Advances in Algebra.
ISSN 0973-6964 Volume 4, Number 2 (2011), pp. 71-79

http://www.ripublication.com/aa.htm

k-Normal Matrix in Minkowski Space

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Abstract

The concept of k-normal matrix in Minkowski space is introduced as a special type of range symmetric matrix in Minkowski space and as an analogue of Complex normal matrix. Equivalent conditions for a matrix to be k-normal in Minkowski space are obtained. Some properties of k-normal matrices are derived.

AMS Subject Classification: 15A57

Keywords: Minkowski space, k-normal matrix, Range symmetric.

Introduction

We shall deal with $C^{n\times n}$, the space of n×n complex matrices. Let C^n be the space of complex n-tuples. Let G be the Minkowski metric tensor defined by

$Gx = (x_1, -x_2, -x_3, \ldots\ldots, -x_n)^T$ for $x = (x_1, x_2, x_3, \ldots\ldots, x_n)^T \in C^n$. Clearly the Minkowski metric matrix

$$(1.1) \quad G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix} \text{ satisfy } G = G^* = G^{-1} \text{ and } G^2 = I_n.$$

In [6], the Minkowski inner product C^n is defined by $(x,y) = [x, Gy]$, where [.,.] denotes the conventional Hilbert space inner product. A space with Minkowski inner product is called a Minkowski space and denoted by ***m***. With respect to the Minkowski inner product, the adjoint of a matrix $A \in C^{n\times n}$, is given by $(Ax,y) = (x, A^{\sim}y)$

(1.2) $A^{\sim} = GA^*G,\ A^* = G A^{\sim} G$

where A^* is the usual Hermitian adjoint.

(1.3) A is ***m***-symmetric if $A = A^{\sim}$

For $A_1, A_2 \in C^{n\times n}$, it can be verified that [4]

(1.4) $(A_1A_2)^{\sim} = A_2^{\sim} A_1^{\sim}$

(1.5) $(A_1+A_2)^{\sim} = A_1^{\sim} + A_2^{\sim}$

(1.6) $(A^{\sim})^{\sim} = A$

Preliminaries

Throughout let 'k' be a fixed product of disjoint transpositions in $S_n = \{1,2,----,n\}$ and K be the associated permutation matrix. A matrix $A = (a_{ij}) \in C^{n\times n}$ is k-hermitian if $a_{ij} = \bar{a}_{k(j)k(i)}$ for $i,j = 1,2,-------,n$. A theory for k-hermitian matrices is developed in[1].

For $x = (x_1, x_2, -----, x_n)^T \in C^n$, let us define the function

$Kx = k(x) = (x_{k(1)}, x_{k(2)}, x_{k(3)}, ------, x_{k(n)})^T \in C^n$.

$K(Gx) = (x_{k(1)}, -x_{k(2)}, -x_{k(3)}, ------, -x_{k(n)})$, $G(KGx) = (x_{k(1)}, x_{k(2)}, x_{k(3)}, ------, x_{k(n)})$

(2.1) $GKG = K$

(2.2) A matrix $A \in C^{n\times n}$, is said to be k-EP in ***m*** $\Leftrightarrow$ GA is k-EP (or) AG is k-EP[5]. A is said to be k-EP$\Leftrightarrow$KA isEP (or) AK is EP[2]. Let R(A),N(A), rk(A) denote the range space, null space and rank of A respectively. K satisfy the following properties:

(P.1) $K = K^T = K^{-1}$

(P.2) $K^{\sim} = (K^{\sim})^T = (K^{\sim})^{-1}$

(P.3) $K^2 = I_n$

Definition 2.1

A Matrix $A \in C^{n\times n}$ is said to be k-normal $\Leftrightarrow$ $A^*KA = AKA^*$ where K is the associated permutation matrix.

Definition 2.2: [3]

$A \in C^{n\times n}$ is said to be range symmetric in ***m*** if and only if $N(A) = N(A^{\sim})$.

Definition 2.3: [4]

For $A \in C^{n\times n}$ and G is the Minkowski tensor of order n, if $Ax = \lambda Gx$ for $x \in C^n$ and λ a scalar then x is said to be a G-eigen vector of A corresponding to the eigen value λ.

k-normal matrices in Minkowski space

In this section we shall introduce the concept of k-normal matrices in Minkowski space m as a generalization of k-normal matrix in unitary space.

Definition 3.1

A matrix $A \in C^{n\times n}$ is said to be k-normal in $m \Leftrightarrow A^{\sim}KGA = AKGA^{\sim}$
where K is the associated permutation matrix and G is the Minkowski metric tensor of order n.

Remark 3.2

If $K = I_n$, the identity transformation then k-normal in m reduces to $A^{\sim}GA = AGA^{\sim}$ that is A is m-normal in m [4].

Lemma 3.3

KG is m-symmetric

Proof

$(KG)^{\sim} = G^{\sim}K^{\sim} = GK^{\sim}$ (by 1.1)

$= G(GK^{*}G)$ (by 1.2)

$= GGKG$

$= G^{2}KG$

$= KG$ (by 1.1)

Then KG is m-symmetric. (by1.3)

Theorem 3.4

For $A \in C^{n\times n}$ in m the following are equivalent:

(i) A is k-normal in m
(ii) GA is k-normal
(iii) AG is k-normal
(iv) $A^{\sim}$ is k-normal in m

Proof

(i) $\Leftrightarrow$ (ii)

A is k-normal in m $\Leftrightarrow A^{\sim}KGA = AKGA^{\sim}$ (by Definition 3.1)

$\Leftrightarrow (GA^{*}G)KGA = AKG(GA^{*}G)$ (by 1.2)

$\Leftrightarrow G(GA^{*}G)KGA = GAKG(GA^{*}G)$

$\Leftrightarrow A^{*}GKGA = GAKA^{*}G$ (by 1.1)

$\Leftrightarrow (GA)^{*}K(GA) = (GA)K(GA)^{*}$ (by 1.1)

$\Leftrightarrow$ GA is k-normal [by Definition 2.1]

(i) $\Leftrightarrow$ (iii)

$$\begin{aligned}
A \text{ is k-normal in } \boldsymbol{m} &\Leftrightarrow A^{\sim}KGA = AKGA^{\sim} \\
&\Leftrightarrow (GA^*G)KGA = AKG(GA^*G) && \text{(by 1.2)} \\
&\Leftrightarrow (GA^*G)KGAG = AKG(GA^*G)G \\
&\Leftrightarrow GA^*(GKG)AG = AG(GKG)GA^* && \text{(by 1.1)} \\
&\Leftrightarrow (AG)^*(GKG)AG = AG(GKG)(AG)^* && \text{(by 1.1)} \\
&\Leftrightarrow (AG)^*K(AG) = (AG)K(AG)^* && \text{(by 2.1)} \\
&\Leftrightarrow AG \text{ is k-normal} && \text{[by Definition 2.1]}
\end{aligned}$$

(i) $\Leftrightarrow$ (iv)

$$\begin{aligned}
A \text{ is k-normal in } \boldsymbol{m} &\Leftrightarrow A^{\sim}KGA = AKGA^{\sim} \\
&\Leftrightarrow A^{\sim}KG(A^{\sim})^{\sim} = (A^{\sim})^{\sim}KGA^{\sim} && \text{(by 1.6)} \\
&\Leftrightarrow A^{\sim} \text{ is k-normal in } \boldsymbol{m}
\end{aligned}$$

Theorem 3.5

Let $\{A_i / i \in I\}$ for $I = \{1,2,----,n\}$ be a class of k-normal matrices in $\boldsymbol{m}$, by satisfying the condition $A_iKGA_j^{\sim} = A_j^{\sim}KGA_i$ for $i \neq j$. Then the sum of k-normal matrices in $\boldsymbol{m}$ is k-normal in $\boldsymbol{m}$.

Proof

Let $\{A_i / i \in I\}$ for $I = \{1,2,----,n\}$ be a class of k-normal matrices in $\boldsymbol{m}$, by satisfying the condition $A_iKGA_j^{\sim} = A_j^{\sim}KGA_i$ for $i \neq j$. We prove the Theorem by induction on n.

Case 1

For $n = 2$, since A_1, A_2 are k-normal matrices in $\boldsymbol{m}$, by Definition 3.1

$A_1KGA_1^{\sim} = A_1^{\sim}KGA_1$ and $A_2KGA_2^{\sim} = A_2^{\sim}KGA_2$. By hypothesis

$A_1KGA_2^{\sim} = A_2^{\sim}KGA_1$ by taking Minkowski adjoint on bothsides yield

$(A_1KGA_2^{\sim})^{\sim} = (A_2^{\sim}KGA_1)^{\sim}$ as $A_2KGA_1^{\sim} = A_1^{\sim}KGA_2$ (by 1.4 & 3.3)

Now consider

$$\begin{aligned}
(A_1+A_2)KG(A_1+A_2)^{\sim} &= (A_1+A_2)KG(A_1^{\sim}+A_2^{\sim}) && \text{(by 1.5)} \\
&= (A_1+A_2)KGA_1^{\sim}+(A_1+A_2)KGA_2^{\sim} \\
&= A_1KGA_1^{\sim}+A_2KGA_1^{\sim}+A_1KGA_2^{\sim}+A_2KGA_2^{\sim} \\
&= A_1^{\sim}KGA_1+A_2KGA_1^{\sim}+A_1KGA_2^{\sim}+A_2^{\sim}KGA_2 && \text{(by Definition 3.1)} \\
&= A_1^{\sim}KGA_1+A_1^{\sim}KGA_2+A_2^{\sim}KGA_1+A_2^{\sim}KGA_2 \\
&= A_1^{\sim}KG(A_1+A_2)+A_2^{\sim}KG(A_1+A_2) \\
&= (A_1^{\sim}+A_2^{\sim})KG(A_1+A_2) \\
&= (A_1+A_2)^{\sim}KG(A_1+A_2)
\end{aligned}$$

Thus A_1+A_2 is k-normal in $\boldsymbol{m}$.

Case 2

For n = 3

Let $B = A_1 + A_2$ by Case 1, B is k-normal in ***m***. To prove $A_1 + A_2 + A_3$ is k-normal in ***m***

That is to prove $B + A_3$ is k-normal in ***m***, it is enough to verify that the k-normal matrices B and A_3 in ***m*** satisfy the condition $BKGA_3^{\sim} = A_3^{\sim}KGB$

$$\begin{aligned} BKGA_3^{\sim} = (A_1+A_2)KG\,A_3^{\sim} &= A_1KGA_3^{\sim} + A_2KGA_3^{\sim} \\ &= A_3^{\sim}KGA_1 + A_3^{\sim}KGA_2 \\ &= A_3^{\sim}KGB \end{aligned}$$

Thus $B + A_3$ is k-normal in ***m***.

Hence Case 1, $A_1 + A_2 + A_3 = B + A_3$ is k-normal in ***m***. Now assume that sum of m-1 k-normal matrices in ***m*** is k-normal in ***m*** and we prove it is true for n=m. Let $A = A_1 + A_2 + A_3 + \text{-------} + A_{m-1}$ By induction A is k-normal in ***m***. Now

$$\begin{aligned} (A+A_m)KG(A+A_m)^{\sim} &= (A+A_m)KG(A^{\sim} + A_m^{\sim}) \\ &= AKGA^{\sim} + A_mKGA^{\sim} + AKGA_m^{\sim} + A_mKGA_m^{\sim} \\ &= A^{\sim}KGA + A_mKGA^{\sim} + AKGA_m^{\sim} + Am^{\sim}KGA_m \\ &\qquad\qquad\qquad \text{(by Definition 3.1)} \\ &= A^{\sim}KGA + A^{\sim}KGA_m + A_m^{\sim}KGA + A_m^{\sim}KGA_m \\ &\qquad\qquad\qquad \text{(by Induction hypothesis)} \\ &= A^{\sim}KG(A+A_m) + A_m^{\sim}KG(A+A_m) \\ &= (A^{\sim}+A_m^{\sim})KG(A+A_m) \\ &= (A+A_m)^{\sim}KG(A+A_m) \end{aligned}$$

Thus $A + A_m$ is k-normal in ***m***.

Therefore $A_1 + A_2 + A_3 + \text{-------} + A_{m-1}$ is k-normal in ***m***.

Hence the theorem.

Theorem 3.6

The product of k-normal matrices in ***m*** is k-normal in ***m*** if each commute with the Minkowski adjoint of the other.

Proof

Let $\{A_i / i \in I\}$ for $I = \{1,2,\text{----},n\}$ be a class of k-normal matrices in ***m***. By hypothesis $A_iA_j^{\sim} = A_j^{\sim}A_i$ for $i \neq j$ we prove the theorem by induction on n.

Case 1

For n = 2

Since A_1, A_2 are k-normal matrices in ***m***, by Definition 3.1 $A_1KGA_1^{\sim} = A_1^{\sim}KGA_1$ and $A_2KGA_2^{\sim} = A_2^{\sim}KGA_2$. By hypothesis $A_1A_2^{\sim} = A_2^{\sim}A_1$ and $A_2A_1^{\sim} = A_1^{\sim}A_2$. Now

$(A_1A_2)^{\sim}KG(A_1A_2) = A_2^{\sim}A_1^{\sim}KGA_1A_2$ (by 1.4)

$$\begin{aligned}
&= A_2^{\sim}(A_1^{\sim}KGA_1)A_2 \\
&= A_2^{\sim}(A_1 KGA_1^{\sim})A_2 && \text{(by Definition 3.1)} \\
&= A_2^{\sim}A_1 KGA_1^{\sim}A_2 \\
&= A_1A_2^{\sim}KG\,A_2A_1^{\sim} && \text{(by hypothesis)} \\
&= A_1A_2KG\,A_2^{\sim}A_1^{\sim} && \text{(by Definition 3.1)} \\
&= (A_1A_2)KG(A_1A_2)^{\sim} && \text{(by 1.4)}
\end{aligned}$$

Thus A_1A_2 is k-normal in ***m***.

Case 2

For n=3 is k-normal in ***m***. To prove $A_1A_2A_3 = BA_3$ is k-normal in ***m***. where B and A_3 are k-normal in ***m***, it is enough to verify the condition $BA_3^{\sim} = A_3^{\sim}B$ and $A_3B^{\sim} = B^{\sim}A_3$ hold by case 1.consider

$$\begin{aligned}
BA_3^{\sim} &= A_1A_2A_3^{\sim} \\
&= A_1A_3^{\sim}A_2 \\
&= A_3^{\sim}A_1A_2 \\
BA_3^{\sim} &= A_3^{\sim}B
\end{aligned}$$

By taking Minkowski adjoint on bothsides yield
$(BA_3^{\sim})^{\sim} = (A_3^{\sim}B)^{\sim}$ as $A_3B^{\sim} = B^{\sim}A_3$ (by 1.4)

Hence by Case 1, $A_1A_2A_3 = BA_3$ is k-normal in ***m***.

Now assume that the product of (n-1) k-normal matrices in ***m*** and we prove that it is true for n.

Let $A = A_1A_2A_3\text{-------}A_{n-1}$. By induction A_1A_2 is k-normal in ***m***.

$AA_n^{\sim} = (A_1A_2A_3\text{-------}A_{n-1})A_n^{\sim}$ where i=1,2,----,n-1and j=n

$= A_1A_2A_3\text{-------}A_n^{\sim}A_{n-1}$ (by hypothesis)

:

:

$= A_n^{\sim}A_1A_2A_3\text{-------}A_{n-1}$.

$AA_n^{\sim} = A_n^{\sim}A$

By taking Minkowski adjoint on bothesides yield
$(AA_n^{\sim})^{\sim} = (A_n^{\sim}A)^{\sim}$ as$A_nA^{\sim} = A^{\sim}A_n$ (by 1.4)

Therefore AA_n is k-normal in ***m*** by Case 1.
That is $A_1A_2A_3\text{-------}A_{n-1}A_n$ is k-normal in ***m***.

Definition 3.7

For $A \in C^{n\times n}$ and G is the Minkowski tensor of order n, if $KAx = \lambda Gx$ and K be the associated permutation matrix for $x \in C^n$ and λ a scalar then x is said to be G-eigen vector of KA corresponding to the eigen value λ.

Remark 3.8

For $A \in C^{n\times n}$, $x \in C^n$, using $G^2=K^2=I_n$ from Definition 3.7 we have the following equivalence:

x is a G-eigen vector of KA in ***m*** $\Leftrightarrow$ x is an eigen vector of GKA $\Leftrightarrow$ Gx is an eigen vector of KAG.

We shall follow the Euclidean norm $\|x\|^2 = [x,x]$ in the unitary space in the following.

Theorem 3.9

For $A \in C^{n\times n}$, the following statements are equivalent:

(i) A is k-normal in ***m***.

(ii) $\|KGAx\| = \|GA^{\sim}Kx\|$

(iii) $(A-\lambda GK)$ is k-normal in ***m***.

Proof

(i)$\Leftrightarrow$(ii)

A is k-normal in ***m*** $\Leftrightarrow$ GA of k-normal (by Theorem 3.4)

$\Leftrightarrow$ KGA is normal (by 2.2)

$\Leftrightarrow \|KGAx\| = \|(KGA)^*x\|$ (by Theorem c; 271 of [7]

$\Leftrightarrow \|KGAx\|^2 = \|(KGA)^*x\|^2$

$\Leftrightarrow [KGAx,KGAx] = [(KGA)^*x, (KGA)^*x]$

$\Leftrightarrow [KGAx,KGAx] = [G(KGA)^{\sim}Gx, (G(KGA)^{\sim}Gx]$ (by 1.2)

$\Leftrightarrow [KGAx,KGAx] = [GA^{\sim}(KG)^{\sim}Gx, GA^{\sim}(KG)^{\sim}Gx]$

$\Leftrightarrow [KGAx,KGAx] = [GA^{\sim}KGGx, GA^{\sim}KGGx]$ (by 3.3)

$\Leftrightarrow [KGAx,KGAx] = [GA^{\sim}Kx, GA^{\sim}Kx]$ (by 1.1)

$\Leftrightarrow \|KGAx\|^2 = \|GA^{\sim}Kx\|^2$

$\Leftrightarrow \|KGAx\| = \|GA^{\sim}Kx\|$

Thus (i)$\Leftrightarrow$(ii) holds.

(i)$\Leftrightarrow$(iii)

A is k-normal in ***m*** $\Leftrightarrow$ GA of k-normal (by Theorem 3.4)

$\Leftrightarrow$ KGA is normal (by 2.2)

$\Leftrightarrow (KGA-\lambda I)$ is normal (by Theorem 62-A[7])

$\Leftrightarrow KG(A-\lambda GK)$ is normal (by using $I=KGGK$ as $G^2=K^2=I_n$)

$\Leftrightarrow G(A-\lambda GK)$ is k-normal

$\Leftrightarrow (A-\lambda GK)$ is k-normal in ***m***. (by Theorem 3.4)

Hence the theorem

Theorem 3.10

Let A be k-normal in ***m***, then x is G-eigen vector of KA in ***m*** with eigen value $\lambda \Leftrightarrow$ x is a G-eigen vector of $(KA)^{\sim}$ with eigen value $\bar{\lambda}$.

Proof

Since A is k-normal in ***m***, by Theorem 3.4, AG is k-normal, KAG is normal (by 2.2). It is well known that (p-290[7]), x is an eigen vector of KAG with eigen value $\lambda \Leftrightarrow$ x is an eigen vector of $(KAG)^* = G(KAG)^{\sim}G = GG^{\sim}(KA)^{\sim}G = (KA)^{\sim}G$ with eigen value $\bar{\lambda} \Leftrightarrow$ Gx is an eigen vector of $(KA)^{\sim}G$ with eigen value $\bar{\lambda}$. By Remark 3.8, it follows that x is a G-eigen vector of KA with eigen value $\lambda \Leftrightarrow$ x is a G-eigen vector of $(KA)^{\sim}$ with eigen value $\bar{\lambda}$. Hence the theorem.

Generalization of k-symmetric matrices in *m*

Since every k-symmetric matrix in ***m*** is k-normal in ***m***, k-normal matrices is a generalization of k-symmetric matrices in ***m***.

In the case of complex matrices, every normal matrix is EP in unitary space. Here we prove that similar result holds for matrices in Minkowski space ***m*** in the following.

Theorem 4.1

If $A \in C^{n\times n}$ is k-normal in ***m***, then A is range symmetric in ***m***.

Proof

Since A is k-normal in ***m*** by Definition 3.1, $A^{\sim}KGA = AKGA^{\sim}$, clearly

$N(A) \subseteq N(A^{\sim}KGA)$. Since G is nonsingular and K be the associated permutation matrix, $rk\,(A^{\sim}KGA) = rk\,(GA^{\sim}KGA) = rk(GGA^*GKGA) = rk(A^*KA) = rk(KA) = rk(A)$. Hence $N(A) = N(A^{\sim}KGA)$. Also $N(A^{\sim}) \subseteq N\,(AKGA^{\sim})$ and $rk(A^{\sim}) = rk(A) = rk(A^{\sim}KGA)$

$= rk\,(AKGA^{\sim})$ it follows that $N(A^{\sim}) = N\,(AKGA^{\sim})$. Hence $N(A^{\sim}) = N\,(AKGA^{\sim}) = N(A^{\sim}KGA) = N(A)$. By Definition 2.2 A is range symmetric in ***m***.

The converse of the above theorem fails. This can be illustrated by the following example.

Example 4.2

$$\text{For } A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} GA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

is non-singular and A is range symmetric in ***m***.

$$(GA)K(GA)^* = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

$$(GA)^*K(GA) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}$$

(GA)K(GA)*≠ (GA)*K(GA). Thus GA is not k-normal and therefore by Theorem 3.3 A is not k-normal in ***m***.

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