = (A, B)$ be a Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$. g is to be connected, if for every pair of vertices, there exist at least one non-zero path, that is, for all $u, v \in V$, the $\mu -$ strength of connectedness

$\mu^{\infty}(\eta, \lambda) = \text{Sup}\{\mu^{1}(u, v)$: for some 1}

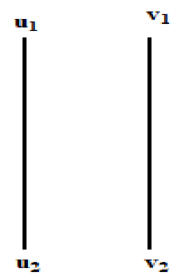

Figure 2.2

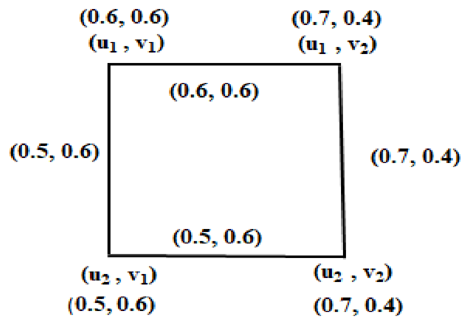

Figure 2.3

and ν - strength of connectedness

$$\nu^{\infty}(\eta,\lambda)=\inf\{\nu^{l}(u,v): \text{for some } l\}$$

Satisfy one of the following conditions:

$\mu^{\infty}(u,v)>0, \nu^{\infty}(u,v)>0$ and

or $\mu^{\infty}(u,v)=0, \nu^{\infty}(u,v)=0$ and

or $\mu^{\infty}(u,v)>0, \nu^{\infty}(u,v)=0$.

Where

$$\mu^{l}(u,v)=\alpha(u\,u_1)\wedge\mu(u_1\,u_2)\wedge.....\mu(u_{l-1}\,v)$$
$$\nu^{l}(\eta,\lambda)=\nu(u\,u_1)\wedge\nu(u_1\,u_2)\wedge.....\nu(u_{l-1}\,v) \text{ for some } l$$

Represents μ - strength of path and ν - strength of path length l, respectively.

Example 2.1.7

Consider a Pythagorean fuzzy graph $g=(A,B)$ as displayed in the following fig for $b,f\in V$. Using definition we have

$$\mu^5=\mu_B(b,c)\wedge\mu_B(c,i)\wedge\mu_B(i,h)\wedge\mu_B(h,g)\wedge\mu_B(g,f)$$
$$=0.6\wedge0.2\wedge0.4\wedge0.8\wedge0.4=0.2,$$

$$\mu^4=\mu_B(b,c)\wedge\mu_B(c,d)\wedge\mu_B(d,e)\wedge\mu_B(e,f)$$
$$=0.6\wedge0.5\wedge0.5\wedge0.5=0.5,$$

$$\mu^4 = \mu_B(b,c) \wedge \mu_B(c,d) \wedge \mu_B(d,g) \wedge \mu_B(g,f)$$
$$= 0.6 \wedge 0.5 \wedge 0.5 \wedge 0.4 = 0.4,$$

This implies

$$\mu^\infty(b,f) = \sup\{\mu^5(b,f), \mu^4(b,f), \mu^4(b,f), \mu^3(b,f)\}$$
$$= \sup\{0.2, 0.5, 0.4, 0.4\} = 0.5 > 0$$

and

$$\nu^5 = \nu_B(b,c) \vee \nu_B(c,i) \vee \nu_B(i,h) \vee \nu_B(h,g) \vee \nu_B(g,f)$$
$$= 0.7 \vee 0.9 \vee 0.8 \vee 0.5 \vee 0.7 = 0.9,$$

$$\nu^4 = \nu_B(b,c) \vee \nu_B(c,d) \vee \nu_B(d,e) \vee \nu_B(e,f)$$
$$= 0.4 \vee 0.7 \vee 0.8 \vee 0.7 = 0.8,$$

$$\nu^4 = \nu_B(b,c) \vee \nu_B(c,d) \vee \nu_B(d,g) \vee \nu_B(g,f)$$
$$= 0.4 \vee 0.7 \vee 0.6 \vee 0.7 = 0.7,$$

$$\nu^3 = \nu_B(b,c) \vee \nu_B(h,g) \vee \nu_B(g,f)$$
$$= 0.4 \vee 0.5 \vee 0.7 = 0.7.$$

This implies

$$\nu^\infty(b,f) = \inf\{\nu^5(b,f), \nu^4(b,f), \nu^4(b,f), \nu^3(b,f)\}$$
$$= \inf\{0.9, 0.8, 0.7, 0.7\} = 0.7 > 0$$

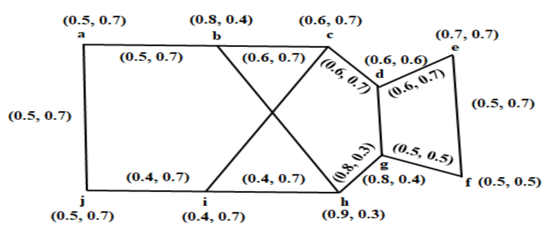

Figure 2.4

That is, there exist a non-zero path between b and f. Therefore, $g = (A, B)$ in fig is connected Pythagorean fuzzy graph.

Theorem 2.1.8

The maximal product $g_1 * g_2$ of two connected SFGs is always a connected SFG.

Proof

Let $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ be two connected Pythagorean Fuzzy Graphs with underlying crisp graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Let $V_1 = \{u_1, u_2, ...u_m\}$ and $V_2 = \{v_1, v_2, ...v_n\}$. Then $\mu_1^{\infty}(u_i\, u_j) > 0$, $\nu_1^{\infty}(u_i\, u_j) > 0$ for all $u_i, u_j \in V_1$ and $\mu_2^{\infty}(v_i\, v_j) > 0$, $\nu_2^{\infty}(v_i\, v_j) > 0$, for all $v_i, v_j \in V_2$.

The maximal product of $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ can be taken as $g = (A, B)$. Consider the k sub graphs of $g = (A, B)$ with the vertex set $V_2 = \{u_i\, v_1, u_i\, v_2, ..., u_i v_n\}$ for $i = 1, 2, 3......m$. Each of these sub graphs of $g = (A, B)$ is connected as u_i 's are same.

Since $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ are connected, each u_i and v_i are adjacent to at least one of the vertices in V_1 and V_2. Therefore, there exists at least one edge between any pair of the above k sub graphs

Thus, we have

$$\mu^{\infty}((u_i\, v_j)(u_k, v_l)) > 0,$$

$$\nu^{\infty}((u_i\, v_j)(u_k, v_l)) > 0$$

For all $(u_i, v_j)(u_k, v_l) \in E$

Hence, $g = (A, B)$ is a connected PFG.

Definition 2.1.9

A Spherical Fuzzy Graph $g = (A, B)$ is said to be a **Complete Spherical Fuzzy Graph** on underlying crisp graph $G = (V, E)$ if

$$\mu_B(u, v) = \mu_A(u) \wedge \mu_A(v),$$

$$\nu_B(u, v) = \nu_A(u) \vee \nu_A(v)$$

for all $u, v \in V$.

Example 2.1.10

Consider a graph $G=(V,E)$ where $V=\{a,b,c,d,e\}$ and

$E=\{ab,ac,ad,ae,bc,bd,be,cd,ce,de\}$. Let A and B be Pythagorean fuzzy vertex set and Pythagorean fuzzy edge set defined on V and $V\times V$, respectively.

$$A=\left\langle\left(\frac{a}{0.25},\frac{b}{0.8},\frac{c}{0.65},\frac{d}{0.4},\frac{e}{0.9}\right)\times\left(\frac{a}{0.8},\frac{b}{0.7},\frac{c}{0.6},\frac{d}{0.7},\frac{e}{0.4}\right)\right\rangle \text{ and}$$

$$B=\left\langle\left(\frac{ab}{0.4},\frac{ac}{0.25},\frac{ad}{0.4},\frac{ae}{0.4},\frac{bc}{0.4},\frac{bd}{0.4},\frac{be}{0.4},\frac{cd}{0.4},\frac{ce}{0.4},\frac{de}{0.4}\right)\right\rangle\times$$

$$\left\langle\left(\frac{ab}{0.7},\frac{ac}{0.8},\frac{ad}{0.7},\frac{ae}{0.7},\frac{bc}{0.7},\frac{bd}{0.4},\frac{be}{0.4},\frac{cd}{0.4},\frac{ce}{0.4},\frac{de}{0.4}\right)\right\rangle.$$

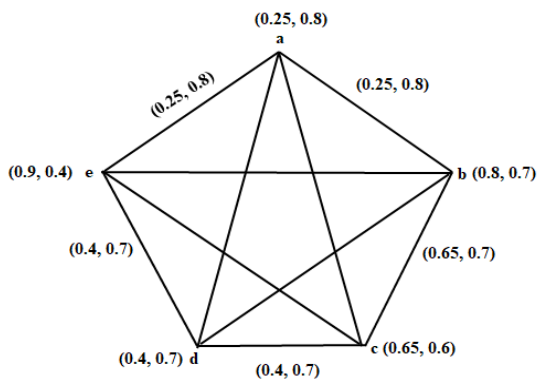

Figure 2.5

By routine calculation, one can see from fig that it is a complete Pythagorean Fuzzy Graph.

Remark 2.1.11

If two Pythagorean fuzzy graphs are complete, their maximal product may not be a complete PFG, as it can be seen in the following example. Consider two PFGs $g_1=(A_1,B_1)$ and $g_2=(A_2,B_2)$ on $V_1=\{u_1,u_2\}$ and $V_2=\{v_1,v_2\}$, respectively, as shown in fig. their maximal product $g_1 * g_2$ is shown in fig.

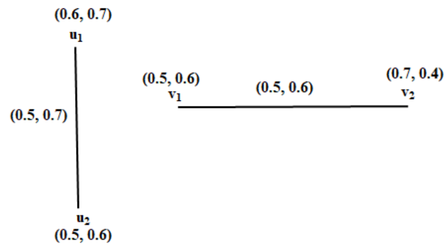

Figure 2.6

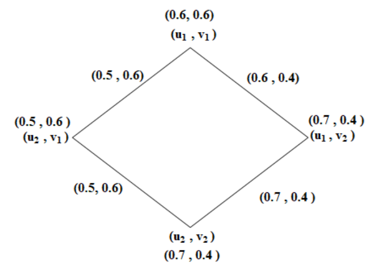

Figure 2.7

By routine calculation, one can seen from fig that g_1 and g_2 are complete PFGs. While notice that $g_1 * g_2$ is not a complete PFG, as the case $u_1 u_2 \in E_1$ and $v_1 v_2 \in E_2$ is not included in the definition of the maximal product. Further one can notice that the maximal product of two complete PFGs is a strong PFG. (Fig)

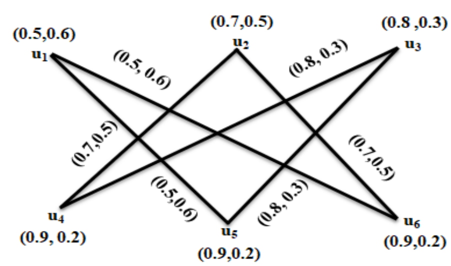

Figure 2.8

Definition 2.1.12

Let $g = (A, B)$ be a Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$. If

$$d_\mu(u) = \sum_{u, v \neq u \in V} \mu_B(u\,v) = k,$$

$$d_\nu(u) = \sum_{u, v \neq u \in V} \nu_B(u\,v) = l \quad \text{for all } u \in V,$$

Then g is said to be regular Pythagorean fuzzy graph of degree $(k,.l)$ or (k, l)-degree regular PFG.

Example 2.1.13

Consider a graph $G=(V,E)$ where $V=\{u_1,u_2,u_3,u_4,u_5,u_6\}$ and $E=\{u_1u_5,u_1u_6,u_2u_4,u_2u_6,u_3u_4,u_3u_5\}$. Let A and B be Pythagorean fuzzy vertex set and Pythagorean fuzzy edge set defined on V and $V\times V$ respectively.

$$A=\left\langle\left(\frac{u_1}{0.5},\frac{u_2}{0.7},\frac{u_3}{0.8},\frac{u_4}{0.9},\frac{u_5}{0.9},\frac{u_6}{0.9}\right)\times\left(\frac{u_1}{0.6},\frac{u_2}{0.5},\frac{u_3}{0.3},\frac{u_4}{0.2},,\frac{u_5}{0.2},\frac{u_6}{0.2}\right)\right\rangle$$

$$B=\left\langle\left(\frac{u_1u_5}{0.3},\frac{u_1u_6}{0.3},\frac{u_2u_4}{0.3},\frac{u_2u_6}{0.3},\frac{u_3u_4}{0.3},\frac{u_3u_5}{0.3}\right)\times\left(\frac{u_1u_5}{0.7},\frac{u_1u_6}{0.7},\frac{u_2u_4}{0.7},\frac{u_2u_6}{0.7},\frac{u_3u_4}{0.7},\frac{u_3u_5}{0.7}\right)\right\rangle$$

Since $d_\mu(u_i)=0.6$ and $d_\nu(u_i)=1.4$ for all $u_i\in V$ and $i=1,2,3,.....,6$, g is a regular Pythagorean fuzzy graph of degree (0.3, 0.7) or (0.3, 0.7) - regular PFG.

Remark 2.1.14

If $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ are two regular PFGs, then their maximal product G_1*G_2 may not be regular PFG as it can be seen in this example. Consider two PFGs $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ on $V_1=\{a,b\}$ and $V2=\{c,d,e,f\}$ respectively as shown in fig. Their maximal product G_1*G_2 is shown in fig

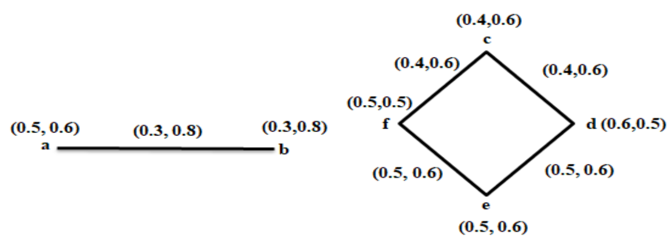

Figure 2.9

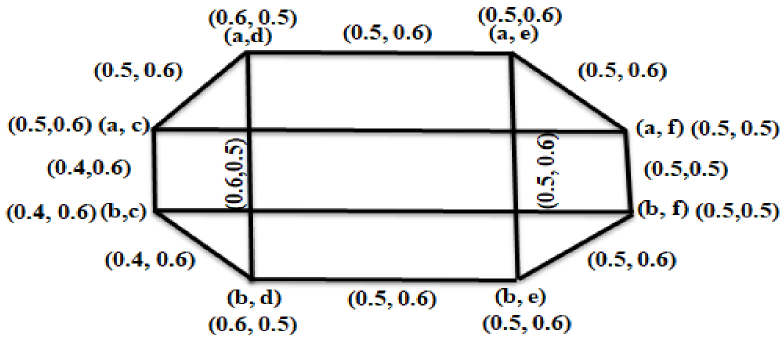

Figure 2.10

By routine calculation, one can see from fig that G_1 and G_2 are regular PFGs. While notice that

$$(d_\mu)_{G_1*G_2}(a,c) = \{\mu_{A_1}(a) \vee \mu_{\beta_2}(cd)\} + \{\mu_{A_1}(a) \vee \mu_{\beta_2}(cf)\} + \{\mu_{\beta_1}(ab) \vee \mu_{A_2}(c)\} = 1.4,$$
$$(d_\nu)_{G_1*G_2}(a,c) = \{\nu_{A_1}(a) \wedge \nu_{\beta_2}(cd)\} + \{\nu_{A_1}(a) \wedge \nu_{\beta_2}(cf)\} + \{\nu_{\beta_1}(ab) \wedge \nu_{A_2}(c)\} = 1.8,$$
$$(d_\mu)_{G_1*G_2}(b,f) = \{\mu_{A_1}(b) \vee \mu_{\beta_2}(fc)\} + \{\mu_{A_1}(b) \vee \mu_{B_2}(fe)\} + \{\mu_{A_1}(ab) \vee \mu_{A_2}(f)\} = 1.2$$
$$(d_\nu)_{G_1*G_2}(b,f) = \{\nu_{A_1}(b) \wedge \nu_{B_2}(fc)\} + \{\nu_{A_1}(b) \wedge \nu_{B_2}(fe)\} + \{\nu_{B_1}(ab) \wedge \mu_{A_2}(f)\} = 1.7$$

That is $(d)_{G_1*G_2}(a,c) \neq (d)_{G_1*G_2}(b,f)$. Therefore, $G_1 * G_2$ is not regular PFG.

Definition 2.1.15

Let $G = (A, B)$ be a Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$. Then, G is said to be a partially regular Pythagorean fuzzy graph if $G = (V, E)$ is a regular graph.

Example 2.1.16

Consider a Pythagorean fuzzy graph $g = (A, B)$ as displayed in the fig. Since $d_\mu(a) = 1.65 \neq 2 = d_\mu(a)$ and $d_\nu(a) = 2.35 \neq 1.8 = d_\nu(b)$. Hence, G is not a regular Pythagorean fuzzy graph but G is a regular graph as the degree of each vertex is equal. Thus, G is a partially regular Pythagorean fuzzy graph.

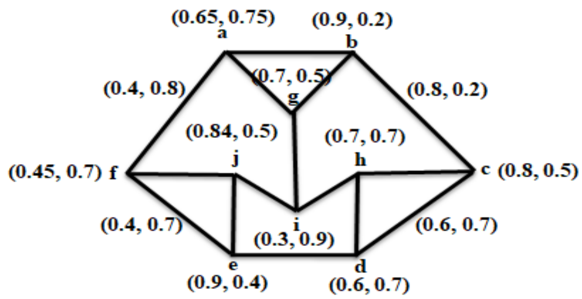

Figure 2.11

The following theorems explain the conditions for the maximal product of two regular PFGs to be regular.

Theorem 2.1.17

If $g_1 = (A_1, B_1)$ is a partially regular PFG and $g_2 = (A_2, B_2)$ is a PFG such that $\mu_{A_1} \leq \mu_{B_2}, \nu_{A_1} \geq \nu_{B_2}$ and μ_{A_2}, ν_{A_2} are constant functions of values c_1 and c_2 respectively, then their maximal product is regular if and only if $g_2 = (A_2, B_2)$ is regular PFG.

Proof

Let $g_1 = (A_1, B_1)$ be a partially regular PFG such that $G_1 = (V_1, E_1)$ is r-regular and $g_2 = (A_2, B_2)$ be any PFG with $\mu_{A_1} \leq \mu_{B_2}, \nu_{A_1} \geq \nu_{B_2}$ then $\mu_{A_2} \geq \mu_{B_1}, \nu_{A_2} \leq \nu_{B_1}$ and μ_{A_2}, ν_{A_2} are constant functions of values c_1 and c_2 respectively.

Now assume that $g_2 = (A_1, B_1)$ is a (k, l)-regular PFG. Then the degree of any vertex in maximal product is given by

$$(d_\mu)_{g_1 * g_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2) \in E_1 \times E_2} (\mu_{B_1} * \mu_{B_2})((u_1, u_2)(v_1, v_2))$$

$$= \sum_{u_1 = v_1, u_2 v_2 \in E_2} \mu_{A_1}(u_1) \vee \mu_{B_2}(u_2 v_2) + \sum_{u_2 = v_2, u_1 v_1 \in E_1} \mu_{B_1}(u_1 v_1) \vee \mu_{A_2}(u_2)$$

$$= \sum_{u_1 = v_1, u_2 v_2 \in V_2} \mu_{B_2}(u_2 v_2) + \sum_{u_2 = v_2, u_1 v_1 \in E_1} \mu_{A_2}(u_2)$$

$$= (d_\mu) g_2(u_2) + d_{G_1}(u_1) \mu_{A_2}(u_2)$$

$= rc_1 + k.$

$$(d_\nu)_{g_1 * g_2}(u_1, u_2) = \sum_{(u_1,u_2)(v_1,v_2) \in E_1 \times E_2} (\nu_{B_1} * \nu_{B_2})((u_1, u_2)(v_1, v_2))$$

$$= \sum_{u_1 = v_1, u_2 v_2 \in E_2} \nu_{A_1}(u_1) \vee \nu_{B_2}(u_2 v_2) + \sum_{u_2 = v_2, u_1 v_1 \in E_1} \nu_{B_1}(u_1 v_1) \vee \nu_{A_2}(u_2)$$

$$= \sum_{u_1 = v_1, u_2 v_2 \in V_2} \nu_{B_2}(u_2 v_2) + \sum_{u_2 = v_2, u_1 v_1 \in E_1} \nu_{A_2}(u_2)$$

$$= (d_\mu) g_2(u_2) + d_{G_1}(u_1) \nu_{A_2}(u_2)$$

$$= rc_2 + l.$$

This is constant for all vertices in $V_1 \times V_2$. Hence $G_1 * G_2$ is a regular PFG.

Conversely, assume that $G_1 * G_2$ is a regular PFG. Then, for any two vertices (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$,

$$(d_\mu)_{g_1 * g_2}(u_1, v_1) = (d_\mu)_{g_1 * g_2}(u_2, v_2)$$

$$(d_\mu) g_2(v_1) + d_{G_1}(u_1) \mu_{A_2}(v_1) = (d_\mu) g_2(v_2) + d_{G_1}(u_2) \mu_{A_2}(v_2)$$

$$rc_1 + (d_\mu)_{g_2}(v_1) = rc_1 + (d_\mu)_{g_2}(v_2)$$

$$(d_\mu)_{g_2}(v_1) = (d_\mu)_{g_2}(v_2)$$

$$(d_\nu)_{g_1 * g_2}(u_1, v_1) = (d_\nu)_{g_1 * g_2}(u_2, v_2)$$

$$(d_\nu) g_2(v_1) + d_{G_1}(u_1) \nu_{A_2}(v_1) = (d_\nu) g_2(v_2) + d_{G_1}(u_2) \nu_{A_2}(v_2)$$

$$rc_2 + (d_\nu)_{g_2}(v_1) = rc_1 + (d_\nu)_{g_2}(v_2)$$

$$(d_\nu)_{g_2}(v_1) = (d_\nu)_{g_2}(v_2)$$

This is true for all vertices in g_2. Hence, g_2 is regular PFG.

Theorem 2.1.18

If $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ are two partially regular PFG such that $\mu_{A_1} \geq \mu_{B_2}$, $\nu_{A_1} \geq \nu_{B_2}$, $\mu_{A_2} \geq \mu_{B_1}$, $\nu_{A_2} \geq \nu_{B_1}$ and μ_{A_2}, ν_{A_2} are constant functions of values c_1 and c_2 respectively, then their maximal product if and only if μ_{A_1} and ν_{A_1} are constant functions.

Proof

Let $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ are two partially regular PFG such that $\mu_{A_1} \geq \mu_{B_2}, \nu_{A_1} \leq \nu_{B_2}$, $\mu_{A_2} \geq \mu_{B_1}, \nu_{A_2} \leq \nu_{B_1}$ and μ_{A_2}, ν_{A_2} are constant functions of values c_1 and c_2, respectively, with G_i is r_i - regular, $i = 1, 2$. Now assume that μ_{A_1} and ν_{A_1} are constant functions of values c_3 and c_4, respectively. Then, the degree of any vertex in maximal product is given by,

$$(d_\mu)_{g_1 * g_2}(u_1, u_2) = \sum_{(u_1, u_2)(v_1, v_2) \in E_1 \times E_2} (\mu_{B_1} * \mu_{B_2})((u_1, u_2)(v_1, v_2))$$

$$= \sum_{u_1 = v_1, u_2 v_2 \in E_1 \times E_2} \mu_{A_1}(u_1) \vee \mu_{B_2}(u_2 v_2) + \sum_{u_2 = v_2, u_1 v_1 \in E_1} \mu_{B_1}(u_1 v_1) \vee \mu_{A_2}(\mu_2)$$

$$= \sum_{u_1 = v_1, u_2 v_2 \in E_2} \mu_{A_1}(u_1) + \sum_{u_2 = v_2, u_1 v_1 \in E_1} \mu_{A_2}(u_2)$$

$$= d_{G_2}(u_2)\mu_{A_1}(u_1) + d_{G_1}(u_1)\mu_{A_2}(u_2)$$

$$= r_1 c_1 + r_2 c_3,$$

$$(d_\nu)_{g_1 * g_2}(u_1, u_2) = \sum_{(u_1, u_2)(v_1, v_2) \in E_1 \times E_2} (\nu_{B_1} * \nu_{B_2})((u_1, u_2)(v_1, v_2))$$

$$= \sum_{u_1 = v_1, u_2 v_2 \in E_1 \times E_2} \nu_{A_1}(u_1) \wedge \nu_{B_2}(u_2 v_2) + \sum_{u_2 = v_2, u_1 v_1 \in E_1} \nu_{B_1}(u_1 v_1) \wedge \nu_{A_2}(\mu_2)$$

$$= \sum_{u_1 = v_1, u_2 v_2 \in E_2} \nu_{A_1}(u_1) + \sum_{u_2 = v_2, u_1 v_1 \in E_1} \nu_{A_2}(u_2)$$

$$= d_{G_2}(u_2)\nu_{A_1}(u_1) + d_{G_1}(u_1)\nu_{A_2}(u_2)$$

$$= r_1 c_2 + r_2 c_4.$$

This is a constant for all vertices in $V_1 \times V_2$. Hence, $g_1 * g_2$ is a regular PFG.

Conversely, assume that $g_1 * g_2$ is a regular PFG. Then, for any two vertices (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$,

$$(d_\mu)_{g_1 * g_2}(u_1, v_1) = (d_\mu)_{g_1 * g_2}(u_2, v_2)$$

$$d_{G_1}(u_1)\mu_{A_2}(v_1) + d_{G_1}(v_1)\mu_{A_1}(u_1) = d_{G_1}(u_2)\mu_{A_2}(v_2) + d_{G_2}(v_2)\mu_{A_1}(u_2)$$

$$rc_1 + r_2\,\mu_{A_1}(u_1) = rc_1 + r_2\,\mu_{A_1}(u_2)$$

$$\mu_{A_1}(u_1) = \mu_{A_1}(u_2).$$

$$(d_\nu)_{g_1 * g_2}(u_1, v_1) = (d_\nu)_{g_1 * g_2}(u_2, v_2)$$

$$d_{G_1}(u_1)\nu_{A_2}(v_1) + d_{G_1}(v_1)\nu_{A_1}(u_1) = d_{G_1}(u_2)\nu_{A_2}(v_2) + d_{G_2}(v_2)\,\nu_{A_1}(u_2)$$

$$rc_1 + r_2\,\nu_{A_1}(u_1) = rc_1 + r_2\,\nu_{A_1}(u_2)$$

$$\nu_{A_1}(u_1) = \nu_{A_1}(u_2).$$

This is true for all vertices in g_1. Hence, μ_{A_1} and ν_{A_1} are constant functions.

Definition 2.1.19

Let $g = (A, B)$ be a Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$. Then g is said to be a **full regular Pythagorean fuzzy graph** if g is both regular and partially regular graph.

Example 2.1.20

Consider a Pythagorean fuzzy graph $g = (A, B)$ as displayed in fig. Since $d_\mu(u_i) = 1.5$ and $d_\nu(u_i) = 2$ for all $u_i \in V$, where i=1,…6. Hence, g is a regular Pythagorean fuzzy graph of degree (1.5,2). Also, G is a regular graph as the degree of each vertex is equal. Thus, g is a full regular Pythagorean fuzzy graph.

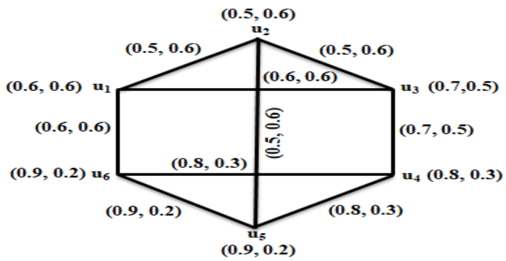

Figure 2.12

Remark 2.1.21

If two Pythagorean fuzzy graphs are full regular, their maximal product may not be full regular PFG as it can be seen in this example. Consider two PFGs $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ on $V_1 = \{a, b\}$ and $V_2 = \{c, d, e\}$ respectively, as shown in fig. Their maximal product $g_1 * g_2$ is shown in fig.

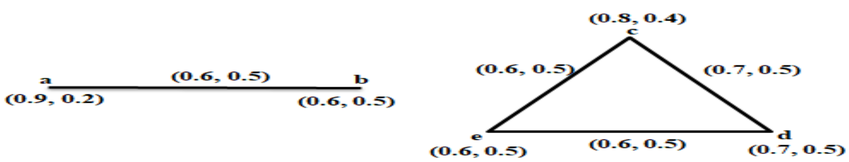

Figure 2.13

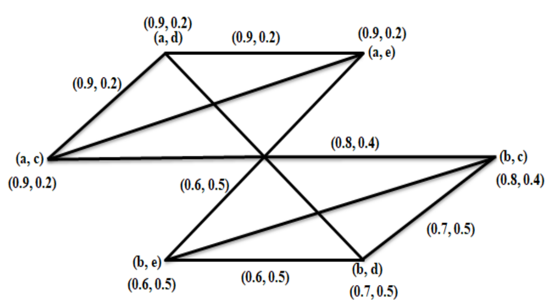

Figure 2.14

By routine calculation, one can see that g_1 and g_2 are full regular PFGs. But $(d)_{g_1 * g_2}(a, d) \neq (d)_{g_1 * g_2}(b, d)$. Hence $g_1 * g_2$ is not full regular PFG.

Remark 2.1.21

The maximal product of two regular PFGs on complete graphs is partially regular. Consider two PFGs $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ on $V_1 = \{u_1, v_1\}$ and $V_2 = \{u_2, v_2\}$ respectively as shown in fig. Their maximal product $g_1 * g_2$ is shown in fig.

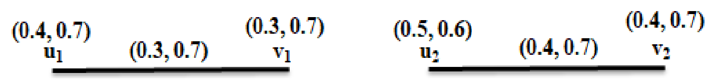

Figure 2.15

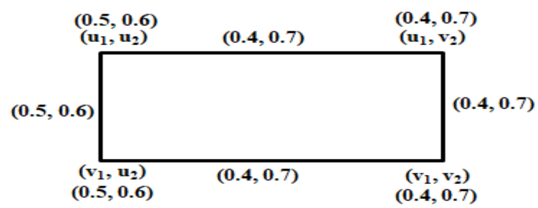

Figure 2.16

By routine calculation, one can see that g_1 and g_2 are regular PFFGs. But $(d)_{g_1 * g_2}(u_1, u_2) \neq (d)_{g_1 * g_2}(v_1, v_2)$. Hence $g_1 * g_2$ is a partially regular PFG as crisp graph is regular.

2.2 Regular residue product in Pythagorean fuzzy graphs

Definition 2.2.1

Let $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ be two PFGS of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. The residue product of g_1 and g_2 is denoted by $g_1 \cdot g_2 = (A_1 \cdot A_2, B_1 \cdot B_2)$ and defined as:

$$\text{(i)} \begin{cases} (\mu_{A_1} \cdot \mu_{A_2})(u_1, u_2) = \mu_{A_1}(u_1) \vee \mu_{A_2}(u_2) \\ (\nu_{A_1} \cdot \nu_{A_2})(u_1, u_2) = \nu_{A}(u_1) \wedge \nu_{A_2}(u_2) \\ \text{for all } (u_1, u_2) \in V_1 \times V_2 \end{cases}$$

$$\text{(ii)} \begin{cases} (\mu_{\beta_1}\cdot\mu_{\beta_2})((u_1,u_2)(v_1,v_2)) = \mu_{B_1}(u_1v_1) \\ (\nu_{\beta_1}\cdot\nu_{\beta_2})((u_1,u_2)(v_1,v_2)) = \nu_{B_1}(u_1v_1) \\ \text{for all } u_1v_1 \in E_1, u_2 \neq v_2. \end{cases}$$

Remark 2.2.2

In general, the residue product of two strong PFGs may not be strong PFG as it is explained in this example. Consider two Pythagorean fuzzy graphs $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ as shows in fig,

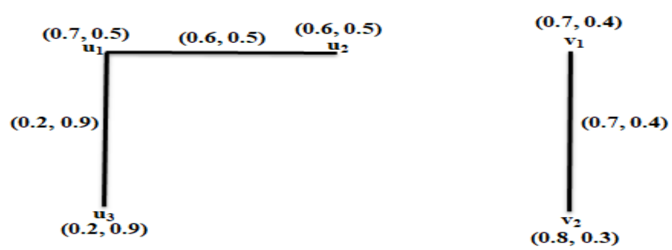

Figure 2.17

Here

$\mu_{B_1}(u_1\,u_2) = \mu_{A_1}(u_1) \wedge \mu_{A_1}(u_2)$

$\nu_{B_1}(u_1\,u_2) = \nu_{A_1}(u_1) \vee \nu_{A_1}(u_2)$ for all $u_1\,u_2 \in E_1$

and

$\mu_{B_1}(v_1\,v_2) = \mu_{A_2}(v_1) \wedge \mu_{A_2}(v_2)$

$\nu_{B_2}(v_1\,v_2) = \nu_{A_2}(v_1) \vee \nu_{A_2}(v_2)$

Hence, g_1 and g_2 are strong PFGs.

But $(\mu_{B_1}\cdot\mu_{B_2})((u_1,v_2)(u_2,v_1)) = 0.6 \neq \mu_{A_1}(u_1,v_2) \wedge \mu_{A_2}(u_2,v_1) = 0.7$ and

$(\nu_{B_1}\cdot\nu_{B_2})((u_1,v_2)(u_2,v_1)) = 0.5 \neq \nu_{A_1}(u_1,v_2) \wedge \nu_{A_2}(u_2,v_1) = 0.4$.

Thus, their residue product $g_1 \cdot g_2$ is not a strong PFG which the residue product is strong.

Theorem 2.2.3

The residue product of a strong PFG $g_1 = (A_1, B_1)$ with any PFG $g_2 = (A_2, B_2)$ is a strong PFG if $\mu_{A_1} \geq \mu_{A_2}, \nu_{A_1} \leq \nu_{A_2}$

Proof

Let $g_1 = (A_1, B_1)$ be a strong PFG and $g_2 = (A_2, B_2)$ be any PFG with $\mu_{A_1} \geq \mu_{A_2}, \nu_{A_1} \leq \nu_{A_2}$. Then

$\mu_{B_1}(u_1 u_2) = \mu_{A_1}(u_1) \wedge \mu_{A_1}(u_2)$,

$\nu_{B_1}(u_1 u_2) = \nu_{A_1}(u_1) \vee \nu_{A_1}(u_2)$ for all $u_1 u_2 \in E_1$

By definition of residue product, we have

If $u_1 u_2 \in E_1$ and $v_1 \neq v_2$, then

$$
\begin{aligned}
&(\mu_{B_1} \cdot \mu_{B_2})((u_1, v_1)(u_2, v_2)) \\
&= \mu_{B_1}(u_1 u_2) \\
&= \mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2) \\
&= \{\mu_{A_1}(u_1) \vee \mu_{A_2}(v_1)\} \wedge \{\mu_{A_1}(u_2) \vee \mu_{A_2}(v_2)\} \\
&= (\mu_{A_1} \cdot \mu_{A_2})(u_1, v_1) \wedge (\mu_{A_1} \cdot \mu_{A_2})(u_2, v_2),
\end{aligned}
$$

$$
\begin{aligned}
&(\nu_{B_1} \cdot \nu_{B_2})((u_1, v_1)(u_2, v_2)) \\
&= \nu_{B_1}(u_1 u_2) \\
&= \nu_{A_1}(u_1) \vee \nu_{A_2}(u_2) \\
&= \{\nu_{A_1}(u_1) \wedge \nu_{A_2}(v_1)\} \vee \{\nu_{A_1}(u_2) \wedge \nu_{A_2}(v_2)\} \\
&= (\nu_{A_1} \cdot \nu_{A_2})(u_1, v_1) \vee (\nu_{A_1} \cdot \nu_{A_2})(u_2, v_2).
\end{aligned}
$$

Hence, $g_1 \cdot g_2$ is a strong PFG.

Theorem 2.2.4

The residue product $g_1 \cdot g_2$ of two connected PFGs is always a connected PFG.

Proof

Let $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ be two connected Pythagorean Fuzzy Graphs with underlying crisp graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Let $V_1 = \{u_1, u_2, ...u_m\}$ and $V_2 = \{v_1, v_2, ...v_n\}$. Then $\mu_1^{\infty}(u_i u_j) > 0$, $\nu_1^{\infty}(u_i u_j) > 0$ for all $u_i, u_j \in V_1$ and $\mu_2^{\infty}(v_i v_j) > 0$, $\nu_2^{\infty}(v_i v_j) > 0$, for all $v_i, v_j \in V_2$.

The residue product of $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ can be taken as $g = (A, B)$. Consider the k sub graphs of $g = (A, B)$ with the vertex set $V_2 = \{u_i v_1, u_i v_2, ..., u_i v_n\}$ for $i = 1, 2, 3......m$. Each of these sub graphs of $g = (A, B)$ is connected as u_i 's are same.

Since $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ are connected, each u_i and v_i are adjacent to at least one of the vertices in V_1 and V_2. Therefore, there exists at least one edge between any pair of the above k sub graphs

Thus, we have

$$\mu^{\infty}((u_i v_j)(u_k, v_l)) > 0,$$

$$\nu^{\infty}((u_i v_j)(u_k, v_l)) > 0$$

for all $(u_i, v_j)(u_k, v_l) \in E$

Hence, $g = (A, B)$ is a connected PFG.

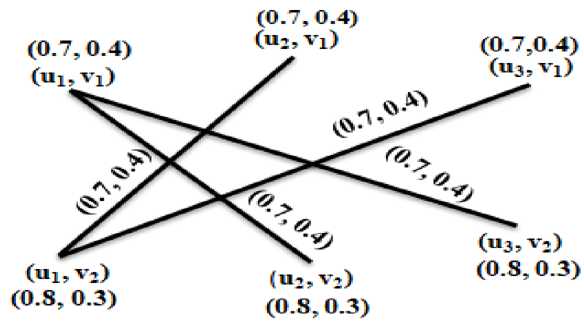

Figure 2.18

Remark 2.2.5

The residue product of two complete PFGs is not a complete PFG. We explain with an example. Consider two PFGs $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ on $V_1 = \{a, b, c\}$ and $V_2 = \{d, e\}$, respectively as shown in fig. their residue product $g_1 \cdot g_2$ shown in fig.

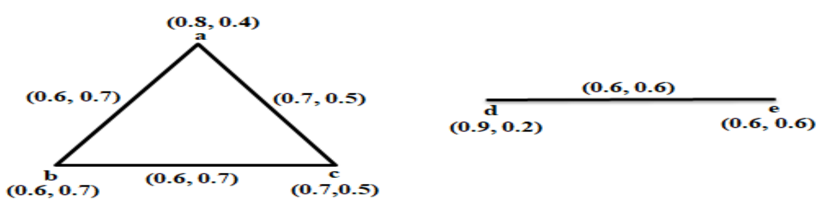

Figure 2.19

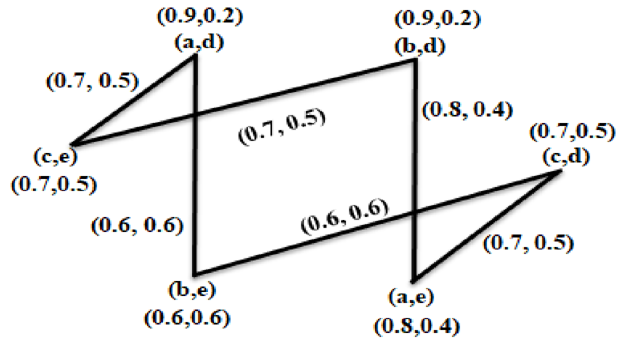

Figure 2.20

By routine calculation one can see from fig that g_1 and g_2 are complete PFGs. while notice that $g_1 \cdot g_2$ is not a complete PFG as the only case $u_1 u_2 \in E_1$ is included in the definition of the residue product.

Remark 2.2.6

If $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ are two regular PFGs, then their residue product $g_1 \cdot g_2$ may not be regular PFG as it is explained in this example. Consider two PFGs $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ on $V_1 = \{a, b, \}$ and $V_2 = \{c, d\}$, respectively as shown in fig. Their residue product $g_1 \cdot g_2$ is shown in fig.

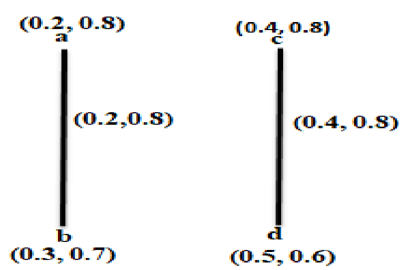

Figure 2.21

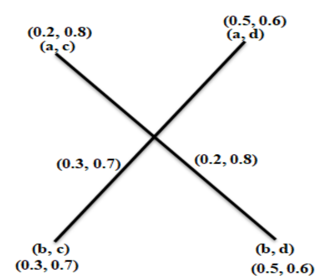

Figure 2.22

By routine calculation one can see from fig that g_1 and g_2 are regular PFGs. While notice that $(d)_{g_1*g_2}(a,d) \neq (d)_{g*g_{21}}(a,c)$. Therefore, $g_1 \cdot g_2$ is not regular PFG. The following theorems explain the conditions for the residue product of two regular PFGs to be regular.

Theorem 2.2.7

The residue product $g_1 \cdot g_2$ of any PFG $g_1 = (A_1, B_1)$ with a PFG $g_2 = (A_2, B_2)$ such that $|V_2| = 1$, is always a PFG with no edge.

Proof

The proof follows from the Definition

Theorem 2.2.8

If $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ are two PFGs such that $|V_2| > 1$, then their residue product is regular if and only if g_1 is regular.

Proof

Let $g_1 = (A_1, B_1)$ be a (k,l)- regular PFG and $g_2 = (A_2, B_2)$ be any PFG with $|V_2| > 1$. If $|V_2| > 1$, then the degree of any vertex in residue product is given by

$$(d_\mu)_{g_1 \cdot g_2}(u_1, u_2) = \sum_{u_1 v_1 \in E_1, u_2 \neq y_2} \mu_{B_1}(u_1 v_1)$$

$$= (d_\mu)_{g_1}(u_1) = k,$$

$$(d_\nu)_{g_1 \cdot g_2}(u_1, u_2) = \sum_{u_1 v_1 \in E_1, u_2 \neq y_2} \nu_{B_1}(u_1 v_1)$$

$$= (d_\nu)_{g_1}(u_1) = l.$$

This is a constant for all vertices in $V_1 \times V_2$. Hence $g_1 \cdot g_2$ is a regular PFG.

Conversely, assume that $g_1 \cdot g_2$ is a regular PFG. Then, for any two vertices (u_1, v_1) and (u_2, v_2) in $V_1 \times V_2$,

$$(d_\mu)_{g_1 \cdot g_2}(u_1, v_1) = (d_\mu)_{g_1 \cdot g_2}(u_2, v_2)$$
$$(d_\mu) g_1(u_1) = (d_\mu)_{g_1}(u_2),$$

$$(d_\nu)_{g_1 \cdot g_2}(u_1, v_1) = (d_\nu)_{g_1 \cdot g_2}(u_2, v_2)$$
$$(d_\nu) g_1(u_1) = (d_\nu)_{g_1}(u_2).$$

This is true for all vertices in V_1.Hence, g_1 is a regular PFG.

Theorem 2.2.9

If $g_1 = (A_1, B_1)$ and $g_2 = (A_2, B_2)$ are two PFGs such that $\mu_{A_1} \geq \mu_{A_2}, \nu_{A_1} \leq \nu_{A_2}$, then the total degree of any vertex in the residue product $g_1 \cdot g_2$ is given as

$$(td_\mu) g_1 \cdot g_2 (u_i, v_j) = \begin{cases} (td_\mu)_{g_1}(u_i) & \text{if } |V_1| > 1, \\ \mu_{A_1}(u_i) & \text{if } |V_1| = 1, \end{cases}$$

$$(td_\nu) g_1 \cdot g_2 (u_i, v_j) = \begin{cases} (td_\nu)_{g_1}(u_i) & \text{if } |V_1| > 1, \\ \nu_{A_1}(u_i) & \text{if } |V_1| = 1. \end{cases}$$

Proof

If $\mu_{A_1} \geq \mu_{A_2}, \nu_{A_1} \leq \nu_{A_2}$ and $|V_2| > 1$, then

$$(td_\mu)_{g_1 \cdot g_2}(u_i, v_i) = \sum_{(u_i, v_i)(u_k, v_l) \in E_1 \times E_2} (\mu_{B_1} \cdot \mu_{B_2})((u_i, v_i)(u_k, v_l)) + (\mu_{A_1} \cdot \mu_{A_2})(u_i, v_i)$$

$$= \sum_{u_i u_k \in E_1, v_j \neq v_l} \mu_{B_1}(u_i u_k) + \mu_{A_1}(u_i) \vee \mu_{A_2}(v_j)$$

$$= \begin{cases} (d_\mu)_{g_1}(u_i) + \mu_{A_1}(u_i) & \text{if } |V_2| > 1, \\ \mu_{A_1}(u_i), & \text{if } |V_2| = 1, \end{cases}$$

$$= \begin{cases} (td_\mu)_{g_1}(u_i), & \text{if } |V_2| > 1, \\ \mu_{A_1}(u_i), & \text{if } |V_2| = 1. \end{cases}$$

$$(td_\nu)_{g_1 \cdot g_2}(u_i, v_i) = \sum_{(u_i, v_i)(u_k, v_l) \in E_1 \times E_2} (\nu_{B_1} \cdot \nu_{B_2})((u_i, v_i)(u_k, v_l)) + (\nu_{A_1} \cdot \nu_{A_2})(u_i, v_i)$$

$$= \sum_{u_i u_k \in E_1, v_j \neq v_l} \nu_{B_1}(u_i u_k) + \nu_{A_1}(u_i) \wedge \nu_{A_2}(v_j)$$

$$= \begin{cases} (d_\nu)_{g_1}(u_i) + \nu_{A_1}(u_i) & \text{if } |V_2| > 1, \\ \nu_{A_1}(u_i), & \text{if } |V_2| = 1, \end{cases}$$

$$= \begin{cases} (td_\nu)_{g_1}(u_i), & \text{if } |V_2| > 1, \\ \nu_{A_1}(u_i), & \text{if } |V_2| = 1. \end{cases}$$

Definition 2.2.10

Let $g = (A, B)$ be a Pythagorean fuzzy graph on underlying crisp graph $G = (V, E)$. If

$$td_\mu(u) = \sum_{u,v \neq u \in V} \mu_B(uv) + \mu(u) = k,$$
$$td_\nu(u) = \sum_{u,v \neq u \in V} \nu_B(uv) + \nu(u) = l \quad \text{for all } u \in V$$

Then g is said to be **totally regular Pythagorean fuzzy graph** of total degree (k, l) or (k, l) - totally regular PFG.

Example 2.2.11

Consider a graph $G = (V, E)$ where $V = \{u_1, u_2, u_3, u_4\}$ and $E = \{u_1u_2, u_1u_3, u_2u_4, u_3u_4\}$. Let A and B be Pythagorean fuzzy vertex set and Pythagorean fuzzy edge set defined on V and $V \times V$ respectively.

$$A = \left\langle \left(\frac{u_1}{0.7}, \frac{u_2}{0.7}, \frac{u_3}{0.7}, \frac{u_4}{0.7} \right) \times \left(\frac{u_1}{0.5}, \frac{u_2}{0.5}, \frac{u_3}{0.5}, \frac{u_4}{0.5} \right) \right\rangle \text{ and}$$

$$B = \left\langle \left(\frac{u_1u_2}{0.6}, \frac{u_1u_3}{0.7}, \frac{u_2u_4}{0.7}, \frac{u_3u_4}{0.6} \right) \times \left(\frac{u_1u_2}{0.6}, \frac{u_1u_3}{0.6}, \frac{u_2u_4}{0.6}, \frac{u_3u_4}{0.6} \right) \right\rangle$$

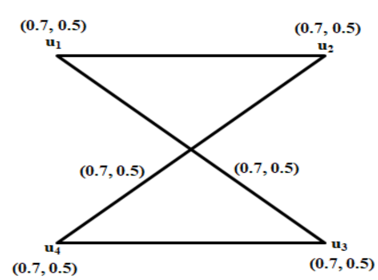

Figure 2.23

Since $td_\mu(u_i)=1.8$ and $td_\nu(u_i)=1.7$ for all $u_i \in V$ and i=1, 2, 3, 4. Hence g is a totally regular Pythagorean fuzzy graph of total degree (1.8, 1.7) or (1.8, 1.7) - totally regular PFG.

Theorem 2.2.12

If $g_1=(A_1,B_1)$ is a totally regular PFG and $g_2=(A_2,B_2)$ is a PFG such that $\mu_{A_1} \geq \mu_{A_2}, \nu_{A_1} \leq \nu_{A_2}$ and $|V_2|>1$, then the residue product is totally regular PFG.

Proof

Let $g_1=(A_1,B_1)$ be a (k, l)-totally regular PFG and $g_2=(A_2,B_2)$ be a PFG such that $\mu_{A_1} \geq \mu_{A_2}, \nu_{A_1} \leq \nu_{A_2}$ and $|V_2|>1$. Then $(td_\mu)_{g_1}(u_i)=k$, $(td_\nu)_{g_1}(u_i)=l$ for all u_i in V_1 and $(\mu_{A_1}\cdot\mu_{A_2})(u_1,v_1)=\mu_{A_1}(u_1)$, $(\nu_{A_1}\cdot\nu_{A_2})(u_1,v_1)=\nu_{A_1}(u_1)$, for all (u_1,v_1) in $V_1\times V_2$.

Now

$$\begin{aligned}
&(td_\mu)_{g_1\cdot g_2}(u_1,v_1)\\
&=(d_\mu)_{g_1\cdot g_2}(u_1,v_1)+(\mu_{A_1}\cdot\mu_{A_2})(u_1,v_1)\\
&=(d_\mu)_{g_1}(u_1)+\mu_{A_1}(u_1)\\
&=(td_\mu)_{g_1}(u_1)\\
&=k,
\end{aligned}$$

$$\begin{aligned}
&(td_\nu)_{g_1\cdot g_2}(u_1,v_1)\\
&=(d_\nu)_{g_1\cdot g_2}(u_1,v_1)+(\nu_{A_1}\cdot\nu_{A_2})(u_1,v_1)\\
&=(d_\nu)_{g_1}(u_1)+\nu_{A_1}(u_1)\\
&=(td_\nu)_{g_1}(u_1)\\
&=l
\end{aligned}$$

This is constant for all vertices in $V_1\times V_2$. Hence $g_1\cdot g_2$ is a (k,l)-totally regular PFG.

CHAPTER 3

CHAPTER – 3

Certain Graphs under Spherical Fuzzy Environment

3.1 Maximal Product in Spherical Fuzzy Graphs

Definition 3.1.1

Let $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ be two spherical fuzzy graph of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. The maximal product of g_1 and g_2 is denoted by $g_1 * g_2 = (m_1 * m_2, n_1 * n_2)$ and defined as

$$\begin{cases} (\alpha_{m_1} * \alpha_{m_2})(\eta_1, \eta_2) = \alpha_{m_1}(\eta_1) \vee \alpha_{m_2}(\eta_2) \\ (\beta_{m_1} * \beta_{m_2})(\eta_1, \eta_2) = \beta_{m_1}(\eta_1) \vee \beta_{m_2}(\eta_2) \\ (\tau_{m_1} * \tau_{m_2})(\eta_1, \eta_2) = \tau_{m_1}(\eta_1) \wedge \tau_{m_2}(\eta_2) \end{cases}$$

for all $(\eta_1, \eta_2) \in V_1 \times V_2$

$$\begin{cases} (\alpha_{n_1} * \alpha_{n_2})((\eta, \eta_2)\,(\eta, \lambda_2)) = \alpha_{m_1}(\eta) \vee \alpha_{n_2}(\eta_2\,\lambda_2) \\ (\beta_{n_1} * \beta_{n_2})((\eta, \eta_2)(\eta, \lambda_2)) = \beta_{m_1}(\eta) \vee \beta_{n_2}(\eta_2\,\lambda_2) \\ (\tau_{n_1} * \tau_{n_2})((\eta, \eta_2)(\eta, \lambda_2)) = \tau_{m_1}(\eta) \wedge \tau_{n_2}(\eta_2\,\lambda_2) \end{cases}$$

for all $\eta \in V_1$ and $\eta_2 \lambda_2 \in E_2$

$$\begin{cases} (\alpha_{n_1} * \alpha_{n_2})((\eta_1, \lambda)(\lambda_1, \lambda)) = \alpha_{n_1}(\eta_1\,\lambda_1) \vee \alpha_{m_2}(z) \\ (\beta_{n_1} * \beta_{n_2})((\eta_1, \lambda)(\lambda_1, \lambda)) = \beta_{n_1}(\eta_1\,\lambda_1) \vee \beta_{m_2}(z) \\ (\tau_{n_1} * \tau_{n_2})((\eta_1, \lambda)(\lambda_1, \lambda)) = \tau_{n_1}(\eta_1\,\lambda_1) \wedge \tau_{m_2}(z) \end{cases}$$

For all $\lambda \in V_2$ and $\eta_1\,\lambda_1 \in E_1$.

Definition 3.1.2

A Spherical Fuzzy Graph $g = (m, n)$ is said to be **Strong Spherical Fuzzy Graph (SSFG)** of underlying crisp graph $G = (V, E)$ if

$$\begin{aligned} \alpha_n(\eta\lambda) &= \alpha_m(\eta) \wedge \alpha_m(\lambda) \\ \beta_n(\eta\lambda) &= \beta_m(\eta) \wedge \beta_m(\lambda) \\ \tau_n(\eta\lambda) &= \tau_m(\eta) \vee \tau_m(\lambda) \end{aligned}$$

for all $\eta\lambda \in E$.

Example 3.1.3

Consider a graph $G = (V, E)$ where $V=\{a, b, c, d, e, f\}$ and $E=\{af, bf, cf, df, ef\}$ Let m and n be Spherical Fuzzy Vertex Set and Spherical Fuzzy Edge Set defined on V and $V\times V$, respectively.

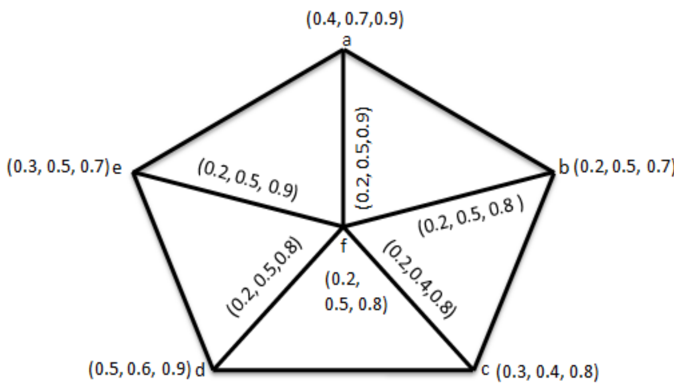

Figure 3.1

$$m = \left\langle \left(\frac{a}{0.4}, \frac{b}{0.2}, \frac{c}{0.3}, \frac{d}{0.5}, \frac{e}{0.3}, \frac{f}{0.2} \right) \times \left(\frac{a}{0.7}, \frac{b}{0.5}, \frac{c}{0.4}, \frac{d}{0.6}, \frac{e}{0.5}, \frac{f}{0.5} \right) \times \left(\frac{a}{0.9}, \frac{b}{0.7}, \frac{c}{0.8}, \frac{d}{0.9}, \frac{e}{0.7}, \frac{f}{0.8} \right) \right\rangle$$

and

$$n = \left\langle \left(\frac{af}{0.2}, \frac{bf}{0.2}, \frac{cf}{0.2}, \frac{df}{0.2}, \frac{ef}{0.2} \right) \times \left(\frac{af}{0.5}, \frac{bf}{0.5}, \frac{cf}{0.4}, \frac{df}{0.5}, \frac{ef}{0.5} \right) \times \left(\frac{af}{0.9}, \frac{bf}{0.8}, \frac{cf}{0.8}, \frac{df}{0.8}, \frac{ef}{0.8} \right) \right\rangle$$

Theorem 3.1.4

If $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ are two SSFG, then their maximal product is also a SSFG

Proof

Let $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ be two SSFG of the graph G_1 and G_2 respectively. Then

$$\alpha_{n_1}(\eta_1 \eta_2) = \alpha_{m_1}(\eta_1) \wedge \alpha_{m_1}(\eta_2),$$

$\beta_{n_1}(\eta_1\eta_2)=\beta_{m_1}(\eta_1)\wedge\beta_{m_1}(\eta_2)$,

$\tau_{n_1}(\eta_1\eta_2)=\tau_{m_1}(\eta_1)\vee\tau_{m_1}(\eta_2)$ for all $\eta_1\eta_2\in E_1$

$\alpha_{n_2}(\lambda_1\lambda_2)=\alpha_{m_2}(\lambda_1)\wedge\alpha_{m_2}(\lambda_2)$

$\beta_{n_2}(\lambda_1\lambda_2)=\beta_{m_2}(\lambda_1)\wedge\beta_{m_2}(\lambda_2)$

$\tau_{n_2}(\lambda_1\lambda_2)=\tau_{m_2}(\lambda_1)\vee\tau_{m_2}(\lambda_2)$ for all $\lambda_1\lambda_2\in E_2$

By the definition of maximal product, we have

If $\eta_1=\eta_2$ and $\lambda_1\lambda_2\in E_2$

$$\begin{aligned}(\alpha_{n_1}*\alpha_{n_2})((\eta_1,\lambda_1)(\eta_2,\lambda_2))&=\alpha_{m_1}(\eta_1)\vee\alpha_{n_2}(\lambda_1\lambda_2)\\ &=\alpha_{m_1}(\eta_1)\vee\{\alpha_{m_2}(\lambda_1)\wedge\alpha_{m_2}(\lambda_2)\}\\ &=\{\alpha_{m_1}(\eta_1)\vee\alpha_{m_2}(\lambda_1)\}\wedge\{\alpha_{m_1}(\eta_1)\vee\alpha_{m_2}(\lambda_2)\}\\ &=(\alpha_{m_1}*\alpha_{m_2})(\eta_1,\lambda_1)\wedge(\alpha_{m_1}*\alpha_{m_2})(\eta_2,\lambda_2)\qquad(\therefore\ (\eta_1=\eta_2)),\end{aligned}$$

$$\begin{aligned}(\beta_{n_1}*\beta_{n_2})((\eta_1,\lambda_1)(\eta_2,\lambda_2))&=\beta_{m_1}(\eta_1)\vee\beta_{n_2}(\lambda_1\lambda_2)\\ &=\beta_{m_1}(\eta_1)\vee\{\beta_{m_2}(\lambda_1)\wedge\beta_{m_2}(\lambda_2)\}\\ &=\{\beta_{m_1}(\eta_1)\vee\beta_{m_2}(\lambda_1)\}\wedge\{\beta_{m_1}(\eta_1)\vee\beta_{m_2}(\lambda_2)\}\\ &=(\beta_{m_1}*\beta_{m_2})(\eta_1,\lambda_1)\wedge(\beta_{m_1}*\beta_{m_2})(\eta_2,\lambda_2)\qquad(\therefore(\eta_1=\eta_2)),\end{aligned}$$

$$\begin{aligned}(\tau_{n_1}*\tau_{n_2})((\eta_1,\lambda_1)(\eta_2,\lambda_2))&=\tau_{m_1}(\eta_1)\wedge\tau_{n_2}(\lambda_1\lambda_2)\\ &=\tau_{m_1}(\eta_1)\wedge\{\tau_{m_2}(\lambda_1)\vee\tau_{m_2}(\lambda_2)\}\\ &=\{\tau_{m_1}(\eta_1)\wedge\tau_{m_2}(\lambda_1)\}\vee\{\tau_{m_1}(\eta_1)\wedge\tau_{m_2}(\lambda_2)\}\\ &=(\tau_{m_1}*\tau_{m_2})(\eta_1,\lambda_1)\vee(\tau_{m_1}*\tau_{m_2})(\eta_2,\lambda_2)\qquad(\therefore(\eta_1=\eta_2))\end{aligned}$$

If $\lambda_1=\lambda_2$ and $\eta_1\eta_2\in E_1$

$$\begin{aligned}(\alpha_{n_1}*\eta\alpha_{n_2})((\eta_1,\lambda_1)(\eta_2,\lambda_2))&=\alpha_{n_1}(\eta_1\lambda_2)\vee\alpha_{m_2}(\lambda_1)\\ &=\{\alpha_{m_1}(\eta_1)\wedge\alpha_{m_1}(\eta_2)\}\vee\alpha_{m_2}(\lambda_1)\\ &=\{\alpha_{m_1}(\eta_1)\vee\alpha_{m_2}(\lambda_1)\}\wedge\{\alpha_{m_1}(\eta_2)\vee\alpha_{m_2}(\lambda_1)\}\\ &=(\alpha_{m_1}*\alpha_{m_2})(\eta_1,\lambda_1)\wedge(\alpha_{m_1}*\alpha_{m_2})(\eta_2,\lambda_2)\qquad(\therefore(\lambda_1=\lambda_2))\end{aligned}$$

$$(\beta_{n_1} * \beta_{n_2})((\eta_1,\lambda_1)(\eta_2,\lambda_2)) = \beta_{n_1}(\eta_1,\lambda_2) \vee \beta_{m_2}(\lambda_1)$$
$$= \{\beta_{m_1}(\eta_1) \wedge \beta_{m_1}(\eta_2)\} \vee \beta_{m_2}(\lambda_1)$$
$$= \{\beta_{m_1}(\eta_1) \vee \beta_{m_2}(\lambda_1)\} \wedge \{\beta_{m_1}(\eta_2) \vee \beta_{m_2}(\lambda_2)\}$$
$$= (\beta_{m_1} * \beta_{m_2})(\eta_1,\lambda_1) \wedge (\beta_{m_1} * \beta_{m_2})(\eta_2,\lambda_2) \qquad (\therefore (\lambda_1 = \lambda_2))$$

$$(\tau_{n_1} * \tau_{n_2})((\eta_1,\lambda_1)(\eta_2,\lambda_2)) = \tau_{n_1}(\eta_1,\lambda_2) \wedge \tau_{m_2}(\lambda_1)$$
$$= \{\tau_{m_1}(\eta_1) \vee \tau_{m_1}(\eta_2)\} \wedge \tau_{m_2}(\lambda_1)$$
$$= \{\tau_{m_1}(\eta_1) \wedge \tau_{m_2}(\lambda_1)\} \vee \{\tau_{m_1}(\eta_2) \wedge \tau_{m_2}(\lambda_2)\}$$
$$= (\tau_{m_1} * \tau_{m_2})(\eta_1,\lambda_1) \vee (\tau_{m_1} * \tau_{m_2})(\eta_2,\lambda_2) \qquad (\therefore (\lambda_1 = \lambda_2))$$

Hence the maximal product $g_1 * g_2$ of two strong SPGs is a strong SFG.

Remark 3.1.5

Converse of the theorem may not be true as it can be seen be in the following example. Consider two spherical fuzzy graphs $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ and their maximal product is also given, Here

$$\alpha_{n_1}(\eta_1,\eta_2) \neq \alpha_{m_1}(\eta_1) \wedge \alpha_{m_1}(\eta_2)$$
$$\beta_{n_1}(\eta_1,\eta_2) \neq \beta_{m_1}(\eta_1) \wedge \beta_{m_1}(\eta_2)$$
$$\tau_{n_1}(\eta_1,\eta_2) \neq \tau_{m_1}(\eta_1) \vee \tau_{m_1}(\eta_2)$$

Hence g_1 and g_2 are not strong spherical fuzzy graph

But $(\alpha_{n_1} * \alpha_{n_2})((\eta,\lambda)(p,q)) = \alpha_{m_1}(\eta,\lambda) \wedge \alpha_{m_2}(p,q)$

$(\beta_{n_1} * \beta_{n_2})((\eta,\lambda)(p,q)) = \beta_{m_1}(\eta,\lambda) \wedge \beta_{m_2}(p,q)$

$(\tau_{n_1} * \tau_{n_2})((\eta,\lambda)(p,q)) = \tau_{m_1}(\eta,\lambda) \vee \tau_{m_2}(p,q)$ for all edge $(\eta\lambda)(p,q)$ in E. Thus their maximal product $g_1 * g_2$ is a strong spherical fuzzy graph.

Definition 3.1.6

Let $g = (m,n)$ be a Spherical Fuzzy Graph (SFG) on underlying crisp graph G= (V, E). g is said to be connected, if every pair of vertices ,there exist at least one non-zero path, that is, for all

$\eta, \lambda \in V$, the α - strength of connectedness.

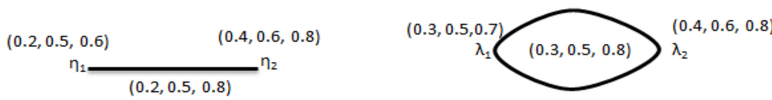

Figure 3.2

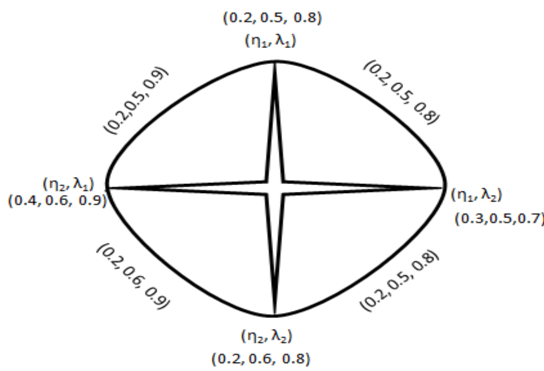

Figure 3.3

$\alpha^{\infty}(\eta,\lambda) = \text{Sup}\{\alpha^{t}(\eta,\lambda)$: for some 1}

and β - strength of connectedness

$\beta^{\infty}(\eta,\lambda) = \text{Sup}\{\beta^{t}(\eta,\lambda)$: for some 1}

and τ - strength of connectedness

$\tau^{t}(\eta,\lambda) = \inf\{\tau^{t}(\eta,\lambda)$: for some 1}

Satisfy one of the following conditions:

$\alpha^{\infty}(\eta,\lambda) > 0$, $\beta^{\infty}(\eta,\lambda) > 0$ and $\tau^{\infty}(\eta,\lambda) > 0$

or $\alpha^{\infty}(\eta,\lambda) = 0$, $\beta^{\infty}(\eta,\lambda) = 0$ and $\tau^{\infty}(\eta,\lambda) > 0$

or $\alpha^{\infty}(\eta,\lambda) > 0$, $\beta^{\infty}(\eta,\lambda) > 0$ and $\tau^{\infty}(\eta,\lambda) = 0$

where,

$$\alpha^{t}(\eta,\lambda)=\alpha(\eta\eta_1)\wedge\alpha(\eta_1\eta_2)\wedge.....\alpha(\eta_{t-1}\lambda)$$
$$\beta^{t}(\eta,\lambda)=\beta(\eta\eta_1)\wedge\beta(\eta_1\eta_2)\wedge.....\beta(\eta_{t-1}\lambda)$$
$$\tau^{t}(\eta,\lambda)=\tau(\eta\eta_1)\vee\tau(\eta_1\eta_2)\vee.......\tau(\eta_{t-1}\lambda)$$

represents α - strength of path, β - strength of path and τ - strength of path of length t, respectively.

Example 3.1.7

Consider a Spherical Fuzzy Graph $g=(m,n)$ as displayed in the following figure. For b, $f \in V$, using the definition, we have

$$\alpha^{5}(b,f)=\alpha_n(b,c)\wedge\alpha_n(c,d)\wedge\alpha_n(d,e)\wedge\alpha_n(e,f)=0.3\wedge0.3\wedge0.3\wedge0.2=0.2$$
$$\alpha^{4}(b,f)=\alpha_n(b,a)\wedge\alpha_n(a,g)\wedge\alpha_n(g,j)\wedge\alpha_n(j,f)=0.2\wedge0.2\wedge0.3\wedge0.2=0.2$$
$$\alpha^{3}(b,f)=\alpha_n(b,h)\wedge\alpha_n(h,i)\wedge\alpha_n(i,j)\wedge\alpha_n(j,f)=0.3\wedge0.4\wedge0.3\wedge0.2=0.2$$

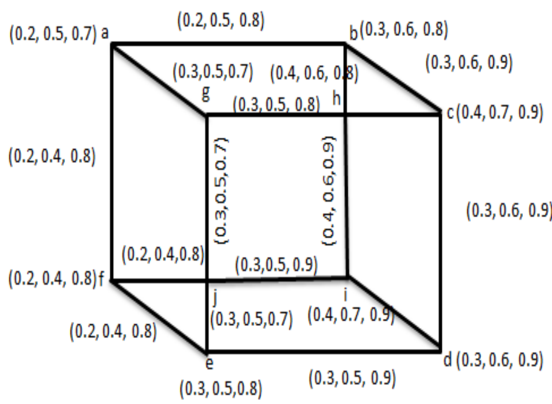

Figure 3.4

This implies

$$\alpha^{\infty}(b,f)=\sup\{\alpha^{5}(b,f),\alpha^{4}(b,f),\alpha^{3}(b,f)\}$$
$$=\sup\{0.2,0.2,0.2\}=0.2>0$$

and

$$\beta^{5}(b,f)=\beta_n(b,c)\wedge\beta_n(c,d)\wedge\beta_n(d,e)\wedge\beta_n(e,f)=0.6\wedge0.6\wedge0.5\wedge0.4=0.4$$

$$\beta^{5}(b,f)=\beta_n(b,a)\wedge\beta_n(a,g)\wedge\beta_n(g,j)\wedge\beta_n(j,f)=0.5\wedge0.5\wedge0.5\wedge0.4=0.4$$

$$\beta^{5}(b,f)=\beta_n(b,h)\wedge\beta_n(h,i)\wedge\beta_n(i,j)\wedge\beta_n(j,f)=0.6\wedge0.6\wedge0.5\wedge0.4=0.4$$

This implies

$$\beta^{\infty}(b,f) = \sup\left\{\beta^{5}(b,f), \beta^{4}(b,f), \beta^{3}(b,f)\right\}$$
$$= \sup\{0.4,0.4,0.4,0.4\} = 0.4 > 0$$

and

$$\tau^{5}(b,f) = \tau_n(b,c) \vee \tau_n(c,d) \vee \tau_n(d,e) \vee \tau_n(e,f) = 0.9 \vee 0.9 \vee 0.9 \vee 0.8 = 0.9$$

$$\tau^{4}(b,f) = \tau_n(b,a) \vee \tau_n(a,g) \vee \tau_n(g,i) \vee \tau_n(j,f) = 0.8 \vee 0.7 \vee 0.7 \vee 0.8 = 0.8$$

$$\tau^{3}(b,f) = \tau_n(b,h) \vee \tau_n(h,i) \vee \tau_n(i,j) \vee \tau_n(j,f) = 0.8 \vee 0.9 \vee 0.9 \vee 0.8 = 0.9$$

This implies

$$\tau^{\infty}(b,f) = \inf\left\{\tau^{5}(b,f), \tau^{4}(b,f), \tau^{3}(b,f)\right\}$$
$$= \inf\{0.9,0.8,0.9\} = 0.8 > 0$$

That is, there exists a non-zero path between b and f. Therefore, $g = (m,n)$ is connected spherical fuzzy graph.

Theorem 3.1.8

The maximal product $g_1 * g_2$ of two connected SFGs is always a connected SFG.

Proof

Let $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ be two connected Spherical Fuzzy Graphs with underlying crisp graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Let $V_1 = \{\eta_1, \eta_2, \eta_3 ......\eta_p\}$ and $V_2 = \{\lambda_1, \lambda_2, \lambda_3 .....\lambda_q\}$. Then $\alpha_1^{\infty}(\eta_i \eta_j) > 0$, $\beta_1^{\infty}(\eta_i \eta_j) > 0$, $\tau_1^{\infty}(\eta_i \eta_j) > 0$ for all $\eta_i, \eta_j \in V_1$ and $\alpha_2^{\infty}(\lambda_i \lambda_j) > 0$, $\beta_2^{\infty}(\lambda_i \lambda_j) > 0$, $\tau_2^{\infty}(\lambda_i \lambda_j) > 0$ for all $\lambda_i, \lambda_j \in V_2$.

The maximal product of $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ can be taken as $g = (m,n)$. Consider the k sub graphs of $g = (m,n)$ with the vertex set $V_2 = \{\eta_i \lambda_1, \eta_i \lambda_2, ..., \eta_i \lambda_q\}$ for i =1,2,….. p. Each of these sub graphs of $g = (m,n)$ is connected as η_i 's are same.

Since $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ are connected, each η_i and λ_i are adjacent to at least one of the vertices in V_1 and V_2. Therefore, there exists at least one edge between any pair of the above k sub graphs

Thus, we have

$\alpha^{\infty}\left(\left(\eta_i\, \lambda_j\right)\left(\eta_k, \lambda_t\right)\right) > 0$,

$\beta^{\infty}\left(\left(\eta_i\, \lambda_j\right)\left(\eta_k, \lambda_t\right)\right) > 0$,

$\tau^{\infty}\left(\left(\eta_i\, \lambda_j\right)\left(\eta_k, \lambda_t\right)\right) > 0$ for all $\left(\eta_i, \lambda_j\right)\left(\eta_k, \lambda_t\right) \in E$

Hence, $g = (m, n)$ is a connected SFG.

Definition 3.1.9

A Spherical Fuzzy Graph $g = (m, n)$ is said to be a **Complete Spherical Fuzzy Graph** on underlying crisp graph G= (V, E) if

$\alpha_n(\eta, \lambda) = \alpha_m(\eta) \wedge \alpha_m(\lambda)$,

$\beta_n(\eta, \lambda) = \beta_m(\eta) \wedge \beta_m(\lambda)$

$\tau_n(\eta, \lambda) = \tau_m(\eta) \vee \tau_m(\lambda)$ for all $\eta, \lambda \in V$.

Example 3.1.10

Consider Classical Graph $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and $E = \{ab, ac, ad, ae, bc, cd, ce, de\}$. Let m and n be spherical fuzzy vertex set and spherical fuzzy edge set defined on V and $V \times V$, respectively.

$$m = \left\langle \left(\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.3}, \frac{d}{0.4}, \frac{e}{0.5}\right) \times \left(\frac{a}{0.7}, \frac{b}{0.5}, \frac{c}{0.6}, \frac{d}{0.5}, \frac{e}{0.6}\right) \times \left(\frac{a}{0.9}, \frac{b}{0.8}, \frac{c}{0.9}, \frac{d}{0.7}, \frac{e}{0.9}\right) \right\rangle$$

$$n = \left\langle \begin{array}{l} \left(\frac{ab}{0.2}, \frac{ac}{0.2}, \frac{ad}{0.2}, \frac{ae}{0.2}, \frac{bc}{0.2}, \frac{bd}{0.2}, \frac{be}{0.2}, \frac{cd}{0.3}, \frac{de}{0.4}\right) \times \\ \left(\frac{ab}{0.5}, \frac{ac}{0.6}, \frac{ad}{0.5}, \frac{ae}{0.6}, \frac{bc}{0.5}, \frac{bd}{0.5}, \frac{be}{0.5}, \frac{cd}{0.5}, \frac{ce}{0.6}, \frac{de}{0.5}\right) \times \\ \left(\frac{ab}{0.9}, \frac{ac}{0.9}, \frac{ad}{0.9}, \frac{ae}{0.9}, \frac{bc}{0.9}, \frac{bd}{0.8}, \frac{be}{0.9}, \frac{cd}{0.9}, \frac{ce}{0.9}, \frac{de}{0.9}\right) \end{array} \right\rangle$$

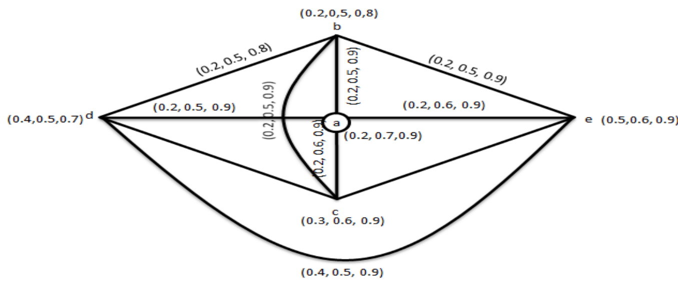

Figure 3.5

Remark 3.1.11

If two spherical fuzzy graphs are complete, their maximal product may not be a complete SFG, as it can be seen in the following example. Consider two SFGs $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ on $V_1 = \{\eta_1, \eta_2\}$ and $g_2 = \{\lambda_1, \lambda_2\}$, respectively. Their maximal product $g_1 * g_2$ is

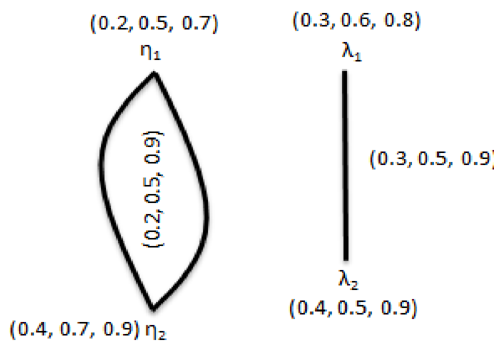

Figure 3.5

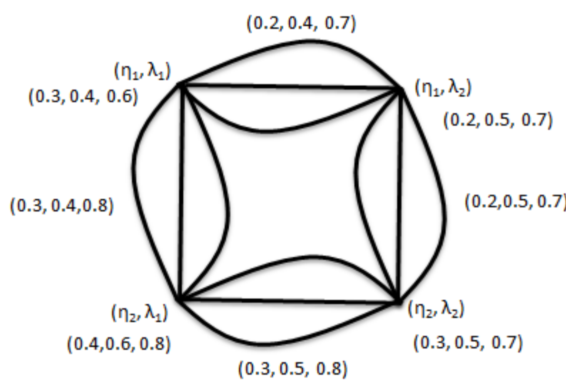

Figure 3.6

By routine calculation, that g_1 and g_2 are complete SFGs. While notice that $g_1 * g_2$ is not a complete SFG, as the case $\eta_1\eta_2 \in E_1$ and $\lambda_1\lambda_2 \in E_2$ is not included in the definition of the maximal product. Further, one can notice that the maximal product of two complete SFGs is a strong SFG.

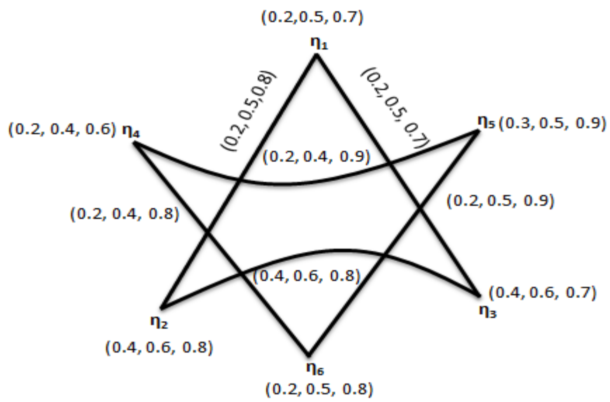

Figure 3.7

3.2 Residue Product in Spherical Fuzzy Graph

Definition 3.2.1

Let $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ be two Spherical Fuzzy Graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively .The Residue Product of g_1 and g_2 is denoted by $g_1 \cdot g_2 = (m_1 \cdot m_2, n_1 \cdot n_2)$ and defined as:

$$\begin{cases} (\alpha_{m_1} \cdot \alpha_{m_2})(\eta_1, \eta_2) = \alpha_{m_1}(\eta_1) \vee \alpha_{m_2}(\eta_2) \\ (\beta_{m_1} \cdot \beta_{m_2})(\eta_1, \eta_2) = \beta_{m_1}(\eta_1) \vee \beta_{m_2}(\eta_2) \\ (\tau_{m_1} \cdot \tau_{m_2})(\eta_1, \eta_2) = \tau_{m_1}(\eta_1) \wedge \tau_{m_2}(\eta_2) \end{cases}$$

for all $(\eta_1, \eta_2) \in V_1 \times V_2$,

$$\begin{cases} (\alpha_{n_1} \cdot \alpha_{n_2})(\eta_1, \eta_2)(\lambda_1, \lambda_2) = \alpha_{n_1}(\eta_1 \lambda_1) \\ (\beta_{n_1} \cdot \beta_{n_2})(\eta_1, \eta_2)(\lambda_1, \lambda_2) = \beta_{n_1}(\eta_1 \lambda_1) \\ (\tau_{n_1} \cdot \tau_{n_2})(\eta_1, \eta_2)(\lambda_1, \lambda_2) = \lambda_{n_1}(\eta_1 \lambda_1) \end{cases}$$

for all $\eta_1\lambda_1 \in E_1$, $\alpha_2 \neq \lambda_2$.

Remark 3.2.2

In general, the residue product of two Strong SFGs may not be Strong SFG as it is explained in the example. Consider two Spherical Fuzzy Graphs $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ as shown in fig

Here

$$\alpha_{n_1}(\eta_1\ \eta_2) = \alpha_{m_{11}}(\eta_1) \wedge \alpha_{m_1}(\eta_2), \beta_{n_1}(\eta_1\ \eta_2) = \beta_{m_1}(\eta_1) \wedge \beta_{ml_1}(\eta_2)$$
$$\tau_{n_1}(\eta_1\ \eta_2) = \tau_{m_{11}}(\eta_1) \vee \tau_{m_1}(\eta_2) \quad \text{for all } \eta_1\ \eta_2 \in E_1$$

And

$$\alpha_{n_2}(\lambda_1, \lambda_2) = \alpha_{m_2}(\lambda_1) \wedge \alpha_{m_2}(\lambda_2), \alpha_{n_2}(\lambda_1, \lambda_2) = \alpha_{m_2}(\lambda_1) \wedge \alpha_{m_2}(\lambda_2)$$
$$\tau_{n_2}(\lambda_1, \lambda_2) = \tau_{m_2}(\lambda_1) \vee \tau_{m_2}(\lambda_2) \text{ for all } \lambda_1\ \lambda_2 \in E_2.$$

Theorem 3.2.3

The residue product of a strong SFG $g_1 = (m_1, n_1)$with any PFG $g_2 = (m_2, n_2)$is a strong SFG if $\alpha_{m_1} \geq \alpha_{m_2}, \beta_{m_1} \geq \beta_{m_2}, \tau_{m_1} \leq \tau_{m_2}$

Proof

Let $g_1 = (m_1, n_1)$ be a strong SFG and $g_2 = (m_2, n_2)$ be any SFG with

$\alpha_{m_1} \geq \alpha_{m_2}, \beta_{m_1} \geq \beta_{m_2}, \tau_{m_1} \leq \tau_{m_2}$. Then

$$\alpha_{n_1}(\eta_1\ \eta_2) = \alpha_{m_1}(\eta_1) \wedge \alpha_{m_2}(\eta_2),$$
$$\beta_{n_1}(\eta_1\ \eta_2) = \beta_{m_1}(\eta_1) \wedge \beta_{m_2}(\eta_2),$$
$$\tau_{n_1}(\eta_1\ \eta_2) = \tau_{m_1}(\eta_1) \vee \tau_{m_2}(\eta_2)$$

for all $\eta_1\ \eta_2 \in E_1$

By the definition of Residue product, we have

If $\eta_1\ \eta_2 \in E_1$ and $\lambda_1 \neq \lambda_2$, then

$$\left(\alpha_{n_1}\cdot\alpha_{n_2}\right)\left((\eta_1,\lambda_1)(\eta_2,\lambda_2)\right)=\alpha_{n_1}(\eta_1\,\eta_2)$$
$$=\alpha_{m_1}(\eta_1)\wedge\alpha_{m_1}(\eta_2)$$
$$=\{\alpha_{m_1}(\eta_1)\vee\alpha_{m_2}(\lambda_1)\}\wedge\{\alpha_{m_1}(\eta_2)\vee\alpha_{m_2}(\lambda_2)\}$$
$$=\left(\alpha_{m_1}\cdot\alpha_{m_2}\right)(\eta_1,\lambda_1)\wedge\left(\alpha_{m_1}\cdot\alpha_{m_2}\right)(\eta_2,\lambda_2),$$

$$\left(\beta_{n_1}\cdot\beta_{n_2}\right)\left((\eta_1,\lambda_1)(\eta_2,\lambda_2)\right)=\beta_{n_1}(\eta_1\,\eta_2)$$
$$=\beta_{m_1}(\eta_1)\wedge\beta_{m_1}(\eta_2)$$
$$=\{\beta_{m_1}(\eta_1)\vee\beta_{m_2}(\lambda_1)\}\wedge\{\beta_{m_1}(\eta_2)\vee\beta_{m_2}(\lambda_2)\}$$
$$=\left(\beta_{m_1}\cdot\beta_{m_2}\right)(\eta_1,\lambda_1)\wedge\left(\beta_{m_1}\cdot\beta_{m_2}\right)(\eta_2,\lambda_2)$$

$$\left(\tau_{n_1}\cdot\tau_{n_2}\right)\left((\eta_1,\lambda_1)(\eta_2,\lambda_2)\right)=\tau_{n_1}(\eta_1\,\eta_2)$$
$$=\tau_{m_1}(\eta_1)\vee\tau_{m_1}(\eta_2)$$
$$=\{\tau_{m_1}(\eta_1)\wedge\tau_{m_2}(\lambda_1)\}\vee\{\tau_{m_1}(\eta_2)\wedge\tau_{m_2}(\lambda_2)\}$$
$$=\left(\tau_{m_1}\cdot\tau_{m_2}\right)(\eta_1,\lambda_1)\vee\left(\tau_{m_1}\cdot\tau_{m_2}\right)(\eta_2,\lambda_2)$$

Hence, $g_1\cdot g_2$ is a Strong SFG.

Theorem 3.2.4

The Residue product $g_1\cdot g_2$ of two connected SFGs is always a connected SFG.

Proof

Let $g_1=(m_1,n_1)$ and $g_2=(m_2,n_2)$ be two connected Spherical Fuzzy Graphs with underlying crisp graph $G_1=(V_1,\ E_1)$ and $G_2=(V_2,\ E_2)$, respectively. Let $V_1=\{\eta_1,\eta_2,\eta_3......\eta_p\}$ and $V_2=\{\lambda_1,\lambda_2,\lambda_3.....\lambda_q\}$. Then $\alpha_1^{\infty}(\eta_i\,\eta_j)>0$, $\beta_1^{\infty}(\eta_i\,\eta_j)>0$, $\tau_1^{\infty}(\eta_i\,\eta_j)>0$ for all $\eta_i,\eta_j\in V_1$ and $\alpha_2^{\infty}(\lambda_i\,\lambda_j)>0$, $\beta_2^{\infty}(\lambda_i\,\lambda_j)>0$, $\tau_2^{\infty}(\lambda_i\,\lambda_j)>0$ for all $\lambda_i,\lambda_j\in V_2$.

The residue product of $g_1=(m_1,n_1)$ and $g_2=(m_2,n_2)$ can be taken as $g=(m,n)$. Consider the k sub graphs of $g=(m,n)$ with the vertex set $V_2=\{\eta_i\,\lambda_1,\eta_i\,\lambda_2,.....,\eta_i\lambda_q\}$ for i =1,2,….. p. Each of these sub graphs of $g=(m,n)$ is connected as η_i's are same.

Since $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ are connected, each η_i and λ_i are adjacent to at least one of the vertices in V_1 and V_2. Therefore, there exists at least one edge between any pair of the above k sub graphs

Thus, we have

$\alpha^{\infty}((\eta_i \lambda_j)(\eta_k, \lambda_t)) > 0$,

$\beta^{\infty}((\eta_i \lambda_j)(\eta_k, \lambda_t)) > 0$,

$\tau^{\infty}((\eta_i \lambda_j)(\eta_k, \lambda_t)) > 0$ For all $(\eta_i, \lambda_j)(\eta_k, \lambda_t) \in E$

Hence, $g = (m, n)$ is a connected SFG.

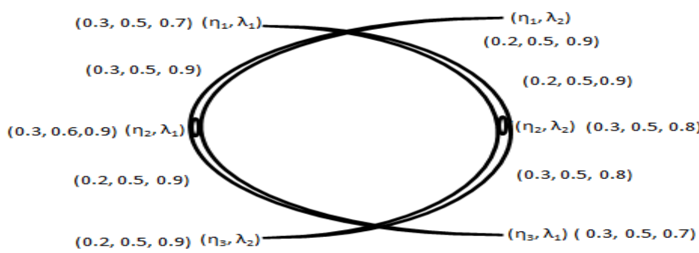

Figure 3.8

Remark 3.2.5

The Residue product of two complete SFG is not a complete SFG. We explain with an example. Consider two SFGs $g_1 = (m_1, n_1)$ and $g_2 = (m_2, n_2)$ on $V_1 = \{a, b, c\}$ and $V_2 = \{d, e\}$, respectively. Their residue product $g_1 \cdot g_2$ is

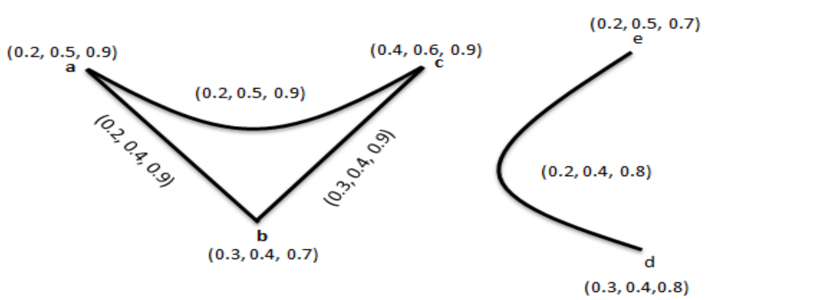

Figure 3.9

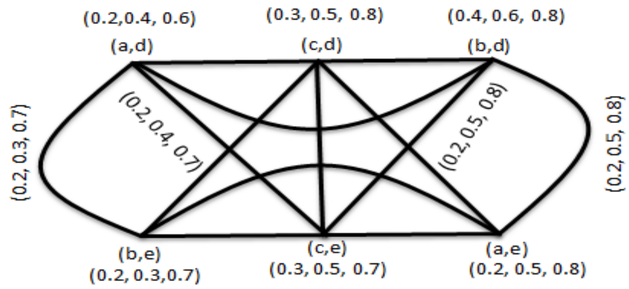

Figure 3.10

By routine calculations, that g_1 and g_2 are complete SFGs. While notice that $g_1 \cdot g_2$ is not complete SFG as the only case $\eta_1 \eta_2 \in E_1$, is included in the definition of the residue product.

SUMMARY AND CONCLUSION

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In this research work, the concept of Pythagorean fuzzy environment is extended to spherical fuzzy environment. We have mentioned two graph product specifically Maximal product and Residue product using these products; we have combined two Spherical fuzzy graphs and studied their properties. And also defined some of the operations like connectedness, completeness and strongness.

In future, we plan to elaborate our research work to more graphs which is used in decision making.

REFERENCES

REFERENCES

1. Akalyadevi, K., and Sudamani Ramaswamy, A.R., (2020), "Bipolar Spherical Fuzzy Graph", *International journal of creative Research Thoughts*, 8(6), 4317-4327.
2. Akalyadevi, K., and Sudamani Ramaswamy A.R., (2020), "An Algorithmic Approach for finding Minimum Spanning Tree in a bipolar spherical Graph", *Waffen-Und Kostumkunde Journal (WOS)*, 11(11), 1-16.
3. Akalyadevi, K., and Sudamani Ramaswamy, A.R., (2020), "Operation on Bipolar Spherical Graph", *Waffen-Und Kostumkunde Journal (WOS)*, 11(11), 78-88.
4. Akalyadevi, K., Sweety, C. A. C., & Ramaswamy, A. S., (2022), "Spherical neutrosophic graph coloring", In *AIP Conference Proceedings*, 2393, 02021.
5. Akram, M., & Akmal, R., (2016), "Operations on intuitionistic fuzzy graph structures", *Fuzzy information and Engineering*, *8*(4), 389-410.
6. Akram, M., & Davvaz, B., (2012), "Strong intuitionistic fuzzy graphs", *Filomat*, *26*(1), 177-196.
7. Akram, M., & Dudek, W. A. (2013), "Intuitionistic fuzzy hypergraphs with applications", *Information Sciences*, *218*, 182-193.
8. Atanassov, K.T., (1986), "Intuitionistic fuzzy sets", Fuzzy Sets and Systems, 20, 87-96.
9. Atanassov, K. T., & Atanassov, K. T., (1999), "Intuitionistic fuzzy sets", 1-137, *Physica-Verlag HD*.
10. Bhattacharya, P., (1987), "Some remarks on fuzzy graphs", *Pattern recognition letters*, *6*(5), 297-302.
11. Dhavudh, S. S., & Srinivasan, R., (2017), "Properties of intuitionistic fuzzy graphs of second type", *Int. J. Comput. Appl. Math*, *12*, 815-823.
12. Garg, H., (2016), "A novel correlation coefficients between Pythagorean fuzzy sets and its applications to decision-making processes", *International Journal of Intelligent Systems*, *31*(12), 1234-1252.

13. Garg, H., (2017), “Confidence levels based Pythagorean fuzzy aggregation operators and its application to decision-making process”, *Computational and Mathematical Organization Theory*, *23*(4), 546-571.
14. Garg, H., (2018), “Some methods for strategic decision-making problems with immediate probabilities in Pythagorean fuzzy environment”, *International Journal of Intelligent Systems*, *33*(4), 687-712.
15. Garg, H., (2018), “Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision-making process”, *International Journal of Intelligent Systems*, *33*(6), 1234-1263.
16. Garg, H., (2018),” Hesitant Pythagorean fuzzy sets and their aggregation operators in multiple attribute decision-making”, *International Journal for Uncertainty Quantification*, *8*(3).
17. Garg, H., (2019), “New logarithmic operational laws and their aggregation operators for Pythagorean fuzzy set and their applications”, *International Journal of Intelligent Systems*, *34*(1), 82-106.
18. Karunambigai, M. G., Rangasamy, P., Atanassov, K., & Palaniappan, N., (2007), “An intuitionistic fuzzy graph method for finding the shortest paths in networks”, *Theoretical advances and applications of fuzzy logic and soft computing*, 3-10.
19. Kutlu Gündoğdu, F., & Kahraman, C., (2019), “Spherical fuzzy sets and spherical fuzzy TOPSIS method”, *Journal of intelligent & fuzzy systems*, *36*(1), 337-352.
20. Kutlu Gündoğdu, F., & Kahraman, C., (2020), “Spherical fuzzy sets and decision making applications”, In *Intelligent and Fuzzy Techniques in Big Data Analytics and Decision Making: Proceedings of the INFUS 2019 Conference, Istanbul, Turkey, July 23-25, 2019* , 979-987.
21. Mordeson, J. N., and Chang-shyh, P., (1994), “Operations on fuzzy graphs”, *Information science,* 79(3-4), 159-170.
22. Akram, M., Dar, J. M., & Naz, S., (2019), “Certain graphs under Pythagorean fuzzy environment”, *Complex & Intelligent Systems*, *5*, 127-144.
23. Naz, S., Ashraf, S., & Akram, M., (2018), “A novel approach to decision-making with Pythagorean fuzzy information”, *Mathematics*, *6*(6), 95.

24. Nagoor Gani, A., Radha, K., (2008), "On regular fuzzy graphs", *J Phys Sci*, 12:33–44.
25. Nirmala, G., & Vijaya, M., (2012), "Fuzzy graphs on composition, tensor and normal products", *International journal of scientific and research publications*, *2*(6), 1-7.
26. Ozceylan, E., Ozkan, B., Kabak, M., & Dagdeviren, M., (2022), "A state-of-the-art survey on spherical fuzzy sets", *Journal of Intelligent & Fuzzy Systems*, *42*(1), 195-212.
27. Parvathi, R., and Karunambigai, M.G., (2006), "Intuitionistic Fuzzy Graphs", *Computational Intelligence Theory and applications,* 139-150.
28. Peng, X., and Yang, Y., (2015), "Some results for Pythagorean fuzzy sets", *International journal of Intelligent Systems,* 30(11), 1132-1160.
29. Rangasamy, P., & Palaniappan, N., (2003), "Some operations on intuitionistic fuzzy sets of second type", *Notes Intuit. Fuzzy Sets*, 10, 1-19.
30. Ren, P., Xu, Z., & Gou, X., (2016), "Pythagorean fuzzy TODIM approach to multi-criteria decision making", *Applied Soft Computing*, 42, 246-259.
31. Rosenfeld, A., (1975), "Fuzzy graphs in Fuzzy sets and their applications" *Academic press Newyork,* 77-95.
32. Sarwar, M., & Akram, M., (2016), "An algorithm for computing certain metrics in intuitionistic fuzzy graphs", *Journal of Intelligent & Fuzzy Systems*, 30(4), 2405-2416.
33. Szmidt, E., & Kacprzyk, J., (2001), "Intuitionistic fuzzy sets in some medical applications", In *Computational Intelligence. Theory and Applications: International Conference, 7th Fuzzy Days Dortmund, Germany*, 148-151.
34. Wang, R., & Li, Y. L., (2018)," A novel approach for group decision-making from intuitionistic fuzzy preference relations and intuitionistic multiplicative preference relations", *Information*, *9*(3), 55.
35. Yager, R. R., & Abbasov, A. M., (2013), "Pythagorean membership grades, complex numbers, and decision making", *International Journal of Intelligent Systems*, *28*(5), 436-452.

36. Yager, R. R., (2013), “Pythagorean membership grades in multicriteria decision making”, *IEEE Transactions on fuzzy systems*, *22*(4), 958-965.

37. Zhang, X., & Xu, Z., (2014), “Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets”, *International Journal of Intelligent Systems*, *29*(12), 1061-1078.

38. Zadeh, L. A., (1965), “Fuzzy sets”, *Information Control*, 8(3):338–353.

39. Zadeh, L. A., (1971), “Similarity relations and fuzzy orderings”, *Information Sciences*, 3(2), 177–200.

PRESENTATION

DETAILS OF PAPER PRESENTATION

Paper Presentation

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